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Topological methods for discontinuous operators and applications

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Topological methods for discontinuous operators and applications

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To Lucía

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Jorge Rodríguez López

Contents

Notation

List of symbols most used throughout the work.

$\mathbb{N}$	Set of natural numbers, that is, $\{1, 2, \dots\}$.
$\mathbb{Z}$	Set of integer numbers.
$\mathbb{Q}$	Set of rational numbers.
$\mathbb{R}$	Set of real numbers.
$\mathbb{R}_+$	Set of nonnegative real numbers.
2^X	*Power set*: set of all subsets of X, also denoted by $\mathcal{P}(X)$.
$\overline{U}$	Closure of the set U.
∂U	Boundary of the set U.
$B_r(x)$	Open ball centered at x and radius r.
$\rho(x, U)$	Distance from the point x to the set U, i.e., $\rho(x, U) = \inf\{d(x, y) \,:\, y \in U\}$.
co	Convex hull.
$\overline{\mathrm{co}}$	Closed convex hull.
$\mathcal{C}(I)$	Space of continuous real functions defined on I.
$\mathcal{C}^n(I)$, $n \in \mathbb{N}$	Space of n-times differentiable real functions defined on I such that the j-th derivative is continuous for $j = 0, 1, \dots, n$.
$\mathcal{AC}(I)$	Space of absolutely continuous functions defined on I.
$L^p(I)$, $1 \le p < \infty$	Lebesgue p space on I, that is, $L^p(I) = \{x : I \to \mathbb{R} \,:\, x$ Lebesgue measurable, $\int_I \lvert x\rvert^p < +\infty\}$.
$L^\infty(I)$	Space of the measurable functions on I which are essentially bounded.
$W^{k,p}(I)$, $k, p \in \mathbb{N}$	Sobolev space $k - p$ on I, that is, $W^{k,p}(I) = \{x \in \mathcal{C}^{k-1}(I) \,:\, x^{k-1)} \in \mathcal{AC}(I),\ x^{k)} \in L^p(I)\}$.
$m(A)$	Lebesgue measure of A.
χ_A	Characteristic function of the set A.
$\mathrm{Fix}(T)$	Set of fixed points of the operator T.
Id	Identity function.

Abstract

This Thesis, entitled *Topological methods for discontinuous operators and applications*, contains most of the research work developed by the author during the last years.

It should be highlighted from the beginning that the keyword in the title is *discontinuous* since it is the lack of continuity of the operators studied along the text what encourages our research.

As a motivation to start with the study of topological methods and its applications to discontinuous differential equations, we present a summary of the chapters included in this manuscript. It may be divided into two parts: the first one, which comprises the first two chapters, focuses on the developing of a fixed point theory for a class of discontinuous operators based on that for multivalued mappings. The second part, which goes from Chapter 3 to Chapter 6, is devoted to the study of differential problems with discontinuous nonlinearities.

The aim of the first part is to build a machinery which applies for discontinuous operators and so it is useful to establish new existence results concerning differential problems. The simplicity of the idea behind the fixed point results may hide their utility and that is why both parts must be seen as a whole instead of as independent items.

Now the main ideas in each chapter are briefly presented. Our results are also collected in several papers, see [61–66, 102–105, 121, 122, 128, 129].

Chapter 1: Degree theory for a class of discontinuous operators

The study of differential problems is frequently addressed by means of the topological degree theory and this implies to deal with the associated fixed point operators defined between normed spaces. In the case of differential problems with discontinuous nonlinearities, which are the type of problems we are concerned with, the operators are not continuous and thus the classical Leray–Schauder's degree theory becomes useless. That is our main motivation to consider a theory for discontinuous operators.

Given an operator $T : D \subset X \longrightarrow X$, not necessarily continuous, we consider its closed–convex envelope $\mathbb{T} : D \subset X \longrightarrow 2^X$ defined as

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\text{co}}\, T\left(\overline{B}_\varepsilon(x) \cap D\right) \quad \text{for every } x \in D,$$

where D is a nonempty subset of a normed space X.

If the set D is the closure of a nonempty, bounded and open subset Ω of a Banach space X, then the topological degree of $\mathbb{T}$ is well–defined as that for upper semicontinuous multivalued mappings [40, 139] under reasonable conditions on T. This fact provides a way to introduce

the notion of degree for a discontinuous operator T by means of that for its closed–convex envelope $\mathbb{T}$.

Definition. *Let Ω be a bounded open subset of a Banach space X and let $T : \overline{\Omega} \longrightarrow X$ be such that $T\overline{\Omega}$ is relatively compact, $Tx \neq x$ for every $x \in \partial\,\Omega$, and*

$$x \in \mathbb{T}x \;\; \textit{implies} \;\;\; x = Tx \;\; \textit{for every} \;\;\; x \in \overline{\Omega}.$$

We define the degree of $Id - T$ on Ω with respect to $0 \in X$ as follows:

$$\deg\left(Id - T, \Omega, 0\right) = \deg\left(Id - \mathbb{T}, \Omega, 0\right).$$

Notice that if T is continuous, then $\mathbb{T}x = Tx$ for every $x \in \overline{\Omega}$ and so the degree defined above coincides with the Leray–Schauder degree. Moreover, observe that the continuity of the operator T is replaced by the more general condition

$$x \in \mathbb{T}x \quad \text{implies} \quad x = Tx \quad \text{for every} \quad x \in \overline{\Omega},$$

which means that all the fixed points of $\mathbb{T}$ are fixed points of T. This assumption lets prove that the new degree inherits the basic properties of the degree for multivalued mappings, namely, homotopy invariance, additivity, normalization and existence properties.

As happens in the classical case, the degree can only be applied to mappings defined in the closure of open sets, what is an important restriction for its applicability to differential problems since it is often convenient to work in sets with empty interior, as for example, some type of cones. To overcome this difficulty, it was introduced the concept of *fixed point index* based on the previously defined degree theory.

Chapter 2: Fixed point theorems for some discontinuous operators

This chapter is devoted to the generalization of some classical fixed point theorems to a class of discontinuous operators by using the same trick that in the definition of the degree: the continuity assumption is replaced by the condition that, exactly, means that the fixed points of a discontinuous operator coincide with those of its closed–convex envelope. Most of these fixed point theorems are obtained as a consequence of the computation of the degree on adequate sets.

A first glance is directed to the celebrated Schauder's fixed point theorem whose extension is now stated.

Theorem. *Let K be a nonempty convex and compact subset of X. Let $T : K \longrightarrow K$ be a mapping satisfying the following condition: $x \in \mathbb{T}x$ implies $x = Tx$ for every $x \in K$. Then T has a fixed point in K.*

The simplicity of the previous theorem turns it into a useful tool when looking for solutions to nonlinear problems. Nevertheless, it is frequently convenient to obtain additional properties about the solutions, as for instance, its *positivity* or *localization*, and then the fixed point theorems in cones acquire great importance. Krasnosel'skiĭ's compression–expansion type fixed point theorems in cones are extensively employed in this direction and here they are adapted to the framework of discontinuous operators.

Theorem. *Let K be a cone, $0 < r_1, r_2 \leq R$, $r_1 \neq r_2$ and let $T : \overline{B}_R(0) \cap K \longrightarrow K$ be a mapping such that $T(\overline{B}_R(0) \cap K)$ is relatively compact and it satisfies the following condition: $x \in \mathbb{T}x$ implies $x = Tx$ for every $x \in \overline{B}_R(0) \cap K$.*

Suppose that

(a) *$\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $\|x\| = r_1$ and all $\lambda \geq 1$,*

(b) *there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \mu w$ for every $\mu \geq 0$ and all $x \in K$ with $\|x\| = r_2$.*

Then T has a fixed point $x \in K$ such that

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}.$$

In a similar fashion, other fixed point theorems are generalized in Chapter 2. Some of them let derive the existence of multiple fixed points, as in the case of Leggett–Williams' three solutions theorem, whereas other ones provide more precise information about their localization, as the vectorial version of Krasnosel'skiĭ's theorem. All of them are applied in the next chapters to obtain new existence results for differential problems.

The main idea behind this new results goes further than the concrete theorems which are generalized in Chapter 2 and gives a method to adapt most of fixed point theorems for compact operators to this class of discontinuous ones.

Chapter 3: First order problems

This chapter deals with the existence of solutions to both scalar and system of first–order problems.

First, the functional initial value problem

$$x'(t) = f(t, x(t)) \text{ for a.a. } t \in I = [0, L], \quad x(0) = F(x),$$

is considered. Here, $F : \mathcal{C}(I) \to \mathbb{R}$ is assumed to be continuous and the nonlinearity f to satisfy the following conditions:

(H1) There exist $r < R$, $N \geq 0$ and $M \in L^1(I)$ such that $N + \|M\|_{L^1} < r$, $|F(x)| \leq N$ if $\|x\|_\infty \leq R$, and for a.a. $t \in I$ and all $x \in [-R, R]$ we have $|f(t, x)| \leq M(t)$.

(H2) Any composition $t \in I \mapsto f(t, x(t))$ is measurable if $x \in \mathcal{C}(I)$ and $\|x\|_\infty \leq R$.

(H3) There exist *admissible discontinuity curves* $\gamma_n : I_n \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ such that for a.a. $t \in I$ the function $x \mapsto f(t, x)$ is continuous on $[-R, R] \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$.

Under these assumptions, it is proven that the previous problem has at least one absolutely continuous solution. Obviously, the discontinuity curves must satisfy some type of *transversality* condition in order to be *admissible* curves. This notion of *admissible discontinuity curves* will be a key ingredient throughout the following chapters. Therefore, conditions like (H3) will be commonly employed in most of the existence results in the present manuscript, what distinguishes them from the classical results concerning Carathéodory nonlinearities.

The existence of solutions will be a consequence of the application of the degree theory introduced in Chapter 1 to the integral operator

$$Tx(t) = F(x) + \int_0^t f(s, x(s))\, ds \quad (t \in I,\ x \in \mathcal{C}(I)).$$

Next, the existence of absolutely continuous solutions to the system

$$x'(t) = f(t, x(t)) \text{ for a.a. } t \in I = [0, L], \quad x(0) = x_0 \in \mathbb{R}^n,$$

is investigated. Again the nonlinearity $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ need not be continuous.

The main hypothesis in the existence result consists in assuming that the function f can be expressed in the form

$$f(t, x) = F(t, g_1(\tau_1(t, x), x), g_2(\tau_2(t, x), x), \dots),$$

where for each $i \in N$,

(i) a) Each function $\tau_i : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable;

b) Each function $g_i : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous in $(\mathbb{R} \setminus A_i) \times \mathbb{R}^n$, where A_i is a null–measure set;

c) For a.a. $t \in I$ and all $x \in \mathbb{R}^n$, the condition $\tau_i(t, x) \in \mathbb{R} \setminus A_i$ for all $i \in N$ implies that $f(t, \cdot)$ is continuous at x.

(ii) For each $(t, x) \in \tau_i^{-1}(A_i)$ we have

$$\nabla \tau_i(t, x) \cdot (1, z) \neq 0 \quad \text{for all } z \in \mathcal{K}f(t, x),$$

where $\mathcal{K}f(t, x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} f\left(t, \overline{B}_\varepsilon(x)\right)$ for every $(t, x) \in I \times \mathbb{R}^n$.

Under additional assumptions on the measurability and boundedness of the nonlinearity f, the existence of solutions is obtained in the following way: first, it is shown that the differential inclusion

$$x'(t) \in \mathcal{K}f(t, x(t)) \text{ for a.a. } t \in I, \quad x(0) = x_0,$$

has at least one solution, and then the assumption stated above allows to prove that it is, in fact, a solution to the former discontinuous differential system.

Moreover, in the scalar case, we can also deduce the existence of extremal solutions. This fact joint with an iterative technique for discontinuous operators and the method of lower and upper solutions are the tools employed in Section 3.3 to establish the existence of extremal absolutely continuous solutions to the functional problem

$$x'(t) = f(t, x(t), x) \text{ for a.a. } t \in I, \quad B(x(0), x) = 0,$$

where both f and B may be discontinuous with respect to all of their variables, even though some monotonicity assumptions are required with respect to the functional arguments.

