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Terminal Value Problem for Differential Equations with Hilfer–Katugampola Fractional Derivative

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Abstract: We present in this work the existence results and uniqueness of solutions for a class of boundary value problems of terminal type for fractional differential equations with the Hilfer–Katugampola fractional derivative. The reasoning is mainly based upon different types of classical fixed point theory such as the Banach contraction principle and Krasnoselskii's fixed point theorem. We illustrate our main findings, with a particular case example included to show the applicability of our outcomes.

Keywords: Hilfer–Katugampola fractional derivative; terminal value problem; existence; uniqueness; fixed point

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1. Introduction

Recently, by means of different tools from nonlinear analysis, many classes of differential equations with Caputo fractional derivative have extensively been studied in books [1–5] and in some papers, for example, [6–11]. In order to solve fractional differential equations, we mention the works [12,13] where the authors propose and prove the equivalence between an initial value problem and the Volterra integral equation.

We consider a new fractional derivative which interpolates the Hilfer, Hilfer–Hadamard, Riemann–Liouville, Hadamard, Caputo, Caputo–Hadamard, generalized and Caputo-type fractional derivatives, as well as the Weyl and Liouville fractional derivatives for particular cases of integration extremes. for more details, see [14–21] and the references therein.

It is well known [22] that the comparison principle for initial value problems of ordinary differential equations is a very useful tool in the study of qualitative and quantitative theory. Recently, attempts have been made to study the corresponding comparison principle for terminal value problems (TVP) [23].

Motivated by the works above, we establish in this paper existence and uniqueness results to the terminal value problem of the following Hilfer–Katugampola type fractional differential equation:

$$\left({}^{\rho}D^{\alpha,\beta}_{a^+}y\right)(t) = f\left(t, y(t), \left({}^{\rho}D^{\alpha,\beta}_{a^+}y\right)(t)\right), \text{ for each , } t \in (a,T], \quad a > 0 \tag{1}$$

$$y(T) = c \in \mathbb{R} \quad (2)$$

where ${}^{\rho}D^{\alpha,\beta}_{a^+}$ is the Hilfer–Katugampola fractional derivative (to be defined below) of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and $f : (a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function. To our knowledge, no papers on terminal value problem for implicit fractional differential equations exist in the literature, in particular for those involving the Hilfer–Katugampola fractional derivative.

This paper is organized as follows. In Section 2, some notations are introduced and we recall some concepts of preliminaries about Hilfer–Katugampola fractional derivative. In Section 3, two results for Equations (1) and (2) are presented: The first one is based on the Banach contraction principle, the second one on Krasnoselskii's fixed point theorem. Finally, in Section 4, we give an example to show the applicability of our main results.

2. Preliminaries

In this part, we present notations and definitions that we will use throughout this paper. Let $0 < a < T$, $J = [a,T]$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm:

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\}$$

We consider the weighted spaces of continuous functions:

$$C_{\gamma,\rho}(J) = \left\{ y : (a,T] \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma y(t) \in C(J,\mathbb{R}) \right\}, \quad 0 \le \gamma < 1$$

and:

$$C^n_{\gamma,\rho}(J) = \left\{ y \in C^{n-1}(J) : y^{(n)} \in C_{\gamma,\rho}(J) \right\}, n \in \mathbb{N},$$
$$C^0_{\gamma,\rho}(J) = C_{\gamma,\rho}(J)$$

with the norms:

$$\|y\|_{C_{\gamma,\rho}} = \sup_{t\in J} \left| \left(\frac{t^\rho - a^\rho}{\rho}\right)^\gamma y(t) \right|$$

and:

$$\|y\|_{C^n_{\gamma,\rho}} = \sum_{k=0}^{n-1} \|y^{(k)}\|_\infty + \|y^{(n)}\|_{C_{\gamma,\rho}}$$

Consider the space $X^p_c(a,b)$, $(c \in \mathbb{R},\ 1 \le p \le \infty)$ of those complex-valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{X^p_c} < \infty$, where the norm is defined by:

$$\|f\|_{X^p_c} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}}, \quad (1 \le p < \infty, c \in \mathbb{R})$$

In particular, when $c = \frac{1}{p}$, the space $X^p_c(a,b)$ coincides with the $L_p(a,b)$ space: $X^p_{\frac{1}{p}}(a,b) = L_p(a,b)$.

