This is a preprint of: "Simultaneity of centers in Z_q-equivariant systems", Jaume Giné, Jaume Llibre, Clàudia Valls, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, vol. 474, 20170811, 2018.
DOI: [10.1098/rspa.2017.0811]

SIMULTANEITY OF CENTERS IN $\mathbb{Z}_q$-EQUIVARIANT SYSTEMS

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ABSTRACT. We study the simultaneous existence of centers for two families of planar $\mathbb{Z}_q$-equivariant systems. First we give a short review about $\mathbb{Z}_q$-equivariant systems. Next we present necessary and sufficient conditions for simultaneous existence of centers for a $\mathbb{Z}_2$-equivariant cubic system and for a $\mathbb{Z}_2$-equivariant quintic system.

1. INTRODUCTION

The second part of Hilbert's 16th problem deals with the existence of a uniform upper bound on the number of limit cycles $H(n)$ of a planar polynomial differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

in function of n where $n = \max(\deg P, \deg Q)$, see for instance [17, 26]. It is well-known that linear polynomial systems have no limit cycles, then $H(1) = 0$ and for $n \geq 2$ the problem remains open. Only lower bounds are known and our objective is to improve these lower bounds. An efficient method is to perturb symmetric Hamiltonian systems. The symmetric Hamiltonian systems are Hamiltonian systems with certain symmetries that allow the existence of a great number of centers whose perturbations give a large number of limit cycles.

A generalization of these symmetric Hamiltonian systems are the $\mathbb{Z}_q$-equivariant systems defined below using a cyclic group $\mathbb{Z}_q$. In this paper we give a survey of the results obtained respect to the local and global bifurcations of limit cycles previously analyzing the center problem for such systems. The $\mathbb{Z}_q$-symmetry allows that analyzing the limit cycles which can bifurcate from one center we are analyzing simultaneously this problem for q centers. Some new particular cases are studied in detail.

Let G be a compact Lie group of transformations acting on $\mathbb{R}^n$. A function $H : \mathbb{R}^n \to \mathbb{R}$ is a G-*equivariant function* if for all $g \in G$ and for all x we

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2010 *Mathematics Subject Classification.* 34C05, 34A34, 34C14.

Key words and phrases. Moon-Rand system, center–focus problem, center manifolds, first integrals.

have that $H(gx) = gH(x)$. Given a G-equivariant map ϕ, the vector field $dx/dt = \phi(x)$ is called a *G-equivariant vector field.*

Given an integer q a group $\mathbb{Z}_q$ is a $\mathbb{Z}_q$ *cyclic group* if it is generated by a planar counter-clockwise rotation of angle $2\pi/q$ around a point p that we can translate to the origin. For instance a $\mathbb{Z}_2$-equivariant vector field in the plane is a vector field whose phase portrait is unchanged after a rotation of angle π around the origin. Its characterization is very straightforward in complex variables. Doing the change of variables $z = x + iy$ and $\bar{z} = x - iy$, system (1) takes the form

$$\dot{z} = F(z, \bar{z}), \qquad \dot{\bar{z}} = \bar{F}(z, \bar{z}). \tag{2}$$

where $F(z, \bar{z}) = P(x, y) + iQ(x, y)$ with $x = (z + \bar{z})/2$ and $y = (z - \bar{z})/(2i)$.

The following result characterizes a $\mathbb{Z}_q$-equivariant complex vector field and it is proved in [33], see also [26, 31].

Theorem 1. *The vector field* (2) *is a $\mathbb{Z}_q$-equivariant complex vector field if and only if the complex function $F(z, \bar{z})$ has the form*

$$F(z, \bar{z}) = \sum_{\ell=1} g_\ell(|z|^2)\bar{z}^{\ell q-1} + \sum_{\ell=0} h_\ell(|z|^2)z^{\ell q+1},$$

where g_ℓ and h_ℓ are polynomials in $|z|^2$ with complex coefficients. Moreover system (2) *is Hamiltonian if and only if $\partial F/\partial z + \partial \bar{F}/\partial \bar{z} = 0$.*

For obtaining some particular $\mathbb{Z}_q$-equivariant complex vector field it is necessary to fix the degree of the system n and the value of q which appears in statement of Theorem 1. As q increases more restrictions have the system because it is invariant under a rotation of smaller angle. In [26] are given the $\mathbb{Z}_5$-equivariant complex vector field of degree 5. However in this classification there are some mistakes. For instance for $q = 4$ appears a term $A_5\bar{z}^5$ that cannot appear attending to the form of $F(z, \bar{z})$. The mistake is also repeated in [31].

