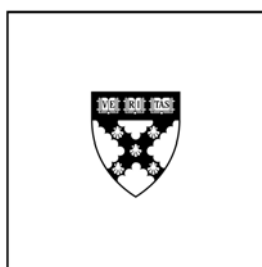# HARVARD | BUSINESS | SCHOOL

Farsighted House Allocation

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Farsighted House Allocation[*]

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Abstract

In this note we study von Neumann-Morgenstern farsightedly stable sets for Shapley and Scarf (1974) housing markets. Kawasaki (2008) shows that the set of competitive allocations coincides with the unique von Neumann-Morgenstern stable set based on a farsighted version of antisymmetric weak dominance (cf., Wako, 1999). We demonstrate that the set of competitive allocations also coincides with the unique von Neumann-Morgensterrn stable set based on a farsighted version of strong dominance (cf., Roth and Postlewaite, 1977) if no individual is indifferent between his endowment and the endowment of someone else.

JEL classification: D63, D70.

Keywords: housing markets, indivisible goods, farsightedness, von Neumann-Morgenstern stable sets, top trading cycles, competitive allocations

1 Introduction

We consider Shapley and Scarf's (1974) housing market model in which each agent is endowed with a house, has (not necessarily strict) preferences over the set of houses in the market, and wishes to consume exactly one house. Recently, this model has been successfully applied to the allocation of student housing with existing tenants (e.g., Abdulkadiroğlu and Sönmez, 1999) and kidney exchange (e.g., Roth et al., 2004).[1]

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[1]An exchange of student housing is identical to a housing market if every student is already assigned to a room (i.e., is a tenant). For kidney exchange, a housing market describes the situation in which every patient (who requires one kidney) also provides a (possibly incompatible) donor (of one kidney). For an overview of the recent literature on allocation and exchange of indivisible goods see Sönmez and Ünver (2009).

The most prominent solution concepts for housing markets are the set of competitive allocations and the core. Shapley and Scarf (1974) show that the set of competitive allocations coincides with the set of outcomes of the *Top Trading Cycles Algorithm* attributed to David Gale by Shapley and Scarf (1974). For strict preferences, Roth and Postlewaite (1977) demonstrate that the unique competitive allocation coincides with the unique allocation in the strong core (i.e., the core defined in terms of weak dominance). In the case of indifferences, the strong core may be empty and the non-empty weak core (i.e., the core based on strong dominance) may exceed the (not necessarily singleton) set of competitive allocations. Wako (1999) proposes a core based on antisymmetric weak dominance and shows that it coincides with the set of competitive allocations. An allocation is antisymmetrically weakly dominated by another allocation if it is weakly dominated *and* agents in the blocking coalition who are indifferent between the old and the new allocation consume the same house in both allocations. Hence, antisymmetric weak dominance is more restrictive than weak dominance but implied by strong dominance. Toda (1997) shows that the set of competitive allocations also coincides with the unique von Neumann-Morgenstern stable set based on antisymmetric weak dominance.

Harsanyi (1974) criticizes the underlying assumption of these notions of (direct) dominance that agents are myopic in the sense that a blocking coalition forms whenever it benefits (weakly or strongly) from redistributing its endowments. A coalition might well enforce a myopically not very attractive outcome (i.e., the redistribution of endowments makes some members of the blocking coalition worse off) in order to set a chain of events in motion that in the end will lead to a preferred outcome for everyone in the coalition. Following this line of thought, Kawasaki (2008) demonstrates that the set of competitive allocations coincides with the unique von Neumann-Morgenstern stable set defined in terms of a farsighted version of Wako's (1999) antisymmetric weak dominance.

In this paper, we show that the set of competitive allocations coincides with the unique von Neumann-Morgenstern stable set defined in terms of a farsighted version of *strong* dominance, *if* no agent is indifferent between his initial endowment and the endowment of another agent. This preference restriction seems to be realistic for standard applications of housing market models. For instance, it is reasonable to assume that a patient/donor pair strictly prefers its own donor kidney to any kidney with the same medical characteristics, while it also strictly prefers any kidney with better medical characteristics to its own. Likewise, a tenant strictly prefers his own room to any room with identical characteristics —he likes to avoid moving—, while he also strictly prefers any room with distinctly better characteristics to his own. Hence, we give a farsighted characterization of the set of competitive allocations in a housing market based on strong dominance (i.e., a stronger dominance relation than the one used in Kawasaki, 2008) on a restricted preference domain. In fact, we demonstrate that this is a maximal domain. More precisely, we provide an example in which (i) one agent is indifferent between his endowment and the endowment of one other agent and (ii) the set of competitive allocations is not a von Neumann-Morgenstern stable set based on our farsighted version of strong dominance.