Definition 1 ([16]). *(Katugampola fractional integral).*
Let $\alpha \in \mathbb{R}_+, c \in \mathbb{R}$ and $g \in X^p_c(a,b)$. The Katugampola fractional integral of order α is defined by:

$$\left({}^{\rho}I^{\alpha}_{a^+} g\right)(t) = \int_a^t s^{\rho-1} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} \frac{g(s)}{\Gamma(\alpha)} ds, \ t > a, \rho > 0$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \ \alpha > 0.$

Definition 2 ([16]). *(Katugampola fractional derivative).*
Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $\rho > 0$. The Katugampola fractional derivative ${}^{\rho}D^{\alpha}_{a+}$ of order α is defined by:

$$\begin{aligned} \left({}^{\rho}D^{\alpha}_{a+}g\right)(t) &= \delta^n_{\rho}({}^{\rho}I^{n-\alpha}_{a+}g)(t) \\ &= \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t s^{\rho-1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{n-\alpha-1}\frac{g(s)}{\Gamma(n-\alpha)}ds,\ t > a, \rho > 0 \end{aligned}$$

where $n = [\alpha]+1$ and $\delta^n_{\rho} = \left(t^{1-\rho}\dfrac{d}{dt}\right)^n$.

Lemma 1 ([24]). *Let $\alpha > 0$, and $0 \le \gamma < 1$. Then, ${}^{\rho}I^{\alpha}_{a+}$ is bounded from $C_{\gamma,\rho}(J)$ into $C_{\gamma,\rho}(J)$.*

Lemma 2 ([24]). *Let $0 < a < T < \infty, \alpha > 0, 0 \le \gamma < 1$ and $y \in C_{\gamma,\rho}(J)$. If $\alpha > \gamma$, then ${}^{\rho}I^{\alpha}_{a+}y$ is continuous on J and*

$$\left({}^{\rho}I^{\alpha}_{a+}y\right)(a) = \lim_{t\to a^+}\left({}^{\rho}I^{\alpha}_{a+}y\right)(t) = 0$$

Lemma 3 ([12]). *Let $x > a$. Then, for $\alpha \ge 0$ and $\beta > 0$, we have:*

$$\begin{aligned} \left[{}^{\rho}I^{\alpha}_{a+}\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\beta-1}\right](t) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha+\beta-1} \\ \left[{}^{\rho}D^{\alpha}_{a+}\left(\frac{s^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\right](t) &= 0,\quad 0 < \alpha < 1 \end{aligned}$$

Lemma 4 ([24]). *Let $\alpha > 0, 0 \le \gamma < 1$ and $g \in C_{\gamma}[a,b]$. Then:*

$$\left({}^{\rho}D^{\alpha}_{a+}\,{}^{\rho}I^{\alpha}_{a+}g\right)(t) = g(t), \quad \textit{for all}\ \ t \in (a,b]$$

Lemma 5 ([24]). *Let $0 < \alpha < 1, 0 \le \gamma < 1$. If $g \in C_{\gamma,\rho}[a,b]$ and ${}^{\rho}I^{1-\alpha}_{a+}g \in C^1_{\gamma,\rho}[a,b]$, then:*

$$\left({}^{\rho}I^{\alpha}_{a+}\,{}^{\rho}D^{\alpha}_{a+}g\right)(t) = g(t) - \frac{\left({}^{\rho}I^{1-\alpha}_{a+}g\right)(a)}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}, \quad \textit{for all}\ \ t \in (a,b]$$

Definition 3 ([24]). *Let order α and type β satisfy $n-1 < \alpha < n$ and $0 \le \beta \le 1$, with $n \in \mathbb{N}$. The Hilfer–Katugampola fractional derivative to t, with $\rho > 0$ of a function $g \in C_{1-\gamma,\rho}[a,b]$, is defined by:*

$$\begin{aligned} \left({}^{\rho}D^{\alpha,\beta}_{a+}g\right)(t) &= \left({}^{\rho}I^{\beta(n-\alpha)}_{a+}\left(t^{\rho-1}\frac{d}{dt}\right)^n {}^{\rho}I^{(1-\beta)(n-\alpha)}_{a+}g\right)(t) \\ &= \left({}^{\rho}I^{\beta(n-\alpha)}_{a+}\delta^n_{\rho}\,{}^{\rho}I^{(1-\beta)(n-\alpha)}_{a+}g\right)(t) \end{aligned}$$

In this paper we consider the case $n = 1$ only, because $0 < \alpha < 1$.

Property 1 ([24]). *The operator ${}^{\rho}D^{\alpha,\beta}_{a+}$ can be written as:*

$${}^{\rho}D^{\alpha,\beta}_{a+} = {}^{\rho}I^{\beta(1-\alpha)}_{a+}\delta_{\rho}\,{}^{\rho}I^{1-\gamma}_{a+} = {}^{\rho}I^{\beta(1-\alpha)}_{a+}\,{}^{\rho}D^{\gamma}_{a+},\ \ \gamma = \alpha+\beta-\alpha\beta$$