Several authors have studied the $\mathbb{Z}_q$-equivariant systems in order to classify their centers and to compute the bifurcations of limit cycles under convenient perturbations. For instance in [26] a method was given to control the parameters in order to obtain as much as possible limit cycles. The method was applied to $\mathbb{Z}_q$-equivariant perturbed polynomial Hamiltonian systems of degree $n = 5$ for $q = 2$ to 6 and with the help of numerical analysis it was proved that at least 24 limit cycles can bifurcate for such systems, [4, 26]. In fact the cases $q = 2$ and $q = 3$ were studied separately in [5, 27] where at least 15 and 23 limit cycles were founded respectively.

In [6, 11, 23, 27, 34, 43, 47] the $\mathbb{Z}_2$-equivariant systems have been studied. In particular the centers, isochronous centers and local critical periods of $\mathbb{Z}_2$-equivariant cubic systems have been studied in [6, 11, 27, 34, 43]. In fact the study of bifurcation of limit cycles of the $\mathbb{Z}_2$-equivariant cubic systems has given rise to the highest lower bound of limit cycles for the cubic systems, see [35] and references therein. The center problem and the bifurcation of limit cycles for $\mathbb{Z}_2$-equivariant quintic systems have been studied in [23, 34, 43]. Finally the study of $\mathbb{Z}_2$-equivariant Liénard systems has been started in [47].

The limit cycles of the $\mathbb{Z}_3$-equivariant near-Hamiltonian systems have been studied in [38, 37]. In [12, 13, 41, 44, 45, 48] the $\mathbb{Z}_4$-equivariant systems have been studied were the most analyzed are the $\mathbb{Z}_4$-equivariant quintic systems. However the $\mathbb{Z}_4$-equivariant cubic systems were treated in [12], and the small limit cycles of the $\mathbb{Z}_4$-equivariant near-Hamiltonian systems have been studied in [45]. Finally the study of $\mathbb{Z}_6$-equivariant quintic systems has been started in [2, 28], and the bifurcations of limit cycles in a $\mathbb{Z}_8$-equivariant planar vector field of degree 7 has been considered in [32].

Recently the limit cycles for $\mathbb{Z}_{2n}$-equivariant systems without infinite singular point have been discussed in [24]. In several papers are classified the phase portraits of some $\mathbb{Z}_q$-equivariant systems but we do not describe here these works because the phase portraits are not the objective of the present paper.

Other interesting systems are the $\mathbb{Z}_2$-symmetric systems.

A system (1) is $\mathbb{Z}_2$-*symmetric* (with respect to the origin) if it is invariant under the involution $(x, y) \to (-x, -y)$, that is, $P(-x, -y) = -P(x, y)$ and $Q(-x, -y) = -Q(x, y)$.

Recently the center problem have been solved for such $\mathbb{Z}_2$-symmetric systems, see [1]. The bifurcation of limit cycles for the $\mathbb{Z}_2$-symmetric cubic systems have been studied in [30] and for the $\mathbb{Z}_2$-symmetric Liénard systems in [46]. The first natural question that arises is what is the relation between $\mathbb{Z}_q$-equivariant systems and $\mathbb{Z}_2$-symmetric systems. Our first result is:

Proposition 2. *Any $\mathbb{Z}_2$-symmetric is a $\mathbb{Z}_2$-equivariant system.*

However there are $\mathbb{Z}_q$-equivariant systems with $q \neq 2$ that are not $\mathbb{Z}_2$-symmetric system. For instance system

$$\dot{x} = -y(A_0 - 4A_3x^3 + 4A_3xy^2), \quad \dot{y} = A_0x + A_3x^4 - 6A_3x^2y^2 + A_3y^4.$$

is a $\mathbb{Z}_4$-equivariant system but is not a $\mathbb{Z}_2$-symmetric system.

An important application of the $\mathbb{Z}_q$-equivariant systems is to obtain lower bounds for the Hilbert numbers $H(n)$. It was known that $H(n) \geq k_1n^2$ for some constant k_1, see for instance [3, 21, 39]. In [8] it was shown that $H(n)$ grows at least as $k_2n^2 \log n$ from perturbing some $\mathbb{Z}_q$-equivariant systems. Some small improvements to this bound have been given in [29, 19, 20, 49] improving the values of the constant k_2. In fact in [36] it was conjectured that $H(n)$ is $O(n^3)$ as $n \to \infty$.

In this paper we study the number of simultaneous centers in planar differential systems. Up to know the simultaneity of centers was investigated only for very few particular families. For instance the existence of two simultaneous centers was studied in [22, 25] for quadratic systems, and in [9, 7] for some particular cubic systems. The simultaneity of centers in planar differential systems is an important goal because perturbations of such systems gives a great number of bifurcations of limit cycles, see [3, 15, 40].