Some second order problems can be reduced to the class of first order problems with functional dependence above and, consequently, new existence results are also deduced for them.

Chapter 4: Second order problems and lower and upper solutions

This chapter concerns the existence and multiplicity of solutions to the following class of second–order boundary value problems

$$\left\{\begin{array}{rcl} x''(t) &=& f(t,x(t),x'(t)) \quad \text{for a.a. } t\in I=[a,b],\\ 0 &=& L_1(x(a),x(b),x'(a),x'(b),x),\\ 0 &=& L_2(x(a),x(b)). \end{array}\right.$$

The method of lower and upper solutions is combined here with the degree theory of Chapter 1 and the fixed point theorems of Schauder's type proven in Chapter 2 in order to get the existence results.

First, we assume the existence of well–ordered lower and upper solutions, that is, the lower solution is smaller than the upper solution, and we establish the existence of a $W^{2,1}$–solution between them. The nonlinearity is assumed to satisfy a Nagumo condition which allows us to obtain *a priori* bounds for the derivative of the solutions. To prove the existence result, we look for fixed points in $\mathcal{C}^1(I)$ of the integral operator associated to a convenient modified problem.

The main difference with respect to other works where this type of problems was studied by means of similar methods is that here the nonlinearity may be discontinuous over the graphs of at most a countable number of *admissible discontinuity curves*. Moreover, the existence of extremal solutions between a lower and an upper solutions is also established.

On the other hand, if there exist a lower and an upper solution, but they are not well–ordered, then it is possible to construct a pair of constant well ordered lower and upper solutions and so the existence of solutions will be a consequence of the previously mentioned results. To do so, we consider a more restrictive problem with $L_2(x,y)=x-y$ and we require stronger assumptions on the nonlinearity.

Several multiplicity results are also obtained as an application of the previous results by assuming the existence of more than a pair of lower and upper solutions with some order relations between them.

Finally, in Section 4.4, we present new existence results to the following second order problem on the half line

$$\left\{\begin{array}{l} x''(t)=f(t,x(t),x'(t)) \quad \text{for a.a. } t\in\mathbb{R}_+,\\ L(x(0),x'(0),x)=0, \quad \lim\limits_{t\to+\infty} x'(t)=B. \end{array}\right.$$

The philosophy employed is similar to that described above in the case of bounded domains, but now we look for fixed points in a different Banach space

$$X=\left\{x\in\mathcal{C}^1(\mathbb{R}_+):\ \lim_{t\to\infty}\frac{x(t)}{1+t}\in\mathbb{R} \text{ and } \lim_{t\to\infty}x'(t)\in\mathbb{R}\right\}$$

endowed with a Bielecki–type norm,

$$\|x\|:=\max\left\{\|x\|_0,\|x\|_1\right\},$$

where

$$\|x\|_0=\sup_{0\le t<\infty}\frac{|x(t)|}{1+t} \quad \text{and} \quad \|x\|_1=\sup_{0\le t<\infty}|x'(t)|.$$

It is remarkable the fact that our results improve those in the literature even in the case of Carathéodory nonlinearities since the usual notion of lower and upper solutions is relaxed and, furthermore, we also prove the existence of extremal solutions between them.

Chapter 5: Positive solutions for second and higher order problems

The existence of *positive* solutions to integral and differential equations acquires a great importance due to the fact that in most applications differential equations model physical or biological processes where the magnitudes cannot attain negative values. In this chapter we look for positive solutions to different second and fourth order problems with discontinuous nonlinearities.

In our setting, the solutions of the differential problem correspond with the fixed points of a Hammerstein–type operator

$$Tu(t) := \int_0^1 G(t,s)g(s)f(s,u(s))\,ds,$$

where G is the Green's function associated to the differential problem. The idea now is to apply the fixed point theorems in cones obtained in Chapter 2 to the operator T. In this way, we deduce the existence of positive solutions to the differential problems. As in the previous chapters, we allow nonlinearity f to be discontinuous over some admissible discontinuity curves.

Notice that the technique based on fixed point theorems in cones requires the construction of a suitable cone of functions (which may vary depending on the differential problem considered). At this point the properties of the Green's function play a crucial role. For the problems studied, there exist a continuous function $\Phi : I \to \mathbb{R}_+$ and a constant $c \in (0,1]$ such that

$$\begin{aligned} G(t,s) &\le \Phi(s) && \text{for all } t,s \in I, \\ c\,\Phi(s) &\le G(t,s) && \text{for all } t \in [a,b],\ s \in I, \end{aligned}$$

where $[a,b] \subset I$. These bounds for the Green's function lets prove that the operator T maps the cone

$$K = \left\{ u \in \mathcal{C}(I) :\ u \ge 0,\ \min_{t\in[a,b]} u(t) \ge c\,\|u\|_\infty \right\},$$

into itself. This cone K is adequate to the application of Krasnosel'skiĭ–type fixed point theorems.

On the other hand, when we look for multiplicity results by means of applying Leggett–Williams–type fixed point theorems we employ the cone of nonnegative continuous functions $P = \{u \in \mathcal{C}(I) :\ u \ge 0\}$.

Chapter 6: Positive solutions for general problems

The aim of this chapter is to obtain the existence of positive solutions for differential problems for which a Green's function may not exist but a *Harnack type inequality* holds. This fact entails the main contrast with respect to the problems studied in the previous chapter. As an example of the type of problems which can be considered we mention ϕ–Laplacian equations.

More precisely, we study the existence of positive solutions for a general problem of the form

$$\begin{cases} Lu(t) = f(t, u(t)) \quad \text{a.e. on } I = [0, 1], \\ u \in \mathcal{B}, \end{cases}$$

where $\mathcal{B} \subset \mathcal{C}(I)$ and $L : D(L) \subset \mathcal{C}(I) \longrightarrow L^1(I)$ is a general operator not necessarily linear.

The method used also differs from that in the previous chapter. Here, we consider first the auxiliary problem

$$\begin{cases} Lu(t) \in \mathcal{K}f(t, u(t)) \quad \text{a.e. on } I, \\ u \in \mathcal{B}. \end{cases}$$

Once the existence of solutions to the differential inclusion is guaranteed, a suitable *transversality* condition on the discontinuities of the nonlinearity f lets prove that they are also solutions to the former problem.

As a consequence of the obtained results, we derive new existence principles for the Dirichlet–Neumann problem involving the ϕ–Laplacian

$$\begin{cases} -(\phi(u'))'(t) = f(t, u(t)) \quad \text{a.e. on } I, \\ u'(0) = u(1) = 0, \end{cases}$$

where $\phi : (-a, a) \to (-b, b)$ is an increasing homeomorphism such that $\phi(0) = 0$ and $0 < a, b \leq \infty$.

Moreover, the localization properties of solutions allow us to obtain some multiplicity results under additional conditions. In the case of nonlinearities with excessive oscillations towards zero or infinity we deduce the existence of infinitely many positive solutions.

Conclusions and further work

Along this thesis we develop a method to deal with differential problems with discontinuous nonlinearities. Such method is essentially based on the fixed point theory for upper semicontinuous mappings and the transversality conditions on the nonlinearities of the differential problems. The notion of admissible discontinuity curves is adapted to a broad number of boundary value problems and, therefore, new existence results are achieved for them. They are illustrated with a variety of examples.

Despite all the theory developed along these pages concerning discontinuous operators and discontinuous differential equations, a huge number of problems still remain open in this field waiting to be solved. There are several directions for further work that we would like to emphasize here and which, in some way, they would complement or improve our present results.

A challenging problem is the definition of a topological degree theory for discontinuous operators without employing that for multivalued mappings. Even in the finite dimensional setting, the extension of the Brouwer degree to a class of discontinuous operators seems to be an arduous task. Obviously, the continuity of the operator must be replaced by another (weaker) condition in order to obtain a meaningful theory.

Our fixed point theory for discontinuous operators depends heavily on the definition of the closed–convex envelope. It is the smaller upper semicontinuous mapping with closed

and convex values which contains the discontinuous operator as a selection. Nevertheless, it could be interesting to analyze another regularizations which may not contain the studied operator but may provide sharper fixed point theorems.

A key notion throughout the text is that of the admissible discontinuity curves. Its definition is given for differential equations, but it is natural to ask oneself whether a similar idea also holds for integral equations. Moreover, in our existence results we assume that it may exist at most a countable number of these curves, so it is reasonable to wonder if with an uncountable number of such curves those results hold true.

Another interesting problem, which was not studied in this thesis, is the uniqueness of solutions for discontinuous differential equations. In the case of first order systems, the paper by Bressan and Shen [25], whose existence result was generalized in Section 3.2, may be a good starting point.

In Chapter 4, the existence of solutions in the case of non well–ordered lower and upper solutions is achieved for a more restrictive class of boundary conditions than in the well–ordered case, so it remains unanswered whether the existence result is true for the general boundary conditions. Moreover, a common extension of the results in Chapter 4 consists in considering the ϕ–Laplacian equations. Nevertheless, in the case of ϕ–Laplacian equations we were not able to check that the fixed points of the integral operator associated to the differential problem coincide with the fixed points of its closed–convex envelope. That is the reason why we employ a different technique in Chapter 6. That technique uses fixed point theory for decomposable mappings, but it seems to be not enough for our purpose and the developing of a degree theory for this class of mappings would be helpful.

Introduction

A usual assumption in the study of the existence of solutions for initial and boundary value problems is the continuity on the right–hand side of the differential equation. More specifically, for an equation of the form

$$Lu = f(t,u) \quad \text{on } I, \tag{Eq}$$

where L stands for a differential operator, the most commons hypotheses concerning the function $f : I \times \mathbb{R} \to \mathbb{R}$ are the continuity in both variables or the Carathéodory conditions. Both of them require to the function f to satisfy the following condition:

(C) For a.a. $t \in I$, the function $u \mapsto f(t,u)$ is continuous on $\mathbb{R}$.

If the condition (C) is weakened or removed, then equation (Eq) is called a *discontinuous differential equation*. Its study often relies on fixed point theorems for monotone operators and thus monotonicity hypotheses about f are imposed, see [78]. Also it is frequent to identify the solutions of discontinuous differential equations with the solutions of some suitable differential inclusions following the early ideas of Filippov [68]. Depending on the multivalued mapping which replaces the function f in (Eq), different notions of *generalized* solutions to (Eq) appear (see the papers [75, 134] and the references therein). They are crucial in the study of some physical processes modeled by discontinuous differential equations as, for instance, dry friction [24, 108].

Our approach throughout this manuscript is essentially different from both mentioned above since in all our existence results for differential problems we look for solutions in the Carathéodory sense and no monotonicity assumptions are required on f. Obviously, not any discontinuous differential problem has a Carathéodory solution, so some transversality condition is necessary at the discontinuity points, see [25, 45, 84]. In this sense, our key ingredient will be the *admissible discontinuity curves* presented initially for second–order ordinary differential equations by Pouso [101] and Figueroa and Infante [58] and which will be extended here to a wide class of ordinary differential problems.

All the differential problems considered along the text can be written as a fixed point problem by means of the inverse of the differential operator L, which is given by an integral operator whose kernel is often the Green's function of the differential problem. However, the usual fixed point theory for compact operators is not applicable to discontinuous differential problems since the integral operators obtained are not continuous. That is the reason why we regularize either the right–hand side in the differential equation or the discontinuous fixed point operator. In both cases, we arrive to a multivalued problem which can be analyzed with the well–known fixed point theorems for upper semicontinuous mappings [49, 50, 144], for instance, Bohnenblust–Karlin's fixed point theorem [17] and Fitzpatrick–Petryshyn's compression–expansion result in cones [69].