Property 2. *The fractional derivative ${}^{\rho}D^{\alpha,\beta}_{\alpha+}$ is an interpolator of the following fractional derivatives: Hilfer ($\rho \to 1$) [14], Hilfer–Hadamard ($\rho \to 0^+$) [25], generalized ($\beta = 0$) [16], Caputo-type ($\beta = 1$), Riemann–Liouville ($\beta = 0, \rho \to 1$) [17], Hadamard ($\beta = 0, \rho \to 0^+$) [17], Caputo ($\beta = 1, \rho \to 1$) [17], Caputo–Hadamard ($\beta = 1, \rho \to 0^+$) [21], Liouville ($\beta = 0, \rho \to 1, a = 0$) [17] and Weyl ($\beta = 0, \rho \to 1, a = -\infty$)[15].*

Definition 4. *We consider the following parameters α, β, γ satisfying:*

$$\gamma = \alpha + \beta - \alpha\beta, \quad 0 < \alpha, \beta, \gamma < 1.$$

Thus, we define the spaces:

$$C^{\alpha,\beta}_{1-\gamma,\rho}(J) = \left\{ y \in C_{1-\gamma,\rho}(J), \, {}^{\rho}D^{\alpha,\beta}_{a^+} y \in C_{1-\gamma,\rho}(J) \right\}$$

and:

$$C^{\gamma}_{1-\gamma,\rho}(J) = \{ y \in C_{1-\gamma,\rho}(J), \, {}^{\rho}D^{\gamma}_{a^+} y \in C_{1-\gamma,\rho}(J) \}$$

Since ${}^{\rho}D^{\alpha,\beta}_{a^+} y = {}^{\rho}I^{\gamma(1-\alpha)}_{a^+} \, {}^{\rho}D^{\gamma}_{a^+} y$, *it follows from Lemma 1 that:*

$$C^{\gamma}_{1-\gamma,\rho}(J) \subset C^{\alpha,\beta}_{1-\gamma,\rho}(J) \subset C_{1-\gamma,\rho}(J)$$

Lemma 6 ([24]). *Let* $0 < \alpha < 1, 0 \leq \beta \leq 1$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $y \in C^{\gamma}_{1-\gamma,\rho}(J)$, *then:*

$${}^{\rho}I^{\gamma}_{a^+} \, {}^{\rho}D^{\gamma}_{a^+} y = {}^{\rho}I^{\alpha}_{a^+} \, {}^{\rho}D^{\alpha,\beta}_{a^+} y$$

and:

$${}^{\rho}D^{\gamma}_{a^+} \, {}^{\rho}I^{\alpha}_{a^+} y = {}^{\rho}D^{\beta(1-\alpha)}_{a^+} y$$

Theorem 1 ([26]). *($PC_{1-\gamma}$ type Arzela–Ascoli Theorem). Let $A \subset PC_{1-\gamma}(J, \mathbb{R})$. A is relatively compact (i.e., $\overline{A}$ is compact) if:*

1. *A is uniformly bounded, i.e., there exists $M > 0$ such that:*

$$|f(x)| < M \text{ for every } f \in A \text{ and } x \in (t_k, t_{k+1}], k = 1, \dots, m$$

2. *A is equicontinuous on $(t_k, t_{k+1}]$, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $x, \overline{x} \in (t_k, t_{k+1}]$, $|x - \overline{x}| \leq \delta$ implies $|f(x) - f(\overline{x})| \leq \epsilon$ for every $f \in A$.*

Theorem 2 ([27]). *(Banach's fixed point theorem). Let C be a non-empty closed subset of a Banach space E, then any contraction mapping T of C into itself has a unique fixed point.*

Theorem 3 ([27]). *(Krasnoselskii's fixed point theorem). Let M be a closed, convex and nonempty subset of a Banach space X, and A, B be the operators such that:*

1. *$Ax + By \in M$ for all $x, y \in M$*
2. *A is compact and continuous*
3. *B is a contraction mapping*

Then there exists $z \in M$ such that $z = Az + Bz$.

3. Existence of Solutions

We consider the following linear fractional differential equation:

$$\left({}^{\rho}D^{\alpha,\beta}_{a^+} y \right)(t) = \varphi(t), \quad t \in (a, T] \tag{3}$$

where $\varphi(\cdot) \in C_{1-\gamma,\rho}(J)$, with the terminal condition:

$$y(T) = c, \; c \in \mathbb{R} \tag{4}$$

The following theorem shows that Equations (3) and (4) have a unique solution given by:

$$y(t) = \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left[c - \frac{1}{\Gamma(\alpha)}\int_a^T \left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(s)ds\right]\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(s)ds \tag{5}$$

Theorem 4. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. If $\varphi : (a, T] \to \mathbb{R}$ is a function such that $\varphi(\cdot) \in C_{1-\gamma,\rho}(J)$, then y satisfies Equations (3) and (4) if and only if it satisfies Equation (5).*