Recently in [34, 43] have been studied the $\mathbb{Z}_2$-equivariant cubic systems of the form

$$\dot{x} = X_1(x, y) + X_3(x, y), \qquad \dot{y} = Y_1(x, y) + Y_3(x, y), \tag{3}$$

where X_i and Y_i are homogeneous polynomials of degree i having to weak centers or focus at the points $(-1, 0)$ and $(1, 0)$. In this work we give necessary and sufficient conditions to have a center and an isochronous center at these singular points. Note that when we have a center at one of the singular points then automatically we have a center at the other one because the system is $\mathbb{Z}_2$-symmetric. Moreover in [43] it was given the necessary and sufficient conditions in order that such centers be isochronous centers. In the present paper we will study the conditions in order to have more centers in system (3). More specifically we will give the condition to have a center at the origin and two more centers in the arbitrary points (a, b) and $(-a, -b)$. We will see that these three more centers appear simultaneously.

In [43] it is also characterized the existence of two centers and isochronous centers at the points $(-1, 0)$ and $(1, 0)$ for the $\mathbb{Z}_2$-equivariant quintic system of the form

$$\dot{x} = X_1(x, y) + X_5(x, y), \qquad \dot{y} = Y_1(x, y) + Y_5(x, y), \tag{4}$$

where X_i and Y_i are homogeneous polynomials of degree i. However in [43] only a particular case is studied. because the general case is computationally unfeasible. The particular case studied has $x = 0$ as an invariant straight line which implies that the origin can not be a center. In the present paper we study the existence of more centers for such systems studying also the simultaneity in its appearance. As before we will give the condition to have a center at the origin and two more centers in the arbitrary points (a, b) and $(-a, -b)$.

2. Definitions and preliminary results

In this section we introduce some definitions and preliminary results which will be used along the work.

By a linear change of coordinates and a time rescaling system (1) with a weak focus can be written into the form

$$\dot{x} = -y + X(x, y) = P(x, y), \qquad \dot{y} = x + Y(x, y) = Q(x, y), \tag{5}$$

where X and Y are polynomials without constant and linear terms. We denote by $\mathcal{X} = P\partial/\partial x + Q\partial/\partial y$ the corresponding vector field associate to system (5). A first integral of system (5) is a nonconstant function H defined in a neighborhood of the origin which is constant along the trajectories, i.e.,

$$\dot{H} = \mathcal{X}H = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0.$$

A function R not identically zero is an *integrating factor* of system (5) if

$$\frac{\partial(RP)}{\partial x} + \frac{\partial(RQ)}{\partial y} = 0.$$

A *first integral* H associated to this integrating factor R is given by

$$H(x,y) = \int RPdy + f(x),$$

where this H must satisfy $\partial H/\partial x = -RQ$. A function V not identically zero is an *inverse integrating factor* of system (5) if it satisfies

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V. \tag{6}$$

This function V defines the integrating factor $R = 1/V$ where it does not vanish. The next results characterize when system (5) has a center at the origin, see for instance [17].

Theorem 3. *System* (5) *has a center at the origin if and only if there exists a local analytic first integral of the form* $H(x,y) = x^2 + y^2 + F(x,y)$ *defined in a neighborhood of the origin, where* F *starts with terms of order higher than* 2*.*

The next theorem is known as Reeb's criterium for the classical center problem, see [16, 42].

Theorem 4. *Let* p *be a focus or a center of system* (5)*. Then* p *is a center if and only if there is a nonzero analytic integrating factor* V *defined in a neighborhood* p *with* $V(p) \neq 0$*.*

For system (5) it is possible to construct a formal first integral of the form $H(x,y) = x^2 + y^2 + \cdots$, such that $\dot{H} = \mathcal{X}H = \sum_{i=1}^{\infty} V_i(x^2+y^2)^{2i}$, where the V_i are polynomials in the parameters of system (5) called the *Poincaré-Liapunov constants*. These constants are the obstructions to the existence of a first integral for system (5). Hence if system (5) has a first integral then all the $V_i = 0$ for all $i \geq 1$. Consequently the simultaneous vanishing of all the Poincaré-Liapunov constants provides the necessary conditions to have a center at the origin of system (5). We define the ideal generate by these constants by $\mathcal{B} = \langle V_1, V_2, \ldots\rangle \subset \mathbb{C}[\lambda]$ where λ are the parameters of system (5). This ideal is called the *Bautin ideal*, and the affine variety $V(\mathcal{B})$ is the *center variety* of system (5).

The Hilbert basis theorem assures the existence of a positive value k such that $\mathcal{B} = \mathcal{B}_k = \langle V_1, V_2, \ldots V_k \rangle$. In fact we always have that $V(\mathcal{B}) \subset V(\mathcal{B}_k)$. The opposite inclusion is satisfied if any point of each component of the irreducible decomposition of $V(\mathcal{B}_k)$ corresponds to a system having a center at the origin. To find the irreducible component of $V(\mathcal{B}_k)$ we use the routine `minAssGTZ` [10] of the computer algebra system `Singular` [18].