In Section 2 we introduce the model. We present our results in Section 3.

2 Housing Markets

We consider housing markets as introduced by Shapley and Scarf (1974). Let $N = \{1, \ldots, n\}$, $n \geq 2$, be the *set of agents*. Each agent $i \in N$ is endowed with one object denoted by $e_i = i$. Thus, N also denotes the *set of objects*.

Allocations An allocation is an assignment of objects such that each agent receives exactly one object, i.e., an *allocation* is a vector $x = (x_i)_{i \in N} \in N^N$ such that

   (i) for each $i \in N$, $x_i \in N$ denotes the object that agent i consumes, e.g., if $x_i = j$, then agent i receives agent j's endowment, and

   (ii) no object is assigned to more than one agent, i.e., $\cup_{i \in N} \{x_i\} = N$.

Let X denote the *set of allocations*.

Trading Cycles An allocation x can be represented by a directed graph g with the set of nodes N and a directed edge from agent i to agent j if agent i consumes the endowment of agent j, i.e., $x_i = j$. A directed edge from i to itself (a loop) thus represents the case that agent i consumes his endowment at allocation x, i.e., $x_i = e_i$. As each agent is assigned to exactly one object, the graph displays a partition of the set of agents N into $L \leq |N| = n$ sets of agents who belong to the same connected component C^l ($l = 1, ..., L$). We call a connected component C^l a *trading cycle* and with some abuse of notation C^l also denotes the set of agents in the cycle. For any $i \in C^l$, we say that agent i trades in trading cycle C^l. Sometimes we denote trading cycle C^l for allocation x and agent $i \in C^l$ by $C^{x,i}$.

Markets Each agent $i \in N$ has complete and transitive preferences R_i over objects. We denote the strict part of R_i by P_i and the indifference part of R_i by I_i. By $\mathcal{R}$ we denote the *set of preferences over N* and by $\mathcal{R}^N = \times_{i \in N} \mathcal{R}$ we denote the *set of (preference) profiles*. Since the set of agents and their endowments remain fixed throughout, $\mathcal{R}^N$ also denotes the *set of housing markets*.

Top Trading Cycles Algorithm The *Top Trading Cycles Algorithm*, attributed to David Gale by Shapley and Scarf (1974), computes a set of allocations that plays a central role in the housing market literature.

Consider a directed graph g^T with set of nodes N and directed edges determined by the following iterative procedure.

Step 1: Each agent points to one of the agents whose endowment he prefers most. Since there is a finite number of agents, there is at least one cycle. A directed edge from agent i to agent j is drawn if i points at j and both are in a cycle. Each agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment. If there is at least one remaining agent, proceed with *Step 2*.

Step k: Each remaining agent points to one of the agents whose endowment he prefers most among the remaining agents. Since there is a finite number of agents, there is at least one cycle. A directed edge from agent i to agent j is drawn if i points at j and both are in a cycle. Each agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment. If there is at least one remaining agent, proceed with *Step $k + 1$*.

As there are finitely many agents, the iterative procedure stops after a finite number of steps and a graph g^T results that partitions the set of agents into trading cycles, which in turn define an allocation. Since preferences are not necessarily strict, an agent may be indifferent between the endowments of several agents (possibly including himself). Clearly, agents pointing at different (indifferent) agents at a given step of the algorithm, might result in a different graph and a different allocation. The set of all allocations that can be generated using the top trading algorithm is called the set of *top trading allocations*. Remarkably, the set of top trading allocations equals the set of *competitive allocations* (Shapley and Scarf, 1974).

To summarize, for given choices (i.e., tie-breaking of indifferences in agents' preferences) in the top trading cycles algorithm, the graph g^T determines an allocation x and a partition of the set of agents N into $L \leq |N|$ *top trading cycles* C^l ($l = 1, ..., L$). Moreover, the iterative procedure provides a partition of the set of agents N into $K \leq L$ sets of agents N^k ($k = 1, ..., K$) with N^k being the set of agents that is removed from the market in Step k. Note that each set N^k is the union of a collection of (top) trading cycles at x.