Proof. $(\Rightarrow)$ Let $y \in C^\gamma_{1-\gamma,\rho}(J)$ be a solution of Equations (3) and (4). We prove that y is also a solution of Equation (5). From the definition of $C^\gamma_{1-\gamma,\rho}(J)$, Lemma 1, and using Definition 2, we have:

$$^\rho I^{1-\gamma}_{a+} y \in C(J) \text{ and } {}^\rho D^\gamma_{a+} y = \delta_\rho\, {}^\rho I^{1-\gamma}_{a+} y \in C_{1-\gamma,\rho}(J) \tag{6}$$

By the Definition of the space $C^n_{1-\gamma,\rho}(J)$, it follows that:

$$^\rho I^{1-\gamma}_{a+} y \in C^1_{1-\gamma,\rho}(J)$$

Using Lemma 5, with $\alpha = \gamma$, we obtain:

$$\left({}^\rho I^\gamma_{a+}\, {}^\rho D^\gamma_{a+} y\right)(t) = y(t) - \frac{\left({}^\rho I^{1-\gamma}_{a+} y\right)(a)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \tag{7}$$

where $t \in (a, T]$. By hypothesis, $y \in C^\gamma_{1-\gamma,\rho}(J)$, using Lemma 6 with Equation (3), we have:

$$\left({}^\rho I^\gamma_{a+}\, {}^\rho D^\gamma_{a+} y\right)(t) = \left({}^\rho I^\alpha_{a+}\, {}^\rho D^{\alpha,\beta}_{a+} y\right)(t) = \left({}^\rho I^\alpha_{a+} \varphi\right)(t) \tag{8}$$

Comparing Equations (7) and (8), we see that:

$$y(t) = \frac{\left({}^\rho I^{1-\gamma}_{a+} y\right)(a)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \left({}^\rho I^\alpha_{a+} \varphi\right)(t) \tag{9}$$

Using Equation (4) we obtain:

$$y(t) = \left(\frac{T^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left[c - \frac{1}{\Gamma(\alpha)}\int_a^T \left(\frac{T^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(s)ds\right]\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(s)ds$$

with $t \in (a, b]$, that is $y(\cdot)$ satisfies Equation (5).

$(\Leftarrow)$ Let $y \in C^\gamma_{1-\gamma,\rho}(J)$, satisfying Equation (5). We show that y also satisfies Equations (3) and (4). Apply operator ${}^\rho D^\gamma_{a+}$ on both sides of Equation (5). Then, from Lemmas 3 and 6 we get:

$$({}^\rho D^\gamma_{a+} y)(t) = \left({}^\rho D^{\beta(1-\alpha)}_{a+} \varphi\right)(t) \tag{10}$$

By Equation (6) we have ${}^\rho D^\gamma_{a+} y \in C_{1-\gamma,\rho}(J)$; then, Equation (10) implies:

$$({}^\rho D^\gamma_{a+} y)(t) = \left(\delta_\rho\, {}^\rho I^{1-\beta(1-\alpha)}_{a+} \varphi\right)(t) = \left({}^\rho D^{\beta(1-\alpha)}_{a+} \varphi\right)(t) \in C_{1-\gamma,\rho}(J) \tag{11}$$

As $\varphi(\cdot) \in C_{1-\gamma,\rho}(J)$ and from Lemma 1, it follows:

$$\left({}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\varphi\right) \in C_{1-\gamma,\rho}(J) \tag{12}$$

From Equations (11) and (12) and by the Definition of the space $C^n_{1-\gamma,\rho}(J)$, we obtain:

$$\left({}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\varphi\right) \in C^1_{1-\gamma,\rho}(J)$$

Applying operator ${}^{\rho}I_{a^+}^{\beta(1-\alpha)}$ on both sides of Equation (11) and using Lemmas 2 and 5, we have:

$$\begin{aligned}\left({}^{\rho}I_{a^+}^{\beta(1-\alpha)}\,{}^{\rho}D_{a^+}^{\gamma}y\right)(t) &= \varphi(t) + \frac{\left({}^{\rho}I_{a^+}^{1-\beta(1-\alpha)}\varphi(t)\right)(a)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\beta(1-\alpha)-1} \\ &= \left({}^{\rho}D_{a^+}^{\alpha,\beta}y\right)(t) = \varphi(t)\end{aligned}$$

that is, Equation (3) holds. Clearly, if $y \in C^{\gamma}_{1-\gamma,\rho}(J)$ satisfies Equation (5), then it also satisfies Equation (4). □

As a consequence of Theorem 4, we have Theorem 5.

