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PERIODIC ORBITS OF THE PLANAR ANISOTROPIC MANEV PROBLEM AND OF THE PERTURBED HYDROGEN ATOM PROBLEM

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Abstract. In this paper we use the averaging theory for studying the periodic solutions of the planar anisotropic Manev problem and of two perturbations of the hydrogen atom problem. When a convenient parameter is sufficiently small we prove that for every value $e \in (0,1)$ a unique elliptic periodic solution with eccentricity e of the Kepler problem can be continued to the mentioned three problems.

1. Introduction and statement of the results

The Manev problem is a two-body problem with a potential of the following form

$$V(r) = \frac{a}{r} + \frac{b}{r^2},$$

where r is the distance between the two particles and a, b are arbitrary constants.

For a, $b > 0$ the problem was first considered by Newton. Manev systems were recently reconsidered in a series of studies having as their starting point Diacu's research [12], see also [2],[6],[17],[20]-[23], [25]-[29],[35]-[38].

Abouelmagd et al. [1] study the periodic solutions for the planar anisotropic Kepler problem using averaging theory. Motivated by this work, we study the anisotropic Manev problem with a and b arbitrary using the averaging theory.

The equations of motion are

$$q_i' = \frac{\partial H}{\partial p_i}, \quad p_i' = -\frac{\partial H}{\partial q_i}, \quad \text{for } i = 1, 2, \tag{1}$$

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(2) $H(q_1, q_2, p_1, p_2) = \dfrac{1}{2}(p_1^2 + p_2^2) - \dfrac{1}{\sqrt{(1+\varepsilon)q_1^2 + q_2^2}} - \dfrac{\varepsilon}{(1+\varepsilon)q_1^2 + q_2^2}.$

Here $|\varepsilon| > 0$ denotes a small parameter and the prime denotes the derivative with respect to the time t. The Hamiltonian system (1) is defined for all $(q_1, q_2, p_1, p_2) \in \mathbb{R}^4$ except in the plane $q_1 = q_2 = 0$.

In this paper we study analytically the periodic solutions of the planar anisotropic Manev problem using the averaging theory, and we get the following result.

Theorem 1. *For $\varepsilon \neq 0$ sufficiently small a unique elliptic periodic solution of the Kepler problem for each value of the eccentricity can be continued to the energy level $h(e)$ of the planar anisotropic Manev problem* (1)*, where*

$$h(e) = -\frac{(1-e^2)(1-\sqrt{1-e^2})^2}{4e^4}.$$

The proof of Theorem 1 is given in Section 2.

The hydrogen atom interaction with a circular polarized microwave field is a quiet well known problem in classical mechanics. There are extensive literatures on this problem, for more details, see [3, 4], [7, 8, 9], [13]-[16], [18, 19], [30]-[33] , [39].

In [31] the authors studied the hydrogen atom problem in a rotation frame with the following Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) - xp_y + yp_x - \frac{1}{\sqrt{x^2+y^2}} + \varepsilon x.$$

We consider the same potential but in fixed coordinates, and instead of the perturbation εx with two more general perturbations $\varepsilon(ax + bx^2)$ and $\varepsilon(ax + by^2)$.

For the perturbation $\varepsilon(ax+bx^2)$ the corresponding Hamiltonian system is

$$\begin{aligned} x' &= p_x, \\ y' &= p_y, \\ p_x' &= -\frac{x}{r^3} - \varepsilon(a + 2bx), \\ p_y' &= -\frac{y}{r^3}, \end{aligned} \tag{3}$$

with Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{\sqrt{x^2+y^2}} + \varepsilon(ax + bx^2).$$

For the perturbation $\varepsilon(ax + by^2)$ the following Hamiltonian system becomes

$$
\begin{aligned}
x' &= p_x, \\
y' &= p_y, \\
p_x' &= -\frac{x}{r^3} - \varepsilon a, \\
p_y' &= -\frac{y}{r^3} - \varepsilon 2by,
\end{aligned} \tag{4}
$$

with Hamiltonian

$$
H = \frac{1}{2}(p_x^2 + p_y^2) - \frac{1}{\sqrt{x^2 + y^2}} + \varepsilon(ax + by^2).
$$

We have the following two results.

Theorem 2. *For $h < 0$, $0 < \left|\dfrac{3ah}{4b}\right| < 1$, $4b + 3ah \neq 0$ and $\varepsilon \neq 0$ sufficiently small a unique elliptic periodic solution with eccentricity $e = |3ah/(4b)|$ of the Kepler problem can be continued to the energy level $H = h$ of the hydrogen atom problem* (3)*.*

Theorem 3. *For $h < 0$, $0 < \left|\dfrac{3ah}{b}\right| < 1$, $b - 3ah \neq 0$ and $\varepsilon \neq 0$ sufficiently small a unique elliptic periodic solution with eccentricity $e = |3ah/b|$ of the Kepler problem can be continued to the energy level $H = h$ of the hydrogen atom problem* (4)*.*

Theorems 2 and 3 are proved in sections 3 and 4, respectively.

