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Research Article

Existence Results for a Coupled System of Nonlinear Singular Fractional Differential Equations with Impulse Effects

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A boundary value problem for the singular fractional differential system with impulse effects is presented. By applying Schauder's fixed point theorem in a suitably Banach space, we obtain the existence of at least one solution for this problem. Two examples are presented to illustrate the main theorem.

1. Introduction

Fractional differential equations have received increasing attention during recent years since the behavior of many physical, chemical, and engineering processes can be properly described by using fractional differential equations theory; see the books [1–3], papers [4, 5] and references therein. For details on the geometric and physical interpretation of the derivatives of noninteger order, see, for example, [6–11]. For some recent works with applications to engineering we refer the reader to [12–15].

For an introduction of the basic theory of impulsive differential equation, we refer the reader to [16]. Among previous research, little is concerned with differential equations with fractional order with impulses [17]. Ahmad and Sivasundaram [18, 19] gave some existence results for two-point boundary value problems involving nonlinear impulsive hybrid differential equations of fractional order $1 < \alpha \leq 2$. Ahmad and Nieto in [20] establish sufficient conditions for the existence of solutions of the antiperiodic boundary value problem for impulsive differential equations with the Caputo derivative of order $q \in (1, 2]$. Some recent results on impulsive initial value problems or boundary value problems for fractional differential equations on a finite interval can be found in [21–23] and references therein. The memory property of fractional calculus makes studies more complicated.

This paper is motivated by [24] in which the following boundary value problem for the fractional differential equation

$$\begin{gathered} D_{0^+}^{\alpha} x(t) = f\left(t, y(t), D_{0^+}^{p} y(t)\right), \quad t \in (0,1), \\ D_{0^+}^{\beta} y(t) = g\left(t, x(t), D_{0^+}^{q} x(t)\right), \quad t \in (0,1), \\ x(0) = 0, \quad y(0) = 0, \quad x(1) - \gamma x(\eta) = 0, \\ y(1) - \gamma y(\eta) = 0 \end{gathered} \tag{1}$$

was studied, where $1 < \alpha, \beta < 2$, $0 < p \leq \beta - 1$ and $0 < q \leq \alpha - 1$, $\gamma > 0$, $1 > \gamma\eta^{\alpha-1}$, $1 > \gamma\eta^{\beta-1}$ and $f, g : [0,1] \times R^2 \to R$ are continuous functions, and D_{0^+} is the Riemann-Liouville fractional derivative. An existence result was proved for BVP (1) in [24]. The growth assumptions imposed on f and g are sublinear cases (see [25, Theorem 3.1]); that is, there exist functions $a, b \in L^1(0,1)$, nonnegative constants $\epsilon_1, \epsilon_2 > 0$, $\delta_1, \delta_2 \geq 0$ and $\rho_1, \rho_2, \sigma_1, \sigma_2 \in (0,1)$ such that

$$\begin{gathered} \left|f(t,x,y)\right| \leq a(t) + \epsilon_1 |x|^{\rho_1} + \epsilon_2 |y|^{\rho_2}, \\ \left|g(t,x,y)\right| \leq b(t) + \delta_1 |x|^{\sigma_1} + \delta_2 |y|^{\sigma_2}. \end{gathered} \tag{2}$$

In [25], the following boundary value problem for the fractional differential equation

$$\begin{aligned} &D_{0^+}^{\alpha}x(t) = f\left(t, y(t), D_{0^+}^{p}y(t)\right), \quad t \in (0,1), \\ &D_{0^+}^{\beta}y(t) = g\left(t, x(t), D_{0^+}^{q}x(t)\right), \quad t \in (0,1), \\ &x(0) = 0, \quad y(0) = 0, \quad x(1) = 0, \quad y(1) = 0 \end{aligned} \tag{3}$$

was studied, where $1 < \alpha, \beta < 2$, $0 < p \le \beta - 1$ and $0 < q \le \alpha - 1$, and $f, g : [0,1] \times R^2 \to R$ are continuous functions, and D_{0^+} is the Riemann-Liouville fractional derivative. The growth assumptions imposed on f and g are sublinear cases (see [25, Theorem 3.1]), that is, there exist functions $a, b \in L^1(0,1)$, nonnegative constants $\epsilon_1, \epsilon_2 > 0$, $\delta_1, \delta_2 \ge 0$, and $\rho_1, \rho_2, \sigma_1, \sigma_2 \in (0,1]$ such that

$$\begin{aligned} &\left|f(t,x,y)\right| \le a(t) + \epsilon_1|x|^{\rho_1} + \epsilon_2|y|^{\rho_2}, \\ &\left|g(t,x,y)\right| \le b(t) + \delta_1|x|^{\sigma_1} + \delta_2|y|^{\sigma_2}, \end{aligned} \tag{4}$$

or sublinear cases, that is, there exist nonnegative constants $\epsilon_1, \epsilon_2 > 0$, $\delta_1, \delta_2 \ge 0$ and $\rho_1, \rho_2, \sigma_1, \sigma_2 \in (1, \infty)$ such that

$$\begin{aligned} &\left|f(t,x,y)\right| \le \epsilon_1|x|^{\rho_1} + \epsilon_2|y|^{\rho_2}, \\ &\left|g(t,x,y)\right| \le \delta_1|x|^{\sigma_1} + \delta_2|y|^{\sigma_2}. \end{aligned} \tag{5}$$

We find that in the superlinear cases, BVP (3) has a pair of solutions $(x, y) = (0, 0)$ without needing any other assumptions. Hence, these cases are trivial ones discussed in [25].

It is interesting to consider the solvability of BVP (1) when the growth assumptions imposed on f, g are superlinear cases. Furthermore, the solvability of BVP (1) is not studied when $q > \alpha - 1$ or $p > \beta - 1$.

In this paper we consider the following nonlinear boundary value problem for the singular multiterm fractional differential equation with impulse effects whose boundary conditions are of integral form

$$\begin{aligned} &D_{0^+}^{\alpha}x(t) = \phi(t) f\left(t, y(t), D_{0^+}^{p}y(t)\right), \\ &\qquad t \in (0,1), \quad t \ne t_1, \\ &D_{0^+}^{\beta}y(t) = \psi(t) g\left(t, x(t), D_{0^+}^{q}x(t)\right), \\ &\qquad t \in (0,1), \quad t \ne t_1, \\ &\lim_{t\to 0} t^{2-\alpha}x(t) = \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p}y(s)\right) ds, \\ &\lim_{t\to 0} t^{2-\beta}y(t) = \int_0^1 v(s) H\left(s, x(s), D_{0^+}^{q}x(s)\right) ds, \\ &x(1) = \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p}y(s)\right) ds, \\ &y(1) = \int_0^1 n(s) N\left(s, x(s), D_{0^+}^{q}x(s)\right) ds, \end{aligned} \tag{6}$$

$$\begin{aligned} &\Delta x(t_1) = \lim_{t\to t_1^+} x(t) - \lim_{t\to t_1^-} x(t) = I\left(t_1, y(t_1), D_{0^+}^{p}y(t_1)\right), \\ &\Delta y(t_1) = \lim_{t\to t_1^+} y(t) - \lim_{t\to t_1^-} y(t) = J\left(t_1, x(t_1), D_{0^+}^{q}x(t_1)\right), \\ &\Delta D_{0^+}^{q}x(t_1) = \lim_{t\to t_1^+} D_{0^+}^{q}x(t) - \lim_{t\to t_1^-} D_{0^+}^{q}x(t) \\ &\qquad = I_1\left(t_1, y(t_1), D_{0^+}^{p}y(t_1)\right), \\ &\Delta D_{0^+}^{p}y(t_1) = \lim_{t\to t_1^+} D_{0^+}^{p}y(t) - \lim_{t\to t_1^-} D_{0^+}^{p}y(t) \\ &\qquad = J_1\left(t_1, x(t_1), D_{0^+}^{q}x(t_1)\right), \end{aligned} \tag{7}$$

where

(a) $1 < \alpha, \beta \le 2$, $0 < p < \beta$ and $0 < q < \alpha$, D_{0^+} is the Riemann-Liouville fractional derivative,

(b) ϕ, $\psi : (0,1) \to R$, f, g defined on $(0,1) \times R^2$,

(c) $m, n, u, v : (0,1) \to R$ with $m, n, u, v \in L^1(0,1)$, G, H, M, N defined on $(0,1) \times R^2$,

(d) $0 = t_0 < t_1 < t_2 = 1$,

(e) $I, I_1, J, J_1 : (0,1) \times R^2 \to R$.

A pair of functions (x, y) defined on $(0,1)$ is called a solution of BVP (1) and BVP (3), if $x|_{(t_k,t_{k+1}]}$, $D_{0^+}^{q}x|_{(t_k,t_{k+1}]}$ and $y|_{(t_k,t_{k+1}]}$, $D_{0^+}^{p}y|_{(t_k,t_{k+1}]}$ $(k = 0, 1)$ are continuous, there exists the limits

$$\begin{aligned} &\lim_{t\to t_k^+} t^{2-\alpha}x(t), \qquad \lim_{t\to t_k^+} t^{2-\beta}y(t), \\ &\lim_{t\to t_k^+} t^{2+q-\alpha}D_{0^+}^{q}x(t), \qquad \lim_{t\to t_k^+} t^{2+p-\beta}D_{0^+}^{p}y(t), \\ &\qquad k = 0, 1, \end{aligned} \tag{8}$$

$D_{0^+}^{\alpha}x, D_{0^+}^{\beta}y \in L^1(0,1)$ and (x, y) satisfies all equations in (6) and (7).

The novelty of this paper is as follows: first, the fractional differential equations in (6) are multiterm ones and their nonlinearities f, g depend on the lower fractional derivatives; second, both ϕ and ψ may be singular at $t = 0$ and $t = 1$, that is, $\phi(t)f(t,x,y)$ and $\psi(t)g(t,x,y)$ may be not continuous functions on $[0,1] \times R^2$, the boundary conditions are integral boundary conditions, and we obtain the results on the existence of at least one solution of BVP (6)-(7); third, $0 < p < \beta$ and $0 < q < \alpha$ are supposed; the growth assumptions imposed on f, g, G, H, M, N and I, I_1, J, J_1 are allowed to be sublinear cases. Finally, two examples are given to illustrate the efficiency of the main theorem.

The remainder of this paper is as follows: in Section 2, we present preliminary results. In Section 3, the main theorem and its proof are given. In Section 4, two examples are given to illustrate the main results.

2. Preliminaries

In this section, we present some background definitions and preliminary results.

Definition 1 (see [1]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $g : (0,\infty) \to R$ is given by

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds, \tag{9}$$

provided that the right-hand side exists.

Definition 2 (see [1]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $g : (0,\infty) \to R$ is given by

$$D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}} ds, \tag{10}$$

where $n-1 \le \alpha < n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 3. $K : (0,1)\times R^2 \to R$ is called a β-Caratheodory function if K satisfies that

(i) $t \to K(t, t^{\beta-2}U, t^{\beta-p-2}V)$ is continuous on $(t_k, t_{k+1}]$ $(k=0,1)$ for every $(U,V) \in R^2$;

(ii) $(U,V) \to K(t, t^{\beta-2}U, t^{\beta-p-2}V)$ is continuous on R^2 for every $t \in (0,1)$;

(iii) for each $r > 0$ there exists a constant $A_r > 0$ such that $|K(t, t^{\beta-2}U, t^{\beta-p-2}V)| \le A_r$, $t \in (0,1)$, $|U|, |V| \le r$.

Definition 4. $Q : (0,1)\times R^2 \to R$ is called a α-Caratheodory function if Q satisfies that

(i) $t \to Q(t, t^{\alpha-2}U, t^{\alpha-q-2}V)$ is continuous on $(t_k, t_{k+1}]$ $(k=0,1)$ for every $(U,V) \in R^2$;

(ii) $(U,V) \to Q(t, t^{\alpha-2}U, t^{\alpha-q-2}V)$ is continuous on R^2 for every $t \in (0,1)$;

(iii) for each $r > 0$ there exists a constant $B_r > 0$ such that $|Q(t, t^{\alpha-2}U, t^{\alpha-q-2}V)| \le B_r$, $t \in (0,1)$, $|U|, |V| \le r$.