Finally we recall some results of the Darboux theory of integrability for polynomials differential systems, see [17] or Chapter 8 of [14] and references therein for more details. An *invariant algebraic curve* $f(x, y) = 0$ of system (5) is given by a polynomial $f(x, y)$ satisfying

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf,$$

where K is called the *cofactor* of the invariant algebraic curve, which is a polynomial of degree at most $n - 1$. A *Darboux first integral* of system (5) is a first integral of the form $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ where f_i are invariant algebraic curves of system (5) and $\alpha_i \in \mathbb{C}$. A Darboux integrating factor of system (5) is an integrating factor of the form $R = f_1^{\beta_1} \cdots f_k^{\beta_k}$ where and $\beta_i \in \mathbb{C}$. Assume that the cofactors of invariant curves $f_1, f_2, \ldots, f_k$ are $K_1, K_2, \ldots, K_k$, then if there exist $\alpha_i \in \mathbb{C}$ for $i = 1, \ldots, k$ such that $\sum_{i=1}^{k} \alpha_i K_i = 0$ then $H = f_1^{\alpha_1} \cdots f_k^{\alpha_k}$ is a Darboux first integral of system (5). Moreover if there exist $\beta_i \in \mathbb{C}$ for $i = 1, \ldots, k$, satisfying $\sum_{i=1}^{k} \alpha_i K_i + \partial P/\partial x + \partial Q/\partial y = 0$, then $R = f_1^{\beta_1} \cdots f_k^{\beta_k}$ is a Darboux integrating factor of system (5).

A *time-reversible system* is a system which has a line through the origin such that this line is a symmetry axis of the phase portrait. More specifically, if this line is given by the straight line through the origin with slope $\tan(\alpha/2)$, then after a rotation of $\alpha/2$ the system is invariant under the symmetry $(x, y, t) \to (x, -y, -t)$. If we know that a singular point on this line is a center or a focus, the presence of this time-reversible symmetry prevents this singularity to be a focus, consequently it must to be a center.

3. Simultaneity of centers for a $\mathbb{Z}_2$-equivariant cubic system

In [34] it was studied what is called the bi-center problem for a $\mathbb{Z}_2$-equivariant cubic system of the form (3) and it was found the necessary and sufficient conditions for existence of two centers at the points $(-1, 0)$ and $(1, 0)$. Imposing that system (3) has a focus-center singular point at such singular points then it takes the form

$$\begin{aligned} \dot{x} = &\ -(c_{21} + 1)y + c_{21}x^2y + c_{12}xy^2 + c_{03}y^3, \\ \dot{y} = &\ -\frac{1}{2}x - d_{01}y + \frac{1}{2}x^3 + d_{01}x^2y + d_{12}xy^2 + d_{03}y^3, \end{aligned} \tag{7}$$

where $c_{i,j}$ and $d_{i,j}$ are real parameters. In Theorem 7 of [34] are given eleven different families of centers that provide the center variety of system (7). In what follows the existence of more centers for such system and the simultaneity in their appearance is studied. First we impose the existence of a singular point at (a,b) with $ab \neq 0$ arbitrary, and after that this point be a focus-center singular point with purely eigenvalues and we get

$$
\begin{aligned}
b^3c_{03} + ab^2c_{12} - b(1+c_{21}) + a^2bc_{21} &= 0,\\
-\frac{a}{2} + \frac{a^3}{2} + bd_{01} - a^2bd_{01} + b^3d_{03} + ab^2d_{12} &= 0,\\
b^2c_{12} + 2abc_{21} = 0, \quad d_{01} - a^2d_{01} + 3b^2d_{03} + 2abd_{12} &= 0,\\
3b^2c_{03} + 2abc_{12} - c_{21} + a^2c_{21} = 0, \quad -\frac{3}{2} + \frac{3a^2}{2} - 2abd_{01} + b^2d_{12} &= 0.
\end{aligned} \tag{8}
$$

The unique real solution of this algebraic system of equations gives the following result.

Theorem 5. *System* (7) *with the additional pair of focus-center singular point* (a,b) *and* $(-a,-b)$ *becomes*

$$
\begin{aligned}
\dot{x} &= \frac{y}{2} - \frac{3x^2y}{2} + \frac{3axy^2}{b} - \frac{(1+3a^2)y^3}{2b^2}.\\
\dot{y} &= -\frac{x}{2} + \frac{x^3}{2} - \frac{3(-1+a^2)xy^2}{2b^2} + \frac{a(-1+a^2)y^3}{b^3}.
\end{aligned} \tag{9}
$$

Moreover system (9) *has five singular points of focus-center type that simultaneously become centers if* $a^2 + b^2 - 1 = 0$.