2. Proof of Theorem 1

We change the equations of motion (1) to the McGehee coordinates (r, θ, v, u) as follows

$$
\begin{aligned}
&(q_1, q_2) = r(\cos\theta, \sin\theta), \\
&r^{-1/2} v = (p_1, p_2) \cdot (\cos\theta, \sin\theta), \\
&r^{-1/2} u = (p_1, p_2) \cdot (-\sin\theta, \cos\theta),
\end{aligned}
$$

for more details on these coordinates see [10, 11, 24]. So in the McGehee coordinates the equations of motion (1) write

$$
\begin{aligned}
r' &= r^{-1/2} v,\\
\theta' &= r^{-3/2} u,\\
v' &= r^{-3/2}\left(u^2 + \frac{1}{2}v^2 + V_1(\theta)\right) + 2r^{-5/2}V_2(\theta),\\
u' &= r^{-3/2}\left(-\frac{1}{2}uv - \frac{dV_1(\theta)}{d\theta}\right) - r^{-5/2}\frac{dV_2(\theta)}{d\theta},
\end{aligned}
\tag{5}
$$

where

$$
V_1(\theta) = -\frac{1}{\sqrt{1+\varepsilon\cos^2\theta}}, \quad V_2(\theta) = -\frac{\varepsilon}{1+\varepsilon\cos^2\theta},
$$

and the energy level $H = h$ becomes

$$
\frac{r}{2}(u^2+v^2) + rV_1(\theta) + V_2(\theta) = r^2 h. \tag{6}
$$

Note that the equations of motion (5) are defined for all $(r,\theta,v,u) \in (0,+\infty)\times\mathbb{S}^1\times\mathbb{R}^2$ with a collision singularity at $r=0$. We can remove this singularity with the change $t \to \tau$ of the independent variable given by $dt/d\tau = r^{5/2}$. The equations of motion (5) in the new time τ become

$$
\begin{aligned}
\dot r &= r^2 v,\\
\dot\theta &= ru,\\
\dot v &= r(u^2 + \tfrac{1}{2}v^2 + V_1(\theta)) + 2V_2(\theta),\\
\dot u &= r\left(-\frac{1}{2}uv - \frac{dV_1(\theta)}{d\theta}\right) - \frac{dV_2(\theta)}{d\theta},
\end{aligned}
\tag{7}
$$

here the dot denotes derivative with respect to τ. Now the equations of motion (7) are defined for all $(r,\theta,v,u) \in [0,+\infty)\times\mathbb{S}^1\times\mathbb{R}^2$.

Since if $\varepsilon = 0$ the Hamiltonian (2) of the anisotropic Manev problem becomes the Hamiltonian of the Kepler problem, we want to detect the periodic solutions of the Kepler problem which can be continued to a fix negative energy level $H = h < 0$ of the anisotropic Manev problem.

Using (6) from the energy level $H = h < 0$ we obtain

$$
r = r(\theta, v, u, h) = f_0 + \varepsilon f_1 + O(\varepsilon^2),
$$

where

$$
\begin{aligned}
f_0 &= \frac{u^2+v^2-2}{2h},\\
f_1 &= \frac{(u^2+v^2-2)(1+\cos 2\theta) - 8h}{4h(u^2+v^2-2)}.
\end{aligned}
$$

Now taking as the new independent variable the variable θ, the equations of motion (7) restricted to the energy level $H = h < 0$ become

$$
\begin{aligned}
\frac{dv}{d\theta} &= \frac{2(u^2+v^2-2)}{2u} + \varepsilon \frac{(u^2+v^2-2)\cos^2\theta - 8h}{2u(u^2+v^2-2)} + O(\varepsilon^2),\\
\frac{du}{d\theta} &= -\frac{v}{2} + \varepsilon \frac{\cos\theta \sin\theta}{u} + O(\varepsilon^2).
\end{aligned} \tag{8}
$$

It is clear that computing periodic solutions of system (8), we are obtaining periodic solutions of system (1) in the energy level $H = h < 0$.

The unperturbed system when $\varepsilon = 0$ is

$$
\frac{dv}{d\theta} = \frac{2u^2+v^2-2}{2u}, \quad \frac{du}{d\theta} = -\frac{v}{2}, \tag{9}
$$

which has the general solution

$$
(v(\theta; e, \theta_0), u(\theta; e, \theta_0)) = \left(\frac{e \sin(\theta - \theta_0)}{\sqrt{1 + e\cos(\theta-\theta_0)}}, \sqrt{1 + e\cos(\theta - \theta_0)} \right). \tag{10}
$$

This is the solution of the planar Kepler problem with eccentricity e and argument of the pericenter θ_0, i.e. θ_0 is the angle which provides the direction of the pericenter. Of course, for $e = 0$ this solution is circular, and for $e \in (0, 1)$ it is elliptic, for more details see [34].