Lemma 5 (the Leray-Schauder nonlinear alternative [23]). *Let X be a Banach space and $T : X \to X$ be a completely continuous operator. Suppose Ω is a nonempty open subset of X centered at zero. Then either there exists $x \in \partial\Omega$ and $\lambda \in (0,1)$ such that $x = \lambda T x$ or there exists $x \in \overline{\Omega}$ such that $x = Tx$.*

Let the gamma and beta functions $\Gamma(\alpha)$ and $\mathbf{B}(p,q)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx,$$
$$\mathbf{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, \tag{11}$$
$$\|m\|_1 = \int_0^1 |m(s)|\, ds \quad \text{for } m \in L^1(0,1).$$

Choose

$$X = \left\{ x : (0,1] \longrightarrow R \;\; \begin{array}{l} x|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}]\ (k=0,1), \\ D_{0^+}^{q} x|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}]\ (k=0,1), \\ \text{there exist the limits} \\ \lim_{t\to t_k^+} t^{2-\alpha} x(t), \\ \lim_{t\to t_k^+} t^{2+q-\alpha} D_{0^+}^{q} x(t) \end{array} \right\},$$

$$Y = \left\{ y : (0,1] \longrightarrow R \;\; \begin{array}{l} y|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}]\ (k=0,1), \\ D_{0^+}^{p} x|_{(t_k,t_{k+1}]} \in C^0(t_k,t_{k+1}]\ (k=0,1), \\ \text{there exist the limits} \\ \lim_{t\to t_k^+} t^{2-\beta} y(t), \\ \lim_{t\to t_k^+} t^{2+p-\beta} D_{0^+}^{p} y(t) \end{array} \right\}. \tag{12}$$

For $x \in X$, define the norm by

$$\begin{aligned}\|x\| &= \|x\|_X \\ &= \max\left\{ \sup_{t\in(0,1)} t^{2-\alpha}|x(t)|,\ \sup_{t\in(0,1)} t^{2+q-\alpha}\left|D_{0^+}^{q} x(t)\right| \right\}.\end{aligned} \tag{13}$$

It is easy to show that X is a real Banach space. For $y \in Y$, define the norm by

$$\begin{aligned}\|y\| &= \|y\|_Y \\ &= \max\left\{ \sup_{t\in(0,1)} t^{2-\beta}|y(t)|,\ \sup_{t\in(0,1)} t^{2+p-\beta}\left|D_{0^+}^{p} y(t)\right| \right\}.\end{aligned} \tag{14}$$

It is easy to show that Y is a real Banach space. Thus, $(X \times Y, \|\cdot\|)$ is a Banach space with the norm defined by $\|(x,y)\| = \max\{\|x\|_X, \|y\|_Y\}$ for $(x,y) \in X \times Y$.

In this paper, we suppose the following:

(A) ϕ satisfies that there exist constants $L_1 > 0$, $k > -1$, $\delta \in (q-\alpha, 0]$ such that $\alpha + 2\delta - q > 0$, $\alpha + k + \delta - q \ge 0$, and $|\phi(t)| \le L_1 t^k (1-t)^{\delta}$ for all $t \in (0,1)$; ψ satisfies that there exist constants $L_2 > 0$, $l > -1$, $\theta \in (p-\beta, 0]$ such that $\beta + 2\theta - p > 0$, $\beta + l + \theta - p \ge 0$, and $|\psi(t)| \le L_2 t^l (1-t)^{\theta}$ for all $t \in (0,1)$.

(B) f, G, M, I, I_1 are β-Caratheodory functions and g, H, N, J, J_1 are α-Caratheodory functions.

Remark 6. Suppose that f is a β-Caratheodory function. For example, $\alpha = 7/4$, $q = 1/8$, choose $k = -1/2$, $\delta = -3/4$ and $\phi(t) = t^k(1-t)^{\delta}$, then $k > -1$, $\delta \in (-\alpha, 0]$ such that $\alpha + 2\delta - q > 0$, $\alpha + k + \delta - q \ge 0$, and $|\phi(t)| \le t^k(1-t)^{\delta}$ for all $t \in (0,1)$. It is easy to see that ϕ is singular at $t = 0$ and $t = 1$.

Lemma 7. *Suppose that $y \in Y$, and (a)–(e), (A)-(B) hold. Then $x \in X$ is a solution of*

$$\begin{gathered} D_{0^+}^{\alpha} x(t) = \phi(t) f\left(t, y(t), D_{0^+}^{p} y(t)\right), \quad t \in (0,1),\ t \neq t_1, \\ \lim_{t\to 0} t^{2-\alpha} x(t) = \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds, \\ x(1) = \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds, \\ \Delta x(t_1) = I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \\ \Delta D_{0^+}^{q} x(t_1) = I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \end{gathered} \tag{15}$$

if and only if $x \in X$ satisfies the integral equation

$$x(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(u) f\left(u, y(u), D_{0^+}^{p} y(u)\right) du \\ -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ \times \int_0^1 (1-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ + t^{\alpha-2} \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ + t^{\alpha-1} \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ + \frac{t^{\alpha-1}}{\Pi} \\ \times \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2} \right) \\ \times I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \\ + \frac{t^{\alpha-1}\left(t_1^{\alpha-2} - t_1^{\alpha-1}\right)}{\Pi} \\ \times I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \quad t \in (0, t_1], \\ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \\ \times \int_0^1 (1-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ + \left(t^{\alpha-2} - t^{\alpha-1}\right) \\ \times \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ + t^{\alpha-1} \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ + \frac{t^{\alpha-1} - t^{\alpha-2}}{\Pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} \\ \times I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \\ + \frac{t^{\alpha-2} - t^{\alpha-1}}{\Pi} t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \\ \qquad t \in (t_1, 1], \end{cases} \tag{16}$$

where

$$\Pi = \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \right) t_1^{2\alpha-q-3}. \tag{17}$$

Proof. If $y \in Y$ is a solution of BVP (15), then

$$\begin{aligned} \|y\| &= \max\left\{ \sup_{t\in(0,1)} t^{2-\beta} \left|y(t)\right|, \sup_{t\in(0,1)} t^{2+p-\beta} \left|D_{0^+}^{p} y(t)\right| \right\} \\ &= r < +\infty, \end{aligned} \tag{18}$$

and x satisfies all equations in (31) From (B), f is a β-Caratheodory function, then there exists $A_r > 0$ such that

$$\begin{aligned} &\left| f\left(t, y(t), D_{0^+}^{p} y(t)\right) \right| \\ &\quad = \left| f\left(t, t^{\beta-2} t^{2-\beta} y(t), t^{\beta-p-2} t^{2+p-\beta} D_{0^+}^{p} y(t)\right) \right| \le A_r. \end{aligned} \tag{19}$$

Similarly we get that there exist constants $A_r', A_r'', B_r', B_r'' > 0$ such that

$$\begin{gathered} \left| G\left(t, y(t), D_{0^+}^{p} y(t)\right) \right| \le A_r', \\ \left| M\left(t, y(t), D_{0^+}^{p} y(t)\right) \right| \le A_r'', \\ t \in (0,1), \\ \left| I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \right| \le B_r', \\ \left| I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \right| \le B_r''. \end{gathered} \tag{20}$$

It follows from (15) that, for $t \in (t_k, t_{k+1}] (k = 0, 1)$, there exist constants $c_k, d_k \in R$ such that

$$\begin{aligned} x(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ & + c_k t^{\alpha-1} + d_k t^{\alpha-2}, \quad t \in (t_k, t_{k+1}], \ k = 0, 1. \end{aligned} \tag{21}$$

From $\lim_{t\to 0} t^{2-\alpha} x(t) = \int_0^1 u(s) G(s, y(s), D_{0^+}^{p} y(s)) ds$, we get

$$d_0 = \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds. \tag{22}$$

From $x(1) = \int_0^1 m(s) M(s, y(s), D_{0^+}^{p} y(s)) ds$, we get

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds + c_1 + d_1 \\ & \quad = \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds. \end{aligned} \tag{23}$$

From $\Delta x(t_1) = I(t_1, y(t_1), D_{0^+}^{p} y(t_1))$, we get

$$(c_1 - c_0) t_1^{\alpha-1} + (d_1 - d_0) t_1^{\alpha-2} = I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right). \tag{24}$$

From $\Delta D_{0^+}^q x(t_1) = I_1(t_1, y(t_1), D_{0^+}^p y(t_1))$, we get

$$(c_1 - c_0)\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} + (d_1 - d_0)\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2} = I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right). \tag{25}$$

It follows that

$$\begin{aligned} c_1 - c_0 &= \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2} I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) - t_1^{\alpha-2} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right) \times (\Pi)^{-1}, \\ d_1 - d_0 &= \left(t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right) \times (\Pi)^{-1}. \end{aligned} \tag{26}$$

Then

$$d_1 = \left(t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right) \times (\Pi)^{-1} + \int_0^1 u(s)\, G\left(s, y(s), D_{0^+}^p y(s)\right) ds. \tag{27}$$

So

$$\begin{aligned} c_1 &= \int_0^1 m(s)\, M\left(s, y(s), D_{0^+}^p y(s)\right) ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi(s)\, f\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\quad - \left(t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right) \times (\Pi)^{-1} \\ &\quad - \int_0^1 u(s)\, G\left(s, y(s), D_{0^+}^p y(s)\right) ds, \\ c_0 &= \int_0^1 m(s)\, M\left(s, y(s), D_{0^+}^p y(s)\right) ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi(s)\, f\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\quad - \left(t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right) \times (\Pi)^{-1} \\ &\quad - \int_0^1 u(s)\, G\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\quad - \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2} I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) - t_1^{\alpha-2} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right) \times (\Pi)^{-1}. \end{aligned} \tag{28}$$

Hence, for $t \in (0, t_1]$, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)\, f\left(u, y(u), D_{0^+}^p y(u)\right) du - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi(s)\, f\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\quad + t^{\alpha-2}\int_0^1 u(s)\, G\left(s, y(s), D_{0^+}^p y(s)\right) ds + t^{\alpha-1}\int_0^1 m(s)\, M\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\quad + \frac{t^{\alpha-1}}{\Pi}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right) \times I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) \\ &\quad + \frac{t^{\alpha-1}\left(t_1^{\alpha-2} - t_1^{\alpha-1}\right)}{\Pi} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right). \end{aligned} \tag{29}$$

And for $t \in (t_1, 1]$, we have

$$\begin{aligned} x(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(s)\, f\left(s, y(s), D_{0^+}^p y(s)\right) ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi(s)\, f\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\quad + \left(t^{\alpha-2} - t^{\alpha-1}\right)\int_0^1 u(s)\, G\left(s, y(s), D_{0^+}^p y(s)\right) ds + t^{\alpha-1}\int_0^1 m(s)\, M\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\quad + \frac{t^{\alpha-1} - t^{\alpha-2}}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} \times I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) \\ &\quad + \frac{t^{\alpha-2} - t^{\alpha-1}}{\Pi}t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right). \end{aligned} \tag{30}$$

Hence, $x \in X$ satisfies (16).

On the other hand, if $y \in Y$ and $x \in X$ is a solution of (16), then we can prove that $x \in X$ is a solution of BVP (6)-(7). The proof is completed. □

Lemma 8. *Suppose that $x \in X$, and (a)–(e), (A)-(B) hold. Then $y \in Y$ is a solution of*

$$\begin{gathered} D_{0^+}^{\beta} y(t) = \psi(t) g\left(t, x(t), D_{0^+}^{q} x(t)\right), \quad t \in (0,1), \; t \neq t_1, \\ \lim_{t \to 0} t^{2-\beta} y(t) = \int_0^1 v(s) H\left(s, x(s), D_{0^+}^{q} x(s)\right) ds, \\ y(1) = \int_0^1 n(s) N\left(s, x(s), D_{0^+}^{q} x(s)\right) ds, \\ \Delta y(t_1) = J\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right), \\ \Delta D_{0^+}^{p} y(t_1) = J_1\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right), \end{gathered} \tag{31}$$

if and only if $y \in Y$ satisfies the integral equation

$$y(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \psi(u) g\left(u, x(u), D_{0^+}^{q} x(u)\right) du \\ \displaystyle -\frac{t^{\beta-1}}{\Gamma(\beta)} \times \int_0^1 (1-s)^{\beta-1} \psi(s) g\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\ \displaystyle +t^{\beta-2} \times \int_0^1 v(s) H\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +t^{\beta-1} \times \int_0^1 n(s) N\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\ \displaystyle +\frac{t^{\beta-1}}{\Xi} \times \left(\frac{\Gamma(\beta)}{\Gamma(\beta-p)} t_1^{\beta-p-1} - \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-2)} t_1^{\beta-p-2}\right) \\ \times J\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right) \\ \displaystyle +\frac{t^{\beta-1}\left(t_1^{\beta-2} - t_1^{\beta-1}\right)}{\Xi} \\ \times J_1\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right), \quad t \in (0, t_1], \\ \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} \psi(s) g\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\ \displaystyle -\frac{t^{\beta-1}}{\Gamma(\beta)} \times \int_0^1 (1-s)^{\beta-1} \psi(s) g\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\ \displaystyle +\left(t^{\beta-2} - t^{\beta-1}\right) \times \int_0^1 v(s) H\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +t^{\beta-1} \times \int_0^1 n(s) N\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\ \displaystyle +\frac{t^{\beta-1} - t^{\beta-2}}{\Xi} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} t_1^{\beta-p-1} \\ \times J\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right) \\ \displaystyle +\frac{t^{\beta-2} - t^{\beta-1}}{\Xi} t_1^{\beta-1} \\ \times J_1\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right), \quad t \in (t_1, 1], \end{cases} \tag{32}$$

where

$$\Xi = \left(\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)} - \frac{\Gamma(\beta)}{\Gamma(\beta-p)}\right) t_1^{2\beta-p-3}. \tag{33}$$

Proof. The proof is similar to that of the proof of Lemma 7 and is omitted.