Proof. The solution of system (8) is

$$
\begin{aligned}
d_{03} = \frac{a(-1+a^2)}{b^3}, \quad c_{03} = -\frac{1+3a^2}{2b^2}, \quad c_{12} = \frac{3a}{b},\\
c_{21} = -\frac{3}{2}, \quad d_{01} = 0, \quad d_{12} = -\frac{3(-1+a^2)}{2b^2}.
\end{aligned}
$$

Introducing these values in system (7) we obtain (9). System (9) has the finite singular points $(0,0)$, $(-1,0)$, $(1,0)$, $(1+a)/2, b/2)$, $(-(1+a)/2, -b/2)$, $((a-1)/2, b/2)$, $(-(a-1)/2, -b/2)$, $(-a,-b)$ and (a,b). It is easy to see that the singular points $(0,0)$, $(-1,0)$, $(1,0)$, $(-a,-b)$ and (a,b) are focus-centers singular points. The first two because system (7) already had them. As we have impose (a,b) as focus-center singular point by the symmetry we obtain that $(-a,-b)$ it is too. The surprise is that the origin becomes also a focus-center singular point. Finally if $a^2 + b^2 - 1 = 0$ then the system is Hamiltonian and all these singular points become centers because system (9) has a polynomial first integral and this happens simultaneously. $\square$

4. Simultaneity of centers for a $\mathbb{Z}_2$-equivariant quintic system

Recently in [43] it was studied the bi-center problem for a $\mathbb{Z}_2$-equivariant quintic system of the form (4) and it was found the necessary and sufficient conditions for the existence of two centers at the points $(-1,0)$ and $(1,0)$. Imposing that system (4) has a focus-center singular point at such singular points then it takes the form

(10)
$$\begin{aligned}\dot{x} = &\ -(1+c_{41})y + c_{41}x^4y + c_{32}x^3y^2 + c_{23}x^2y^3 + c_{14}xy^4 + c_{05}y^5,\\ \dot{y} = &\ -\frac{x}{4} + \frac{x^5}{4} + d_{01}y - d_{01}x^4y + d_{32}x^3y^2 + d_{23}x^2y^3 + d_{14}xy^4 + d_{05}y^5,\end{aligned}$$

where $c_{i,j}$ and $d_{i,j}$ are real parameters. In Theorem 4.2 of [43] are given four different families of centers that provide the center variety of system (7) but only in the particular case $c_{41} = -1$ and $c_{05} = 0$. Under these last conditions the origin cannot be a center because the line $x = 0$ is an invariant straight line of system (10). In the following the existence of more centers for the general system (10) and the simultaneity in their appearance is studied. First we impose the existence of a singular point at (a,b) with $ab \neq 0$ arbitrary with the condition that this point be a focus-center singular point and we obtain

$$\begin{aligned} b^5c_{05} + ab^4c_{14} + a^2b^3c_{23} + a^3b^2c_{32} - b(1+c_{41}) + a^4bc_{41} &= 0,\\ -\frac{a}{4} + \frac{a^5}{4} + bd_{01} - a^4bd_{01} + b^5d_{05} + ab^4d_{14} + a^2b^3d_{23} + a^3b^2d_{32} &= 0,\\ b^4c_{14} + 2ab^3c_{23} + 3a^2b^2c_{32} + 4a^3bc_{41} &= 0,\\ d_{01} - a^4d_{01} + 5b^4d_{05} + 4ab^3d_{14} + 3a^2b^2d_{23} + 2a^3bd_{32} &= 0,\\ 5b^4c_{05} + 4ab^3c_{14} + 3a^2b^2c_{23} + 2a^3bc_{32} - c_{41} + a^4c_{41} &= 0,\\ -\frac{5}{4} + \frac{5a^4}{4} - 4a^3bd_{01} + b^4d_{14} + 2ab^3d_{23} + 3a^2b^2d_{32} &= 0.\end{aligned} \tag{11}$$

The unique real solution of this algebraic system of equations is

$$\begin{aligned} d_{05} &= -\frac{a - a^5 + 4ab^4d_{14} + 2a^2b^3d_{23}}{6b^5}, \quad c_{41} = -\frac{5}{4}, \quad d_{01} = 0,\\ c_{05} &= -\frac{3 + 5a^4 + 8ab^3c_{14} + 4a^2b^2c_{23}}{12b^4}, \quad c_{32} = -\frac{-5a^3 + b^3c_{14} + 2ab^2c_{23}}{3a^2b},\\ d_{32} &= -\frac{-5 + 5a^4 + 4b^4d_{14} + 8ab^3d_{23}}{12a^2b^2}.\end{aligned}$$

However we do not introduce all these values in system (10) because we must to compute the Poincaré-Liapunov constants or focal values in the point $(1,0)$ and with these substitutions the computations become harder to be computed. Hence we only impose the condition $c_{41} = -5/4$ and $d_{01} = 0$. To compute the focal values at the point $(1,0)$ we first move this point to the origin applying the transformation $u = x - 1$ and $v = y$. Computing these focal values and doing the decomposition of the ideal generated by the Poincaré-Liapunov constants we can establish the following theorem.