We are interested in knowing what are the periodic solutions of the Kepler problem which can be continued to periodic solutions of the anisotropic Manev problem, i.e. what solutions (10) with eccentricity $e \in [0, 1)$ can be extended.

In order to apply the averaging theory to system (8) we identify our variables with those of averaging theory given in [5], and we obtain

$$
z = \begin{pmatrix} e \\ \theta_0 \end{pmatrix}, \quad \begin{pmatrix} t \\ T \end{pmatrix} = \begin{pmatrix} \theta \\ 2\pi \end{pmatrix},
$$

$$
F_0 = \begin{pmatrix} F_{01} \\ F_{02} \end{pmatrix}, \quad F_1 = \begin{pmatrix} F_{11} \\ F_{12} \end{pmatrix},
$$

$$
x(t; z, 0) = \begin{pmatrix} v(\theta; e, \theta_0) \\ u(\theta; e, \theta_0) \end{pmatrix},
$$

where

$$
\begin{aligned}
F_{01} &= \frac{2u^2+v^2-2}{2u}, & F_{11} &= \frac{-16h + (-4+2u^2+2v^2)\cos^2\theta}{4u(-2+u^2+v^2)},\\
F_{02} &= -\frac{v}{2}, & F_{12} &= \frac{\cos\theta\sin\theta}{u}.
\end{aligned}
$$

The first variational equation of the unperturbed system (9) along the periodic solution (10) with $e \in [0,1)$ is

$$
\begin{pmatrix} y_1' & y_2' \\ y_3' & y_4' \end{pmatrix} = J \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, \tag{11}
$$

where

$$
J = D_x F_0(t, x(t;z,0)) = \left. \begin{pmatrix} \dfrac{\partial F_{01}}{\partial v} & \dfrac{\partial F_{01}}{\partial u} \\ \dfrac{\partial F_{02}}{\partial u} & \dfrac{\partial F_{02}}{\partial v} \end{pmatrix} \right|_{v=v(\theta;e,\theta_0),u=u(\theta;e,\theta_0)},
$$

and consequently

$$
J = \begin{pmatrix} \dfrac{e\sin(\theta-\theta_0)}{1+e\cos(\theta-\theta_0)} & \dfrac{3}{2}+\dfrac{1-e^2}{2(1+e\cos(\theta-\theta_0))^2} \\ -\dfrac{1}{2} & 0 \end{pmatrix}.
$$

The Fundamental matrix $M_z(\theta)$ of system (11) such that $M_z(0)$ is the identity matrix is

$$
M_z(\theta) = \begin{pmatrix} y_1(\theta;e,\theta_0) & y_2(\theta;e,\theta_0) \\ y_3(\theta;e,\theta_0) & y_4(\theta;e,\theta_0) \end{pmatrix},
$$

where

$$
\begin{aligned}
y_1(\theta;e,\theta_0) &= \frac{\sqrt{1+e\cos\theta_0}[4\cos\theta + e[\cos(2\theta-\theta_0)+3\cos\theta_0]\,]}{4[1+e\cos(\theta-\theta_0)]^{3/2}}, \\
y_2(\theta;e,\theta_0) &= \frac{q(\theta)}{4[1+e\cos(\theta-\theta_0)]^{3/2}\sqrt{1+e\cos\theta_0}}, \\
y_3(\theta;e,\theta_0) &= -\frac{\sin\theta\sqrt{1+e\cos\theta_0}}{2\sqrt{1+e\cos(\theta-\theta_0)}}, \\
y_4(\theta;e,\theta_0) &= \frac{e\sin\theta\sin\theta_0 + 2\cos\theta[1+e\cos\theta_0]}{2\sqrt{(1+e\cos\theta_0)(1+e\cos(\theta-\theta_0))}},
\end{aligned}
$$

where

$$
q(\theta) = 8\sin\theta + 8e(2+\cos\theta)\cos\frac{\theta-\theta_0}{2}\sin\frac{\theta}{2} + e^2(\cos\theta + 3\cos(\theta-2\theta_0)\sin\theta).
$$

Now we compute the functions $(G_1, G_2) = M_z^{-1}(\theta)F_1(\theta, x(t; z, 0))$ which appear in the next integral

$$(g_1, g_2) = F(z) = \frac{1}{T}\int_0^T M_z^{-1}(t)F_1(t, x(t; z, 0))dt, \tag{12}$$

and we obtain

$$\begin{aligned} G_1 &= \frac{p_1(\theta)}{8(1+e\cos\theta_0)^{3/2}(e^2-1)(1+e\cos(\theta-\theta_0))}, \\ G_2 &= \frac{p_2(\theta)}{16(e^2-1)(1+e\cos(\theta-\theta_0))\sqrt{1+e\cos\theta_0}}, \end{aligned}$$