Now, we define the operator T on $X \times Y$ by $T(x,y)(t) = ((T_1 y)(t), (T_2 x)(t))$ with

$$(T_1 y)(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(u) f\left(u, y(u), D_{0^+}^{p} y(u)\right) du \\ \displaystyle -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +t^{\alpha-2} \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +t^{\alpha-1} \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +\frac{t^{\alpha-1}}{\Pi} \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2}\right) \\ \times I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \\ \displaystyle +\frac{t^{\alpha-1}\left(t_1^{\alpha-2} - t_1^{\alpha-1}\right)}{\Pi} \\ \times I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \quad t \in (0, t_1], \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle -\frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +\left(t^{\alpha-2} - t^{\alpha-1}\right) \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +t^{\alpha-1} \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\ \displaystyle +\frac{t^{\alpha-1} - t^{\alpha-2}}{\Pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \\ \displaystyle +\frac{t^{\alpha-2} - t^{\alpha-1}}{\Pi} t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \\ \qquad t \in (t_1, 1], \end{cases} \tag{34}$$

$$(T_2x)(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(u)\, g\left(u, x(u), D_{0^+}^q x(u)\right) du \\ \displaystyle -\frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\psi(s)\, g\left(s, x(s), D_{0^+}^q x(s)\right) ds \\ \displaystyle +t^{\beta-2}\int_0^1 v(s)\, H\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ \displaystyle +t^{\beta-1}\int_0^1 n(s)\, N\left(s, x(s), D_{0^+}^q x(s)\right) ds \\ \displaystyle +\frac{t^{\beta-1}}{\Xi}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-p)}t_1^{\beta-p-1} - \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-2)}t_1^{\beta-p-2}\right) \\ \times J\left(t_1, x(t_1), D_{0^+}^q x(t_1)\right) \\ \displaystyle +\frac{t^{\beta-1}\left(t_1^{\beta-2}-t_1^{\beta-1}\right)}{\Xi} J_1\left(t_1, x(t_1), D_{0^+}^q x(t_1)\right), \\ \qquad\qquad t \in (0, t_1], \\ \displaystyle\int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\psi(s)\, g\left(s, x(s), D_{0^+}^q x(s)\right) ds \\ \displaystyle -\frac{t^{\beta-1}}{\Gamma(\beta)}\int_0^1 (1-s)^{\beta-1}\psi(s)\, g\left(s, x(s), D_{0^+}^q x(s)\right) ds \\ \displaystyle +\left(t^{\beta-2}-t^{\beta-1}\right)\int_0^1 v(s)\, H\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ \displaystyle +t^{\beta-1}\int_0^1 n(s)\, N\left(s, x(s), D_{0^+}^q x(s)\right) ds \\ \displaystyle +\frac{t^{\beta-1}-t^{\beta-2}}{\Xi}\frac{\Gamma(\beta)}{\Gamma(\beta-p)}t_1^{\beta-p-1} \\ \times J\left(t_1, x(t_1), D_{0^+}^q x(t_1)\right) \\ \displaystyle +\frac{t^{\beta-2}-t^{\beta-1}}{\Xi}t_1^{\beta-1} J_1\left(t_1, x(t_1), D_{0^+}^q x(t_1)\right), \\ \qquad\qquad t \in (t_1, 1]. \end{cases} \tag{35}$$

□

Remark 9. By Lemmas 7 and 8, $(x, y) \in X \times Y$ is a solution of BVP (6)-(7) if and only if $(x, y) \in X \times Y$ is a fixed point of the operator T.

Lemma 10. *Suppose that (a)–(e) and (A)-(B) hold. Then $T : X \times Y \rightarrow X \times Y$ is well defined and is completely continuous.*

Proof. The proof is very long, so we list the steps. First, we prove that T is well defined; second, we prove that T is continuous, and, finally, we prove that T is compact. So T is completely continuous. Thus, the proof is divided into three steps.

Step 1. Prove that $T : X \times Y \rightarrow X \times Y$ is well defined.

For $(x, y) \in X \times Y$, we have $\|(x, y)\| = r > 0$. Then

$$\max\left\{\sup_{t\in(0,1)} t^{2-\alpha}|x(t)|, \sup_{t\in(0,1)} t^{2+q-\alpha}\left|D_{0^+}^q x(t)\right|\right\} \le r < +\infty,$$

$$\max\left\{\sup_{t\in(0,1)} t^{2-\beta}|y(t)|, \sup_{t\in(0,1)} t^{2+p-\beta}\left|D_{0^+}^p y(t)\right|\right\} \le r < +\infty. \tag{36}$$

From (B), f, G, M, I, I_1 are β-Caratheodory functions, then there exist constants $A_r > 0$ such that

$$\begin{gathered} \left|f\left(t, y(t), D_{0^+}^p y(t)\right)\right| \le A_r, \quad t \in (0,1), \\ \left|G\left(t, y(t), D_{0^+}^p y(t)\right)\right| \le A_r, \quad t \in (0,1), \\ \left|M\left(t, y(t), D_{0^+}^p y(t)\right)\right| \le A_r, \quad t \in (0,1), \\ \left|I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right| \le A_r, \quad t \in (0,1), \\ \left|I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right| \le A_r, \quad t \in (0,1). \end{gathered} \tag{37}$$

Hence,

$$\begin{aligned} &\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)\, f\left(u, y(u), D_{0^+}^p y(u)\right) du\right| \\ &\quad\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|\phi(u)\, f\left(u, y(u), D_{0^+}^p y(u)\right)\right| du \\ &\quad\le A_r L_1 \frac{\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)} < \infty. \end{aligned} \tag{38}$$

From (34), (37), and (38), we see that $(T_1 y)(t)$ is defined on $(0, 1]$, continuous on $(0, t_1]$ and $(t_1, 1]$, respectively. One sees that

$$\begin{aligned} &\lim_{t\rightarrow 0} t^{2-\alpha}(T_1 y)(t) \\ &\quad= \lim_{t\rightarrow 0}\left[t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)\, f\left(u, y(u), D_{0^+}^p y(u)\right) du\right. \\ &\qquad - \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi(s)\, f\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\qquad + \int_0^1 u(s)\, G\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\qquad + t\int_0^1 m(s)\, M\left(s, y(s), D_{0^+}^p y(s)\right) ds \\ &\qquad + \frac{t}{\Pi}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right) \\ &\qquad \times I\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right) \\ &\qquad \left.+\frac{t\left(t_1^{\alpha-2}-t_1^{\alpha-1}\right)}{\Pi} I_1\left(t_1, y(t_1), D_{0^+}^p y(t_1)\right)\right] \\ &\quad= \int_0^1 u(s)\, G\left(s, y(s), D_{0^+}^p y(s)\right) ds, \end{aligned} \tag{39}$$

and there exits the limit $\lim_{t\rightarrow t_1^+}(T_1 y)(t)$.

On the other hand, we have

$$
\begin{aligned}
&D_{0^+}^{q}\left(T_1 y\right)(t) \\
&= \begin{cases}
\displaystyle\int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} \phi(u) f\left(u, y(u), D_{0^+}^{p} y(u)\right) du \\
\displaystyle -\frac{t^{\alpha-q-1}}{\Gamma(\alpha-q)} \\
\displaystyle \times \int_0^1 (1-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + t^{\alpha-q-2} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \\
\displaystyle \times \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + t^{\alpha-q-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\
\displaystyle \times \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + \frac{t^{\alpha-q-1}}{\Pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\
\displaystyle \times \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2}\right) \\
\displaystyle \times I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \\
\displaystyle + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \frac{t^{\alpha-q-1}\left(t_1^{\alpha-2} - t_1^{\alpha-1}\right)}{\Pi} \\
\displaystyle \times I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \qquad t \in (0, t_1], \\
\displaystyle\int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle -\frac{t^{\alpha-q-1}}{\Gamma(\alpha-q)} \\
\displaystyle \times \int_0^1 (1-s)^{\alpha-1} \phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + \left(t^{\alpha-q-2} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} - t^{\alpha-q-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\right) \\
\displaystyle \times \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + t^{\alpha-q-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + \frac{1}{\Pi} \left(t^{\alpha-q-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} - t^{\alpha-q-2} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\right) \\
\displaystyle \times \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \\
\displaystyle + \frac{1}{\Pi} \left(t^{\alpha-q-2} \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} - t^{\alpha-q-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\right) \\
\displaystyle \times t_1^{\alpha-1} I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right), \qquad t \in (t_1, 1],
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
&D_{0^+}^{p}\left(T_2 x\right)(t) \\
&= \begin{cases}
\displaystyle\int_0^t \frac{(t-s)^{\beta-p-1}}{\Gamma(\beta-p)} \psi(u) g\left(u, x(u), D_{0^+}^{q} x(u)\right) du \\
\displaystyle -\frac{t^{\beta-p-1}}{\Gamma(\beta-p)} \\
\displaystyle \times \int_0^1 (1-s)^{\beta-1} \psi(s) g\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\
\displaystyle + t^{\beta-p-2} \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)} \\
\displaystyle \times \int_0^1 v(s) H\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + t^{\beta-p-1} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \\
\displaystyle \times \int_0^1 n(s) N\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\
\displaystyle + \frac{t^{\beta-p-1}}{\Xi} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \\
\displaystyle \times \left(\frac{\Gamma(\beta)}{\Gamma(\beta-p)} t_1^{\beta-p-1} - \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-2)} t_1^{\beta-p-2}\right) \\
\displaystyle \times J\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right) \\
\displaystyle + \frac{t^{\beta-p-1}\left(t_1^{\beta-2} - t_1^{\beta-1}\right)}{\Xi} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \\
\displaystyle \times J_1\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right), \qquad t \in (0, t_1], \\
\displaystyle\int_0^t \frac{(t-s)^{\beta-p-1}}{\Gamma(\beta-p)} \psi(s) g\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\
\displaystyle -\frac{t^{\beta-p-1}}{\Gamma(\beta)} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \\
\displaystyle \times \int_0^1 (1-s)^{\beta-1} \psi(s) g\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\
\displaystyle + \left(t^{\beta-p-2} \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)} - t^{\beta-p-1} \frac{\Gamma(\beta)}{\Gamma(\beta-p)}\right) \\
\displaystyle \times \int_0^1 v(s) H\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
\displaystyle + t^{\beta-p-1} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \int_0^1 n(s) N\left(s, x(s), D_{0^+}^{q} x(s)\right) ds \\
\displaystyle + \frac{1}{\Xi} \left(t^{\beta-p-1} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} - t^{\beta-p-2} \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}\right) \\
\displaystyle \times \frac{\Gamma(\beta)}{\Gamma(\beta-p)} t_1^{\beta-p-1} J\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right) \\
\displaystyle + \frac{1}{\Xi} \left(t^{\beta-p-2} \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)} - t^{\beta-p-1} \frac{\Gamma(\beta)}{\Gamma(\beta-p)}\right) \\
\displaystyle \times t_1^{\beta-1} J_1\left(t_1, x(t_1), D_{0^+}^{q} x(t_1)\right), \qquad t \in (t_1, 1].
\end{cases}
\end{aligned}
\tag{40}
$$

It is easy to see that

$$\left|\int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\phi(u) f\left(u, y(u), D_{0^+}^{p} y(u)\right) du\right| \leq A_r L_1 \frac{\mathbf{B}(\alpha+\delta-q, k+1)}{\Gamma(\alpha-q)} < \infty. \tag{41}$$

From (37) and (41), we see that $D_{0^+}^{q}(T_1 y)(t)$ is defined on $(0,1]$, continuous on $(0,t_1]$ and $(t_1,1]$, respectively. One sees that

$$\begin{aligned}
&\lim_{t\to 0} t^{2+q-\alpha} D_{0^+}^{q}(T_1 y)(t) \\
&\quad = \lim_{t\to 0}\Bigg[t^{2+q-\alpha} \\
&\qquad \times \int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\phi(u) f\left(u, y(u), D_{0^+}^{p} y(u)\right) du \\
&\qquad - \frac{t}{\Gamma(\alpha-q)} \\
&\qquad \times \int_0^1 (1-s)^{\alpha-1}\phi(s) f\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
&\qquad + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
&\qquad + t\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \int_0^1 m(s) M\left(s, y(s), D_{0^+}^{p} y(s)\right) ds \\
&\qquad + \frac{t}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\
&\qquad \times \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2}\right) \\
&\qquad \times I\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right) \\
&\qquad + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\frac{t\left(t_1^{\alpha-2} - t_1^{\alpha-1}\right)}{\Pi} \\
&\qquad \times I_1\left(t_1, y(t_1), D_{0^+}^{p} y(t_1)\right)\Bigg] \\
&\quad = \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \int_0^1 u(s) G\left(s, y(s), D_{0^+}^{p} y(s)\right) ds,
\end{aligned} \tag{42}$$

and there exits the limit $\lim_{t\to t_1^+} D_{0^+}^{q}(T_1 y)(t)$.