Theorem 5. *Let $\gamma = \alpha + \beta - \alpha\beta$ where $0 < \alpha < 1$ and $0 \le \beta \le 1$; let $f : (a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function such that $f(\cdot, y(\cdot), u(\cdot)) \in C_{1-\gamma,\rho}(J)$ for any $y, u \in C_{1-\gamma,\rho}(J)$.*

If $y \in C^{\gamma}_{1-\gamma,\rho}(J)$, then y satisfies Equations (1) and (2) if and only if y is the fixed point of the operator $N : C_{1-\gamma,\rho}(J) \to C_{1-\gamma,\rho}(J)$ defined by:

$$Ny(t) = M\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s)ds, \quad t \in (a, T] \tag{13}$$

where:

$$M := \left(\frac{T^{\rho}-a^{\rho}}{\rho}\right)^{1-\gamma}\left[c - \frac{1}{\Gamma(\alpha)}\int_a^T \left(\frac{T^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}g(s)ds\right]$$

and $g : (0, T] \to \mathbb{R}$ be a function satisfying the functional equation:

$$g(t) = f(t, y(t), g(t))$$

Clearly, $g \in C_{1-\gamma,\rho}(J)$. In addition, by Lemma 1, $Ny \in C_{1-\gamma,\rho}(J)$.

Suppose that the function $f : (a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies the conditions:

$(H1)$ The function $f : (a, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is such that:

$$f(\cdot, u(\cdot), v(\cdot)) \in C^{\beta(1-\alpha)}_{1-\gamma,\rho} \text{ for any } u, v \in C_{1-\gamma,\rho}(J)$$

$(H2)$ There exist constants $K > 0$ and $0 < L < 1$ such that:

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \le K|u - \bar{u}| + L|v - \bar{v}|$$

for any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in (a, T]$.

Now, we state and prove our existence result for Equations (1) and (2) based on Banach's fixed point.

Theorem 6. *Assume $(H1)$ and $(H2)$ hold. If:*

$$\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^\alpha<\frac{1}{2} \tag{14}$$

then the Equations (1) and (2) has unique solution in $C^\gamma_{1-\gamma,\rho}(J)\subset C^{\alpha,\beta}_{1-\gamma,\rho}(J)$.

Proof. The proof will be given in two steps:

Step 1: We show that the operator N defined in Equation (13) has a unique fixed point y^* in $C_{1-\gamma,\rho}(J)$. Let $y,u\in C_{1-\gamma,\rho}(J)$ and $t\in(a,T]$, then, we have:

$$\begin{aligned}
&|Ny(t)-Nu(t)|\\
\leq\ &\frac{1}{\Gamma(\alpha)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\int_a^T\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)-h(s)|ds\\
+\ &\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)-h(s)|ds
\end{aligned}$$

where $g,h\in C_{1-\gamma,\rho}(J)$ such that:

$$\begin{aligned}
g(t)&=f(t,y(t),g(t))\\
h(t)&=f(t,u(t),h(t))
\end{aligned}$$

By (H2), we have:

$$\begin{aligned}
|g(t)-h(t)|&=|f(t,y(t),g(t))-f(t,u(t),h(t))|\\
&\leq K|y(t)-u(t)|+L|g(t)-h(t)|
\end{aligned}$$

Then:

$$|g(t)-h(t)|\leq\frac{K}{1-L}|y(t)-u(t)|$$

Hence, for each $t\in(a,T]$:

$$\begin{aligned}
&|Ny(t)-Nu(t)|\\
\leq\ &\frac{K}{(1-L)\Gamma(\alpha)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\int_a^T\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|y(s)-u(s)|ds\\
+\ &\frac{K}{(1-L)\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|y(s)-u(s)|ds\\
\leq\ &\frac{K}{(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\|y-u\|_{C_{1-\gamma,\rho}}\left({}^\rho I^\alpha_{a^+}\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right)(T)\\
+\ &\frac{K}{(1-L)}\left(I^\alpha_{a^+}\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right)(t)\|y-u\|_{C_{1-\gamma,\rho}}
\end{aligned}$$

By Lemma 3, we have:

$$\begin{aligned}
|Ny(t)-Nu(t)|\ \leq\ &\left[\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right.\\
+\ &\left.\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\right]\|y-u\|_{C_{1-\gamma,\rho}},
\end{aligned}$$

hence:

$$\begin{aligned}\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(Ny(t)-Nu(t))\right| &\leq \left[\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\right.\\ &+\left.\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right]\|y-u\|_{C_{1-\gamma,\rho}}\\ &\leq \frac{2K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\|y-u\|_{C_{1-\gamma,\rho}},\end{aligned}$$

which implies that:

$$\|Ny-Nu\|_{C_{1-\gamma,\rho}} \leq \frac{2K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\|y-u\|_{C_{1-\gamma,\rho}}.$$

By Equation (14), the operator N is a contraction. Hence, by Banach's contraction principle, N has a unique fixed point $y^* \in C_{1-\gamma,\rho}(J)$.