3 Von Neumann-Morgenstern Farsightedly Stable Sets

We first describe how a coalition of agents can enforce an allocation starting from another allocation. We need the following notation. The set of feasible reallocations of objects among the members of coalition $S \subseteq N$ is denoted by $X_S = \{(x_i)_{i \in S} \in \times_{i \in S} N : \cup_{i \in S} \{x_i\} = S\}$. Let $y \in X$ and $S \subseteq N$. Then, by $y_S = (y_i)_{i \in S}$ we denote the restriction of allocation y to coalition S. Moreover, $y(S) = \cup_{i \in S} \{y_i\}$ denotes the set of objects that coalition S consumes at allocation y.

Enforceability We adopt Kawasaki's (2008) enforceability notion. A *coalition S of agents can enforce an allocation y starting from an allocation x*, denoted by $x \rightarrow_S y$, if

(E1) $y_S \in X_S$;
(E2) $y_i = x_i$ for all $i \in N \backslash S$ with $x(S) \cap C^{x,i} = \emptyset$;
(E3) $y_i = e_i$ for all $i \in N \backslash S$ with $x(S) \cap C^{x,i} \neq \emptyset$.

That is, at allocation y, coalition S reallocates its endowment among itself (E1). If no object in a trade cycle at x is involved in the reallocation, then the trade cycle is not affected, i.e., it is part of allocation y (E2). If some object in a trade cycle at x is involved in the reallocation, then all agents in the trade cycle that are not in S consume their endowments (E3).

An allocation y *indirectly dominates* allocation x, denoted by $y \gg x$, if there exists a sequence of allocations $x = x^1, \ldots, x^L = y$ and a sequence of coalitions $\{S^1, \ldots, S^{L-1}\}$ such that for all $l \in \{1, \ldots, L-1\}$, $x^l \rightarrow_{S^l} x^{l+1}$ and for all $i \in S^l$, $y_i P_i x_i^l$.[2] We refer to such a sequence of coalitions as an *indirect dominance path of coalitions* (from x to y). Furthermore, we refer to the sequence of allocations $x = x^1, \ldots, x^L = y$ as an *indirect dominance path of allocations* (from x to y).[3]

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[2]Hence, we follow the literature on myopic blocking dynamics (e.g., Roth and Vande Vate, 1990; Diamantoudi et al., 2004; Serrano and Volij, 2008) and assume that in the sequence of allocations from x to y, coalition S^l can enforce allocation x^{l+1} starting from allocation x^l.

[3]Note that an allocation y that indirectly dominates an allocation x also indirectly dominates allocation x according to Definition 3 in Kawasaki (2008). The converse, however, is not true (see Example 1).

A set of allocations $V \subseteq X$ is *farsightedly internally stable* if for all $x, y \in V$, $y \not\gg x$. Note that every set of allocations V with cardinality $|V| = 1$ is farsightedly internally stable.

A set of allocations $V \subseteq X$ is *farsightedly externally stable* if for all allocations $x \in X \backslash V$ there exists an allocation $y \in V$ such that $y \gg x$. Note that X is farsightedly externally stable.

A set of allocations $V \subseteq M$ is a *von Neumann-Morgenstern (vNM) farsightedly stable set* if it is farsightedly internally and externally stable.

Note that the vNM farsightedly stable sets represent Greenberg's (1990) optimistic stable standard of behavior (OSSB) based on the indirect dominance relation.

Lemma 1. *The set of competitive allocations is farsightedly internally stable on $\mathcal{R}^N$*
For any housing market $R \in \mathcal{R}^N$, the set of competitive allocations is farsightedly internally stable.

To prove Lemma 1, it would suffice to prove that no competitive allocation farsightedly dominates another competitive allocation. We in fact prove a slightly stronger result by showing that any competitive allocation is farsightedly undominated.

Proof. Let x be a competitive allocation. We show that there is no allocation y with $y \gg x$.