From the above discussion, we have $(T_1 y) \in X$. Similarly, we can show that $(T_2 x) \in Y$. Hence, $((T_1 y), (T_2 x)) \in X \times Y$. Then $T : X \times Y \to X \times Y$ is well defined.

Step 2. We prove that T is continuous. Let $(x_n, y_n) \in X \times Y$ with $(x_n, y_n) \to (x_0, y_0)$ as $n \to \infty$. We will show that $T(x_n, y_n) \to T(x_0, y_0)$ as $n \to \infty$, that is, prove that $T_1 y_n \to T_1 y_0$ and $T_2 x_n \to T_2 x_0$ as $n \to \infty$.

In fact, we have $r > 0$ such that $\|(x_n, y_n)\| = r > 0$. Then

$$\begin{aligned}
&\max\left\{\sup_{t\in(0,1)} t^{2-\alpha}\left|x_n(t)\right|, \sup_{t\in(0,1)} t^{2+q-\alpha}\left|D_{0^+}^{q} x_n(t)\right|\right\} \\
&\quad \leq r < +\infty, \quad n = 0, 1, 2, \ldots, \\
&\max\left\{\sup_{t\in(0,1)} t^{2-\beta}\left|y_n(t)\right|, \sup_{t\in(0,1)} t^{2+p-\beta}\left|D_{0^+}^{p} y_n(t)\right|\right\} \\
&\quad \leq r < +\infty, \quad n = 0, 1, 2, \ldots.
\end{aligned} \tag{43}$$

From (B), f, G, M, I, I_1 are β-Caratheodory functions, then there exist constants $A_r > 0$ such that

$$\begin{gathered}
\left|f\left(t, y_n(t), D_{0^+}^{p} y_n(t)\right)\right| \leq A_r, \\
t \in (0,1), \quad n = 0, 1, 2, \ldots, \\
\left|G\left(t, y_n(t), D_{0^+}^{p} y_n(t)\right)\right| \leq A_r, \\
t \in (0,1), \quad n = 0, 1, 2, \ldots, \\
\left|M\left(t, y_n(t), D_{0^+}^{p} y_n(t)\right)\right| \leq A_r, \\
t \in (0,1), \quad n = 0, 1, 2, \ldots, \\
\left|I\left(t_1, y_n(t_1), D_{0^+}^{p} y_n(t_1)\right)\right| \leq A_r, \\
t \in (0,1), \quad n = 0, 1, 2, \ldots, \\
\left|I_1\left(t_1, y_n(t_1), D_{0^+}^{p} y_n(t_1)\right)\right| \leq A_r, \\
t \in (0,1), \quad n = 0, 1, 2, \ldots, \\
\sup_{t\in(0,1)} t^{2-\alpha}\left|x_n(t) - x_0(t)\right| \longrightarrow 0, \\
\sup_{t\in(0,1)} t^{2-\beta}\left|y_n(t) - y_0(t)\right|, \\
\sup_{t\in(0,1)} t^{2+q-\alpha}\left|D_{0^+}^{q} x_n(t) - D_{0^+}^{q} x_0(t)\right| \longrightarrow 0, \\
\sup_{t\in(0,1)} t^{2+p-\beta}\left|D_{0^+}^{p} y_n(t) - D_{0^+}^{p} y_0(t)\right| \longrightarrow 0,
\end{gathered} \tag{44}$$

as $n \to \infty$. We have

$$D^q_{0^+}(T_1 y_n)(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\phi(u) f\left(u, y_n(u), D^p_{0^+} y_n(u)\right) du \\ \displaystyle -\frac{t^{\alpha-q-1}}{\Gamma(\alpha-q)} \\ \displaystyle \times \int_0^1 (1-s)^{\alpha-1}\phi(s) f\left(s, y_n(s), D^p_{0^+} y_n(s)\right) ds \\ \displaystyle +t^{\alpha-q-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \\ \displaystyle \times \int_0^1 u(s) G\left(s, y_n(s), D^p_{0^+} y_n(s)\right) ds \\ \displaystyle +t^{\alpha-q-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\ \displaystyle \times \int_0^1 m(s) M\left(s, y_n(s), D^p_{0^+} y_n(s)\right) ds \\ \displaystyle +\frac{t^{\alpha-q-1}}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\ \displaystyle \times\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2}\right) \\ \times I\left(t_1, y_n(t_1), D^p_{0^+} y_n(t_1)\right) \\ \displaystyle +\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\frac{t^{\alpha-q-1}\left(t_1^{\alpha-2} - t_1^{\alpha-1}\right)}{\Pi} \\ \times I_1\left(t_1, y_n(t_1), D^p_{0^+} y_n(t_1)\right), \quad t \in (0, t_1], \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\phi(s) f\left(s, y_n(s), D^p_{0^+} y_n(s)\right) ds \\ \displaystyle -\frac{t^{\alpha-q-1}}{\Gamma(\alpha-q)} \\ \displaystyle \times \int_0^1 (1-s)^{\alpha-1}\phi(s) f\left(s, y(s), D^p_{0^+} y(s)\right) ds \\ \displaystyle +\left(t^{\alpha-q-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} - t^{\alpha-q-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\right) \\ \displaystyle \times \int_0^1 u(s) G\left(s, y_n(s), D^p_{0^+} y_n(s)\right) ds \\ \displaystyle +t^{\alpha-q-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\ \displaystyle \times \int_0^1 m(s) M\left(s, y_n(s), D^p_{0^+} y_n(s)\right) ds \\ \displaystyle +\frac{1}{\Pi}\left(t^{\alpha-q-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} - t^{\alpha-q-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\right) \\ \displaystyle \times\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} I\left(t_1, y_n(t_1), D^p_{0^+} y_n(t_1)\right) \\ \displaystyle +\frac{1}{\Pi}\left(t^{\alpha-q-2}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} - t^{\alpha-q-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\right) \\ \times t_1^{\alpha-1} I_1\left(t_1, y_n(t_1), D^p_{0^+} y_n(t_1)\right), \quad t \in (t_1, 1]. \end{cases} \tag{45}$$

From the Lebesgue dominated convergence theorem, we get

$$\begin{gathered} \sup_{t\in(0,1)} t^{2-\beta}\left|(T_1 y_n)(t) - (T_1 y_0)(t)\right|, \\ \sup_{t\in(0,1)} t^{2+p-\beta}\left|D^p_{0^+}(T_1 y_n)(t) - D^p_{0^+}(T_1 y_0)(t)\right| \longrightarrow 0, \end{gathered} \tag{46}$$

as $n \to \infty$. Similarly, we can show that

$$\begin{gathered} \sup_{t\in(0,1)} t^{2-\alpha}\left|(T_2 x_n)(t) - (T_2 x_0)(t)\right| \longrightarrow 0, \\ \sup_{t\in(0,1)} t^{2+q-\alpha}\left|D^q_{0^+}(T_2 x_n)(t) - D^q_{0^+}(T_2 x_0)(t)\right| \longrightarrow 0, \end{gathered} \tag{47}$$

as $n \to \infty$. It follows from (46) and (47) that T is continuous.

Step 3. We prove that T is compact, that is, for each nonempty open bounded subset Ω of $X\times Y$, prove that $T(\overline{\Omega})$ is relatively compact. We must prove that $T(\overline{\Omega})$ is uniformly bounded, equicontinuous on each subinterval $[a, b] \subseteq (t_k, t_{k+1}]$ $(k = 0, 1)$, $T(\overline{\Omega})$ is equiconvergent as $t \to 0$, and equiconvergent as $t \to t_1$.

Let Ω be a bounded open subset of Y. We have $r > 0$ such that

$$\begin{aligned} &\max\left\{\sup_{t\in(0,1)} t^{2-\beta}\left|y(t)\right|, \sup_{t\in(0,1)} t^{2+p-\beta}\left|D^p_{0^+} y(t)\right|\right\} \\ &\quad \leq r < +\infty, \quad y \in \overline{\Omega}. \end{aligned} \tag{48}$$

From (B), f, G, M, I, I_1 are β-Caratheodory functions, then there exist constants $A_r > 0$ such that

$$\begin{gathered} \left|f\left(t, y(t), D^p_{0^+} y(t)\right)\right| \leq A_r, \quad t \in (0,1), \\ \left|G\left(t, y(t), D^p_{0^+} y(t)\right)\right| \leq A_r, \quad t \in (0,1), \\ \left|M\left(t, y(t), D^p_{0^+} y(t)\right)\right| \leq A_r, \quad t \in (0,1), \\ \left|I\left(t_1, y(t_1), D^p_{0^+} y(t_1)\right)\right| \leq A_r, \quad t \in (0,1), \\ \left|I_1\left(t_1, y(t_1), D^p_{0^+} y(t_1)\right)\right| \leq A_r, \quad t \in (0,1). \end{gathered} \tag{49}$$

Substep 3.1. Prove that $T(\overline{\Omega})$ is uniformly bounded.

In fact, for $t \in (0, t_1]$, use (49), we have

$$\begin{aligned} &t^{2-\alpha}\left|(T_1 y)(t)\right| \\ &\quad \leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|\phi(u) f\left(u, y(u), D^p_{0^+} y(u)\right)\right| du \\ &\qquad + \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|\phi(s) f\left(s, y(s), D^p_{0^+} y(s)\right)\right| ds \\ &\qquad + \int_0^1 \left|u(s) G\left(s, y(s), D^p_{0^+} y(s)\right)\right| ds \\ &\qquad + t\int_0^1 \left|m(s) M\left(s, y(s), D^p_{0^+} y(s)\right)\right| ds \end{aligned}$$

$$\begin{aligned}
&+\frac{t}{\Pi}\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1}-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right| \\
&\times\left|I\left(t_1,y\left(t_1\right),D_{0^+}^{p}y\left(t_1\right)\right)\right| \\
&+\frac{t\left(t_1^{\alpha-2}-t_1^{\alpha-1}\right)}{\Pi}\left|I_1\left(t_1,y\left(t_1\right),D_{0^+}^{p}y\left(t_1\right)\right)\right| \\
\le{}& A_rL_1\frac{\mathbf{B}(\alpha+\delta,k+1)}{\Gamma(\alpha)}+\frac{A_rL_1}{\Gamma(\alpha)}\mathbf{B}(\alpha+\delta,k+1) \\
&+A_r\|u\|_1+A_r\|m\|_1 \\
&+\frac{A_r}{\Pi}\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1}-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right| \\
&+\frac{\left(t_1^{\alpha-2}-t_1^{\alpha-1}\right)}{\Pi}A_r<\infty.
\end{aligned}\tag{50}$$

Similarly, we can get for $t\in(t_1,1]$ that

$$\begin{aligned}
&t^{2-\alpha}\left|\left(T_1y\right)(t)\right| \\
&\le A_rL_1\frac{\mathbf{B}(\alpha+\delta,k+1)}{\Gamma(\alpha)}+\frac{A_rL_1}{\Gamma(\alpha)}\mathbf{B}(\alpha+\delta,k+1) \\
&\quad+A_r\|u\|_1+A_r\|m\|_1+\frac{A_r}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} \\
&\quad+\frac{A_r}{\Pi}t_1^{\alpha-1}<\infty.
\end{aligned}\tag{51}$$

Furthermore, we have for $t\in(0,t_1]$ that

$$\begin{aligned}
&t^{2+q-\alpha}\left|D_{0^+}^{q}\left(T_1y\right)(t)\right| \\
&\le A_rL_1\frac{\mathbf{B}(\alpha+\delta-q,k+1)}{\Gamma(\alpha-q)} \\
&\quad+\frac{A_rL_1}{\Gamma(\alpha-q)}\mathbf{B}(\alpha+\delta,k+1) \\
&\quad+\frac{A_r\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\|u\|_1+\frac{A_r\Gamma(\alpha)}{\Gamma(\alpha-q)}\|m\|_1 \\
&\quad+\frac{A_r}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\
&\quad\times\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1}-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right| \\
&\quad+\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\frac{A_r\left(t_1^{\alpha-2}-t_1^{\alpha-1}\right)}{\Pi}<\infty,
\end{aligned}\tag{52}$$

and for $t\in(t_1,1]$ that

$$\begin{aligned}
&t^{2+q-\alpha}\left|D_{0^+}^{q}\left(T_1y\right)(t)\right| \\
&\le A_rL_1\frac{\mathbf{B}(\alpha+\delta-q,k+1)}{\Gamma(\alpha-q)}+\frac{A_rL_1}{\Gamma(\alpha-q)}\mathbf{B}(\alpha+\delta,k+1) \\
&\quad+\left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\right)A_r\|u\|_1 \\
&\quad+\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}A_r\|m\|_1 \\
&\quad+\frac{1}{\Pi}\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}+\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\right)\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1}A_r \\
&\quad+\frac{1}{\Pi}\left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\right)t_1^{\alpha-1}A_r<\infty.
\end{aligned}\tag{53}$$

Hence,

$$\max\left\{\sup_{t\in(0,1)}t^{2-\alpha}\left|(Ty)(t)\right|,\ \sup_{t\in(0,1)}t^{2+q-\alpha}\left|D_{0^+}^{q}(Ty)(t)\right|\right\}<+\infty,\quad y\in\overline{\Omega}.\tag{54}$$

Similarly, we can show that

$$\max\left\{\sup_{t\in(0,1)}t^{2-\beta}\left|(Tx)(t)\right|,\ \sup_{t\in(0,1)}t^{2+p-\beta}\left|D_{0^+}^{p}(Tx)(t)\right|\right\}<+\infty,\quad y\in\overline{\Omega}.\tag{55}$$

It is easy to see that $T(\overline{\Omega})$ is uniformly bounded.