where

$$\begin{aligned} p_1(\theta) =& [e^2-1-16h+(e^2-1)\cos 2\theta - 16eh\cos(\theta-\theta_0)]\cdot \\ & [2\cos\theta(1+e\cos\theta_0)+e\sin\theta\sin\theta_0]\cdot(1+e\cos(\theta-\theta_0)) - \\ & \cos\theta\sin\theta(e^2-1)[16(1+e\cos\theta_0)\sin\theta + \\ & e((e+4\cos\theta_0+3e\cos 2\theta_0)\sin 2\theta + \\ & 16(2+\cos\theta)\sin^2\frac{\theta}{2}\sin\theta_0)+6e^2\sin^2\theta\sin 2\theta_0], \\ p_2(\theta) =& \sin\theta\{10(e^2-1)-16h(e^2+2)+10(e^2-1)\cos 2\theta + \\ & e[2(5e^2-32h-5)\cos(\theta-\theta_0)-16eh\cos(2\theta-2\theta_0) + \\ & (e^2-1)(3\cos(3\theta-\theta_0)+7\cos(\theta+\theta_0))]. \end{aligned}$$

Computing the integrals (12), we obtain

$$\begin{aligned} g_1(e,\theta_0) &= -\frac{p_3(e,\theta_0)}{16\pi e^3(1+e\cos\theta_0)^{3/2}(1-e^2)^{3/2}}, \\ g_2(e,\theta_0) &= -\frac{p_4(e,\theta_0)}{8e^3(1-e^2)\sqrt{1+e\cos\theta_0}}, \end{aligned}$$

where

$$\begin{aligned} p_3 =& -8e^4\sqrt{1-e^2}h\pi[4\cos\theta_0+e(\cos 2\theta_0+3)] + \\ & \pi(e^2-1)[e^3(\sqrt{1-e^2}+e^2-1)+4e^2(\sqrt{1-e^2}+e^2-1)\cos\theta_0 - \\ & 6e(2-2\sqrt{1-e^2}+e^2(\sqrt{1-e^2}-2))\cos 2\theta_0-(4-4\sqrt{1-e^2}+ \\ & e^2(e^2+3\sqrt{1-e^2}-5))(4\cos 3\theta_0+e\cos 4\theta_0)], \\ p_4 =& e^2(1-\sqrt{1-e^2}+e^2(\sqrt{1-e^2}-8h-1))\sin\theta_0 + \\ & (e^2-1)(4-4\sqrt{1-e^2}+e^2(\sqrt{1-e^2}-3))\sin 3\theta_0. \end{aligned}$$

Let

$$m(e,\theta_0)=\det\begin{pmatrix}\dfrac{\partial g_1(e,\theta_0)}{\partial e} & \dfrac{\partial g_1(e,\theta_0)}{\partial \theta_0}\\ \dfrac{\partial g_2(e,\theta_0)}{\partial e} & \dfrac{\partial g_2(e,\theta_0)}{\partial \theta_0}\end{pmatrix}.$$

Now we have to study the solutions (e,θ_0) of system $g_1(e,\theta_0)=0$, $g_2(e,\theta_0)=0$ with $e\in(0,1)$. Note that if $e=0$ then $g_1(e,\theta_0)\equiv 0$ and $g_2(e,\theta_0)\equiv 0$, and that in this case the averaging theory does not provide information on the periodic solutions of system (8).

We first solve the equation $g_2(e,\theta_0)$ with respect to θ_0, and we obtain ten solutions as follows:

$$\begin{array}{ll}
\theta_0^1=0, & \theta_0^2=\pi,\\
\theta_0^3=-\arccos(-F(e)), & \theta_0^4=-\arccos(-F(e))+\pi,\\
\theta_0^5=\arccos(-F(e)), & \theta_0^6=\arccos(-F(e))+\pi,\\
\theta_0^7=-\arccos F(e), & \theta_0^8=-\arccos F(e)+\pi,\\
\theta_0^9=\arccos F(e), & \theta_0^{10}=\arccos F(e)+\pi,
\end{array}$$

where

$$F(e)=\frac{\sqrt{2[(1+\sqrt{1-e^2})(1-16h)-e^2(1+\sqrt{1-e^2}-4h(3+\sqrt{1-e^2}))]}}{2e\sqrt{1-e^2}}.$$

Of course, $g_1(\theta_0,e)=0$ if and only if $p_3(\theta_0,e)=0$.

Case 1: $\theta_0=\theta_0^1=0$. Substituting θ_0^1 into $p_3(e,\theta_0)$, we obtain

$$\bar{p}_3(e)=16(e-1)^2(e+1)^3-8(1+e)\sqrt{1-e^2}(2-3e^2+e^4+4e^4h).$$

We solve $\bar{p}_3(e)=0$ with respect to h, we have

$$h(e)=-\frac{(1-e^2)(1-\sqrt{1-e^2})^2}{4e^4}. \tag{13}$$

By simple computation, we have

$$h'(e)=\frac{4-4\sqrt{1-e^2}+e^2(\sqrt{1-e^2}-3)}{2e^5}. \tag{14}$$

Let $f(e)=4-4\sqrt{1-e^2}+e^2(\sqrt{1-e^2}-3)$, we have

$$f'(e)=\frac{3e(1-\sqrt{1-e^2})^2}{\sqrt{1-e^2}}. \tag{15}$$

Note that $f(0)=0$, then by (15), we deduce that $f(e)>0$ for all $0<e<1$. Then from (14), we have $h'(e)>0$ for all $0<e<1$, so $h(e)$ is a strictly monotone function on $(0,1)$. Since $h(1)=0$, $h(e)<0$ in the interval $(0,1)$.