Theorem 6. *System* (10) *with* $c_{41} = -5/4$ *and* $d_{01} = 0$ *has a center at the points* $(-1, 0)$ *and* $(1, 0)$ *if and only if one of the following conditions holds:*

(a) $d_{05} = d_{23} = c_{14} = c_{32} = 0$,
(b) $c_{32} - d_{23} = c_{14} - d_{05} = 2d_{32} + 1 = 4c_{05} + 1 = 4c_{23} - 4d_{14} + 3 = 0$,
(c) $c_{32} + d_{23} = c_{23} + 2d_{14} = c_{14} + 5d_{05} = 2d_{32} - 5 = 0$,
(d) $c_{14} + d_{23} + 2d_{05} = c_{23} - c_{05} + d_{32} - d_{14} + 1 = c_{32} + 3d_{05} = 2d_{32}d_{05} - d_{23} - 2d_{05} = c_{05}d_{23} + d_{23}d_{14} - c_{05}d_{05} + 3d_{14}d_{05} - d_{23} - 4d_{05} = 2c_{05}d_{32} + 2d_{32}d_{14} + 2c_{23} - 5c_{05} - d_{14} = 0$.

Proof. We computed the first eight nonzero focal values using the method described in Section 2. Their expressions are extremely long and we only write here the first two.

$$V_1 = 2c_{32} - 3d_{23} - 2c_{32}d_{32},$$

$$\begin{aligned} V_2 = {} & 96c_{14} - 34c_{32} + 144c_{23}c_{32} - 40c_{32}^3 - 120d_{05} - 160c_{32}d_{14} + 291d_{23} \\ & - 36c_{23}d_{23} + 60c_{32}^2d_{23} - 48c_{14}d_{32} + 410c_{32}d_{32} - 104c_{23}c_{32}d_{32} + 40c_{32}^3d_{32} \\ & - 240d_{23}d_{32} - 536c_{32}d_{32}^2 + 492d_{23}d_{32}^2 + 328c_{32}d_{32}^3. \end{aligned}$$

Next we compute the irreducible decomposition of the variety $V(\mathcal{B}_8) = V(\langle V_1, V_2, V_3, V_4, V_5, V_6, V_{17}, V_8\rangle)$ of these Poincaré-Liapunov constants using the routine `minAssGTZ` of the computer algebra system `Singular` over the field of the rational numbers and we obtain the families given in the statement of the theorem. Now we prove the sufficiency.

Case (a). Under condition (a) of Theorem 6 system (10) with $c_{41} = -5/4$, and $d_{01} = 0$ and with the point $(1, 0)$ at the origin of coordinates takes the form

$$\begin{aligned} \dot{u} = {} & -v + 5uv - \frac{15u^2v}{2} + 5u^3v - \frac{5u^4v}{4} + c_{23}v^3(1 - 2u + u^2) + c_{05}v^5, \\ \dot{v} = {} & u - \frac{5u^2}{2} + \frac{5u^3}{2} - \frac{5u^4}{4} + \frac{u^5}{4} - d_{32}v^2(1 - 3u + 3u^2 - u^3) - d_{14}v^4(1 - u). \end{aligned}$$

This system is invariant by the symmetry $(u, v, t) \to (u, -v, -t)$, hence it is a time-reversible system and it has a center at the points $(1, 0)$ and $(-1, 0)$.

Case (b). System (10) with $c_{41} = -5/4$ and $d_{01} = 0$ under the conditions of statement (b) of Theorem 6 and with the point $(1, 0)$ at the origin of coordinates takes the form

$$\begin{aligned} \dot{u} = {} & \frac{1}{4}\, v\,[-4 + 20u - 30u^2 + 20u^3 - 5u^4 + 4d_{23}(u - 1)^3v \\ & +(4d_{14} - 3)(u - 1)^2v^2 + 4d_{05}(u - 1)v^3 - v^4], \\ \dot{v} = {} & \frac{1}{4}\,(u - 2)(u - 1)u(2 + (u - 2)u) - \frac{1}{2}(u - 1)^3v^2 \\ & +d_{23}(u - 1)^2v^3 + d_{14}(u - 1)v^4 + d_{05}v^5. \end{aligned} \tag{12}$$

System (12) has an inverse integrating factor of the form $V = (1 - 2u + u^2 + v^2)^3$, hence by Reeb's theorem, see [42], system (12) has a center at the points $(1, 0)$ and $(-1, 0)$. However if we go back to the origin the inverse integrating factor takes the form $V = (x^2 + y^2)^3$ and the Reeb's theorem cannot be applied to this point.