Substep 3.2. Prove that $T(\overline{\Omega})$ is equicontinuous on each subinterval $[a,b]\subseteq(t_k,t_{k+1}]$ $(k=0,1)$.

For each $[a,b]\subseteq(t_0,t_1]$, and $s_1,s_2\in[a,b]$ with $s_2<s_1$, we have

$$\begin{aligned}
&\left|s_1^{2-\alpha}(Ty)\left(s_1\right)-s_2^{2-\alpha}(Ty)\left(s_2\right)\right| \\
&\le\left|s_1^{2-\alpha}\int_0^{s_1}\frac{\left(s_1-s\right)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)\,f\left(u,y(u),D_{0^+}^{p}y(u)\right)du\right. \\
&\qquad\left.-s_2^{2-\alpha}\int_0^{s_2}\frac{\left(s_2-s\right)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)\,f\left(u,y(u),D_{0^+}^{p}y(u)\right)du\right| \\
&\quad+\left|s_1-s_2\right|A_r \\
&\quad\times\left[\frac{L_1\mathbf{B}(\alpha+\delta,k+1)}{\Gamma(\alpha)}+\|m\|_1\right. \\
&\qquad+\frac{1}{\Pi}\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1}-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right| \\
&\qquad\left.+\frac{t_1^{\alpha-2}-t_1^{\alpha-1}}{\Pi}\right].
\end{aligned}\tag{56}$$

Note that $|\tau_1^{\varrho}-\tau_2^{\varrho}|\le|\tau_1-\tau_2|^{\varrho}$ for all $\tau_1,\tau_2\ge0$ and $\varrho\in(0,1)$.

Since

$$\begin{aligned}&\left|s_1^{2-\alpha}\int_0^{s_1}\frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)f\left(u,y(u),D_{0^+}^{p}y(u)\right)du\right.\\&\left.-s_2^{2-\alpha}\int_0^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}\phi(u)f\left(u,y(u),D_{0^+}^{p}y(u)\right)du\right|\\&\leq\left|s_1^{2-\alpha}-s_2^{2-\alpha}\right|A_rL_1\frac{\mathbf{B}(\alpha+\delta,k+1)}{\Gamma(\alpha)}\\&\quad+b^{2-\alpha}A_rL_1s_1^{\alpha+k+\delta}\int_{s_2/s_1}^{1}\frac{(1-w)^{\alpha+\delta-1}w^kdw}{\Gamma(\alpha)}\\&\quad+\left|s_1-s_2\right|^{\alpha-1}\frac{A_rL_1b^{2-\alpha}}{\Gamma(\alpha)}\int_0^b s^k(1-s)^{\delta}ds\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{57}$$

It follows that

$$\begin{aligned}&\left|s_1^{2-\alpha}(Ty)(s_1)-s_2^{2-\alpha}(Ty)(s_2)\right|\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{58}$$

For $[a,b]\subseteq(t_1,1]$, and $s_1,s_2\in[a,b]$ with $s_2<s_1$, we can prove similarly that

$$\begin{aligned}&\left|s_1^{2-\alpha}(Ty)(s_1)-s_2^{2-\alpha}(Tyx)(s_2)\right|\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{59}$$

On the other hand, for $[a,b]\subseteq(t_0,t_1]$, and $s_1,s_2\in[a,b]$ with $s_2<s_1$, we have

$$\begin{aligned}&\left|s_1^{2+q-\alpha}D_{0^+}^{q}(T_1y)(s_1)-s_2^{2+q-\alpha}D_{0^+}^{q}(T_1y)(s_2)\right|\\&\leq\left|s_1^{2+q-\alpha}\int_0^{s_1}\frac{(s_1-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\phi(u)f\left(u,y(u),D_{0^+}^{p}y(u)\right)du\right.\\&\quad\left.-s_2^{2+q-\alpha}\int_0^{s_2}\frac{(s_2-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}\phi(u)f\left(u,y(u),D_{0^+}^{p}y(u)\right)du\right|\\&\quad+\left|s_2-s_1\right|A_r\left[\frac{L_1B(\alpha+\delta,k+1)}{\Gamma(\alpha-q)}+\frac{\Gamma(\alpha)\|m\|_1}{\Gamma(\alpha-q)}\right.\\&\qquad+\frac{1}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\\&\qquad\times\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1}-\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right|\\&\qquad\left.+\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\frac{\left|t_1^{\alpha-2}-t_1^{\alpha-1}\right|}{\Pi}\right].\end{aligned}\tag{60}$$

It is easy to see that

$$\begin{aligned}&\left|s_1^{2+q-\alpha}-s_2^{2+q-\alpha}\right|A_rL_1\int_0^1\frac{(1-w)^{\alpha+\delta-q-1}w^kds}{\Gamma(\alpha-q)}\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2,\end{aligned}\tag{61}$$

$$\begin{aligned}&b^{2+q-\alpha}A_rL_1\int_{s_2/s_1}^{1}\frac{(1-w)^{\alpha-q-1}w^kdw}{\Gamma(\alpha-q)}\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{62}$$

For the third term, if $\alpha-q-1\geq 0$, use $|\tau_1^{\varrho}-\tau_2^{\varrho}|\leq|\tau_1-\tau_2|^{\varrho}$, then

$$\begin{aligned}&\int_0^{s_2}\frac{\left|(s_1-s)^{\alpha-q-1}-(s_2-s)^{\alpha-q-1}\right|}{\Gamma(\alpha-q)}s^k(1-s)^{\delta}ds\\&\leq\left|s_1-s_2\right|^{\alpha-q-1}\int_0^b\frac{1}{\Gamma(\alpha-q)}s^k(1-s)^{\delta}ds\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{63}$$

If $\alpha-q-1<0$, use $|\tau_1^{\varrho}-\tau_2^{\varrho}|\leq|\tau_1-\tau_2|^{\varrho}$, then

$$\begin{aligned}&\int_0^{s_2}\frac{\left|(s_1-s)^{\alpha-q-1}-(s_2-s)^{\alpha-q-1}\right|}{\Gamma(\alpha-q)}s^k(1-s)^{\delta}ds\\&=\int_0^{s_2}\frac{(s_2-s)^{\alpha-q-1}-(s_1-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}s^k(s_1-s)^{\delta}ds\\&\leq\left|s_2^{\alpha+\delta+k-q}-s_1^{\alpha+\delta+k-q}\right|\frac{\mathbf{B}(\alpha+\delta+k-q,k+1)}{\gamma(\alpha-q)}\\&\quad+b^{\alpha+k+\delta-q}\frac{\mathbf{B}(\alpha+2\delta-q,k+1)}{\Gamma(\alpha-q)}\left|1-\frac{s_1}{s_2}\right|^{-\delta}\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{64}$$

For $[a,b]\subseteq(t_1,1]$, and $s_1,s_2\in[a,b]$ with $s_2<s_1$, we can prove similarly that

$$\begin{aligned}&\left|s_1^{2+q-\alpha}D_{0^+}^{q}(T_1y)(s_1)-s_2^{2+q-\alpha}D_{0^+}^{q}(T_1y)(s_2)\right|\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{65}$$

Similarly, we can show that for each $[a,b]\subseteq(t_0,t_1]$, and $s_1,s_2\in[a,b]$ with $s_2<s_1$, we have

$$\begin{aligned}&\left|s_1^{2-\beta}(Tx)(s_1)-s_2^{2-\beta}(Tx)(s_2)\right|\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{66}$$

For $[a,b]\subseteq(t_1,1]$, and $s_1,s_2\in[a,b]$ with $s_2<s_1$, we can prove similarly that

$$\begin{aligned}&\left|s_1^{2-\beta}(Tx)(s_1)-s_2^{2-\beta}(Tx)(s_2)\right|\longrightarrow 0\\&\qquad\text{uniformly as}\quad s_1\longrightarrow s_2.\end{aligned}\tag{67}$$

For each $[a,b] \subseteq (t_0, t_1]$, and $s_1, s_2 \in [a,b]$ with $s_2 < s_1$, we have

$$\left| s_1^{2+p-\beta} D_{0^+}^p (Tx)(s_1) - s_2^{2+p-\beta} D_{0^+}^p (Tx)(s_2) \right| \longrightarrow 0 \quad \text{uniformly as } s_1 \longrightarrow s_2. \tag{68}$$

For $[a,b] \subseteq (t_1, 1]$, and $s_1, s_2 \in [a,b]$ with $s_2 < s_1$, we can prove similarly that

$$\left| s_1^{2+p-\beta} D_{0^+}^p (Tx)(s_1) - s_2^{2+p-\beta} D_{0^+}^p (Tx)(s_2) \right| \longrightarrow 0 \quad \text{uniformly as } s_1 \longrightarrow s_2. \tag{69}$$

So $T(\overline{\Omega})$ is equicontinuous on each subinterval $[a,b] \subseteq (t_k, t_{k+1}] (k = 0, 1)$.

Substep 3.3. Prove that $T(\overline{\Omega})$ is equiconvergent as $t \to 0$, and equiconvergent as $t \to t_1$.

We have

$$\begin{aligned} &\left| t^{2-\alpha} (Ty)(t) - \int_0^1 u(s) G\left(s, y(s), D_{0^+}^p y(s)\right) ds \right| \\ &\leq A_r L_1 t^{2+k+\delta} \frac{\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)} \\ &\quad + \frac{t}{\Gamma(\alpha)} A_r L_1 \mathbf{B}(\alpha+\delta, k+1) \\ &\quad + t A_r \|m\|_1 \\ &\quad + \frac{t}{\Pi} \left| \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2} \right| A_r \\ &\quad + \frac{t \left| t_1^{\alpha-2} - t_1^{\alpha-1} \right|}{\Pi} A_r. \end{aligned} \tag{70}$$

It follows that

$$\left| t^{2-\alpha} (Ty)(t) - \int_0^1 u(s) G\left(s, y(s), D_{0^+}^p y(s)\right) ds \right| \longrightarrow 0 \quad \text{uniformly as } t \longrightarrow 0. \tag{71}$$

Similarly, we can show that $t^{2-\alpha}(Ty)(t)$ is equiconvergent at $t = t_1$. On the other hand, we have

$$\begin{aligned} &\left| t^{2+q-\alpha} D_{0^+}^q (Ty)(t) - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \times \int_0^1 u(s) G\left(s, y(s), D_{0^+}^p y(s)\right) ds \right| \\ &\leq A_r L_1 t^{2+k+\delta} \frac{\mathbf{B}(\alpha+\delta-q, k+1)}{\Gamma(\alpha-q)} \\ &\quad + \frac{t}{\Gamma(\alpha-q)} A_r L_1 \mathbf{B}(\alpha+\delta, k+1) \\ &\quad + t \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} A_r \|m\|_1 + \frac{t}{\Pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\ &\quad \times \left| \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2} \right| A_r \\ &\quad + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \frac{t \left| t_1^{\alpha-2} - t_1^{\alpha-1} \right|}{\Pi} A_r. \end{aligned} \tag{72}$$

It follows that

$$\left| t^{2+q-\alpha} D_{0^+}^q (Ty)(t) - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \times \int_0^1 u(s) G\left(s, y(s), D_{0^+}^p y(s)\right) ds \right| \longrightarrow 0 \quad \text{uniformly as } t \longrightarrow 0. \tag{73}$$

Similarly, we can show that $t^{2+q-\alpha} D_{0^+}^q (Ty)(t)$ is equiconvergent at $t = t_1$.

Similarly we can prove that

$$\left| t^{2-\beta} (Tx)(t) - \int_0^1 v(s) H\left(s, x(s), D_{0^+}^q x(s)\right) ds \right| \longrightarrow 0 \quad \text{uniformly as } t \longrightarrow 0, \tag{74}$$

and $t^{2+p-\beta} D_{0^+}^p (Tx)(t)$ is equiconvergent at $t = 0$, both $t^{2-\beta}(Tx)(t)$ and $t^{2+p-\beta} D_{0^+}^p (Tx)(t)$ are equiconvergent at $t = t_1$.

Hence, $T(\overline{\Omega})$ is equiconvergent as $t \to 0$ and $T(\overline{\Omega})$ is equiconvergent as $t \to t_1$.

So $T(\overline{\Omega})$ is relatively compact. Then T is completely continuous. The proofs are completed. □

3. Main Result

In this section, we will establish the existence of at least one solution of BVP (6)-(7).

Definition 11 (see [26]). An odd homeomorphism Φ of the real line $\mathbb{R}$ onto itself is called a *pseudo-sub-multiplicative function* if there exists a homeomorphism ω of $[0, \infty)$ onto itself which *supports* Φ in the sense that for all $v_1, v_2 \geq 0$ we have $\Phi(v_1 v_2) \geq \omega(v_1)\Phi(v_2)$. ω is called the *supporting function* of Φ.