Next we can compute $m(e,\theta_0^1)$ and we obtain

$$-\frac{4e^2(5-3\sqrt{1-e^2})+e^4(\sqrt{1-e^2}-5)+16(\sqrt{1-e^2}-1)}{2e^7(1+e)(\sqrt{1-e^2})}. \tag{16}$$

Let $g(e)=4e^2(5-3\sqrt{1-e^2})+e^4(\sqrt{1-e^2}-5)+16(\sqrt{1-e^2}-1)$, we have

$$g'(e)=-\frac{5e(8-8e^2+e^4-8\sqrt{1-e^2}+4e^2\sqrt{1-e^2})}{\sqrt{1-e^2}}. \tag{17}$$

Let $y(e)=8-8e^2+e^4-8\sqrt{1-e^2}+4e^2\sqrt{1-e^2}$, we have

$$y'(e)=\frac{4e(1-\sqrt{1-e})^3}{\sqrt{1-e^2}}>0, \tag{18}$$

for $e\in(0,1)$. Note that $y(0)=0$, and from (17) and (18) we can deduce that $y(e)>0$ and $g'(e)<0$ in the interval $(0,1)$. Then, since $g(0)=0$ we have $g(e)<0$ in the interval $(0,1)$. So, from (16), we have $m(e,0)>0$ for $e\in(0,1)$. Therefore, from the averaging theory (see [5]) for every value of the eccentricity $e\in(0,1)$ and for ε sufficiently small the periodic solution $(v(\theta;e,0),u(\theta;e,0))$ (given in (10)) of the Kepler problem such that $(v(0;e,0),u(0;e,0))=(0,\sqrt{1+e})$, can be continued to the energy level $h=h(e)<0$ of the anisotropic Manev problem (1).

Case 2: $\theta_0=\theta_0^2=\pi$. Working as in the previous case we obtain that the periodic solution $(v(\theta;e,\pi),u(\theta;e,\pi))$ which can be continued from the Kepler problem to the anisotropic Manev problem is the same that in Case 1, because $(v(0;e,\pi),u(0;e,\pi))=(0,\sqrt{1+e})$, i.e. both periodic solutions have the same initial conditions.

Case 3: $\theta_0=\theta_0^3$. Now solving $\bar{p}_3(e)=p_3(e,\theta_0^3)=0$ with respect to h, we obtain the two solutions

$$h_\pm(e)=s_\pm(e)\frac{1-e^2}{4e^6},$$

where $s_\pm(e)$ is given by

$$-e^2[4+e^4-4\sqrt{1-e^2}+e^2(3\sqrt{1-e^2}-5)]\pm$$
$$\sqrt{-[e^6+e^2(48-32\sqrt{1-e^2})+6e^4(\sqrt{1-e^2}-3)+32(\sqrt{1-e^2}-1)]}.$$

Since substituting $\theta_0=\theta_0^3$ and $h=h_\pm(e)$ into $m(e,\theta_0)$, we found $m(e,\theta_0^3)=0$ for all $e\in(0,1)$, the averaging theory does not provide information if the corresponding periodic solution of the Kepler problem $(v(\theta;e,\theta_0^3),u(\theta;e,\theta_0^3))$ can be continued or not to the anisotropic Manev problem (1).

For the cases θ_0^j for $j = 4, \ldots, 10$, we also found that $m(e, \theta_0^j) = 0$ for all $e \in (0, 1)$, so again the averaging theory does not provide information about the possible continuation of this periodic solution to the anisotropic Manev problem (1).

In conclusion the averaging theory provides a unique periodic solution of the Kepler problem for each value of the eccentricity $e \in (0, 1)$ which can be continued to the anisotropic Kepler problem (1). This completes the proof of Theorem 1.

3. Proof of Theorem 2

In the McGehee coordinates the equations of motion (3) become

$$
\begin{aligned}
r' &= r^{-1/2}v,\\
\theta' &= r^{-3/2}u,\\
v' &= r^{-3/2}\left(u^2 + \frac{1}{2}v^2 - 1\right) + \varepsilon r^{\frac{1}{2}}(a\cos\theta + 2br\cos^2\theta),\\
u' &= r^{-3/2}(-\frac{1}{2}uv) + \varepsilon r^{\frac{1}{2}}(a\sin\theta + 2br\cos\theta\sin\theta),
\end{aligned} \tag{19}
$$

and the energy level $H = h$ writes

$$
\frac{1}{2}(u^2 + v^2) - 1 + \varepsilon(ar^2\cos\theta + br^3\cos^2\theta) = rh. \tag{20}
$$