After obtaining a fixed point for the multivalued problem, we prove that in fact it is a fixed point of the single–valued discontinuous operator and so a solution of the discontinuous differential problem. That is the point where the transversality condition on the discontinuity set of f plays a decisive role. In this way, the step of going to the multivalued problem can be seen as an auxiliary problem for, later, coming back and achieving new results concerning the former discontinuous single–valued one.

In what follows, we describe and compare briefly the two approaches employed in the thesis to regularize the discontinuous differential problem and study it by using multivalued analysis.

The first one consists in considering the discontinuous single–valued fixed point operator and 'convexifying' it as follows: given the mapping $T : D \subset X \longrightarrow X$, we associate to it the multivalued operator $\mathbb{T} : D \subset X \longrightarrow 2^X$ defined as

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\text{co}}\, T\left(\overline{B}_\varepsilon(x) \cap D\right) \quad \text{for every } x \in D,$$

where X is a Banach space. Under suitable conditions on T, the set–valued operator $\mathbb{T}$ is upper semicontinuous with nonempty, closed and convex values and maps bounded sets into relatively compact sets, so it is under the hypotheses of the topological degree for multivalued mappings [40, 139]. Moreover, if T is continuous at the point x, then the associated multivalued map $\mathbb{T}$ satisfies that $\mathbb{T}x = \{Tx\}$. This lets us develop a degree theory for a class of single-valued mappings for which the condition $\text{Fix}(\mathbb{T}) \subset \text{Fix}(T)$ holds and, subsequently, some fixed point theorems for this class of mappings are deduced. This method allows to deal with a lot of different differential problems just as in the continuous case with the unique difference of showing that T satisfies that $\text{Fix}(\mathbb{T}) \subset \text{Fix}(T)$ instead of proving that it is continuous.

In the second approach we regularize the right–hand side in the differential equation instead of the fixed point operator. Hence, in order to study the solutions of (Eq), we consider the differential inclusion

$$Lu \in \mathcal{K}f(t,u) \quad \text{on } I,$$

where $\mathcal{K}f : I \times \mathbb{R} \to 2^{\mathbb{R}}$ is the multivalued mapping defined by

$$\mathcal{K}f(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}} f\left(t, \overline{B}_\varepsilon(x)\right) \quad \text{for every } (t,x) \in I \times \mathbb{R}.$$

First, we look for solutions to the differential inclusion, which are known as *Krasovskij solutions*, and, later, we prove that they are solutions to (Eq).

Both techniques are not equivalent as shown by the following example.

Example. *Consider the autonomous Cauchy problem*

$$x' = f(x) = \begin{cases} 1, & \textit{if } x \in \mathbb{Q}, \\ -1, & \textit{otherwise}, \end{cases} \quad \textit{for a.a. } t \in I = [0,1],\ x(0) = 0.$$

It is obvious that $x(t) = t$ *for all* $t \in I$ *is a solution to*

$$x' \in \mathcal{K}f(x) \quad \textit{for a.a. } t \in I,\ x(0) = 0,$$

since $\mathcal{K}f(x) = [-1, 1]$ *for all* $x \in \mathbb{R}$.

On the other hand, consider the integral operator $T : \mathcal{C}(I) \to \mathcal{C}(I)$ *given by*

$$Tx(t) = \int_0^t f(x(s))\, ds.$$

Notice that finding solutions of the Cauchy problem is equivalent to find fixed points of the operator T*. Moreover, any fixed point of* T *must belong to the closed and convex set*

$$K = \{y \in \mathcal{C}(I) :\ y(0) = 0,\ |y(t) - y(s)| \le |t - s|\}.$$

Consider the multivalued mapping $\mathbb{T}$ *associated to the operator* $T : K \to K$ *defined as above.*

We shall show that the function $x(t) = t$ *is not a fixed point of the mapping* $\mathbb{T}$.

It is clear that $\mathbb{T}K \subset K$*. Hence if* $x \in \mathbb{T}x$*, it must exist a sequence of functions* $\{y_n\}_{n\in\mathbb{N}} \subset K$ *such that* $\|y_n - x\|_\infty \to 0$ *and* $\|Ty_n - x\|_\infty \to 0$*. Denote* $A_n = y_n^{-1}(\mathbb{Q})$ *and notice that*

$$Ty_n(1) = \int_0^1 f(y_n(s))\, ds = \int_{A_n} 1\, ds + \int_{[0,1]\setminus A_n} -1\, ds = m(A_n) - m([0,1] \setminus A_n).$$

Given an arbitrary sequence of functions $\{y_n\}_{n\in\mathbb{N}} \subset K$ *such that* $\|y_n - x\|_\infty \le 1/n$ *for each* $n \in \mathbb{N}$*, we shall show that* $m(A_n) \le 2/n$ *and thus* $x \notin \mathbb{T}x$*. Indeed, since* $\mathbb{Q}$ *is a null measure set, it is known that* $y_n' = 0$ *almost everywhere in* A_n *and then*

$$\begin{aligned} 1 - \frac{2}{n} \le y_n(1) - y_n(0) &= \int_0^1 y_n'(s)\, ds = \int_{[0,1]\setminus A_n} y_n'(s)\, ds + \int_{A_n} y_n'(s)\, ds \\ &\le m([0,1] \setminus A_n) = 1 - m(A_n), \end{aligned}$$

so the conclusion is obtained.

The proof of the condition $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$ have been done for a variety of integral operators T coming from first, second and higher order ordinary differential equations under certain conditions on the nonlinearities throughout this manuscript. As the reader may see, the proof is laborious and requires some technical results from measure theory. It was adapted to different kinds of problems by using as a starting point that given by Pouso in [101].

The main advantage of this method is, roughly speaking, that it allows to investigate a large number of differential problems just as in the case of continuous nonlinearities with the exception of proving that $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$ instead of showing that T is continuous, as a consequence of the fixed point machinery developed in Chapters 1 and 2. Also, compared to the second approach, it avoids to study the upper semicontinuity and the compactness of the Nemytskii operator associated to the multivalued mapping $\mathcal{K}f$ which is not a trivial matter, see [85].

However, when the differential operator is not linear, for instance in the case of ϕ–Laplacian problems, we noticed that similar reasonings to those for linear operators are not helpful to prove the mentioned condition $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$. That is the reason why we consider then the second method (consisting in regularizing the right–hand side in the differential equation), which becomes really useful in Chapter 6 in order to obtain existence results for differential problems involving the ϕ–Laplacian.

Chapter 1

Degree theory for a class of discontinuous operators

Degree theory is a fundamental tool in nonlinear analysis, especially in the study of existence of solutions to many types of problems; see [73, 100, 113]. Readers interested in the history and the development of degree theory are referred to the expository paper by Mawhin [110].

As a well–known fact, *continuity* is a basic assumption in degree theory and the clearest limitation of its applicability. As an important particular case, we point out the usual degree-theory-based proofs of existence of solutions to boundary value problems, which consist on turning the former problems into fixed point problems of integral operators for which degree theory applies. However, most *discontinuous* differential equations, see [38, 78], fall outside that scope simply because the corresponding fixed point operators are not continuous.

On the other hand, the analysis of discontinuous differential equations usually leans on fixed points results for monotone operators, and therefore the corresponding existence results lean, to some extent, on monotonicity conditions imposed on the nonlinear parts of the considered problems.

In this chapter we introduce a new definition of topological degree, which coincides with the usual degree in the continuous case, and it is also suitable for a wide class of operators which need not be continuous. As a consequence, this new degree is proved to be useful in the study of discontinuous differential equations and, moreover, it yields new existence results which do not require monotonicity at all. Using multivalued analysis in the study of discontinuous problems is a classical approach and the novelty here is the use of multivalued analysis to obtain results for single–valued operators. The applicability of our abstract results will be clear in the next chapters, where we prove existence of solutions to differential equations which are allowed to be discontinuous over countable families of curves, similar to Filippov discontinuity sets [68].

Nevertheless, results based on degree theory need the operator to be defined in a suitable set with nonempty interior and this can be quite restrictive for certain applications. For instance, when we are looking for positive solutions of boundary value problems in a certain cone of continuous functions, this cone could have empty interior and so degree–based techniques become inapplicable. In the literature, this inconvenience is usually overcome by using the so-called *fixed point index* [4, 69], which allows us to obtain new fixed point results as for example the well–known Krasnosel'skiĭ's fixed point theorem in cones. Therefore, in the last section of this chapter, we define a fixed point index theory, which applies to discontinuous operators, by using the fixed point theory for discontinuous operators that was developed in Section 1.1 and a retraction trick, following the line of [4, 69].

The main results of this chapter are part of the papers [64] and [66].

1.1 A topological degree for discontinuous operators

Here and henceforth, we consider a normed space $(X, \|\cdot\|)$ and operators

$$T : D \longrightarrow X, \qquad \emptyset \neq D \subset X,$$

not necessarily continuous, for which we want to define a degree.

The main idea in our definition of degree consists in replacing T by the following multivalued version of it for which a degree theory is already available under certain additional assumptions to be detailed.

Definition 1.1.1. *The closed–convex envelope of an operator $T : D \subset X \longrightarrow X$ is the multivalued mapping $\mathbb{T} : D \longrightarrow 2^X$ given by*

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\text{co}}\, T\left(\overline{B}_\varepsilon(x) \cap D\right) \quad \text{for every } x \in D, \tag{1.1.1}$$

where $\overline{B}_\varepsilon(x)$ denotes the closed ball centered at x and radius ε, and $\overline{\text{co}}$ means closed convex hull.

In other words, we say that $y \in \mathbb{T}x$ if for every $\varepsilon > 0$ and every $\rho > 0$ there exist $m \in \mathbb{N}$ and a finite family of vectors $x_i \in \overline{B}_\varepsilon(x) \cap D$ and coefficients $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, m)$ such that $\sum \lambda_i = 1$ and

$$\left\| y - \sum_{i=1}^{m} \lambda_i T x_i \right\| < \rho.$$

The following properties are straightforward consequences of the previous definition.

Proposition 1.1.2. *In the conditions of Definition 1.1.1 the following statements are true:*

1. *$\mathbb{T}x$ is closed and convex, and $Tx \in \mathbb{T}x$ for all $x \in D$;*

2. *If $T\,D \subset K$ for some closed and convex set $K \subset X$, then $\mathbb{T}\,D \subset K$.*

Closed–convex envelopes (cc–envelopes, for short) need not be upper semicontinuous (usc, for short), see [50, Example 1.2], unless some additional assumptions are imposed on T. In the following proposition we show that compactness in some sense is a sufficient condition.

Proposition 1.1.3. *Let T be an operator in the conditions of Definition 1.1.1. If T maps bounded sets into relatively compact sets, then $\mathbb{T}$ assumes compact values and it is usc. Moreover, if $T\,D$ is relatively compact, then $\mathbb{T}\,D$ is relatively compact.*

Proof. Let $x \in D$ be fixed and let us prove that $\mathbb{T}x$ is compact. We know that $\mathbb{T}x$ is closed, so it suffices to show that it is contained in a compact set. To do so, we note that

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\text{co}}\, T\left(\overline{B}_\varepsilon(x) \cap D\right) \subset \overline{\text{co}}\, T\left(\overline{B}_1(x) \cap D\right) \subset \overline{\text{co}}\, \overline{T\left(\overline{B}_1(x) \cap D\right)},$$

and $\overline{\text{co}}\,\overline{T\left(\overline{B}_1(x)\cap D\right)}$ is compact because it is the closed convex hull of a compact subset of a Banach space; see [3, Theorem 5.35]. Hence $\mathbb{T}x$ is compact for every $x \in D$, and this property allows us to check that $\mathbb{T}$ is usc by means of sequences, see Proposition A.2. To do so, let $x_n \to x$ in D and let $y_n \in \mathbb{T}x_n$ for all $n \in \mathbb{N}$ be such that $y_n \to y$, then we have to prove that $y \in \mathbb{T}x$. Let $\varepsilon > 0$ be fixed and take $N \in \mathbb{N}$ such that $\overline{B}_\varepsilon(x_n) \subset \overline{B}_{2\varepsilon}(x)$ for all $n \geq N$. Then we have

$$y_n \in \overline{\text{co}}\,T(\overline{B}_\varepsilon(x_n)\cap D) \subset \overline{\text{co}}\,T(\overline{B}_{2\varepsilon}(x)\cap D) \text{ for all } n \geq N,$$

which implies that $y \in \overline{\text{co}}\,T(\overline{B}_{2\varepsilon}(x)\cap D)$. Since $\varepsilon > 0$ was arbitrary, we conclude that $y \in \mathbb{T}x$.