Remark 12. Note that any submultiplicative function is a pseudo-submultiplicative function. Also any function of the form $\Phi(u) := \sum_{j=0}^{k} c_j |u|^j u$, $u \in \mathbb{R}$ is pseudo-sup-multiplicative, provided that $c_j \geq 0$. Here, a supporting function is defined by $\omega(u) := \min\{u^{k+1}, u\}$, $u \geq 0$.

Remark 13. It is clear that a pseudo-submultiplicative function Φ and any corresponding supporting function ω are increasing functions vanishing at zero; moreover, their inverses Φ^{-1} and ν, respectively, are increasing and for all $v_1, v_2 \geq 0$, we have $\Phi^{-1}(v_1 v_2) \leq \nu(v_1)\Phi^{-1}(v_2)$.

Theorem 14. *Suppose that (a)–(e) and (A)-(B) hold, $\Phi : R \to R$ is a submultiplicative-like function with the supporting function ω, its inverse function is denoted by $\Phi^{-1} : R \to R$ with the supporting function ν. Furthermore, suppose that*

(i) *there exist nonnegative numbers $C_f, B_f, A_f, C_G, B_G, A_G, C_M, B_M$, and A_M such that*

$$\begin{aligned} &\left|f\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \\ &\quad \le C_f + B_f\Phi^{-1}(|U|) + A_f\Phi^{-1}(|V|), \\ &\left|G\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \\ &\quad \le C_G + B_G\Phi^{-1}(|U|) + A_G\Phi^{-1}(|V|), \\ &\left|M\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \\ &\quad \le C_M + B_M\Phi^{-1}(|U|) + A_M\Phi^{-1}\left(\left|\Phi^{-1}(|U|)\right|\right), \end{aligned} \tag{75}$$

holds for all $(U, V) \in R^2$, $t \in (0, 1]$.

(ii) *there exist nonnegative numbers $C_g, B_g, A_g, C_H, B_H, A_H, C_N, B_N$, and A_N such that*

$$\begin{aligned} &\left|g\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \le C_g + B_g\Phi(U) + A_g\Phi(V), \\ &\left|H\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \le C_H + B_H\Phi(U) + A_H\Phi(V), \\ &\left|N\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \le C_N + B_N\Phi(U) + A_N\Phi(V), \end{aligned} \tag{76}$$

hold for all $(U, V) \in R^2$, $t \in (0, 1]$.

(iii) *there exist the nonnegative numbers $C_I, B_I, A_I, C_{1,I}, B_{1,I}$, and $A_{1,I}$ such that*

$$\begin{aligned} &\left|I\left(t_1, t_1^{\alpha-2}U, t_1^{\alpha-q-2}V\right)\right| \le C_I + B_I\Phi^{-1}(|U|) + A_I\Phi^{-1}(|V|), \\ &\left|I_1\left(t_1, t_1^{\alpha-2}U, t_1^{\alpha-q-2}V\right)\right| \\ &\quad \le C_{1,I} + B_{1,I}\Phi^{-1}(|U|) + A_{1,I}\Phi^{-1}(|V|), \end{aligned} \tag{77}$$

hold for all $(U, V) \in R^2$.

(iv) *there exist the nonnegative numbers $C_J, B_J, A_J, C_{1,J}, B_{1,J}$, and $A_{1,J}$ such that*

$$\begin{aligned} &\left|J\left(t_1, t_1^{\beta-2}U, t_1^{\beta-p-2}V\right)\right| \le C_J + B_J\Phi(U) + A_J\Phi(V), \\ &\left|J_1\left(t_1, t_1^{\beta-2}U, t_1^{\beta-p-2}V\right)\right| \le C_{1,J} + B_{1,J}\Phi(U) + A_{1,J}\Phi(V), \end{aligned} \tag{78}$$

hold for all $(U, V) \in R^2$.

Then BVP (6)-(7) has at least one solution if

$$\begin{aligned} &\max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}\, \nu\left(2\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}\right) < 1 \text{ or} \\ &\frac{\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}}{w\left(\left(2\max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}\right)^{-1}\right)} < 1, \end{aligned} \tag{79}$$

where

$$\begin{aligned} \Theta_1 = {} & L_1\frac{\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)}C_f + \frac{L_1\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)}C_f \\ & + \|u\|_1 C_G + \|m\|_1 C_M \\ & + \frac{1}{\Pi}\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right|C_I \\ & + \frac{\left|t_1^{\alpha-2} - t_1^{\alpha-1}\right|}{\Pi}C_{1,I}, \\ \Theta_2 = {} & L_1\frac{\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)}\left[B_f + A_f\right] \\ & + \frac{L_1\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)}\left[B_f + A_f\right] \\ & + \|u\|_1\left[B_G + A_G\right] + \|m\|_1\left[B_M + A_M\right] \\ & + \frac{1}{\Pi}\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right| \\ & \times\left[B_I + A_I\right] + \frac{\left|t_1^{\alpha-2} - t_1^{\alpha-1}\right|}{\Pi}\left[B_{1,I} + A_{1,I}\right], \\ \Theta_3 = {} & L_1\frac{\mathbf{B}(\alpha+\delta-q, k+1)}{\Gamma(\alpha-q)}C_f \\ & + \frac{L_1}{\Gamma(\alpha-q)}\mathbf{B}(\alpha+\delta, k+1)C_f \\ & + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\|u\|_1 C_G + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\|m\|_1 C_M \\ & + \frac{1}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \\ & \times\left|\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)}t_1^{\alpha-q-2}\right|C_I \\ & + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\frac{\left|t_1^{\alpha-2} - t_1^{\alpha-1}\right|}{\Pi}C_{1,I}, \\ \Theta_4 = {} & L_1\frac{\mathbf{B}(\alpha+\delta-q, k+1)}{\Gamma(\alpha-q)}\left[B_f + A_f\right] \\ & + \frac{L_1}{\Gamma(\alpha-q)}\mathbf{B}(\alpha+\delta, k+1)\left[B_f + A_f\right] \\ & + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\|u\|_1\left[B_G + A_G\right] \\ & + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}\|m\|_1\left[B_M + A_M\right] \\ & + \frac{1}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \end{aligned}$$

$$
\begin{aligned}
&\times \left| \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} - \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-2)} t_1^{\alpha-q-2} \right| \\
&\times [B_I + A_I] \\
&+ \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \frac{\left| t_1^{\alpha-2} - t_1^{\alpha-1} \right|}{\Pi} [B_{1,I} + A_{1,I}], \\
\Sigma_1 &= L_1 \frac{\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)} C_f + \frac{L_1 \mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)} C_f \\
&+ \|u\|_1 C_G + \frac{1}{\Pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} C_I + \frac{1}{\Pi} t_1^{\alpha-1} C_{1,I}, \\
\Sigma_2 &= L_1 \frac{\mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)} \left[B_f + A_f\right] \\
&+ \frac{L_1 \mathbf{B}(\alpha+\delta, k+1)}{\Gamma(\alpha)} \left[B_f + A_f\right] + \|u\|_1 [B_G + A_G] \\
&+ \frac{1}{\Pi} \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} [B_I + A_I] + \frac{1}{\Pi} t_1^{\alpha-1} [B_{1,I} + A_{1,I}], \\
\Sigma_3 &= L_1 \frac{\mathbf{B}(\alpha+\delta-q, k+1)}{\Gamma(\alpha-q)} C_f \\
&+ \frac{L_1}{\Gamma(\alpha-q)} \mathbf{B}(\alpha+\delta, k+1) C_f \\
&+ \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \right) \|u\|_1 C_G \\
&+ \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \|m\|_1 C_M \\
&+ \frac{1}{\Pi} \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \right) \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} C_I \\
&+ \frac{1}{\Pi} \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \right) t_1^{\alpha-1} C_{1,I}, \\
\Sigma_4 &= L_1 \frac{\mathbf{B}(\alpha+\delta-q, k+1)}{\Gamma(\alpha-q)} \left[B_f + A_f\right] \\
&+ \frac{L_1}{\Gamma(\alpha-q)} \mathbf{B}(\alpha+\delta, k+1) \left[B_f + A_f\right] \\
&+ \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \right) \|u\|_1 [B_G + A_G] \\
&+ \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \|m\|_1 [B_M + A_M] \\
&+ \frac{1}{\Pi} \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} + \frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} \right) \\
&\times \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} t_1^{\alpha-q-1} [B_I + A_I] \\
&+ \frac{1}{\Pi} \left(\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} \right) \\
&\times t_1^{\alpha-1} [B_{1,I} + A_{1,I}], \\
\Upsilon_1 &= L_2 \frac{\mathbf{B}(\beta+\theta, l+1)}{\Gamma(\beta)} C_g \\
&+ \frac{L_2 \mathbf{B}(\beta+\theta, l+1)}{\Gamma(\beta)} C_g + \|v\|_1 C_H + \|n\|_1 C_N \\
&+ \frac{1}{\Xi} \left| \frac{\Gamma(\beta)}{\Gamma(\beta-p)} t_1^{\beta-p-1} - \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-2)} t_1^{\beta-p-2} \right| C_J \\
&+ \frac{\left| t_1^{\beta-2} - t_1^{\beta-1} \right|}{\Xi} C_{1,J}, \\
\Upsilon_2 &= L_2 \frac{\mathbf{B}(\beta+\theta, l+1)}{\Gamma(\beta)} \left[B_g + A_g\right] \\
&+ \frac{L_2 \mathbf{B}(\beta+\theta, l+1)}{\Gamma(\beta)} \left[B_g + A_g\right] \\
&+ \|b\|_1 [B_H + A_H] + \|n\|_1 [B_N + A_N] \\
&+ \frac{1}{\Xi} \left| \frac{\Gamma(\beta)}{\Gamma(\beta-p)} t_1^{\beta-p-1} - \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-2)} t_1^{\beta-p-2} \right| \\
&\times [B_J + A_J] + \frac{\left| t_1^{\beta-2} - t_1^{\beta-1} \right|}{\Xi} [B_{1,J} + A_{1,J}], \\
\Upsilon_3 &= L_2 \frac{\mathbf{B}(\beta+\theta-p, l+1)}{\Gamma(\beta-p)} C_g \\
&+ \frac{L_2}{\Gamma(\beta-p)} \mathbf{B}(\beta+\theta, l+1) C_g \\
&+ \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)} \|v\|_1 C_H + \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \|n\|_1 C_N \\
&+ \frac{1}{\Xi} \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \\
&\times \left| \frac{\Gamma(\beta)}{\Gamma(\beta-p)} t_1^{\beta-p-1} - \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-2)} t_1^{\beta-p-2} \right| C_J \\
&+ \frac{\Gamma(\beta)}{\Gamma(\beta-p)} \frac{\left| t_1^{\beta-2} - t_1^{\beta-1} \right|}{\Xi} C_{1,J}, \\
\Upsilon_4 &= L_2 \frac{\mathbf{B}(\beta+\theta-p, l+1)}{\Gamma(\beta-p)} \left[B_g + A_g\right] \\
&+ \frac{L_2}{\Gamma(\beta-p)} \mathbf{B}(\beta+\theta, l+1) \left[B_g + A_g\right] \\
&+ \frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)} \|v\|_1 [B_H + A_H]
\end{aligned}
\tag{80}
$$

$$+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\|n\|_1\left[B_N+A_N\right]$$

$$+\frac{1}{\Xi}\frac{\Gamma(\beta)}{\Gamma(\beta-p)}$$

$$\times\left|\frac{\Gamma(\beta)}{\Gamma(\beta-p)}t_1^{\beta-p-1}-\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-2)}t_1^{\beta-p-2}\right|\left[B_J+A_J\right]$$

$$+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\frac{\left|t_1^{\beta-2}-t_1^{\beta-1}\right|}{\Xi}\left[B_{1,J}+A_{1,J}\right],$$

$$\Lambda_1=L_2\frac{\mathbf{B}(\beta+\theta,l+1)}{\Gamma(\beta)}C_g+\frac{L_2\mathbf{B}(\beta+\theta,l+1)}{\Gamma(\beta)}C_g$$

$$+\|v\|_1C_H+\frac{1}{\Xi}\frac{\Gamma(\beta)}{\Gamma(\beta-p)}t_1^{\beta-p-1}C_J+\frac{1}{\Xi}t_1^{\beta-1}C_{1,J},$$

$$\Lambda_2=L_2\frac{\mathbf{B}(\beta+\theta,l+1)}{\Gamma(\beta)}\left[B_g+A_g\right]$$

$$+\frac{L_2\mathbf{B}(\beta+\theta,l+1)}{\Gamma(\beta)}\left[B_g+A_g\right]+\|v\|_1\left[B_H+A_H\right]$$

$$+\frac{1}{\Xi}\frac{\Gamma(\beta)}{\Gamma(\beta-p)}t_1^{\beta-p-1}\left[B_J+A_J\right]+\frac{1}{\Xi}t_1^{\beta-1}\left[B_{1,J}+A_{1,J}\right],$$

$$\Lambda_3=L_2\frac{\mathbf{B}(\beta+\theta-p,l+1)}{\Gamma(\beta-p)}C_g$$

$$+\frac{L_2}{\Gamma(\beta-p)}\mathbf{B}(\beta+\theta,l+1)C_g$$

$$+\left(\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\right)\|v\|_1C_H$$

$$+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\|n\|_1C_N$$

$$+\frac{1}{\Xi}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-p)}+\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}\right)\frac{\Gamma(\beta)}{\Gamma(\beta-p)}t_1^{\beta-p-1}C_J$$

$$+\frac{1}{\Xi}\left(\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\right)t_1^{\beta-1}C_{1,J},$$

$$\Lambda_4=L_2\frac{\mathbf{B}(\beta+\theta-p,l+1)}{\Gamma(\beta-p)}\left[B_g+A_g\right]$$

$$+\frac{L_2}{\Gamma(\beta-p)}\mathbf{B}(\beta+\theta,l+1)\left[B_g+A_g\right]$$

$$+\left(\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\right)\|v\|_1\left[B_H+A_H\right]$$

$$+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\|n\|_1\left[B_N+A_N\right]$$

$$+\frac{1}{\Xi}\left(\frac{\Gamma(\beta)}{\Gamma(\beta-p)}+\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}\right)$$

$$\times\frac{\Gamma(\beta)}{\Gamma(\beta-p)}t_1^{\beta-p-1}\left[B_J+A_J\right]$$

$$+\frac{1}{\Xi}\left(\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}+\frac{\Gamma(\beta)}{\Gamma(\beta-p)}\right)t_1^{\beta-1}\left[B_{1,J}+A_{1,J}\right]. \tag{81}$$

Proof. To apply Lemma 5, we should define an open bounded subset Ω of $X\times Y$ centered at zero such that assumptions in Lemma 5 hold.