Suppose there is an allocation y with $y \gg x$. Then, there exists a sequence of allocations $x = x^1, \ldots, x^L = y$ and a sequence of coalitions $\{S^1, \ldots, S^{L-1}\}$ such that for all $l \in \{1, \ldots, L-1\}$, $x^l \to_{S^l} x^{l+1}$ and for all $i \in S^l$, $y_i P_i x_i^l$. Recall that N^k $(k = 1, ..., K)$ denotes the set of agents that have been removed from the market after Step k in the top trading cycles algorithm that generated allocation x. We prove that for each $k = 1, ..., K$,

$$N^k \cap (S^1 \cup \cdots \cup S^{L-1}) = \emptyset \text{ and } y_{N^k} = x_{N^k}. \tag{1}$$

Then, since $N = \cup_{k=1,...,K} N^k$, we obtain $y = x$, which contradicts $y \gg x$.

Let $k = 1$. Suppose $N^k \cap (S^1 \cup \cdots \cup S^{L-1}) = N^1 \cap (S^1 \cup \cdots \cup S^{L-1}) \neq \emptyset$. Let l^* be the index of the first coalition on the indirect dominance path in which some member of N^1 participates, i.e., $l^* := \min\{l : N^1 \cap S^l \neq \emptyset\}$. Let $i \in N^1 \cap S^{l^*}$. Then, $y_i P_i x_i^{l^*}$. Recall that agents in N^1 only trade among themselves at x. Hence, from the definition of l^* and (E2), for all steps $l < l^*$, $x_i^{l^*} = x_i^{l^*-1} = \cdots = x_i^1 = x_i$. Hence, $y_i P_i x_i$, which contradicts that in the top trading cycles algorithm all agents in N^1 are assigned one of their most preferred objects. Hence, $N^1 \cap (S^1 \cup \cdots \cup S^{L-1}) = \emptyset$. Then, by (E2), for all steps $l = 1, ..., L-1$ and for all $i \in N^1$, $y_i = x_i^{L-1} = x_i^{L-2} = \cdots = x_i^1 = x_i$. So, (1) holds for $k = 1$.

Suppose now that for some k', $1 < k' \leq K$, (1) holds for all $k < k'$ (induction hypothesis). We show that (1) also holds for $k = k'$. Assume that $N^{k'} \cap (S^1 \cup \cdots \cup S^{L-1}) \neq \emptyset$. Let l^* be the index of the first coalition on the indirect dominance path in which some member of $N^{k'}$ participates, i.e., $l^* := \min\{l : N^{k'} \cap S^l \neq \emptyset\}$. Let $i \in N^{k'} \cap S^{l^*}$. Then, $y_i P_i x_i^{l^*}$. Recall that agents in $N^{k'}$ only trade among themselves at x. Hence, from the definition of l^* and (E2), for all steps $l < l^*$, $x_i^{l^*} = x_i^{l^*-1} = \cdots = x_i^1 = x_i$. Hence, $y_i P_i x_i$. Since x is a competitive allocation, $y_i \in N^1 \cup \cdots \cup N^{k'-1}$. Hence, for some $j \in N^1 \cup \cdots \cup N^{k'-1}$, $y_j \neq x_j$, which contradicts the induction hypothesis. So, $N^{k'} \cap (S^1 \cup \cdots \cup S^{L-1}) = \emptyset$. Then, from (E2) applied to steps $l = 1, ..., L-1$, we obtain that for all $i \in N^{k'}$, $y_i = x_i^{L-1} = x_i^{L-2} = \cdots = x_i^1 = x_i$. So, (1) holds for $k = k'$ as well. $\qquad \square$

For $i \in N$, let $\mathcal{R}'_i$ be the set of preferences of agent i over N where for any object $j \neq i$ his initial endowment is either strictly better or strictly worse than j. Formally, for $R_i \in \mathcal{R}_i$, $R_i \in \mathcal{R}'_i$ if and only if [for all $j \neq i$, $i\, P_i\, j$ or $j\, P_i\, i$]. Let $\mathcal{R}'^N = \times_{i \in N}\mathcal{R}'_i$. Note that if for each $i \in N$, $R_i \in \mathcal{R}_i$ is a (strict) linear order, then $\times_{i \in N} R_i \in \mathcal{R}'^N$.

Lemma 2. *The set of competitive allocations is farsightedly externally stable on $\mathcal{R}'^N$*
For any housing market $R \in \mathcal{R}'^N$, the set of competitive allocations is farsightedly externally stable.