Arguments are similar for the second part of the proposition. For every $x \in D$ and $\varepsilon > 0$ we have

$$\overline{\text{co}}\,T(\overline{B}_\varepsilon(x)\cap D) \subset \overline{\text{co}}\,\overline{T\,D},$$

and therefore $\mathbb{T}x \subset \overline{\text{co}}\,\overline{T\,D}$ for all $x \in D$. Hence, $\overline{\mathbb{T}\,D}$ is compact because it is a closed subset of the compact set $\overline{\text{co}}\,\overline{T\,D}$. $\square$

Our next proposition shows that $\mathbb{T}$ is the smallest closed and convex–valued usc operator which has T as a selection.

Proposition 1.1.4. *Let $\mathbb{T}$ be the cc–envelope of an operator $T : D \longrightarrow X$.*

If $\tilde{\mathbb{T}} : D \longrightarrow 2^X$ is an usc operator which assumes closed and convex values and, moreover, $Tx \in \tilde{\mathbb{T}}x$ for all $x \in D$, then $\mathbb{T}x \subset \tilde{\mathbb{T}}x$ for all $x \in D$.

Proof. Let $\tilde{\mathbb{T}} : D \longrightarrow 2^X$ be an operator in the conditions of the statement, let $x \in D$ be fixed and take $y \in \mathbb{T}x$; we have to show that $y \in \tilde{\mathbb{T}}x$.

First, we fix $r > 0$ and we consider the open set

$$V = \bigcup_{u \in \tilde{\mathbb{T}}x} B_{r/2}(u).$$

Obviously, we have $\tilde{\mathbb{T}}x \subset V$ and $\rho(z, \tilde{\mathbb{T}}x) < r/2$ for all $z \in V$, where ρ denotes the metric induced by the norm in X. Furthermore, as $\tilde{\mathbb{T}}x$ is convex, we have that

$$\rho(z, \tilde{\mathbb{T}}x) < r/2 \quad \text{for all } z \in \text{co}\, V. \tag{1.1.2}$$

Since $\tilde{\mathbb{T}}$ is upper semicontinuous, there exists $\varepsilon_0 > 0$ such that $\tilde{\mathbb{T}}(\overline{B}_{\varepsilon_0}(x)\cap D) \subset V$. Since T is a selection of $\tilde{\mathbb{T}}$, we also have that $T(\overline{B}_{\varepsilon_0}(x)\cap D) \subset V$, and then

$$y \in \mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\text{co}}\,T\left(\overline{B}_\varepsilon(x)\cap D\right) \subset \overline{\text{co}}\,T(\overline{B}_{\varepsilon_0}(x)\cap D) \subset \overline{\text{co}}\,V.$$

Hence we can find $z_i \in V$ and $\lambda_i \in [0,1]$, for $i = 1, 2, \ldots, m$, such that $\sum \lambda_i = 1$ and

$$\left\| y - \sum_{i=1}^m \lambda_i z_i \right\| < \frac{r}{2}.$$

Since $\sum \lambda_i z_i \in \operatorname{co} V$, we can use (1.1.2) to obtain that

$$\rho(y, \tilde{\mathbb{T}}x) \le \rho\left(\sum_{i=1}^{m} \lambda_i z_i, \tilde{\mathbb{T}}x\right) + \left\| y - \sum_{i=1}^{m} \lambda_i z_i \right\| < r,$$

which implies that $y \in \tilde{\mathbb{T}}x$ because $r > 0$ can be arbitrarily small and $\tilde{\mathbb{T}}x$ is closed. □

As a remarkable corollary of the previous result we obtain the following.

Corollary 1.1.5. *If $T : D \longrightarrow X$ is continuous, then $\mathbb{T}x = \{Tx\}$ for all $x \in D$.*

We can also deduce the following useful property concerning cc-envelopes.

Lemma 1.1.6. *Let $T, S : D \longrightarrow X$ be mappings in the conditions of Definition 1.1.1. The following properties hold:*

i) If $F = T + S$ and S is a continuous mapping, then $\mathbb{F}x = \mathbb{T}x + Sx$ for all $x \in D$.

ii) If $F = \lambda T$ with $\lambda \in \mathbb{R}$, then $\mathbb{F}x = \lambda \mathbb{T}x$ for all $x \in D$.

Proof. Let us prove the first statement. Observe that the mapping $F = T + S$ is a selection of $\mathbb{T} + S$, which is usc with closed and convex values, and thus Proposition 1.1.4, applied to F, implies that $\mathbb{F}x \subset \mathbb{T}x + Sx$ for all $x \in D$.

On the other hand, T is a selection of $\mathbb{F} - S$, which is also usc with closed and convex values, so again Proposition 1.1.4 guarantees that $\mathbb{T}x \subset \mathbb{F}x - Sx$ for all $x \in D$. Hence, we have that $\mathbb{T}x + Sx \subset \mathbb{F}x$ for all $x \in D$.

The second part follows immediately from the definition of cc-envelopes. □

We are already in a position to define a topological degree for some class of discontinuous operators. In this case we replace continuity by condition (1.1.3), which just means that every fixed point of $\mathbb{T}$ must be a fixed point of T. In the sequel, Id denotes the identity map.

Definition 1.1.7. *Let Ω be a bounded open subset of a Banach space X and let $T : \overline{\Omega} \longrightarrow X$ be such that $T\overline{\Omega}$ is relatively compact, $Tx \neq x$ for every $x \in \partial\,\Omega$, and*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for every } x \in \overline{\Omega}, \tag{1.1.3}$$

where $\mathbb{T}$ is the cc–envelope of T.

We define the degree of $Id - T$ on Ω with respect to $0 \in X$ as follows:

$$\deg\left(Id - T, \Omega, 0\right) = \deg\left(Id - \mathbb{T}, \Omega, 0\right), \tag{1.1.4}$$

where the degree in the right–hand side is that of usc multivalued operators (see Appendix A).

Remark 1.1.8. *As we mentioned above, condition* (1.1.3) *means that every fixed point of $\mathbb{T}$ in $\overline{\Omega}$ is a fixed point of T. Clearly, it is equivalent to the following*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for every } x \in \overline{\Omega} \cap \mathbb{T}\overline{\Omega},$$

since the fixed points of the operator $\mathbb{T}$ are expected to be located in its image. This simple fact helps to check condition (1.1.3) *in applications.*

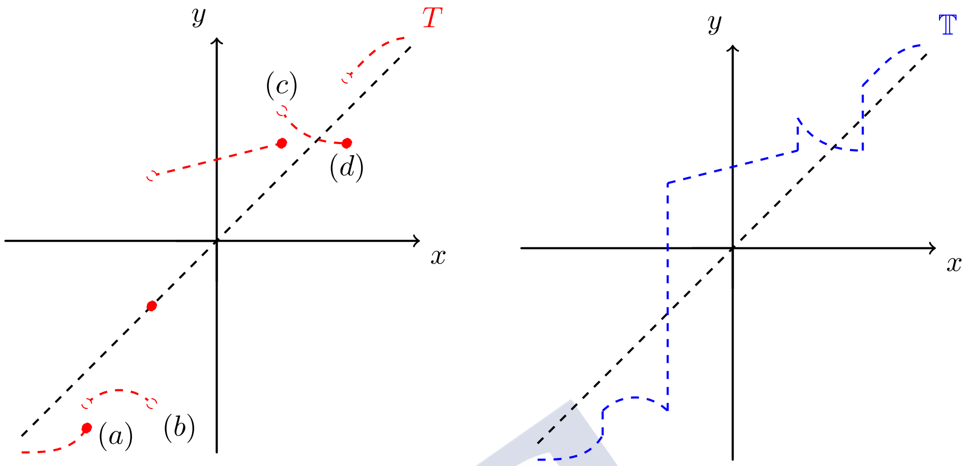

Figure 1.1.1: Function T and its convex–closed envelope $\mathbb{T}$

Let us see that $\deg\left(Id - \mathbb{T}, \Omega, 0\right)$ is well–defined in the conditions of Definition 1.1.7. First, we know from Proposition 1.1.3 that $\mathbb{T}\,\overline{\Omega}$ is relatively compact. Second, if $x \in \mathbb{T}x$ for some $x \in \partial\,\Omega$, then

$$\{x\} \cap \mathbb{T}x = \{x\} \quad \text{and } x \in \overline{\Omega} \cap \mathbb{T}\overline{\Omega},$$

which, together with condition (1.1.3), yields $x = Tx$, a contradiction with the assumptions on T. Therefore, $\deg\left(Id - \mathbb{T}, \Omega, 0\right)$ is well defined and Definition 1.1.7 makes sense.

Moreover, Definition 1.1.7 reduces to the usual Leray–Schauder degree when T is continuous. Indeed, when T is continuous we have $\mathbb{T}x = \{Tx\}$ for all $x \in \overline{\Omega}$, so condition (1.1.3) is trivially satisfied, and (1.1.4) is just

$$\deg\left(Id - T, \Omega, 0\right) = \deg\left(Id - \{T\}, \Omega, 0\right),$$

where $\deg\left(Id - \{T\}, \Omega, 0\right)$ is the degree for multivalued operators in the particular case of a single–valued completely continuous operator T, which coincides with the Leray–Schauder degree [95].

In what follows, as usual, we shall simplify notation and write $\deg\left(Id - T, \Omega\right)$ instead of $\deg\left(Id - T, \Omega, 0\right)$. The definition of $\deg\left(Id - T, \Omega, p\right)$ for any $p \in X$ reduces to the case $p = 0$, see [113].

Finally, we explain the geometrical meaning of condition (1.1.3) with a graphical example in the one–dimensional case. See Figure 1.1.1 and observe that at the discontinuity points (a) and (c) the jump in the graph of the function remains below and above the graph of the identity function, respectively. Hence, $\{x\} \cap \mathbb{T}x$ is just the empty set at both points, and so condition $\{x\} \cap \mathbb{T}x \subset \{Tx\}$ holds trivially there. In general, the behavior of the function “does not matter” when its graph does not cross the line $y = x$. On the other hand, the point (b) is a fixed point of the function T and thus condition $\{x\} \cap \mathbb{T}x \subset \{Tx\}$ is satisfied. Finally, the point (d) is a fixed point of $\mathbb{T}$ and not a fixed point of T, which means that this type of discontinuity points is not admissible (condition $\{x\} \cap \mathbb{T}x \subset \{Tx\}$ is not satisfied there). In short, when the graph of T crosses the graph of the identity function, then it must do so at a fixed point of T.

1.2 Basic properties of the degree

The degree we defined in the previous section provides, as we have seen, a generalization of Leray–Schauder degree for some kind of discontinuous operators. As we said at the beginning, Leray–Schauder degree is a very powerful tool that has been extensively used in many contexts, particularly for guaranteeing the existence of solutions of differential equations, and this is possible because of its topological and algebraic properties. Now we will show that our new degree also fulfills these properties, and this will be a consequence of the properties of degree for multivalued mappings.

Proposition 1.2.1. *Let Ω be a bounded open subset of a Banach space X.*

The degree introduced in Definition 1.1.7 satisfies the following properties:

1. *(Homotopy invariance) Let $H : \overline{\Omega} \times [0,1] \longrightarrow X$ be a mapping such that $H\left(\overline{\Omega} \times [0,1]\right)$ is relatively compact, $x \neq H(x,t)$ for all $x \in \partial\Omega$ and all $t \in [0,1]$, and for each $(x,t) \in \overline{\Omega} \times [0,1]$ we have*

$$\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\}, \tag{1.2.1}$$

where $\mathbb{H}_t$ is the cc-envelope of the operator $H_t(x) := H(x,t)$ for all $x \in \overline{\Omega}$.