Let $\Omega_1=\{(x,y)\in X\times Y: (x,y)=\lambda T(x,y)$ for some $\lambda\in(0,1)\}$. We prove that Ω_1 is bounded. For $(x,y)\in\Omega_1$, we get $(x,y)=\lambda T(x,y)$. It follows that $x=\lambda T_1y$ and $y=\lambda T_2x$.

For $t\in(0,t_1]$, we obtain $t^{2-\alpha}|x(t)|\le t^{2-\alpha}|(T_1y)(t)|\le\Theta_1+\Theta_2\Phi^{-1}(\|y\|)$.

For $t\in(t_1,1]$,

$$\begin{aligned}
&t^{2-\alpha}|x(t)|\\
&\quad\le t^{2-\alpha}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left|\phi(s)f\left(s,y(s),D_{0^+}^py(s)\right)\right|ds\\
&\qquad+\frac{t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\left|\phi(s)f\left(s,y(s),D_{0^+}^py(s)\right)\right|ds\\
&\qquad+(1-t)\int_0^1\left|u(s)G\left(s,y(s),D_{0^+}^py(s)\right)\right|ds\\
&\qquad+t\int_0^1\left|m(s)M\left(s,y(s),D_{0^+}^py(s)\right)\right|ds\\
&\qquad+\frac{1-t}{\Pi}\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}t_1^{\alpha-q-1}\left|I\left(t_1,y(t_1),D_{0^+}^py(t_1)\right)\right|\\
&\qquad+\frac{1-t}{\Pi}t_1^{\alpha-1}\left|I_1\left(t_1,y(t_1),D_{0^+}^py(t_1)\right)\right|\\
&\quad\le\Sigma_1+\Sigma_2\Phi^{-1}(\|y\|).
\end{aligned}\tag{82}$$

It follows that

$$\sup_{t\in(0,1)}t^{2-\alpha}|x(t)|\le\max\{\Theta_1,\Sigma_1\}+\max\{\Theta_2,\Sigma_2\}\Phi^{-1}(\|y\|). \tag{83}$$

Similarly, we have for $t\in(0,t_1]$ that

$$t^{q+2-\alpha}\left|D_{0^+}^qx(t)\right|\le\Theta_3+\Theta_4\Phi^{-1}(\|y\|) \tag{84}$$

and for $t\in(0,t_1]$

$$t^{q+2-\alpha}\left|D_{0^+}^qx(t)\right|\le\Sigma_3+\Sigma_4\Phi^{-1}(\|y\|). \tag{85}$$

It follows that

$$\begin{aligned}
&\sup_{t\in(0,1)}t^{2+q-\alpha}\left|D_{0^+}^qx(t)\right|\\
&\quad\le\max\{\Theta_3,\Sigma_3\}+\max\{\Theta_4,\Sigma_4\}\Phi^{-1}(\|y\|).
\end{aligned}\tag{86}$$

Hence,

$$\|x\| \leq \max\{\Theta_1, \Sigma_1, \Theta_3, \Sigma_3\} + \max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}\Phi^{-1}(\|y\|). \tag{87}$$

Similar to the above discussion we can prove that

$$\|y\| \leq \max\{\Upsilon_1, \Lambda_1, \Upsilon_3, \Lambda_3\} + \max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}\Phi(\|x\|). \tag{88}$$

Case 1. Consider $(\max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}\nu(2\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}) < 1)$.

With out loss of generality, suppose that

$$\|x\| \geq \Phi^{-1}\left(\frac{\max\{\Upsilon_1, \Lambda_1, \Upsilon_3, \Lambda_3\}}{\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}}\right). \tag{89}$$

Then use Remark 13, and the previous inequalities to get

$$\|x\| \leq \max\{\Theta_1, \Sigma_1, \Theta_3, \Sigma_3\} + \max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}\nu(2\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\})\|x\|. \tag{90}$$

It follows that there exists a constant $W > 0$ such that $\|x\| \leq W$. Thus

$$\|x\| \leq \max\left\{W, \Phi^{-1}\left(\frac{\max\{\Upsilon_1, \Lambda_1, \Upsilon_3, \Lambda_3\}}{\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}}\right)\right\}. \tag{91}$$

Then

$$\begin{aligned}\|y\| \leq{}& \max\{\Upsilon_1, \Lambda_1, \Upsilon_3, \Lambda_3\} \\ &+ \max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}\Phi \\ &\times\left(\max\left\{W, \Phi^{-1}\left(\frac{\max\{\Upsilon_1, \Lambda_1, \Upsilon_3, \Lambda_3\}}{\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}}\right)\right\}\right).\end{aligned} \tag{92}$$

It follows that Ω_1 is bounded.

Case 2. Consider $((\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}/w((2\max\{\Theta_2,\Sigma_2, \Theta_4, \Sigma_4\})^{-1})) < 1)$.

Without loss of generality, suppose that

$$\|y\| \geq \Phi\left(\frac{\max\{\Theta_1, \Sigma_1, \Theta_3, \Sigma_3\}}{\max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}}\right). \tag{93}$$

Then using Remark 12 and the previous inequalities, we get

$$\|y\| \leq \max\{\Upsilon_1, \Lambda_1, \Upsilon_3, \Lambda_3\} + \frac{\max\{\Upsilon_2, \Lambda_2, \Upsilon_4, \Lambda_4\}}{w\left((2\max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\})^{-1}\right)}\|y\|. \tag{94}$$

It follows that there exists a constant $W > 0$ such that $\|y\| \leq W$. We get

$$\|y\| \leq \max\left\{W, \Phi\left(\frac{\max\{\Theta_1, \Sigma_1, \Theta_3, \Sigma_3\}}{\max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}}\right)\right\}. \tag{95}$$

Then

$$\begin{aligned}\|x\| \leq{}& \max\{\Theta_1, \Sigma_1, \Theta_3, \Sigma_3\} \\ &+ \max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}\Phi^{-1} \\ &\times\left(\max\left\{W, \Phi\left(\frac{\max\{\Theta_1, \Sigma_1, \Theta_3, \Sigma_3\}}{\max\{\Theta_2, \Sigma_2, \Theta_4, \Sigma_4\}}\right)\right\}\right).\end{aligned} \tag{96}$$

It follows that Ω_1 is bounded.

To apply Lemma 5, let Ω be a nonempty open bounded subset of X such that $\Omega \supset \overline{\Omega_1}$ centered at zero.

It is easy to see from Lemma 8 that T is a completely continuous operator. One can see that

$$(x, y) \neq \lambda T(x, y) \quad \forall (x, y) \in \partial\Omega, \lambda \in (0, 1). \tag{97}$$

Thus, from Lemma 5, $(x, y) = T(x, y)$ has at least one solution $(x, y) \in \overline{\Omega}$. So (x, y) is a pair of solutions of BVP (3) and BVP (6). The proof of Theorem 14 is complete. □

4. Two Examples

To illustrate the usefulness of our main result, we present two examples that Theorem 14 can readily apply.

Example 15. Consider the following impulsive boundary value problem:

$$\begin{aligned}&D_{0^+}^{8/5}x(t) = t^{-1/5}(1-t)^{-1} \\ &\quad\times\left(c + bt^{6/5}[y(t)]^3 + at^{9/5}\left[D_{0^+}^{1/5}y(t)\right]^3\right), \\ &\qquad t \in (0,1), \quad t \neq \frac{1}{2}, \\ &D_{0^+}^{9/5}y(t) \\ &\quad = t^{-1/5}(1-t)^{-1} \\ &\quad\times\left(c_0 + b_0t^{1/15}[x(t)]^{1/3} + a_0t^{2/15}\left[D_{0^+}^{1/5}x(t)\right]^{1/3}\right), \\ &\qquad t \in (0,1), \quad t \neq t_1, \\ &\lim_{t\to 0}t^{2/5}x(t) = G, \qquad \lim_{t\to 0}t^{1/5}y(t) = H, \\ &x(1) = M, \qquad y(1) = N, \\ &\Delta x\left(\frac{1}{2}\right) = c_I, \qquad \Delta y\left(\frac{1}{2}\right) = c_J, \\ &\Delta D_{0^+}^{1}x\left(\frac{1}{2}\right) = c_{1,I}, \qquad \Delta D_{0^+}^{1}y\left(\frac{1}{2}\right) = c_{1,J},\end{aligned} \tag{98}$$

where $c, b, a, c_0, b_0, a_0, G_0, H_0, M_0, N_0, C_I, C_J, C_{1,I}, C_{1,J}$ are constants.

Corresponding to BVP (1), we have

(a) $\alpha = 8/5$, $\beta = 9/5$, $p = q = 1/5$,

(b) $\phi(t) = \psi(t) = t^{-1/5}(1-t)^{-1/5}$, $f(t,U,V) = c + bt^{6/5}U^3 + at^{9/5}V^3$ and $g(t,U,V) = c_0 + b_0t^{1/15}U^{1/3} + a_0t^{2/15}V^{1/3}$ defined on $(0,1) \times R^2$,

(c) $u(t) = v(t) = m(t) = n(t) \equiv 1$, $G(t,U,V) = G_0$, $H(t,U,V) = H_0$, $M(t,U,V) = M_0$, $N(t,U,V) = N_0$,

(d) $0 = t_0 < t_1 = (1/2) < t_2 = 1$,

(e) $I(t,U,V) = c_I$, $I_1(t,U,V) = c_{1,I}$, $J(t,U,V) = c_J$, $J_1(t,U,V) = c_{1,J}$.

It is easy to show that

(A) ϕ satisfies $\alpha + 2\delta - q > 0$, $\alpha + k + \delta - q \geq 0$, and $|\phi(t)| \leq L_1 t^k (1-t)^\delta$ for all $t \in (0,1)$ with $L_1 = 1$ and $k = -(1/5) = \delta$;

ψ satisfies $\eta + 2\theta - p > 0$, $\beta + l + \theta - p \geq 0$, and $|\psi(t)| \leq L_2 t^l (1-t)^\theta$ for all $t \in (0,1)$ with $L_2 = 1$ and $l = -(1/5) = \theta$;

(B) f, G, M, I, I_1 are β-Caratheodory functions and g, H, N, J, J_1 are α-Caratheodory functions.