Proof. First, for any $S \subseteq N$ and $z \in X_S$, we introduce the following two directed graphs. The "reference" (or red) graph $g^r(S, z)$ has nodes S and there is a directed edge from $i \in S$ to $j \in S$ if $z_i = j$. In other words, in $g^r(S, z)$ each agent points to the object he consumes at z. The "best object" (or blue) graph $g^b(S, z)$ has the same set of nodes S and from each $i \in S$ there is a unique directed edge to $p_i := p_i(S, z)$ where

$$p_i(S, z) := \begin{cases} z_i & \text{if for all } h, \ z_i\, R_i\, h; \\ j & \text{if } [j\, P_i\, z_i \text{ and for all } h, \ j\, R_i\, h] \text{ and } [[j'\, P_i\, z_i \text{ and for all } h, \ j'\, R_i\, h] \Rightarrow j' \geq j]; \end{cases}$$

i.e., $p_i(S, z)$ is the object that agent i consumes at allocation z, unless there is a strictly preferred object, in which case $p_i(S, z)$ is the most preferred object with lowest index.[4] We refer to an agent i with $p_i(S, z) = i$ as a loop in $g^b(S, z)$.

Let x be an allocation that is not competitive. Then, we construct a competitive allocation y such that $y \gg x$. The construction is based on the alternative application of two procedures called Top and Trade, which are now introduced.

Procedure **Top***.*
Input: (S, z) where $S \subseteq N$ and $z \in X_S$.
Let $\hat{S} := S$ and $\hat{z} := z$. Set $\hat{y} = \emptyset$. As long as there is a cycle C that is present in both $g^r(\hat{S}, \hat{z})$ and $g^b(\hat{S}, \hat{z})$, set $\hat{y} := (\hat{y}, \hat{z}_C)$, $\hat{z} := \hat{z}_{\hat{S} \backslash C}$, and $\hat{S} := \hat{S} \backslash C$.
Output: $\text{Top}(S, z) := (\hat{S}, \hat{z}, \hat{y})$.

In other words, given a coalition S and a feasible reallocation z for S, procedure Top finds all top trading cycles at z and creates a vector $\hat{y}$ that records the assigned objects accordingly. Note that for $(\hat{S}, \hat{z}, \hat{y}) = \text{Top}(S, z)$, $\hat{z} \in X_{\hat{S}}$ and there is no cycle in $g^b(\hat{S}, \hat{z})$ that is also present in $g^r(\hat{S}, \hat{z})$.

Procedure **Trade***.*[5]
Input: (S, z) where $S \subseteq N$ and $z \in X_S$ such that there is no cycle in $g^b(S, z)$ that is also present in $g^r(S, z)$.
Let $S^* := \cup\{C : C \text{ is a cycle in } g^b(S, z)\} \neq \emptyset$. Let $S^0 := \cup\{i : i \text{ is a loop in } g^b(S, z)\}$. Let $S^{**} := \{i \in S^* : p_i \neq z_i\} = \{i \in S^* : p_i\, P_i\, z_i\} \neq \emptyset$. Let $F^r(S^*)$ be the set of objects that can be reached through a directed path in the graph $g^r(S, z)$ starting from some object in S^*. (Note $S^* \subseteq F^r(S^*)$.) For each $i \in S$ define

$$z_i^1 := \begin{cases} i & \text{if } i \in F^r(S^*), \\ z_i & \text{if } i \in S \backslash F^r(S^*). \end{cases}$$

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[4]One easily verifies that any other tie-breaking rule would also work.
[5]See Figure 1 for an illustration. The graph g^b is given by the broken edges. The other edges belong to g^r.

Finally, for each $i \in S$ define

$$z_i^2 := \begin{cases} p_i & \text{if } i \in S^*, \\ i & \text{if } i \in F^r(S^*)\backslash S^*, \\ z_i & \text{if } i \in S\backslash F^r(S^*). \end{cases}$$

Output: $\text{Trade}(S, z) := (S^*, S^{**}, S^0, z^1, z^2)$.