If for each $(x,t) \in \overline{\Omega} \times [0,1]$ and all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x, t) > 0$ such that

$$s \in [0,1],\ |t-s| < \delta \implies \|H(z,t) - H(z,s)\| < \varepsilon \ \ \forall\, z \in \overline{B}_\delta(x) \cap \overline{\Omega}, \tag{1.2.2}$$

then $\deg(Id - H_t, \Omega)$ does not depend on t.

In the next three properties, we assume that $T : \overline{\Omega} \longrightarrow X$ is a mapping in the conditions of Definition 1.1.7.

2. *(Additivity) Let Ω_1 and Ω_2 be open, disjoint and such that $\Omega_1 \cup \Omega_2 \subset \Omega$.*

If $0 \notin (Id - T)(\overline{\Omega} \backslash (\Omega_1 \cup \Omega_2))$, then we have

$$\deg(Id - T, \Omega) = \deg(Id - T, \Omega_1) + \deg(Id - T, \Omega_2).$$

3. *(Excision) Let $A \subset \overline{\Omega}$ be a closed set such that $0 \notin (Id - T)(\partial\Omega) \cup (Id - T)(A)$. Then*

$$\deg(Id - T, \Omega) = \deg(Id - T, \Omega \backslash A).$$

4. *(Existence) If $\deg(Id - T, \Omega) \neq 0$, then there exists $x \in \Omega$ such that $Tx = x$.*

5. *(Normalization) $\deg(Id, \Omega) = 1$ if and only if $0 \in \Omega$.*

Proof.

1. We define $\mathbb{H}$ as the following multivalued mapping:

$$\mathbb{H}(x,t)=\bigcap_{\varepsilon>0}\overline{\text{co}}\,H\left(\overline{B}_\varepsilon(x)\cap\overline{\Omega},t\right).$$

Observe that $\mathbb{H}(x,t)=\mathbb{H}_t(x)$, where $\mathbb{H}_t$ is as in the statement.
Since $H\left(\overline{\Omega}\times[0,1]\right)$ is a relatively compact set, $\mathbb{H}\left(\overline{\Omega}\times[0,1]\right)$ is relatively compact. In addition, the multivalued mapping $\mathbb{H}$ is convex and closed valued. Let us prove that $\mathbb{H}:\overline{\Omega}\times[0,1]\to 2^X$ is an upper semicontinuous operator. To see this, it suffices to prove that if $x_n\to x$ in $\overline{\Omega}$, $t_n\to t$ in $[0,1]$ and $y_n\in\mathbb{H}(x_n,t_n)$ with $y_n\to y$, then $y\in\mathbb{H}(x,t)$. Let $\varepsilon>0$ and $\mu>0$ be fixed; we have to find $x_i\in\overline{B}_\varepsilon(x)\cap\overline{\Omega}$ and $\lambda_i\in[0,1]$ $(i=1,\dots,m)$ such that $\displaystyle\sum_{i=1}^m\lambda_i=1$ and

$$\left\|y-\sum_{i=1}^m\lambda_iH(x_i,t)\right\|<\mu. \tag{1.2.3}$$

We can assume without loss of generality that $\varepsilon<\delta(\mu/4,x,t)$, where δ is as in 1., and so we can take $N\in\mathbb{N}$ such that

$$\begin{aligned}&x_N\in\overline{B}_{\varepsilon/2}(x)\cap\overline{\Omega},\\&\|y-y_N\|<\frac{\mu}{2},\\&\|H(z,t_N)-H(z,t)\|<\frac{\mu}{4}\quad\forall\,z\in\overline{B}_\varepsilon(x)\cap\overline{\Omega}.\end{aligned}$$

As $y_N\in\mathbb{H}(x_N,t_N)$ we know that there exist $x_i\in\overline{B}_{\varepsilon/2}(x_N)\cap\overline{\Omega}$ and $\lambda_i\in[0,1]$ $(i=1,\dots,m)$ with $\displaystyle\sum_{i=1}^m\lambda_i=1$ and

$$\left\|y_N-\sum_{i=1}^m\lambda_iH(x_i,t_N)\right\|<\frac{\mu}{4}.$$

Hence, we have

$$\|x-x_i\|\le\|x-x_N\|+\|x_N-x_i\|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

so $x_i\in\overline{B}_\varepsilon(x)\cap\overline{\Omega}$. Moreover, by triangle inequality

$$\begin{aligned}\left\|y-\sum_{i=1}^m\lambda_iH(x_i,t)\right\|&\le\|y-y_N\|+\left\|y_N-\sum_{i=1}^m\lambda_iH(x_i,t_N)\right\|\\&\quad+\left\|\sum_{i=1}^m\lambda_i\left(H(x_i,t_N)-H(x_i,t)\right)\right\|\\&<\frac{\mu}{2}+\frac{\mu}{4}+\frac{\mu}{4}=\mu,\end{aligned}$$

and thus (1.2.3) is satisfied.

On the other hand, condition (1.2.1) along with the fact that $x \neq H(x,t)$ for every $(x,t) \in \partial\,\Omega \times [0,1]$, imply that $x \notin \mathbb{H}(x,t)$ for all $(x,t) \in \partial\,\Omega \times [0,1]$, and so the degree $\deg\,(Id - H_t, \Omega) = \deg\,(Id - \mathbb{H}_t, \Omega)$ is well–defined and it is independent of $t \in [0,1]$, by homotopy property of degree for multivalued mappings, see Theorem A.11.

2. Condition (1.1.3) and the hypothesis $0 \notin (Id - T)\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$ imply that condition $x \notin \mathbb{T}x$ holds for all $x \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$. Then, by direct application of the additivity property of degree for multivalued mappings we conclude that

$$\begin{aligned}\deg\,(Id - T, \Omega) &= \deg\,(Id - \mathbb{T}, \Omega) = \deg\,(Id - \mathbb{T}, \Omega_1) + \deg\,(Id - \mathbb{T}, \Omega_2)\\ &= \deg\,(Id - T, \Omega_1) + \deg\,(Id - T, \Omega_2)\,.\end{aligned}$$

3. As $0 \notin (Id - T)(A) \cup (Id - T)(\partial\,\Omega)$, condition (1.1.3) implies that

$$0 \notin (Id - \mathbb{T})(A) \cup (Id - \mathbb{T})(\partial\,\Omega),$$

and so the conclusion follows from the excision property of the degree for multivalued mappings.

4. As $\deg(Id - \mathbb{T}, \Omega) = \deg(Id - T, \Omega) \neq 0$, then there exists $x \in \Omega$ such that $x \in \mathbb{T}x$, and so condition (1.1.3) implies that $x = Tx$.

5. Since $\deg(Id, \Omega) = \deg(Id - 0, \Omega)$, and the operator 0 is continuous, our degree coincides with Leray–Schauder's one, and the normalization property is fulfilled. $\square$

Remark 1.2.2. *Note that condition* (1.1.3) *is not essential in order to define* $\deg(Id - T, \Omega)$ *in terms of* $\deg(Id - \mathbb{T}, \Omega)$*; in fact, to this end it suffices to require that* $\{x\} \cap \mathbb{T}x \subset \{Tx\}$ *in* $\partial\,\Omega$*. However, we need this condition to be satisfied in the whole of* $\overline{\Omega} \cap \mathbb{T}\overline{\Omega}$ *to guarantee the desirable existence property. As an example, the reader can consider the mapping* $T : (-1,1) \longmapsto (-1,1)$ *defined by* $\frac{1}{2}(\chi_{(-1,0]} - \chi_{(0,1)})$*. Thus defined,* $\{x\} \cap \mathbb{T}x = \emptyset$ *for* $x \in \{-1,1\}$ *and* $\deg(Id - \mathbb{T}, (-1,1)) \neq 0$ *(as a consequence of the multivalued version of Borsuk's Theorem [139]), but* T *has no fixed point in* $(-1,1)$.

The homotopy invariance property that we proved above becomes not very useful in practice. It is due to the instability of condition (1.2.1) requested for all $t \in [0,1]$ and all $x \in \overline{\Omega}$, because the set of functions satisfying this condition is not very well–behaved, as we show in the following example.

Example 1.2.3. *Let* $T : [0,1] \longrightarrow [0,1]$ *be the piecewise constant function given by*

$$T(x) = \begin{cases} 1/3 & \textit{if} \quad 0 \leq x \leq 1/3, \\ 2/3 & \textit{if} \quad 1/3 < x \leq 2/3, \\ 1 & \textit{if} \quad 2/3 < x \leq 1. \end{cases}$$

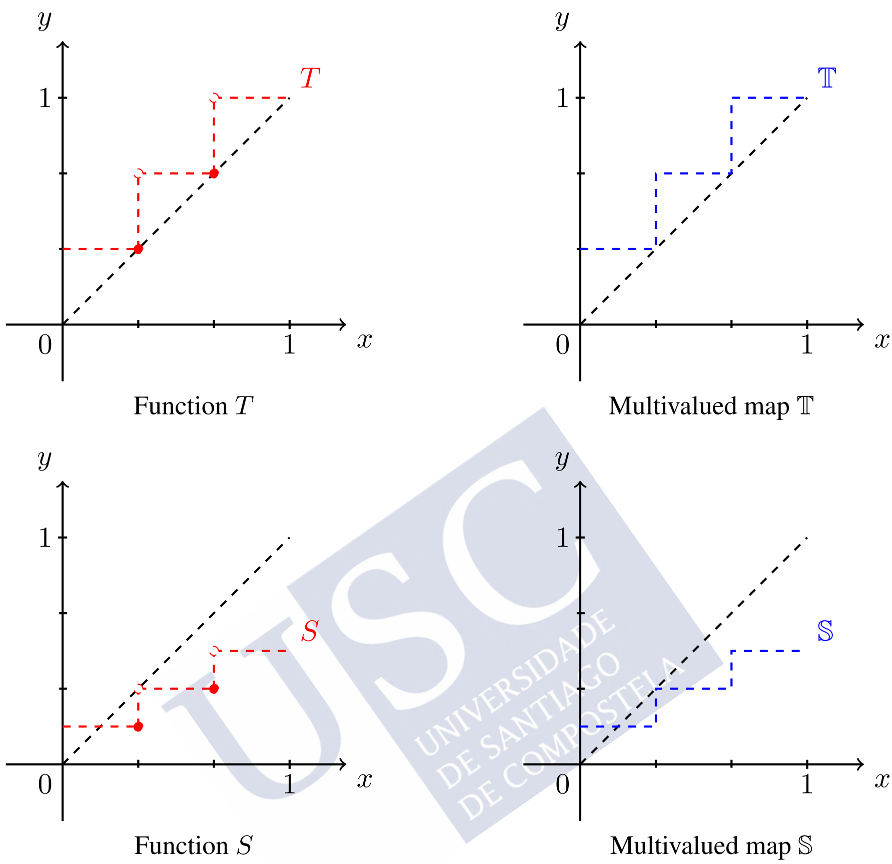

Figure 1.2.1: Functions T and S with their associated multivalued maps $\mathbb{T}$ and $\mathbb{S}$

Then it is easy to check that condition (1.1.3) *holds for all* $x \in [0,1]$ *(see Figure 1.2.1) but this is not true for the mapping* $S = \frac{1}{2}T$ *at the point* $x = 1/3$*. Indeed, in this case we have*

$$\left\{\frac{1}{3}\right\} \bigcap \mathbb{S}\left(\frac{1}{3}\right) = \left\{\frac{1}{3}\right\} \bigcap \left[\frac{1}{6}, \frac{1}{3}\right] = \left\{\frac{1}{3}\right\} \not\subset \left\{\frac{1}{6}\right\} = \left\{S\left(\frac{1}{3}\right)\right\}.$$

The previous example shows that even for linear homotopies condition (1.2.1) can fail. To overcome this difficulty improve on the previous proposition in order to avoid requesting condition (1.2.1) for all t.