Furthermore, we have $\Phi^{-1}(x) = x^3$ and $\Phi(x) = x^{1/3}$ with $w(x) = x^{1/3}$ and $\nu(x) = x^3$. It is easy to see that

(i) the inequalities

$$\begin{gathered}\left|f\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \leq C_f + B_f \Phi^{-1}(|U|) + A_f \Phi^{-1}(|V|), \\ \left|G\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \leq C_G + B_G \Phi^{-1}(|U|) + A_G \Phi^{-1}(|V|), \\ \left|M\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \\ \leq C_M + B_M \Phi^{-1}(|U|) + A_M \Phi^{-1}\left(\left|\Phi^{-1}(|U|)\right|\right)\end{gathered} \tag{99}$$

hold for all $(U,V) \in R^2$, $t \in (0,1]$ with $C_f = |c|$, $B_f = |b|$, $A_f = |a|$, $C_G = |G_0|$, $B_G = 0$, $A_G = 0$ and $C_M = |M_0|$, $B_M = 0$, $A_M = 0$;

(ii) the inequalities

$$\begin{gathered}\left|g\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \leq C_g + B_g \Phi(U) + A_g \Phi(V), \\ \left|H\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \leq C_H + B_H \Phi(U) + A_H \Phi(V), \\ \left|N\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \leq C_N + B_N \Phi(U) + A_N \Phi(V)\end{gathered} \tag{100}$$

hold for all $(U,V) \in R^2$, $t \in (0,1]$ with $C_g = |c_0|$, $B_g = |b_0|$, $A_g = |a_0|$, $C_H = |H_0|$, $B_H = A_H = 0$, $C_N = |N_0|$, $B_N = A_N = 0$;

(iii) the inequalities

$$\begin{gathered}\left|I\left(t_1, t_1^{\alpha-2}U, t_1^{\alpha-q-2}V\right)\right| \leq C_I + B_I \Phi^{-1}(|U|) + A_I \Phi^{-1}(|V|), \\ \left|I_1\left(t_1, t_1^{\alpha-2}U, t_1^{\alpha-q-2}V\right)\right| \\ \leq C_{1,I} + B_{1,I}\Phi^{-1}(|U|) + A_{1,I}\Phi^{-1}(|V|)\end{gathered} \tag{101}$$

hold for all $(U,V) \in R^2$ with $C_I = |c_I|$, $B_I = A_I = 0$, $C_{1,I} = |c_{1,I}|$, $B_{1,I} = A_{1,I} = 0$;

(iv) the inequalities

$$\begin{gathered}\left|J\left(t_1, t_1^{\beta-2}U, t_1^{\beta-p-2}V\right)\right| \leq C_J + B_J \Phi(U) + A_J \Phi(V), \\ \left|J_1\left(t_1, t_1^{\beta-2}U, t_1^{\beta-p-2}V\right)\right| \leq C_{1,J} + B_{1,J}\Phi(U) + A_{1,J}\Phi(V)\end{gathered} \tag{102}$$

hold for all $(U,V) \in R^2$ with $C_J = |c_J|$, $B_J = A_J = 0$, $C_{1,J} = |c_{1,J}|$, $B_{1,J} = A_{1,J} = 0$.

By direct computation, we know that

$$\begin{gathered}\Theta_2 = 2\frac{\mathbf{B}(7/5,4/5)}{\Gamma(8/5)}[|b| + |a|], \\ \Sigma_2 = 2\frac{\mathbf{B}(7/5,4/5)}{\Gamma(8/5)}[|b| + |a|], \\ \Theta_4 = \left(\frac{\mathbf{B}(6/5,4/5)}{\Gamma(7/5)} + \frac{\mathbf{B}(7/5,4/5)}{\Gamma(7/5)}\right)[|b| + |a|], \\ \Sigma_4 = \left(\frac{\mathbf{B}(6/5,4/5)}{\Gamma(7/5)} + \frac{\mathbf{B}(7/5,4/5)}{\Gamma(7/5)}\right)[|b| + |a|], \\ \Upsilon_2 = 2\frac{\mathbf{B}(8/5,4/5)}{\Gamma(9/5)}[|b_0| + |a_0|], \\ \Upsilon_4 = \left(\frac{\mathbf{B}(7/5,4/5)}{\Gamma(8/5)} + \frac{\mathbf{B}(8/5,4/5)}{\Gamma(8/5)}\right)[|b_0| + |a_0|], \\ \Lambda_2 = 2\frac{\mathbf{B}(8/5,4/5)}{\Gamma(9/5)}[|b_0| + |a_0|], \\ \Lambda_4 = \left(\frac{\mathbf{B}(7/5,4/5)}{\Gamma(8/5)} + \frac{\mathbf{B}(8/5,4/5)}{\Gamma(8/5)}\right)[|b_0| + |a_0|].\end{gathered} \tag{103}$$

Then Theorem 14 implies that the existence of at least one solution if

$$\begin{gathered}\max\left\{2\frac{\mathbf{B}(8/5,4/5)}{\Gamma(9/5)}, \frac{\mathbf{B}(7/5,4/5)}{\Gamma(8/5)} + \frac{\mathbf{B}(8/5,4/5)}{\Gamma(8/5)}\right\} \\ \times\left(\max\left\{2\frac{\mathbf{B}(7/5,4/5)}{\Gamma(8/5)}, \frac{\mathbf{B}(6/5,4/5)}{\Gamma(7/5)} + \frac{\mathbf{B}(7/5,4/5)}{\Gamma(7/5)}\right\}\right)^{1/3} \\ \times[|b_0| + |a_0|][|b| + |a|]^{1/3} < \frac{1}{\sqrt[3]{2}}.\end{gathered} \tag{104}$$

Example 16. Consider the following boundary value problem without impulse effects:

$$\begin{gathered}D_{0^+}^{7/4} x(t) = t^{-1/4}(1-t)^{-1/4} \\ \times\left(C + Bt^{3/4}[y(t)]^3 + At^{15/4}\left[D_{0^+}^{1} y(t)\right]^3\right), \\ t \in (0,1),\end{gathered}$$

$$D_{0^+}^{5/4} y(t) = t^{-1/8}(1-t)^{-1/8} \times \left(C_0 + B_0 t^{1/4}[x(t)]^{1/3} + A_0 t^{7/12}\left[D_{0^+}^{1/4}x(t)\right]^{1/3}\right), \quad t \in (0,1),$$

$$\lim_{t\to 0} t^{1/4}x(t) = 0, \qquad \lim_{t\to 0} t^{3/4}y(t) = 0,$$

$$x(1) = 0, \qquad y(1) = 0, \tag{105}$$

where C, B, A, C_0, B_0, and A_0 are constants.

Corresponding to BVP (1), we have

(a) $\alpha = 7/4$, $\beta = 5/4$, $p = 1$ and $q = 1/4$,

(b) $\phi(t) = t^{-1/4}(1-t)^{-1/4}$, $\psi(t) = t^{-1/8}(1-t)^{-1/8}$, f, g defined on $(0,1) \times R^2$, $f(t,U,V) = C + Bt^{1/12}U^3 + At^{5/12}V^3$ and $g(t,U,V) = C_0 + B_0 t^{1/4}U^{1/3} + A_0 t^{7/12}V^{1/3}$,

(c) $m(t) = n(t) = u(t) = v(t) \equiv 0$, $G(t,U,V) = H(t,U,V) = M(t,U,V) = N(t,U,V) \equiv 0$,

(d) there exists no impulse point,

(e) $I(t,U,V) = I_1(t,U,V) = J(t,U,V) = J_1(t,U,V) \equiv 0$.

It is easy to show that

(A) ϕ satisfies $\alpha + 2\delta - q > 0$, $\alpha + k + \delta - q > 0$, $|\phi(t)| \le L_1 t^k (1-t)^\delta$ for all $t \in (0,1)$ with $L_1 = 1$, $k = -(1/4) = \delta$;

ψ satisfies $\beta + 2\theta - p > 0$, $\beta + l + \theta - p \ge 0$, and $|\psi(t)| \le L_2 t^l (1-t)^\theta$ for all $t \in (0,1)$ with $L_2 = 1$, $l = -(1/8) = \theta$;

(B) f, G, M, I, I_1 are β-Caratheodory functions and g, H, N, J, J_1 are α-Caratheodory functions.

Furthermore, $\Phi(x) = x^{1/3}$ and $\Phi^{-1}(x) = x^3$, we have $w(x) = x^{1/3}$ and $\nu(x) = x^3$, and

(i) the inequalities

$$\left|f\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \le C_f + B_f \Phi^{-1}(|U|) + A_f \Phi^{-1}(|V|),$$

$$\left|G\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \le C_G + B_G \Phi^{-1}(|U|) + A_G \Phi^{-1}(|V|),$$

$$\left|M\left(t, t^{\alpha-2}U, t^{\alpha-q-2}V\right)\right| \le C_M + B_M \Phi^{-1}(|U|) + A_M \Phi^{-1}\left(\left|\Phi^{-1}(|U|)\right|\right) \tag{106}$$

hold for all $(U,V) \in R^2$, $t \in (0,1)$ with $C_G = B_G = A_G = C_M = B_M = A_M = 0$, $C_f = |C|$, $B_f = |B|$ and $A_f = |A|$;

(ii) the inequalities

$$\left|g\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \le C_g + B_g \Phi(U) + A_g \Phi(V),$$

$$\left|H\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \le C_H + B_H \Phi(U) + A_H \Phi(V),$$

$$\left|N\left(t, t^{\beta-2}U, t^{\beta-p-2}V\right)\right| \le C_N + B_N \Phi(U) + A_N \Phi(V) \tag{107}$$

hold for all $(U,V) \in R^2$, $t \in (0,1)$ with $C_H = B_H = A_H = C_N = B_N = A_N = 0$, $C_g = |C_0|$, $B_g = |B_0|$ and $A_g = |A_0|$;

(iii) the inequalities

$$\begin{aligned} &\left|I\left(t_1, t_1^{\alpha-2}U, t_1^{\alpha-q-2}V\right)\right| \le C_I + B_I \Phi^{-1}(|U|) + A_I \Phi^{-1}(|V|), \\ &\left|I_1\left(t_1, t_1^{\alpha-2}U, t_1^{\alpha-q-2}V\right)\right| \le C_{1,I} + B_{1,I} \Phi^{-1}(|U|) + A_{1,I} \Phi^{-1}(|V|) \end{aligned} \tag{108}$$

hold for all $(U,V) \in R^2$ with $C_I = B_I = A_I = C_{1,I} = B_{1,I} = A_{1,I} = 0$;

(iv) there exist the nonnegative numbers $A_{i,k}, B_{i,k}, C_{i,k}$ $(i = 1,2)$ such that

$$\left|J\left(t_1, t_1^{\beta-2}U, t_1^{\beta-p-2}V\right)\right| \le C_J + B_J \Phi(U) + A_J \Phi(V),$$

$$\left|J_1\left(t_1, t_1^{\beta-2}U, t_1^{\beta-p-2}V\right)\right| \le C_{1,J} + B_{1,J} \Phi(U) + A_{1,J} \Phi(V) \tag{109}$$

hold for all $(U,V) \in R^2$ with $C_J = B_J = A_J = C_{1,J} = B_{1,J} = A_{1,J} = 0$.

By direct computation, we know that

$$\begin{aligned} \Theta_2 &= 2\frac{\mathbf{B}(3/2, 3/4)}{\Gamma(7/4)}[|B| + |A|], \\ \Sigma_2 &= 2\frac{\mathbf{B}(3/2, 3/4)}{\Gamma(7/4)}[|B| + |A|], \\ \Theta_4 &= \left(\frac{\mathbf{B}(5/4, 3/4)}{\Gamma(3/2)} + \frac{\mathbf{B}(3/2, 3/4)}{\Gamma(3/2)}\right)[|B| + |A|], \\ \Sigma_4 &= \left(\frac{\mathbf{B}(5/4, 3/4)}{\Gamma(3/2)} + \frac{\mathbf{B}(3/2, 3/4)}{\Gamma(3/2)}\right)[|B| + |A|], \\ \Upsilon_2 &= 2\frac{\mathbf{B}(9/8, 7/8)}{\Gamma(5/4)}\left[|B_0| + |A_0|\right], \\ \Upsilon_4 &= \left(\frac{\mathbf{B}(1/8, 7/8)}{\Gamma(\beta - p)} + \frac{\mathbf{B}(9/8, 7/8)}{\Gamma(1/4)}\right)\left[|B_0| + |A_0|\right], \\ \Lambda_2 &= 2\frac{\mathbf{B}(9/8, 7/8)}{\Gamma(5/4)}\left[|B_0| + |A_0|\right], \\ \Lambda_4 &= \left(\frac{\mathbf{B}(1/8, 7/8)}{\Gamma(1/4)} + \frac{\mathbf{B}(9/8, 7/8)}{\Gamma(1/4)}\right)\left[|B_0| + |A_0|\right]. \end{aligned} \tag{110}$$

Then Theorem 14 implies the existence of at least one solution if

$$\max\left\{2\frac{\mathbf{B}(3/2,3/4)}{\Gamma(7/4)},\frac{\mathbf{B}(5/4,3/4)}{\Gamma(3/2)}+\frac{\mathbf{B}(3/2,3/4)}{\Gamma(3/2)}\right\}$$
$$\times\left(\max\left\{2\frac{\mathbf{B}(9/8,7/8)}{\Gamma(5/4)},\frac{\mathbf{B}(1/8,7/8)}{\Gamma(\beta-p)}+\frac{\mathbf{B}(9/8,7/8)}{\Gamma(1/4)}\right\}\right)^3$$
$$\times\left[\left|B_0\right|+\left|A_0\right|\right]^3\left[|B|+|A|\right]<\frac{1}{8}. \quad (111)$$

Remark 17. It is easy to see that the previous boundary value problems have at least one solution for sufficiently small $|B_1|, |B_2|$ and $|A_0|, |B_0|, |a|, |b|, |a_0|$ and $|b_0|$. They cannot be solved by the theorems in [24, 25].

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