Step 2: We show that such a fixed point $y^* \in C_{1-\gamma,\rho}(J)$ is actually in $C^{\gamma}_{1-\gamma,\rho}(J)$.
Since y^* is the unique fixed point of operator N in $C_{1-\gamma,\rho}(J)$, then, for each $t \in (a, T]$, we have:

$$\begin{aligned} y^*(t) &= \left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left[c-\frac{1}{\Gamma(\alpha)}\int_a^T\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s,y^*(s),g(s))ds\right]\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\\ &+ {}^\rho I^{\alpha}_{a+}f(s,y^*(s),g(s))\end{aligned}$$

Applying ${}^\rho D^{\gamma}_{a+}$ to both sides and by Lemmas 3 and 6, we have:

$$\begin{aligned}{}^\rho D^{\gamma}_{a+}y^*(t) &= \left({}^\rho D^{\gamma}_{a+}\,{}^\rho I^{\alpha}_{a+}f(s,y^*(s),g(s))\right)(t)\\ &= \left({}^\rho D^{\beta(1-\alpha)}_{a+}f(s,y^*(s),g(s))\right)(t)\end{aligned}$$

Since $\gamma \geq \alpha$, by (H1), the right hand side is in $C_{1-\gamma,\rho}(J)$ and thus ${}^\rho D^{\gamma}_{a+}y^* \in C_{1-\gamma,\rho}(J)$, which implies that $y^* \in C^{\gamma}_{1-\gamma,\rho}(J)$. As a consequence of Steps 1 and 2 together with Theorem 5, we can conclude that Equations (1) and (2) have a unique solution in $C^{\gamma}_{1-\gamma,\rho}(J)$. □

We present now the second result, which is based on Krasnoselskii fixed point theorem.

Theorem 7. *Assume* (H1) *and* (H2) *hold. If:*

$$\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha} < 1 \tag{15}$$

then Equations (1) and (2) have at least one solution.

Proof. Consider the set:

$$B_{\eta^*} = \{y \in C_{1-\gamma,\rho}(J) : ||y||_{C_{1-\gamma,\rho}} \leq \eta^*\}$$

where:

$$\eta^* \geq \frac{\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left[|c|+\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\right]}{1-\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}}$$

and $f^* = \sup_{t\in J}|f(t,0,0)|$.
We define the operators P and Q on B_{η^*} by:

$$Py(t) = \left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left[c-\frac{1}{\Gamma(\alpha)}\int_a^T\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}g(s)ds\right]\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1} \tag{16}$$

$$Qy(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} g(s) ds \tag{17}$$

Then the fractional integral Equation (13) can be written as the operator equation:

$$Ny(t) = Py(t) + Qy(t), \;\; y \in C_{1-\gamma,\rho}(J)$$

The proof will be given in several steps:

Step 1: We prove that $Py + Qu \in B_{\eta^*}$ for any $y, u \in B_{\eta^*}$. For operator P, multiplying both sides of Equation (16) by $\left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma}$, we have:

$$\left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} Py(t) = \left(\frac{T^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left[c - \frac{1}{\Gamma(\alpha)} \int_a^T \left(\frac{T^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} g(s) ds \right]$$

then:

$$\left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} Py(t) \right| \leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left[|c| + \frac{1}{\Gamma(\alpha)} \int_a^T \left(\frac{T^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} |g(s)| ds \right] \tag{18}$$

By (H3), we have for each $t \in (a, T]$:

$$\begin{aligned} |g(t)| &= |f(t, y(t), g(t)) - f(t, 0, 0) + f(t, 0, 0)| \\ &\leq |f(t, y(t), g(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\ &\leq K|y(t)| + L|g(t)| + f^* \end{aligned}$$

Multiplying both sides of the above inequality by $\left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma}$, we get:

$$\begin{aligned} \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} g(t) \right| &\leq \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} f^* + K \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} y(t) \right| \\ &+ L \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} g(t) \right| \\ &\leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{1-\gamma} f^* + K\eta^* + L \left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} g(t) \right| \end{aligned}$$

Then, for each $t \in (a, T]$, we have:

$$\left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} g(t) \right| \leq \frac{\left(\frac{T^\rho - a^\rho}{\rho} \right)^{1-\gamma} f^* + K\eta^*}{1 - L} := M \tag{19}$$

Thus, Equation (18) and Lemma 3, imply:

$$\left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} Py(t) \right| \leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left[|c| + \frac{M\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\gamma-1} \right]$$

This gives:

$$||Py||_{C_{1-\gamma,\rho}} \leq \left(\frac{T^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left[|c| + \frac{M\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} \left(\frac{T^\rho - a^\rho}{\rho} \right)^{\alpha+\gamma-1} \right] \tag{20}$$

Using Equation (19) and Lemma 3, we have:

$$|Q(u)(t)| \leq \left[\frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}+\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\right]\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}$$