Case (c). Under the conditions of statement (c) of Theorem 6 system (10) with $c_{41} = -5/4$, $d_{01} = 0$, and with the point $(1, 0)$ at the origin of coordinates takes the form

$$\begin{aligned}
\dot{u} = & \ \frac{1}{4} v\,[-4 + 20u - 30u^2 + 20u^3 - 5u^4 - 4d_{23}(u-1)^3 v \\
& -8d_{14}(u-1)^2 v^2 - 20d_{05}(u-1)v^3 + 4c_{05}v^4], \\
\dot{v} = & \ \frac{1}{4}(u-2)(u-1)u(2 + (-2+u)u) + \frac{5}{2}(u-1)^3 v^2 \\
& +d_{23}(u-1)^2 v^3 + d_{14}(u-1)v^4 + d_{05}v^5.
\end{aligned} \tag{13}$$

System (14) is Hamiltonian and then it has a center at the singular points $(1, 0)$ and $(-1, 0)$.

Case (d). Under the conditions of statement (c) of Theorem 6 system (10) with $c_{41} = -5/4$, $d_{01} = 0$, and with the point $(1, 0)$ at the origin of coordinates takes the form

$$\begin{aligned}
\dot{u} = & \ \frac{1}{4} v\,[-4 + 20u - 30u^2 + 20u^3 - 5u^4 - 12d_{05}(u-1)^3 v + 4c_{23}(u-1)^2 v^2 \\
& -8d_{05}d_{32}(u-1)v^3 - (1 - c_{23} - d_{32} - 2c_{23}d_{32} - 2d_{32}^2)v^4], \\
\dot{v} = & \ \frac{1}{4}\,[(u-2)(u-1)u(2 + (u-2)u) + 4d_{32}(u-1)^3 v^2 \\
& +8d_{05}(d_{32} - 1)(u-1)^2 v^3 \\
& -((1 + d_{32})(2d_{32} - 5) + c_{23}(2d_{32} - 3))(u-1)v^4 + 4d_{05}v^5].
\end{aligned}$$

This system has an inverse integrating factor of the form $V = (1 - 2u + u^2 + v^2)^{5/2 - d_{32}}$, hence by Reeb's theorem system (12) has a center at the points $(1, 0)$ and $(-1, 0)$. However if we go back to the origin the inverse integrating factor takes the form $V = (x^2 + y^2)^{5/2 - d_{32}}$ and the Reeb's theorem cannot be applied at $(0, 0)$. □

Now we study if the families of centers given in Theorem 6 having also center at the origin of coordinates.

Theorem 7. *Any system* (10) *with* $c_{41} = -5/4$ *and* $d_{01} = 0$ *satisfying conditions one of the conditions (a), (b) and (d) of Theorem 6 has always a center at the origin of coordinates. Moreover satisfying condition (b) of Theorem 6 has a center at the origin if and only if* $3d_{05} + d_{23} = 0$.

Proof. For the case (a) the system (10) takes the form

$$\begin{aligned}\dot{x} = &\ \frac{y}{4} - \frac{5x^4y}{4} + c_{23}x^2y^3 + c_{05}y^5,\\ \dot{y} = &\ -\frac{x}{4} + \frac{x^5}{4} + d_{32}x^3y^2 + d_{14}xy^4.\end{aligned}$$

Therefore is also time-reversible and consequently has a center at $(0,0)$.

For the case (b) although the system has an inverse integrating factor given by $V = (x^2+y^2)^3$ we have that $V(0,0) = 0$ and the Reeb's theorem cannot be applied at the origin. In fact if we construct the first integral associated to this inverse integrating factor we obtain a first integral which is not analytic at the origin. Moreover computing the first focal value at the origin we obtain $V_4 = 3d_{05} + d_{23}$. If we vanish this constant we get the first integral

$$H(x,y) = (x^2+y^2)\, e^{\dfrac{1+8x^2y^2+16d_{05}xy^3+(5-4d_{14})y^4}{2(x^2+y^2)^2}}.$$

or, the first integral

$$\begin{aligned}G(x,y) &= \log H^2(x,y)\\ &= \log\left((x^2+y^2)^2\, e^{\frac{1+8x^2y^2+16d_{05}xy^3+(5-4d_{14})y^4}{(x^2+y^2)^2}}\right)\\ &= \frac{1+P(x,y)+(x^2+y^2)^2\log(x^2+y^2)^2}{(x^2+y^2)^2},\end{aligned}$$

where

$$P(x,y) = 8x^2y^2 + 16d_{05}xy^3 + (5-4d_{14})y^4.$$

Then the first integral

$$F(x,y) = \frac{1}{G(x,y)}$$

is well defined at the origin. So the origin is a center.