Theorem 1.2.4. *Let* $H : \overline{\Omega} \times [0,1] \longrightarrow X$ *be a map satisfying the following conditions:*

(a) *for each* $(x,t) \in \overline{\Omega} \times [0,1]$ *and all* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, x, t) > 0$ *such that*

$$s \in [0,1],\ |t-s| < \delta \implies \|H(z,t) - H(z,s)\| < \varepsilon \ \ \forall\, z \in \overline{B}_\delta(x) \cap \overline{\Omega};$$

(b) $H\left(\overline{\Omega}\times[0,1]\right)$ *is relatively compact;*

(c) $\{x\}\cap\mathbb{H}_t(x)\subset\{H_t(x)\}$ *is satisfied for all* $x\in\overline{\Omega}\cap\mathbb{H}_t\overline{\Omega}$ *when* $t=0$ *and* $t=1$.

If $x\notin\mathbb{H}(x,t)$ *for all* $(x,t)\in\partial\,\Omega\times[0,1]$, *then*

$$\deg\left(Id-H_0,\Omega\right)=\deg\left(Id-H_1,\Omega\right).$$

Proof. It is possibly to prove that the degree for multivalued mappings is well defined for

$$\mathbb{H}_t(x)=\bigcap_{\varepsilon>0}\overline{\text{co}}\left(H_t\left(\overline{B}_\varepsilon(x)\cap\overline{\Omega}\right)\right),$$

for every $t\in[0,1]$, in a similar way that for homotopy invariance property above. Therefore, the homotopy invariance property of the degree for multivalued mappings guarantees in particular that

$$\deg\left(Id-H_0,\Omega\right)=\deg\left(Id-\mathbb{H}_0,\Omega\right)=\deg\left(Id-\mathbb{H}_1,\Omega\right)=\deg\left(Id-H_1,\Omega\right),$$

which ends the proof. □

We finish this section by introducing two classical results in the context of Leray–Schauder degree that remain true when considering our new degree for discontinuous operators satisfying (1.1.3). The first one is the well–known fact that for degree "only what happens in the boundary matters," and the second one is the natural extension of Borsuk's Theorem in our setting.

Proposition 1.2.5. *Let* $T,S:\overline{\Omega}\longrightarrow X$ *be two mappings in the conditions of Definition 1.1.7. If* $\mathbb{T}x=\mathbb{S}x$ *for all* $x\in\partial\,\Omega$ *and* $0\notin(Id-T)(\partial\,\Omega)$, *then*

$$\deg(Id-T,\Omega)=\deg(Id-S,\Omega).$$

Proof. The degree for the multivalued mappings $\mathbb{T}$ and $\mathbb{S}$ in Ω is well defined because T and S are in the conditions of Definition 1.1.7, so $\mathbb{T}x=\mathbb{S}x$ for all $x\in\partial\,\Omega$ implies that $\deg\left(Id-\mathbb{T},\Omega\right)=\deg\left(Id-\mathbb{S},\Omega\right)$, see Proposition A.13. Therefore, by Definition 1.1.7, we conclude that $\deg\left(Id-T,\Omega\right)=\deg\left(Id-S,\Omega\right)$. □

As the proof of the following result is similar to the previous one with the obvious changes we will omit it.

Theorem 1.2.6. *(Borsuk's) Assume that* $0\in\Omega$ *and that* $x\in\Omega$ *implies* $-x\in\Omega$, *and let* $T:\overline{\Omega}\longrightarrow X$ *be a mapping in the conditions of Definition 1.1.7. If* $0\notin(Id-T)(\partial\,\Omega)$ *and* $\mathbb{T}(x)=-\mathbb{T}(-x)$ *for all* $x\in\partial\,\Omega$, *then* $\deg(Id-T,\Omega)$ *is odd.*

1.3 Fixed point index for discontinuous operators

In this section we introduce a fixed point index for discontinuous operators satisfying condition (1.1.3) and then we study its properties.

Definition 1.3.1. *Let K be a nonempty closed and convex subset of a Banach space $(X, \|\cdot\|)$, U a relatively open subset of K, and let $T : \overline{U} \subset K \longrightarrow K$ be an operator such that $T\,\overline{U}$ is relatively compact, T has no fixed points on $\partial_K U$ and*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \textit{for every } x \in \overline{U}, \tag{1.3.1}$$

where $\mathbb{T}$ is the cc–envelope of T.

We define the fixed point index of T in K over U as

$$i_K(T, U) = \deg(Id - T \circ r, r^{-1}(U) \cap B_R(0)),$$

where r is a continuous retraction of X onto K and $B_R(0)$ is an open ball centered at 0 and radius $R > 0$ such that $\overline{T\overline{U}} \subset B_R(0)$.

Notice that the retraction r exists by virtue of the following well–known result by Dugundji [54, Theorem 4.1].

Theorem 1.3.2 (Dugundji)**.** *Suppose that A is a closed subset of a metric space B, and let L be a normed linear space. Every continuous function $f : A \longrightarrow L$ has a continuous extension $g : B \longrightarrow L$ such that $g(B) \subset \operatorname{co} f(A)$.*

Next we will show the number $\deg(Id - T \circ r, r^{-1}(U) \cap B_R(0))$ is well–defined.

Proposition 1.3.3. *Let T be as in the conditions of Definition 1.3.1. Then the mapping*

$$F = T \circ r : \overline{r^{-1}(U)} \longrightarrow K$$

satisfies that $F\left(\overline{r^{-1}(U)}\right)$ is relatively compact, $Fx \neq x$ for all $x \in \partial\left(r^{-1}(U) \cap B_R(0)\right)$ and

$$\{x\} \cap \mathbb{F}x \subset \{Fx\} \quad \textit{for all } x \in \overline{r^{-1}(U)},$$

where $\mathbb{F}$ is the cc–envelope of F.

Proof. First, the set $F\left(\overline{r^{-1}(U)}\right)$ is a relatively compact subset of X because it is contained in $T\,\overline{U}$, which is relatively compact.

Second, if for some $x \in \partial\left(r^{-1}(U) \cap B_R(0)\right)$ we have $x = F\,x \in K$, then $r(x) = x$. Hence $x = Tx$, and by hypothesis T is fixed point free on $\partial_K U$, which implies that x belongs to $U \cap B_R(0)$. Since $U \cap B_R(0) \subset r^{-1}(U) \cap B_R(0)$ and $r^{-1}(U) \cap B_R(0)$ is open, we deduce that $x \notin \partial\left(r^{-1}(U) \cap B_R(0)\right)$, a contradiction.

Finally we will see that $\{x\} \cap \mathbb{F}x \subset \{Fx\}$ for all $x \in \overline{r^{-1}(U)}$. Clearly, it suffices to check this for all $x \in \overline{r^{-1}(U)} \cap \mathbb{F}\left(\overline{r^{-1}(U)}\right)$. As $F\left(\overline{r^{-1}(U)}\right) \subset T\,\overline{U} \subset K$, and K is a convex and closed set, we obtain $\mathbb{F}\left(\overline{r^{-1}(U)}\right) \subset K$ and then

$$\overline{r^{-1}(U)} \cap \mathbb{F}\left(\overline{r^{-1}(U)}\right) \subset \overline{r^{-1}(U)} \cap K \subset \overline{U}.$$

Let $x \in \overline{U}$, then $F\,x = T\,x$. Moreover, since r is continuous at x and $r(x) = x$ we have that for every $\varepsilon > 0$ there exists $\delta > 0$ (we can choose $\delta < \varepsilon$) such that

$$r\left(\overline{B}_\delta(x) \cap \overline{r^{-1}(U)}\right) \subset \overline{B}_\varepsilon\left(r(x)\right) \cap \overline{U} = \overline{B}_\varepsilon\left(x\right) \cap \overline{U}.$$

Hence we deduce

$$\begin{aligned}\mathbb{F}\,x &= \bigcap_{\delta>0} \overline{\mathrm{co}}\,F\left(\overline{B}_\delta(x) \cap \overline{r^{-1}(U)}\right) = \bigcap_{\delta>0} \overline{\mathrm{co}}\,(T \circ r)\left(\overline{B}_\delta(x) \cap \overline{r^{-1}(U)}\right) \\ &\subset \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\,T\left(\overline{B}_\varepsilon(x) \cap \overline{U}\right) = \mathbb{T}\,x.\end{aligned}$$

Therefore $\{x\} \cap \mathbb{F}\,x \subset \{x\} \cap \mathbb{T}\,x \subset \{T\,x\} = \{F\,x\}$ for all $x \in \overline{U} \supset \overline{r^{-1}(U)} \cap \mathbb{F}\left(\overline{r^{-1}(U)}\right)$ and this ends the proof. $\square$

Let us show now that Definition 1.3.1 does not depend on the retraction or the ball $B_R(0)$ chosen.

Proposition 1.3.4. *In the conditions of Definition 1.3.1, $i_K(T, U)$ does not depend on the retraction or the ball $B_R(0)$ chosen.*

Proof. Let $r_1 : X \longrightarrow K$ and $r_2 : X \longrightarrow K$ be two retractions and let $B_{R_1}(0)$ and $B_{R_2}(0)$ two open balls in the conditions of Definition 1.3.1. Assume that $R_1 < R_2$. For $j = 1, 2$, the additivity property of the degree yields

$$\begin{aligned}\deg\left(Id - T \circ r_j, r_j^{-1}(U) \cap B_{R_2}(0)\right) &= \deg\left(Id - T \circ r_j, r_j^{-1}(U) \cap B_{R_1}(0)\right) \\ &\quad + \deg\left(Id - T \circ r_j, r_j^{-1}(U) \cap B_{R_2}(0) \setminus \overline{B_{R_1}(0)}\right).\end{aligned}$$

Since $T \circ r_j$ has no fixed point outside $\overline{B_{R_1}(0)}$, we have

$$\deg\left(Id - T \circ r_j, r_j^{-1}(U) \cap B_{R_2}(0)\right) = \deg\left(Id - T \circ r_j, r_j^{-1}(U) \cap B_{R_1}(0)\right).$$

It remains to prove that

$$\deg\left(Id - T \circ r_1, r_1^{-1}(U) \cap B_{R_1}(0)\right) = \deg\left(Id - T \circ r_2, r_2^{-1}(U) \cap B_{R_1}(0)\right). \tag{1.3.2}$$

Now consider the set $V = r_1^{-1}(U) \cap r_2^{-1}(U) \cap B_{R_1}(0)$. We have

$$\begin{aligned}\deg\left(Id - T \circ r_j, r_j^{-1}(U) \cap B_{R_1}(0)\right) &= \deg\left(Id - T \circ r_j, r_j^{-1}(U) \cap B_{R_1}(0) \setminus \overline{V}\right) \\ &\quad + \deg\left(Id - T \circ r_j, V\right) \\ &= \deg\left(Id - T \circ r_j, V\right),\end{aligned}$$

because $T \circ r_j$ has no fixed point in $r_j^{-1}(U) \cap B_{R_1}(0) \setminus \overline{V}$.

Therefore, proving (1.3.2) reduces to check that

$$\deg\left(Id - T \circ r_1, V\right) = \deg\left(Id - T \circ r_2, V\right).$$

Consider the homotopy $H : \overline{V} \times [0,1] \longrightarrow K$ given by

$$H(x,t) = tF_1\,x + (1-t)F_2\,x,$$

where $F_j\,x = T \circ r_j(x)$, $j = 1, 2$. Since the F_j's are bounded, condition (1.2.2) is immediately satisfied.

Let us check now that condition

$$\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\} \tag{1.3.3}$$

is satisfied for every $x \in \overline{V} \cap \mathbb{H}_t(\overline{V})$ and all $t \in [0,1]$.