In other words, procedure Trade shifts the objects owned by coalition S^* following the directed cycles in $g^b(S, z)$. This is done in two steps. First, coalition $S^{**} \subseteq S^*$ enforces allocation z^1 over allocation z by giving each agent in $F^r(S^*)$ his endowment. Next, coalition $S^*\backslash S^0$ enforces allocation z^2 over allocation z^1 by trading the objects according to the cycles in $g^b(S, z)$ – at allocation z^1 agents in S^0 already trade objects according to loops (self-cycles) in $g^b(S, z)$ and do not change their assignment from z^1 to z^2. The enforceability of the allocations z^1 and z^2 follows from the following claim.
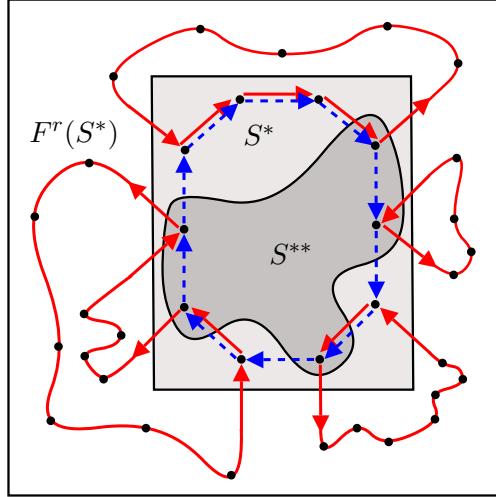

Figure 1: Illustration of procedure Trade

CLAIM 1: $z \rightarrow_{S^{**}} z^1$ and $z^1 \rightarrow_{S^*\backslash S^0} z^2$.
PROOF: We first prove that $z \rightarrow_{S^{**}} z^1$. Since $F^r(S^*)$ is a union of cycles in $g^r(S, z)$, it is sufficient to show that

$$\text{for each } j \in F^r(S^*) \text{ there is } i \in S^{**} \text{ with } j \in F^r(\{i\}). \tag{2}$$

Suppose that (2) is not true. Then, there is a cycle $C \subseteq F^r(S^*)$ in $g^r(S, z)$ with $C \cap S^{**} = \emptyset$.
  Since $C \subseteq F^r(S^*)$, there is $i_1 \in C \cap S^*$. Since $C \cap S^{**} = \emptyset$, $i_1 \notin S^{**}$. Let (i_1, i_2) be the edge in $g^r(S, z)$ that starts in i_1, i.e., $i_2 = z_{i_1}$. By definition of C and $i_1 \in C$, $i_2 \in C$. Since $i_1 \notin S^{**}$, $p_{i_1} = z_{i_1}$. Hence, $p_{i_1} = i_2$ and (i_1, i_2) is also an edge in $g^b(S, z)$. Hence, by definition of S^* and $i_1 \in S^*$, $i_2 \in S^*$. So, $i_2 \in C \cap S^*$. We can now repeat the previous arguments to obtain that C is a cycle in $g^b(S, z)$. However, by assumption (of the input of the procedure Trade), there is no cycle in $g^b(S, z)$ that is also present in $g^r(S, z)$. This contradiction leads to the conclusion that (2) is true.

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It remains to show that $z^1 \rightarrow_{S^* \backslash S^0} z^2$. This, however, follows immediately from $(S^* \backslash S^0) \subseteq F^r(S^*)$ being a collection of singletons in $g^r(S, z^1)$ and a collection of cycles in $g^b(S, z)$. More precisely, at allocation z^1 coalition $S^* \backslash S^0$ carries out the trades induced by the directed cycles in $g^b(S, z)$ and can do this without affecting any other agent since $S^* \backslash S^0$ is a set of singletons in $g^r(S, z^1)$. ∎

Next, we show that if (S, z) satisfies the assumption in the procedure Trade, then for $(S^*, S^{**}, S^0, z^1, z^2) = \text{Trade}(S, z)$, coalitions S^{**} and $S^* \backslash S^0$ would indeed be willing to enforce allocations z^1 (over z) and z^2 (over z^1), respectively, as long as z^2 is the final allocation of the indirect dominance path.

CLAIM 2:   (i) For all $i \in S^{**}$, $z_i^2 \, P_i \, z_i$. (ii) For all $i \in S^* \backslash S^0$, $z_i^2 \, P_i \, z_i^1$.

PROOF: Part (i). Since $i \in S^{**} \subseteq S^*$, $z_i^2 = p_i$. Since $i \in S^{**}$, $p_i \, P_i \, z_i$. Hence, $z_i^2 \, P_i \, z_i$.