The case (c) is Hamiltonian and consequently all its focus-center singular points are centers because in this case system (10) has a polynomial first integral.

Finally the case (d) also has the inverse integrating factor $V = (x^2+y^2)^{5/2-d_{32}}$ with $V(0,0) = 0$, and the Reeb's theorem cannot also be applied at the origin. But the associated first integral to V is

$$H(x,y) = (x^2+y^2)^{-(3/2)+d_{32}} f,$$

where $f = 1+2d_{32}+3x^4-2d_{32}x^4+4(3+d_{32}-2d_{32}^2)x^2y^2+4d_{05}(3-4(d_{32}-1)d_{32})xy^3-(2d_{32}-3)(1-c_{23}-d_{32}-2c_{23}d_{32}-2d_{32}^2)y^4$. Hence this first integral or its inverse is always analytic at the origin. Therefore the origin of system (10) is a center. $\square$

The next result gives when the families of centers given in Theorem 6 have a center at the singular point (a, b).

Theorem 8. *Any system* (10) *with* $c_{41} = -5/4$ *and* $d_{01} = 0$ *satisfying conditions one of the conditions (a), (b), (c) or (d) of Theorem 6 has the additional centers at the singular point* (a, b) *and* $(-a, -b)$ *with* $ab \neq 0$ *if, and only if, for case (a)* $-1 + a^4 + 5a^2b^2 = 0$*, for cases (b), (c) and (d) always.*

Proof. We take system (10) and impose the conditions of statement (a). Next we impose that the system has the singular point (a, b) and system (10) becomes

$$\begin{aligned} \dot{x} =& \quad \frac{y}{4} - \frac{5x^4y}{4} + \frac{5a^2x^2y^3}{2b^2} - \frac{(1+5a^4)y^5}{4b^4}, \\ \dot{v} =& \quad -\frac{x}{4} + \frac{x^5}{4} - \frac{(a^4-1)x^3y^2}{2a^2b^2} - \frac{(1-a^4)xy^4}{4b^4}. \end{aligned} \tag{14}$$

The next step is to move the point (a, b) to the origin applying the transformation $u = x - a$ and $v = y - b$, and compute the focal values at this point. The first two non–zero focal values are

$$V_1 = \frac{a}{2b^3}(1 + 2a^4 + 2a^2b^2)(-1 + a^4 + 5a^2b^2)$$

and

$$\begin{aligned} V_2 = &- \frac{1}{96ab^7}(1 + 2a^4 + 2a^2b^2)(-1 + a^4 + 5a^2b^2) \\ &\big(164a^4 - 424a^8 + 164a^12 - 247a^2b^2 + 64a^6b^2 \\ &+ 2220a^{10}b^2 + 141b^4 + 1736a^4b^4 + 7020a^8b^4 + 6500a^6b^6\big). \end{aligned}$$

Then we have that the unique common factor of V_1 and V_2 with real roots is $-1 + a^4 + 5a^2b^2$, which are $b = \pm\sqrt{1 - a^4}/(\sqrt{5}a)$.

For the cases (b) and (d) we have that $V = (x^2 + y^2)^3$ and $V = (x^2 + y^2)^{5/2-d_{32}}$ are inverse integrating factors respectively. Therefore at the point (a, b) with $ab \neq 0$ we have the inverse integrating factor $V = ((u+a)^2 + (v+b)^2)^3$ and $V = ((u+a)^2+(v+b)^2)^{5/2-d_{32}}$ both with $V(0, 0) \neq 0$ respectively. Consequently applying Reeb's theorem, see [42], in both cases we have a center at the points (a, b).

Finally, the case (c) is Hamiltonian and any focus-center singular point must be a center. □

5. Acknowledgements

The first author is partially supported by a MINECO/ FEDER grant number MTM2014-53703-P and an AGAUR (Generalitat de Catalunya) grant number 2014SGR 1204. The second author is partially supported by a

FEDER-MINECO grant MTM2016-77278-P, a MINECO grant MTM2013-40998-P, and an AGAUR grant number 2014SGR-568. The third author is supported by Portuguese national funds through FCT - Fundação para a Ciência e a Tecnologia within the project PEst-OE/EEI/LA0009/2013 (CAMGSD).

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