Since K is convex we have that $H_t\left(\overline{V}\right) \subset K$ and since K is closed, it follows that $\mathbb{H}_t\left(\overline{V}\right) \subset K$. Hence, $\overline{V} \cap \mathbb{H}_t(\overline{V}) \subset \overline{V} \cap K$. Let $x \in \overline{V} \cap K$ and $t \in [0,1]$ be fixed, then

$$H_t(x) = t\,T\,x + (1-t)T\,x = T\,x.$$

By Proposition 1.3.3, we deduce $\mathbb{F}_j\,x \subset \mathbb{T}\,x$, $j = 1, 2$. In a similar way we obtain that

$$\mathbb{H}_t(x) \subset \mathbb{T}\,x,$$

so we have

$$\{x\} \cap \mathbb{H}_t(x) \subset \{x\} \cap \mathbb{T}\,x \subset \{T\,x\} = \{H_t(x)\}\,.$$

If we prove that $x \neq H(x,t)$ for all $(x,t) \in \partial V \times [0,1]$, then we can conclude, by means of condition (1.3.3), that $x \notin \mathbb{H}(x,t)$ for all $(x,t) \in \partial V \times [0,1]$. So, assume that $x = H(x,t)$ for some $x \in \partial V$ and $t \in [0,1]$, then

$$x = t\,T(r_1(x)) + (1-t)T(r_2(x)) \in K,$$

and so $r_1(x) = x = r_2(x)$. Hence $x = T\,x$, a contradiction.

Therefore, invariance under homotopy guarantees that

$$\deg\left(Id - T \circ r_2, V\right) = \deg\left(Id - H_0, V\right) = \deg\left(Id - H_1, V\right) = \deg\left(Id - T \circ r_1, V\right),$$

and we conclude. $\square$

As a direct consequence of Proposition 1.2.1, the following properties of the fixed point index hold.

Theorem 1.3.5. *Let T be a mapping with the conditions of Definition 1.3.1. Then the following properties are satisfied*

1. *(Homotopy invariance) Let $H : \overline{U} \times [0,1] \longrightarrow K$ be a mapping such that:*

   (a) for each $(x,t) \in \overline{U} \times [0,1]$ and all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x, t) > 0$ such that

   $$s \in [0,1],\ |t-s| < \delta \implies \|H(z,t) - H(z,s)\| < \varepsilon\ \ \forall\, z \in \overline{B}_\delta(x) \cap \overline{U};$$

   (b) $H\left(\overline{U} \times [0,1]\right)$ is relatively compact;

(c) $\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\}$ *for all* $t \in [0,1]$ *and all* $x \in \overline{U} \cap \mathbb{H}_t\overline{U}$*, where* $\mathbb{H}_t$ *denotes the cc–envelope of operator* $H_t(x) = H(t,x)$*.*

If $x \neq H(x,t)$ *for all* $(x,t) \in \partial U \times [0,1]$*, then the index* $i_K(H_t, U)$ *does not depend on* $t \in [0,1]$*.*

2. *(Additivity) Let* U_1 *and* U_2 *be open, disjoint and such that* $U_1 \cup U_2 \subset U$*.*
   If $0 \notin (Id - T)\left(\overline{U} \setminus (U_1 \cup U_2)\right)$*, then*
   $$i_K(T,U) = i_K(T,U_1) + i_K(T,U_2).$$
3. *(Excision) Let* $A \subset \overline{U}$ *be a closed set. If* $0 \notin (Id-T)(\partial U) \cup (Id-T)(A)$*, then*
   $$i_K(T,U) = i_K(T, U \setminus A).$$
4. *(Existence) If* $i_K(T,U) \neq 0$*, then there exists* $x \in U$ *such that* $Tx = x$*.*
5. *(Normalization) For every constant map* T *such that* $T\,\overline{U} \subset U$*,* $i_K(T,U) = 1$*.*

Note that Theorem 1.3.5, when specialized to continuous operators, gives the usual properties of the Leray–Schauder fixed point index, see [4].

Proposition 1.3.6. *Let* $H : \overline{U} \times [0,1] \longrightarrow K$ *be a map satisfying the following conditions:*

(a) *For each* $(x,t) \in \overline{U} \times [0,1]$ *and all* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, x, t) > 0$ *such that*
$$s \in [0,1],\ |t-s| < \delta \implies \|H(z,t) - H(z,s)\| < \varepsilon \ \ \forall z \in \overline{B}_\delta(x) \cap \overline{U};$$

(b) $H\left(\overline{U} \times [0,1]\right)$ *is relatively compact;*

(c) $\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\}$ *holds for all* $x \in \overline{U} \cap \mathbb{H}_t\overline{U}$ *when* $t = 0$ *and* $t = 1$*.*

If $x \notin \mathbb{H}(x,t)$ *for all* $(x,t) \in \partial U \times [0,1]$*, then*
$$i_K(H_0, U) = i_K(H_1, U).$$

Remark 1.3.7. *Continuity in the time variable is a natural assumption for an homotopy, since its idea consists in producing a continuous deformation of a function* H_0 *in time* $t = 0$ *to another function* H_1 *in time* $t = 1$*. However, for every fixed t, the function* H_t *may be discontinuous.*

Proposition 1.3.8. *Let* T *be a mapping in the conditions of Definition 1.3.1. The fixed point index of* T *satisfies that*
$$i_K(T,U) = i_K(\mathbb{T}, U),$$
where the right–hand index is the fixed point index defined for multivalued mappings, see [69].

Proof. First we have by Definitions 1.3.1 and 1.1.7 that

$$i_K(T,U) = \deg\left(Id - T\circ r, r^{-1}(U)\cap B_R(0)\right) = \deg\left(Id - \mathbb{F}, r^{-1}(U)\cap B_R(0)\right),$$

where $\mathbb{F}$ is the cc–envelope of $F = T\circ r$.

On the other hand, the fixed point index for the multivalued mapping $\mathbb{T}$ reads as

$$i_K(\mathbb{T},U) = \deg\left(Id - \mathbb{T}\circ r, r^{-1}(U)\cap B_R(0)\right).$$

Since $T\circ r(x) \in \mathbb{T}\circ r(x)$ for all $x \in r^{-1}(U)$, $\mathbb{T}\circ r$ is a mapping with convex closed values and, in addition, it is upper semicontinuous because it is the composition of two upper semicontinuous mappings, Proposition 1.1.4 guarantees $\mathbb{F}\,x \subset \mathbb{T}(r(x))$ for all $x \in r^{-1}(U)$.

Now we consider the homotopy $H : \overline{r^{-1}(U)\cap B_R(0)} \times [0,1] \longrightarrow 2^K$ given by

$$H(x,t) = t\,\mathbb{F}\,x + (1-t)\mathbb{T}\circ r(x).$$

We will see that this homotopy is admissible, so assume on the contrary that there exists $x \in \partial\left(r^{-1}(U)\cap B_R(0)\right)$ and $t\in[0,1]$ such that $x \in H(x,t)$. Then,

$$x \in t\,\mathbb{F}\,x + (1-t)\mathbb{T}\circ r(x) \subset t\,\mathbb{T}\circ r(x) + (1-t)\mathbb{T}\circ r(x) = \mathbb{T}\circ r(x),$$

so $x \in K$, and this implies that $x \in \mathbb{T}x$. By virtue of the continuity of the retraction r it follows that $x = r(x) \in \partial\,U$, a contradiction. Therefore, the homotopy invariance of the topological degree for multivalued mappings guarantees

$$\deg\left(Id - \mathbb{F}, r^{-1}(U)\cap B_R(0)\right) = \deg\left(Id - \mathbb{T}\circ r, r^{-1}(U)\cap B_R(0)\right),$$

and the proof is over. □

By means of the previous proposition and the properties of fixed point index for multivalued mappings [69, 116], it is possible to deduce the following result.

Proposition 1.3.9. *Let S be a convex and closed subset of X and let T be a mapping in the conditions of Definition 1.3.1 satisfying $T\,\overline{U} \subset S$. Then,*

$$i_K(T,U) = i_S(T, U\cap S).$$

Proof. First, since $x \neq Tx$ for all $x \in \partial\,U$ and $S \subset K$, it follows $\partial_S(U\cap S) \subset \partial_K U$, and so $x \neq Tx$ for all $x \in \partial_S(U\cap S)$. Hence, both hands of the previous equality are well–defined.

Now, as S is convex and closed and $T\,\overline{U} \subset S$, then $\mathbb{T}\,\overline{U} \subset S$. Therefore we have

$$i_K(\mathbb{T},U) = i_S(\mathbb{T}, U\cap S),$$

and the conclusion is obtained by application of Proposition 1.3.8. □

Remark 1.3.10. *In Definition 1.3.1, $T\overline{U}$ is required to be a relatively compact set, but weaker conditions about the compactness of this image can be sufficient in order to define a fixed point index. Related to this, in [69, 137, 139] a fixed point index for continuous condensing mappings is considered. The previous theory could be easily extended to that class of operators, following the paper by Väth [137], but it is not necessary for our applications so, for simplicity, we omit it here.*

Chapter 2
Fixed point theorems for some discontinuous operators

Fixed point theorems are powerful tools for proving existence, uniqueness or multiplicity of solutions for a wide range of equations which, in particular, include integral equations derived from nonlinear differential problems. Furthermore, choosing adequate domains makes it possible to deduce qualitative properties of the solutions of such problems, such as positivity or localization. Having as much information on the solutions as we may is particularly important when the considered problems are mathematical models coming from biology, physics, economy or other applied sciences.

Two main types of assumptions are at the base of most fixed point theorems, namely, *continuity* and *monotonicity*. By obvious reasons, discontinuous differential equations cannot be studied by means of fixed point theorems for continuous operators, and the use of fixed point theorems for monotone operators introduces rather strong monotonicity assumptions on the nonlinear parts of the equations. In this chapter we use the degree theory introduced in Chapter 1 to prove discontinuous analogues of well–known fixed point theorems for continuous operators which can be applied in the analysis of many non–monotone discontinuous differential equations.

In particular, we prove discontinuous versions of Schauder [133] and Krasnosel'skiĭ fixed point theorems [91], whose classical versions have been extensively applied and generalized to different classes of maps and spaces in the literature [56, 100, 144]. In addition, we shall improve on two recent modifications of Krasnosel'skiĭ fixed point theorem in cones. The first one is due to Cabada, Cid, Infante and collaborators [28–30, 43], who obtained fixed point theorems in cones by imposing compression or expansion conditions only at one boundary of a conical shell instead of imposing conditions on the two boundaries, as in the original result. The price they paid was the addition of a monotonicity hypothesis on the operator and the assumption of the existence of an upper (or lower) fixed point.

The second modification of Krasnosel'skiĭ's result concerns systems of operator equations and it is due to Precup [118, 119]. It gives upper and lower bounds for the fixed points in each variable independently. Moreover, this componentwise approach of the compression–expansion type results allows different behaviors (that is, compression or expansion) in each component of the system.

We remark that while some of our fixed point theorems are not particularly surprising corollaries of the analogous results for upper semicontinuous multivalued mappings, they have a much more general applicability range than their classical single–valued continuous versions. This fact allows us to give new existence results for differential problems where the classical assumptions about the nonlinearities are weaker, as shown in the following chapters.

The fixed point theorems studied here were presented in the papers [61, 64, 66, 103, 128].

2.1 Schauder type fixed point theorems

One of the most typical applications of degree theory is that of looking for topological spaces which satisfy the so called *fixed point property*, that is, topological spaces M such that any continuous mapping $T : M \longrightarrow M$ has a fixed point. It was proved by Dugundji [54] that in every infinite–dimensional space there exists a fixed–point–free mapping from its closed unit ball into itself, and so some extra–assumption regarding compactness of images is required in the infinite–dimensional case. In this section we will use our new degree to extend these results for operators which are not necessarily continuous but satisfy condition (1.1.3).

The investigation of this question was initiated in the papers by Pouso [101] and Figueroa and Infante [58]. There, fixed point theorems are presented as direct consequences of the analogous results for upper semicontinuous multivalued maps, namely, Bohnenblust–Karlin fixed point theorem [49]. Nevertheless, here we shall prove our fixed point results directly, by using the fixed point index defined in the previous chapter.