Then doing the change of time $t \to \tau$ given by $dt/d\tau = r^{3/2}$, the equations of motion (19) in the new time τ are

$$
\begin{aligned}
\dot{r} &= rv,\\
\dot{\theta} &= u,\\
\dot{v} &= u^2 + \tfrac{1}{2}v^2 - 1 + \varepsilon(ar^2\cos\theta + 2br^3\cos^2\theta),\\
\dot{u} &= -\frac{1}{2}uv - \varepsilon(ar^2\sin\theta + 2br^3\cos\theta\sin\theta).
\end{aligned} \tag{21}
$$

Note that these equations of motion are defined for all $(r, \theta, v, u) \in [0, +\infty) \times \mathbb{S}^1 \times \mathbb{R}^2$.

Computing r from the energy level $H = h < 0$, where H is given in (20) we have

$$
r = r(\theta, v, u, h) = f_0 + \varepsilon f_1 + O(\varepsilon^2),
$$

where

$$
\begin{aligned}
f_0 =& \frac{u^2 + v^2 - 2}{2h},\\
f_1 =& \frac{(u^2 + v^2 - 2)^2\cos\theta[2ah + b(u^2 + v^2 - 2)\cos\theta]}{8h^4}.
\end{aligned}
$$

Then the equations of motion (21) in the energy level $H = h < 0$ become

(22)

$$
\begin{aligned}
\frac{dv}{d\theta} &= \frac{2u^2+v^2-2}{2u} - \varepsilon \frac{(u^2+v^2-2)^2[ah+b(u^2+v^2-2)\cos\theta]\cos\theta}{4h^3u} + O(\varepsilon^2),\\
\frac{du}{d\theta} &= -\frac{v}{2} + \varepsilon \frac{(u^2+v^2-2)^2[ah+b(u^2+v^2-2)\cos\theta]\sin\theta}{4h^3u} + O(\varepsilon^2).
\end{aligned}
$$

Since hydrogen atom problem (3) for $\varepsilon = 0$ is the Kepler problem (9), as before we take h negative because we want to continue the periodic solutions of the Kepler problem to periodic solutions of the hydrogen atom problem (3).

As in the previous problem we shall use the averaging theory for studying the periodic solutions of system (9) which can be continued to system (22).

We use the notation of [5] for applying the averaging theory, thus we have from system (22) that

$$
\begin{aligned}
F_{01} &= \frac{2u^2+v^2-2}{2u}, \quad & F_{11} &= \frac{-16h+(-4+2u^2+2v^2)\cos^2\theta}{4u(-2+u^2+v^2)},\\
F_{02} &= -\frac{v}{2}, & F_{12} &= \frac{\cos\theta\sin\theta}{u}.
\end{aligned}
$$

Now we compute the functions $(G_1, G_2) = M_z^{-1}(\theta) F_1(\theta, x(t; z, 0))$ which appear in the integral (12) and we obtain

$$
\begin{aligned}
G_1 &= \frac{p_1(\theta)}{8h^3(1+e\cos(\theta-\theta_0))^4(1+e\cos\theta_0)^{\frac{3}{2}}},\\
G_2 &= \frac{p_2(\theta)}{8h^3(1+e\cos(\theta-\theta_0))^4\sqrt{1+e\cos\theta_0}},
\end{aligned}
$$

where

$$
\begin{aligned}
p_1(\theta) =& (e^2-1)^2(b(e^2-1)\cos\theta + ah(1+e\cos(\theta-\theta_0))\cdot\\
& (\cos 2\theta - 3 - 2e\cos\theta_0(2+\cos\theta(1-\cos\theta+e\cos\theta_0))+\\
& e(\cos\theta - 4 - 3e\cos\theta_0)\sin\theta\sin\theta_0),\\
p_2(\theta) =& (e^2-1)^2(b(e^2-1)\cos\theta + ah(1+e\cos(\theta-\theta_0))(\cos\theta+e\cos\theta_0)\sin\theta,
\end{aligned}
$$

Computing the integrals (12) for the hydrogen atom problem (22) we get

$$g_1(e,\theta_0) = \frac{\sqrt{1-e^2}(7be^2+12ah+4e(4b+3ah)\cos\theta_0+9be^2\cos 2\theta_0)}{32h^3(1+e\cos\theta_0)^{\frac{3}{2}}},$$

$$g_2(e,\theta_0) = \frac{be\sin\theta_0\sqrt{1-e^2}}{16h^3\sqrt{1+e\cos\theta_0}}.$$

Now we have to study the solutions (e,θ_0) of the system $g_1(e,\theta_0) = 0$, $g_2(e,\theta_0) = 0$ with $e \in (0,1)$, as before e cannot be zero.