Part (ii). First note that since $i \in S^* \backslash S^0$, $z_i^2 = p_i \neq i$. By definition of p_i, $p_i \, R_i \, i$. Since $R_i \in \mathcal{R}_i'$, $p_i \neq i$, and $p_i \, R_i \, i$, it follows that $p_i \, P_i \, i$. By definition of z^1, $z_i^1 = i$. Summarizing, $z_i^2 = p_i \, P_i \, i = z_i^1$. ∎

Finally, the following algorithm constructively finds a competitive allocation y that indirectly dominates x. Set $t := 1$, $\hat{N} := N$, and $\hat{x} := x$.

Step 0: Let $(\hat{N}, \hat{x}, y) := \text{Top}(\hat{N}, \hat{x})$ and go to Step 1.a. (Note that $\hat{N} \neq \emptyset$ and $y \in X_{N \backslash \hat{N}}$.)[6]

Step t.a: If $\hat{N} \neq \emptyset$, then let $(S^*, S^{**}, S^0, z^1, z^2) := \text{Trade}(\hat{N}, \hat{x})$, set $y := (y, z_{S^*}^2)$, $\hat{N} := \hat{N} \backslash S^*$, and $\hat{x} := z_{\hat{N}}^2$, and go to Step t.b. Otherwise, stop.

Step t.b: If $\hat{N} \neq \emptyset$, then let $(\hat{N}, \hat{x}, \hat{y}) = \text{Top}(\hat{N}, \hat{x})$, set $y := (y, \hat{y})$ and $t := t + 1$, and go to Step t.a. Otherwise, stop.

In Steps 0 and t.b, the algorithm augments the allocation y by adding all top trading cycles that are present in $\hat{x} \in X_{\hat{N}}$. In Steps t.a, the algorithm finds an indirect dominance path from $\hat{x}$ to z^2 via z^1 (see Claims 1 and 2).

The algorithm terminates in a finite number of steps since in Steps t.a, $S^* \neq \emptyset$ (see description of Trade), which implies that $|\hat{N}|$ strictly decreases. (By definition of Top, in Steps t.b, $|\hat{N}|$ does not increase.)

When the algorithm stops we have an allocation $y \in X_N$. It follows directly from the definition of Top and S^* (in Trade) that y is a competitive allocation.

Note that a change (augmentation) in y during the algorithm is definite. Hence, it follows directly from the definition of Top (augmentation of y by adding a part of $\hat{x}$) and Claims 1 and 2 for Trade (indirect dominance path from $\hat{x}$ to z^2 and adding the new trade cycles to y) that $y \gg x$. □

Our main result is that the set of competitive allocations is the unique vNM farsightedly stable set on $\mathcal{R}'^N$.

Theorem 1. *For any housing market $R \in \mathcal{R}'^N$, the set of competitive allocations is the unique vNM farsightedly stable set.*

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[6]If $\hat{N} = \emptyset$ then x is a competitive allocation, which is not the case.

Proof. Recall that in the proof of Lemma 1 we show that any competitive allocation is farsightedly undominated. Hence, by farsightedly external dominance, the set of competitive allocations has to be part of any vNM farsightedly stable set. Thus, by Lemma 2, the set of competitive allocations is the *unique* vNM farsightedly stable set. $\qquad\square$

The following example shows that Theorem 1 is a maximal domain result. More precisely, the example demonstrates that the set of competitive allocations need not be the unique vNM farsightedly stable set if there exists one agent i with preferences $R_i \in \mathcal{R}_i \backslash \mathcal{R}_i'$, i.e., agent i is indifferent between his endowment and the endowment of some other agent.

Example 1. *Maximality of domain $\mathcal{R}'^N$*
Let R be such that $2\,P_1\,3\,P_1\,1$, $3\,P_2\,2\,P_2\,1$, and $1\,I_3\,3\,P_3\,2$. Hence, agent 3's preferences violate the assumption in Theorem 1, i.e., $R_3 \notin \mathcal{R}_3'$. Straightforward calculations show that the set of competitive allocations is $Y = \{y^1, y^2\}$ with $y^1 = (2,3,1)$ and $y^2 = (1,2,3)$. It follows from Lemma 1 that Y is farsightedly internally stable. We now show that Y is not farsightedly externally stable. More precisely, we demonstrate that $x = (3,2,1)$ is not indirectly dominated by y^1 or y^2.