Therefore:

$$\begin{aligned}\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}Qu(t)\right| &\leq \left[\frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\right.\\ &+ \left.\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\right]\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha},\\ &\leq \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}\\ &+ \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\end{aligned}$$

Thus:

$$\|Qu\|_{C_{1-\gamma,\rho}} \leq \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}+\frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha} \tag{21}$$

Linking Equations (20) and (21), for every $y, u \in B_{\eta^*}$ we obtain:

$$\begin{aligned}\|Py+Qu\|_{C_{1-\gamma,\rho}} &\leq \max\left\{\|Py\|_{C_{1-\gamma,\rho}}, \|Qu\|_{C_{1-\gamma,\rho}}\right\}\\ &\leq \left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left[|c|+\frac{M\Gamma(\gamma)}{\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha+\gamma-1}\right]\\ &= \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha}\\ &+ \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}+\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}|c|\end{aligned}$$

Since:

$$\eta^* \geq \frac{\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left[|c|+\frac{\Gamma(\gamma)f^*}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\right]}{1-\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}}$$

we have:

$$\|Py+Qu\|_{PC_{1-\gamma,\rho}} \leq \eta^*$$

which infers that $Py+Qu \in B_{\eta^*}$.

Step 2: P is a contraction.
Let $y, u \in C_{1-\gamma,\rho}(J)$ and $t \in (a, T]$; then, we have:

$$\begin{aligned}&|Py(t)-Pu(t)|\\ \leq\ &\frac{1}{\Gamma(\alpha)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\int_a^T\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)-h(s)|ds\end{aligned}$$

where $g, h \in C_{1-\gamma,\rho}(J)$ such that:

$$\begin{aligned}g(t) &= f(t, y(t), g(t))\\ h(t) &= f(t, u(t), h(t))\end{aligned}$$

By (H2), we have:

$$\begin{aligned} |g(t)-h(t)| &= |f(t,y(t),g(t))-f(t,u(t),h(t))| \\ &\le K|y(t)-u(t)|+L|g(t)-h(t)| \end{aligned}$$

Then,

$$|g(t)-h(t)| \le \frac{K}{1-L}|y(t)-u(t)|$$

Therefore, for each $t \in (a,T]$:

$$\begin{aligned} & |Py(t)-Pu(t)| \\ \le\ & \frac{K}{(1-L)\Gamma(\alpha)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\int_a^T\left(\frac{T^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|y(s)-u(s)|ds \\ \le\ & \frac{K}{(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\|y-u\|_{C_{1-\gamma,\rho}}\left({}^\rho I_{a+}^\alpha\left(\frac{s^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right)(T). \end{aligned}$$

By Lemma 3, we have:

$$|Py(t)-Pu(t)| \le \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\|y-u\|_{C_{1-\gamma,\rho}},$$

hence:

$$\left|\left(\frac{t^\rho-a^\rho}{\rho}\right)^{1-\gamma}(Py(t)-Pu(t))\right| \le \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\|y-u\|_{C_{1-\gamma,\rho}},$$

which implies that:

$$\|Py-Pu\|_{C_{1-\gamma,\rho}} \le \frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}\|y-u\|_{C_{1-\gamma,\rho}}.$$

By Equation (15) the operator P is a contraction.

Step 3: Q is compact and continuous.

The continuity of Q follows from the continuity of f. Next we prove that Q is uniformly bounded on B_{η^*}.

Let any $u \in B_{\eta^*}$. Then by Equation (21) we have:

$$\|Qu\|_{PC_{1-\gamma,\rho}} \le \frac{\Gamma(\gamma)f^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{1-\gamma+\alpha} + \frac{K\Gamma(\gamma)\eta^*}{(1-L)\Gamma(\alpha+\gamma)}\left(\frac{T^\rho-a^\rho}{\rho}\right)^{\alpha}$$

This means that Q is uniformly bounded on B_{η^*}. Next, we show that QB_{η^*} is equicontinuous. Let any $u \in B_{\eta^*}$ and $0 < a < \tau_1 < \tau_2 \le T$. Then:

$$\begin{aligned} & \left|\left(\frac{\tau_2^\rho-a^\rho}{\rho}\right)^{1-\gamma}Q(y)(\tau_2)-\left(\frac{\tau_1^\rho-a^\rho}{\rho}\right)^{1-\gamma}Q(y)(\tau_1)\right| \le \\ & \frac{\left(\frac{\tau_2^\rho-a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)}\int_{\tau_1}^{\tau_2}\left(\frac{\tau_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)|ds \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{\Gamma(\alpha)} \int_a^{\tau_1} \left| \left[\left(\frac{\tau_2^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left(\frac{\tau_2^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \right.\right. \\
&- \quad \left.\left. \left(\frac{\tau_1^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left(\frac{\tau_1^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \right] \right| |g(s)| ds \\
&\leq \quad \frac{M\Gamma(\gamma) \left(\frac{\tau_2^\rho - a^\rho}{\rho} \right)^{1-\gamma}}{\Gamma(\alpha+\gamma)} \left(\frac{\tau_2^\rho - \tau_1^\rho}{\rho} \right)^{\alpha+\gamma-1} \\
&+ \quad \frac{M}{\Gamma(\alpha)} \int_a^{\tau_1} \left| \left[\left(\frac{\tau_2^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left(\frac{\tau_2^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \right.\right. \\
&- \quad \left.\left. \left(\frac{\tau_1^\rho - a^\rho}{\rho} \right)^{1-\gamma} \left(\frac{\tau_1^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \right] \right| \left(\frac{s^\rho - a^\rho}{\rho} \right)^{\gamma-1} ds
\end{aligned}
$$

Note that:

$$
\left| \left(\frac{\tau_2^\rho - a^\rho}{\rho} \right)^{1-\gamma} Q(y)(\tau_2) - \left(\frac{\tau_1^\rho - a^\rho}{\rho} \right)^{1-\gamma} Q(y)(\tau_1) \right| \to 0 \quad \text{as} \quad \tau_2 \to \tau_1
$$

This shows that Q is equicontinuous on J. Therefore, Q is relatively compact on B_{η^*}. By $C_{1-\gamma}$, type Arzela–Ascoli Theorem Q is compact on B_{η^*}.

As a consequence of Krasnoselskii's fixed point theorem, we conclude that N has at least a fixed point $y^* \in C_{1-\gamma,\rho}(J)$ and by the same way of the proof of Theorem 6, we can easily show that $y^* \in C^{\gamma}_{1-\gamma,\rho}(J)$. Using Lemma 5, we conclude that Equations (1) and (2) have at least one solution in the space $C^{\gamma}_{1-\gamma,\rho}(J)$. □

4. An Example

Consider the following terminal value problem:

$$
{}^{\frac{1}{2}}D_{1^+}^{\frac{1}{2},0} y(t) = \frac{2 + |y(t)| + \left| {}^{\frac{1}{2}}D_{0^+}^{\frac{1}{2},0} y(t) \right|}{108 e^{-t+3} \left(1 + |y(t)| + \left| {}^{\frac{1}{2}}D_{0^+}^{\frac{1}{2},0} y(t) \right| \right)} + \frac{\ln(\sqrt{t}+1)}{3\sqrt{\sqrt{t}-1}}, \quad t \in (1,2] \tag{22}
$$

$$
y(2) = c \in \mathbb{R} \tag{23}
$$

Set:

$$
f(t,u,v) = \frac{2+u+v}{108 e^{-t+3}(1+u+v)} + \frac{\ln(\sqrt{t}+1)}{3\sqrt{t}}, \; t \in (1,2], \; u,v \in [0,+\infty)
$$

We have:

$$
C^{\beta(1-\alpha)}_{1-\gamma,\rho}([1,2]) = C^{0}_{\frac{1}{2},\frac{1}{2}}([1,2]) = \left\{ h : (1,2] \to \mathbb{R} : \sqrt{2} \left(\sqrt{t} - 1 \right)^{\frac{1}{2}} h \in C([1,2]) \right\}
$$

with $\gamma = \alpha = \rho = \frac{1}{2}$ and $\beta = 0$. Clearly, the function $f \in C_{\frac{1}{2},\frac{1}{2}}([1,2])$.

Hence condition (H1) is satisfied.

For each $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in (1,2]$:

$$
\begin{aligned}
|f(t,u,v)-f(t,\bar{u},\bar{v})| &\leq \frac{1}{108e^{-t+3}}(|u-\bar{u}|+|v-\bar{v}|)\\
&\leq \frac{1}{108e}(|u-\bar{u}|+|v-\bar{v}|)
\end{aligned}
$$

Therefore, (H2) is verified with $K = L = \frac{1}{108e}$.

The condition:

$$\frac{K\Gamma(\gamma)}{\Gamma(\alpha+\gamma)(1-L)}\left(\frac{T^\rho - a^\rho}{\rho}\right)^\alpha \approx 0.0055 < 1$$

is satisfied with with $T = 2$ and $a = 1$. It follows from Theorem 7 that Equations (22) and (23) have a solution in the space $C^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}([1,2])$.

5. Conclusions

We have provided sufficient conditions ensuring the existence and uniqueness of solutions to a class of terminal value problem for differential equations with the Hilfer–Katugampola type fractional derivative. The arguments are based on the classical Banach contraction principle, and the Krasnoselskii's fixed point theorem. An example is included to show the applicability of our results.

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