First we solve the equation $g_2(e,\theta_0)$ with respect to θ_0, and we obtain two solutions

$$\theta_0^1 = 0, \qquad \theta_0^2 = \pi.$$

Case 1: $\theta_0 = \theta_0^1 = 0$. Substituting θ_0^1 into $g_1(e,\theta_0)$, we have

$$\bar{g}_1(e) = \frac{\sqrt{1-e^2}[16be^2+12ah+4e(4b+3ah)]}{32h^3(1+e)^{\frac{3}{2}}}.$$

We solve $\bar{g}_1(e) = 0$ with respect to e, we have

$$e = 1, \qquad e = -\frac{3ah}{4b}.$$

The solution $e = 1$ should be eliminated since we study which periodic solutions of the Kepler problem can be continued to periodic solutions of the perturbed hydrogen atom problem, and for$e = 1$ the solutions of the Kepler problem are parabolas.

By direct computation we have $m\left(-\frac{3ah}{4b}, 0\right) = -\frac{3ab}{512h^5}\left|\frac{4b+3ah}{b}\right|$. Therefore, from the averaging theory we obtain that if the parameters a and b of the perturbation satisfy $0 < -3ah/4b < 1$, $m(-3ah/4b, 0) \neq 0$ and $\varepsilon \neq 0$ is sufficiently small, then the periodic solution

$$(v(\theta; -3ab/(512h^5), 0), u(\theta; -3ab/(512h^5), 0))$$

of the Kepler problem can be continued to the energy level $h < 0$ of the perturbed hydrogen atom problem (3).

Case 2: $\theta_0 = \theta_0^2 = \pi$. Substituting θ_0^2 into $g_1(e,\theta_0)$, we obtain

$$\bar{g}_1(e) = \frac{\sqrt{1-e^2}[16be^2+12ah-4e(4b+3ah)]}{32h^3(1-e)^{\frac{3}{2}}}.$$

We solve $\bar{g}_1(e) = 0$ with respect to e and we get

$$e = -1, \qquad e = \frac{3ah}{4b}.$$

Since $m\left(\frac{3ah}{4b},\pi\right) = \frac{3ab}{512h^5}\left|\frac{4b+3ah}{b}\right|$, as before from the averaging theory we obtain that if the parameters a and b of the perturbation satisfy $3ah/4b \in (0,1)$, $m(3ah/(4b),\pi) \neq 0$ and for $\varepsilon \neq 0$ is sufficiently small, then the periodic solution of the Kepler problem

$$(v(\theta; 3ab/(4b),\pi), u(\theta; 3ab/(4b),\pi))$$

can be continued to the energy level $h < 0$ of the perturbed hydrogen atom problem (3). This completes the proof of Theorem 2.

4. Proof of Theorem 3

The proof of Theorem 3 is similar to the proof of Theorem 2.

We also take McGehee coordinates for the equations of motion (4) and doing the change of time $t \to \tau$ given by $dt/d\tau = r^{3/2}$, the equations of motion (4) in the new time τ become

$$\begin{aligned} \dot r &= rv, \\ \dot\theta &= u, \\ \dot v &= u^2 + \tfrac{1}{2}v^2 - 1 + \varepsilon(ar^2\cos\theta + 2br^3\sin^2\theta), \\ \dot u &= -\frac{1}{2}uv - \varepsilon(ar^2\sin\theta - 2br^3\cos\theta\sin\theta). \end{aligned} \tag{23}$$

and the energy level $H = h$ becomes

$$\frac{1}{2}(u^2+v^2) - 1 + \varepsilon(ar^2\cos\theta + br^3\sin^2\theta) = rh. \tag{24}$$

From this energy relation $H = h < 0$ we can compute

$$r = r(\theta, v, u, h) = f_0 + \varepsilon f_1 + O(\varepsilon^2),$$

where

$$\begin{aligned} f_0 &= \frac{u^2+v^2-2}{2h}, \\ f_1 &= \frac{(u^2+v^2-2)^2[2ah\cos\theta + b(u^2+v^2-2)\sin^2\theta]}{8h^4}. \end{aligned}$$

Then the equations of motion (23) restricted to the energy level $H = h < 0$ are

$$\frac{dv}{d\theta} = \frac{2u^2+v^2-2}{2u} - \varepsilon\frac{(u^2+v^2-2)^2[ah\cos\theta + b(u^2+v^2-2)\sin^2\theta]}{4h^3u} + O(\varepsilon^2),$$

$$\frac{du}{d\theta} = -\frac{v}{2} + \varepsilon\frac{(u^2+v^2-2)^2[ah - b(u^2+v^2-2)\cos\theta]\sin\theta}{4h^3u} + O(\varepsilon^2).$$

The unperturbed system when $\varepsilon = 0$ is again the Kepler problem given in (9) having the general solution provided in (10).