First, observe that no agent strictly prefers y^2 to x. Hence, $y^2 \not\gg x$. Assume that $y^1 \gg x$. Let $x = x^1, \ldots, x^L = y^1$ be an indirect dominance path from x to y^1 and let $\{S^1, \ldots, S^{L-1}\}$ be the associated indirect dominance path of coalitions. As $x_3 = y_3^1$, $S_1 \subseteq \{1,2\}$. W.l.o.g. we may assume that $x^2 \neq x^1$. Hence, $S^1 \neq \{2\}$. So, $S_1 = \{1\}$ or $S_1 = \{1,2\}$. By (E1) for $x^1 \to_{S^1} x^2$, $y_{S^1} \in X_{S^1}$. So, $x^2 = (1,2,3)$ or $x^2 = (2,1,3)$. In either case, $x_3^2 = 3$. Since for all j, $3\,R_3\,j$, we have for all $l = 2, \ldots, L-1$, $3 \notin S^l$. In particular, $3 = x_3^2 = x_3^3 = \ldots = x_3^L = y_3^1 = 1$; a contradiction. Hence, $y^1 \not\gg x$.[7] $\qquad\diamond$

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[7]The set $V = \{y^1, y^2, x\}$ is vNM farsightedly stable.
Proof. **Farsightedly internal stability:** x is not indirectly dominated by y^1 and y^2 (see the discussion of Example 1) and y^1 and y^2 are not indirectly dominated by any allocation according to the proof of Lemma 1.
Farsightedly external stability: Note first that $X \backslash V = \{(1,3,2), (2,1,3), (3,1,2)\}$. Now observe that $x \gg (1,3,2)$ because $(1,3,2) \to_{\{1,3\}} x$ with $3\,P_1\,1$ and $1\,P_3\,2$, $y^2 \gg (2,1,3)$ because $(2,1,3) \to_{\{2\}} y^2$ with $2\,P_2\,1$, and $y^2 \gg (3,1,2)$ because $(3,1,2) \to_{\{2\}} y^2$ with $2\,P_2\,1$. Hence, any allocation in $X \backslash V$ is indirectly dominated by an allocation in V. $\qquad\square$

References

Abdulkadiroğlu, A. and Sönmez, T. (1999): "House Allocation with Existing Tenants." *Journal of Economic Theory*, 88: 233–260.

Diamantoudi, E., Miyagawa, E., and Xue, L. (2004): "Random Paths to Stability in the Roommate Problem." *Games and Economic Behavior*, 48: 18–28.

Greenberg, J. (1990): *The Theory of Social Situations*. Cambridge University Press, Cambridge.

Harsanyi, J. C. (1974): "An Equilibrium-Point Interpretation of Stable Sets and a Proposed Alternative Definition." *Management Science*, 20: 1472–1495.

Kawasaki, R. (2008): "Farsighted Stability of the Competitive Allocations in an Exchange Economy with Indivisible Goods." Mimeo (October).

Roth, A. E. and Postlewaite, A. (1977): "Weak versus Strong Domination in a Market with Indivisible Goods." *Journal of Mathematical Economics*, 4: 131–137.

Roth, A. E., Sönmez, T., and Ünver, U. (2004): "Kidney Exchange." *Quarterly Journal of Economics*, 119: 457–488.

Roth, A. E. and Vande Vate, J. (1990): "Random Paths to Stability in Two-Sided Matching." *Econometrica*, 58: 1475–1480.

Serrano, R. and Volij, O. (2008): "Mistakes in Cooperation: the Stochastic Stability of Edgeworth's Recontracting." *Economic Journal*, 118: 1719–1741.

Shapley, L. and Scarf, H. (1974): "On Cores and Indivisibility." *Journal of Mathematical Economics*, 1: 23–37.

Sönmez, T. and Ünver, U. (2009): "Matching, Allocation, and Exchange of Discrete Resources." Forthcoming in Handbook of Social Economics.

Toda, M. (1997): "Implementation and Characterizations of the Competitive Solution with Indivisibility." Mimeo.

Wako, J. (1999): "Coalition-Proofness of the Competitive Allocations in an Indivisble Goods Market." *Fields Institute Communications*, 23: 277–283.