Using the notation of [5] we have for system (23) that

$$\begin{aligned} F_{01} &= \frac{2u^2+v^2-2}{2u}, \qquad F_{11} = -\frac{v}{2}, \\ F_{02} &= -\frac{(u^2+v^2-2)^2[ah\cos\theta + b(u^2+v^2-2)\sin^2\theta]}{4h^3u}, \\ F_{12} &= \frac{(u^2+v^2-2)^2[ah - b(u^2+v^2-2)\cos\theta]\sin\theta}{4h^3u}. \end{aligned}$$

Now we compute the functions $(G_1, G_2) = M_z^{-1}(\theta)F_1(\theta, x(t; z, 0))$ which appear in the integral (12) and we obtain

$$\begin{aligned} G_1 &= \frac{p_1(\theta)}{32h^3(1+e\cos(\theta-\theta_0))^5(1+e\cos\theta_0)^{\frac{3}{2}}}, \\ G_2 &= \frac{p_2(\theta)}{16h^3(1+e\cos(\theta-\theta_0))^4\sqrt{1+e\cos\theta_0}}, \end{aligned}$$

where

$$\begin{aligned} p_1(\theta) =& (1-e^2)^2(-4(ah\cos\theta(1+e\cos(\theta-\theta_0))^2 + b(e^2-1)\sin^2\theta)\cdot \\ & (2\cos\theta(1+e\cos\theta_0) + e\sin\theta\sin\theta_0) - ((b(1-e^2)\cos\theta + \\ & ah(1+e\cos(\theta-\theta_0)))\sin\theta(16(1+e\cos\theta_0)\sin\theta + \\ & e((e+4\cos\theta_0+3e\cos 2\theta_0)\sin 2\theta + 16(2+\cos\theta)\sin^2\frac{\theta}{2}\sin\theta_0) + \\ & 6e^2\sin^2\theta\sin 2\theta_0))), \\ p_2(\theta) =& (e^2-1)^2(3b - 3be^2 + a(e^2+2)h\cos\theta - b(e^2-1)\cos 2\theta + \\ & ae^2h\cos(\theta-2\theta_0) + 3be\cos(\theta-\theta_0) - 3be^3\cos(\theta-\theta_0) + \\ & aeh\cos(2\theta-\theta_0) + 3aeh\cos\theta_0 + be\cos(\theta+\theta_0) - \\ & be^3\cos(\theta+\theta_0))\sin\theta, \end{aligned}$$

Computing the integrals (12) for our system (23) we get

$$\begin{aligned} g_1 &= \frac{\sqrt{1-e^2}[2e(b-3ah)\cos\theta_0 + 3be^2\cos 2\theta_0 - be^2 - 6ah]}{16h^3(1+e\cos\theta_0)^{3/2}}, \\ g_2 &= -\frac{be\sin\theta_0\sqrt{1-e^2}}{4h^3\sqrt{1+e\cos\theta_0}}. \end{aligned}$$

Now we have to study the solutions (e, θ_0) of the system $g_1(e, \theta_0) = 0$, $g_2(e, \theta_0) = 0$ with $e \in (0, 1)$. We first solve the equation $g_2(e, \theta_0)$

with respect to θ_0, and we obtain the two solutions

$$\theta_0^1 = 0, \qquad \theta_0^2 = \pi.$$

Case 1: $\theta_0 = \theta_0^1 = 0$. Substituting θ_0^1 into $g_1(e, \theta_0)$, we get

$$\bar{g}_1(e) = \frac{\sqrt{1-e^2}(2be^2 + 2e(b-3ah) - 6ah)}{16h^3(1+e)^{\frac{3}{2}}}.$$

We solve $\bar{g}_1(e) = 0$ with respect to e, we have

$$e = 1, \qquad e = \frac{3ah}{b}.$$

By direct computations we have $m\left(\frac{3ah}{b}, 0\right) = -\frac{3ab}{32h^5}\left|\frac{b-3ah}{b}\right|$. Therefore for every $e = 3ah/b \in (0,1)$, $m(3ah/b, 0) \neq 0$ and for $\varepsilon \neq 0$ sufficiently small the periodic solution of the Kepler problem

$$(v(\theta; 3ab/b, 0), u(\theta; 3ab/b, 0))$$

can be continued to the energy level $h < 0$ of the perturbed hydrogen atom problem (4).

Case 2: $\theta_0 = \theta_0^2 = \pi$. Substituting θ_0^2 into $g_1(e, \theta_0)$, we obtain

$$\bar{g}_1(e) = \frac{\sqrt{1-e^2}(2be^2 - 2e(b-3ah) - 6ah)}{16h^3(1-e)^{\frac{3}{2}}}.$$

We solve $\bar{g}_1(e) = 0$ with respect to e, we have

$$e = -1, \qquad e = -\frac{3ah}{b}.$$

Since $m\left(-\frac{3ah}{b}, \pi\right) = \frac{3ab}{32h^5}\left|\frac{b-3ah}{b}\right|$, for every value of the eccentricity $e = -3ah/b \in (0,1)$, $m(-3ah/b, \pi) \neq 0$ and for $\varepsilon \neq 0$ sufficiently small the periodic solution of the Kepler problem

$$(v(\theta; 3ab/b, \pi), u(\theta; 3ab/b, \pi))$$

can be continued to the energy level $h < 0$ of the perturbed hydrogen atom problem (4). This completes the proof of Theorem 3.

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