Theorem 2.1.1. *Let M be a nonempty closed and convex subset of a Banach space X. Let $T : M \longrightarrow M$ be a mapping satisfying that*

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for all } x \in M, \tag{2.1.1}$$

where $\mathbb{T}$ is as in (1.1.1) *with $D = M$, and such that TM is relatively compact. Then $i_M(T, M) = 1$.*

Proof. Since M is closed and convex, $i_M(T, M)$ is well–defined. Let $x_0 \in M$ be fixed and consider the homotopy $H : M \times [0, 1] \longrightarrow M$ given by

$$H(x, t) = (1 - t)x_0 + t\,Tx.$$

Since TM is relatively compact, $H(M \times [0, 1])$ is relatively compact too. Finally, taking into account that $\partial_M M = \emptyset$ and, by assumption, $\{x\} \cap \mathbb{H}_t x \subset \{H_t x\}$ for all $x \in M$ when $t = 0, 1$, the properties of fixed point index imply $i_M(T, M) = i_M(x_0, M) = 1$. □

Theorem 2.1.2. *Let M be a nonempty closed and convex subset of a Banach space X. Let $T : M \longrightarrow M$ be a mapping satisfying condition* (2.1.1) *and such that TM is relatively compact. Then the operator T has at least a fixed point in M.*

Proof. It is a straightforward consequence of Theorem 2.1.1 and the existence property of the fixed point index. □

As a consequence of the previous result we obtain a generalization of the usual statement of Schauder's fixed point theorem.

Theorem 2.1.3. *Let M be a nonempty convex and compact subset of X. Let $T : M \longrightarrow M$ be a mapping satisfying* (2.1.1)*. Then T has a fixed point in M.*

Proof. Since $TM \subset M$ and M is compact, TM is a relatively compact subset of X. As an application of Theorem 2.1.2, the operator T has a fixed point in M. □

2.2 Krasnosel'skiĭ's compression–expansion type fixed point theorems in cones

The classical compression–expansion fixed point theorem of Krasnosel'skiĭ, which provides the existence of fixed points of compact operators inside conical shells (see [4] or [144]), is a well–known tool of nonlinear analysis and it has been extensively used in literature in order to obtain existence of positive solutions for some kind of nonlinear problems. In this section we prove a generalization of that theorem where continuity will be replaced by our condition (2.1.1).

For the sake of completeness, we recall some basic definitions about cones. A closed and convex subset K of a Banach space $(X, \|\cdot\|)$ is a cone if it satisfies the following conditions:

(i) if $x \in K$, then $\lambda x \in K$ for all $\lambda \geq 0$;

(ii) if $x \in K$ and $-x \in K$, then $x = 0$.

A cone K defines the partial ordering in X given by $x \preceq y$ if and only if $y - x \in K$. For $x, y \in X$, with $x \preceq y$, the set

$$[x, y] = \{z \in X : x \preceq z \preceq y\}$$

is said to be an order interval.

The cone K is called normal with a normal constant $c > 0$ if

$$\|x\| \leq c\,\|y\| \text{ for all } x, y \in X \text{ with } 0 \preceq x \preceq y.$$

Moreover, if K has nonempty interior, then it is said to be a solid cone.

For a given $\rho > 0$, we will denote

$$K_\rho = \{x \in K : \|x\| < \rho\}.$$

Now we introduce the main results in this section.

Theorem 2.2.1. *Let $R > 0$ and $0 \in \Omega_i \subset K_R$ relatively open subsets of K $(i = 1, 2)$. Let $T : \overline{K}_R \longrightarrow K$ be a mapping such that $T\,\overline{K}_R$ is relatively compact and it fulfills condition* (2.1.1) *in $\overline{K}_R$.*

(a) *If $\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $x \in \partial\,\Omega_1$ and all $\lambda \geq 1$, then $i_K(T, \Omega_1) = 1$.*

(b) *If there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \lambda w$ for every $\lambda \geq 0$ and all $x \in K$ with $x \in \partial\,\Omega_2$, then $i_K(T, \Omega_2) = 0$.*

Proof.

(a) We define the homotopy $H : \overline{\Omega}_1 \times [0, 1] \longrightarrow K$ given by

$$H(x, t) = t\,Tx.$$

The set $T\,\overline{K}_R$ is relatively compact and, then, so is $H\left(\overline{\Omega}_1 \times [0,1]\right)$. By assumption, we have

$$\{x\} \cap \mathbb{H}_t(x) \subset \{H_t(x)\} \text{ for } t = 0, 1 \text{ and all } x \in \overline{\Omega}_1.$$

Therefore, if there exists $(x,t) \in \partial\,\Omega_1 \times [0,1]$ such that $x \in \mathbb{H}(t,x)$, then it implies that $\dfrac{1}{t}x \in \mathbb{T}x$ for some $t \in (0,1]$ and $x \in \partial\,\Omega_1$, a contradiction.

By Proposition 1.3.6 and the normalization property of the index, we have

$$i_K(T,\Omega_1) = i_K(0,\Omega_1) = 1.$$

(b) Assume on the contrary that $i_K(T,\Omega_2) \neq 0$. Since $T\,\overline{K}_R$ is relatively compact, we can take $\mu > 0$ such that $\|y\| < \mu$ for every $y \in \mathbb{T}\,x$ and all $x \in \overline{\Omega}_2$. Choose $\lambda > (R+\mu)/\,\|w\|$ and consider the homotopy given by

$$H(x,t) = \mathbb{T}\,x + t\lambda w.$$

We have by assumption that $x \notin H(x,t)$ for all $x \in \partial\,\Omega_2$ and every $t \in [0,1]$. Hence, by virtue of the homotopy invariance property of fixed point index for multivalued mappings, Theorem A.11, we obtain

$$i_K(\mathbb{T},\Omega_2) = i_K(\mathbb{T} + \lambda w, \Omega_2).$$

Now, since $i_K(\mathbb{T},\Omega_2) = i_K(T,\Omega_2) \neq 0$ (see Proposition 1.3.8), there exists $x \in \Omega_2$ such that $x \in \mathbb{T}\,x + \lambda w$. Then,

$$\|x - \lambda w\| \geq \lambda\,\|w\| - \|x\| > (R+\mu) - \|x\| > \mu > \|y\|\,,$$

for all $y \in \mathbb{T}x$, a contradiction. Therefore, $i_K(T,\Omega_2) = 0$. □

In particular, if we take $\Omega_1 = K_{r_1}$ and $\Omega_2 = K_{r_2}$ for some constants $r_1, r_2 > 0$, then we obtain a new version of the classical Krasnosel'skiĭ theorem in conical shells.

Theorem 2.2.2. *Let $0 < r_i \leq R$ $(i = 1,2)$ and let $T : \overline{K}_R \longrightarrow K$ be a mapping such that $T\,\overline{K}_R$ is relatively compact and it fulfills condition* (2.1.1) *in $\overline{K}_R$.*

(a) *If $\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $\|x\| = r_1$ and all $\lambda \geq 1$, then $i_K(T,K_{r_1}) = 1$.*

(b) *If there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \lambda w$ for every $\lambda \geq 0$ and all $x \in K$ with $\|x\| = r_2$, then $i_K(T,K_{r_2}) = 0$.*

The following corollary is an immediate consequence of Theorem 2.2.2, where conditions are now written in terms of the partial ordering of the cone.

Corollary 2.2.3. *Let $0 < r_i \leq R$ $(i = 1,2)$ and let $T : \overline{K}_R \longrightarrow K$ be a mapping such that $T\,\overline{K}_R$ is relatively compact and it fulfills condition* (2.1.1) *in $\overline{K}_R$.*

(i) *If $y \not\geq x$ for all $y \in \mathbb{T}\,x$ and all $x \in K$ with $\|x\| = r_1$, then $i_K(T,K_{r_1}) = 1$.*

(ii) *If $y \not\leq x$ for all $y \in \mathbb{T}\,x$ and all $x \in K$ with $\|x\| = r_2$, then $i_K(T,K_{r_2}) = 0$.*

Proof. It suffices to show that all the conditions in Theorem 2.2.2 are satisfied. First, we prove that condition (i) implies condition (a) in Theorem 2.2.2. To do so, let $x \in K$ be such that $\|x\| = r_1$ and let $\lambda \geq 1$, and assume by contradiction that $y = \lambda x \in \mathbb{T}x$. Then we have

$$y - x = (\lambda - 1)x \in K \quad \text{(because } \lambda - 1 \geq 0\text{)},$$

which implies that $y \succeq x$, a contradiction with condition (i).

On the other hand, to show that condition (ii) implies (b) in Theorem 2.2.2, we again assume by contradiction that for every $w \in K$ such that $\|w\| \neq 0$ we can find $x \in \partial K_{r_2}$ and $\lambda \geq 0$ such that $x \in \mathbb{T}x + \lambda w$, i.e., there exists $y \in \mathbb{T}\,x$ such that $x = y + \lambda w$. Hence, $x - y = \lambda w \in K$, which contradicts (ii). □

These computations for fixed point index allow us to deduce some compression–expansion type fixed point theorems.

Theorem 2.2.4. *Let $R > 0$ and $\Omega_i \subset K_R$ relatively open subsets of K $(i = 1, 2)$ such that either $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ or $0 \in \Omega_2 \subset \overline{\Omega}_2 \subset \Omega_1$. Let $T : \overline{K}_R \longrightarrow K$ be a mapping such that $T\,\overline{K}_R$ is relatively compact and it fulfills condition* (2.1.1) *in $\overline{K}_R$. Suppose that*

(a) $\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $x \in \partial\,\Omega_1$ and all $\lambda \geq 1$,

(b) there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \lambda w$ for every $\lambda \geq 0$ and all $x \in K$ with $x \in \partial\,\Omega_2$.

Then T has a fixed point which belongs either to $\Omega_2 \setminus \overline{\Omega}_1$ (if $\Omega_1 \subset \Omega_2$) or to $\Omega_1 \setminus \overline{\Omega}_2$ (if $\Omega_2 \subset \Omega_1$).

Proof. From Theorem 2.2.1, the conclusion is obtained by the additivity and existence properties of the fixed point index. □

Theorem 2.2.5. *Let $0 < r_1, r_2 \leq R$, $r_1 \neq r_2$ and let $T : \overline{K}_R \longrightarrow K$ be a mapping such that $T\,\overline{K}_R$ is relatively compact and it fulfills condition* (2.1.1) *in $\overline{K}_R$. Suppose that*

(a) $\lambda x \notin \mathbb{T}x$ for all $x \in K$ with $\|x\| = r_1$ and all $\lambda \geq 1$,

(b) there exists $w \in K$ with $\|w\| \neq 0$ such that $x \notin \mathbb{T}x + \lambda w$ for every $\lambda \geq 0$ and all $x \in K$ with $\|x\| = r_2$.

Then T has a fixed point $x \in K$ such that

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}\,.$$

Remark 2.2.6. *If $r_1 > r_2$, then conditions (a)–(b) are usually called compression type conditions. On the other hand, if $r_1 < r_2$, they are said cone–expansion conditions.*

Remark 2.2.7. *Observe that Theorem 2.2.5 does not remain true if we replace $\mathbb{T}$ by T in the assumptions, as we show in the following example.*

Example 2.2.8. *In $X = \mathbb{R}^2$ we consider the cone $K = \mathbb{R}^2_+ = \left\{(x,y) \in \mathbb{R}^2 : x, y \geq 0\right\}$.*
Let $0 < r < R$ and define a mapping $T : K \longrightarrow K$ in polar coordinates as

$$T(\rho,\theta) = \begin{cases} (0,0), & \text{if } \rho \neq r, \\ \left(r, \dfrac{\pi}{2}\right), & \text{if } \theta \in \left[0, \dfrac{\pi}{4}\right),\ \rho = r, \\ (r,0), & \text{if } \theta \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right],\ \rho = r. \end{cases}$$

Note that $\mathbb{T}x = \{Tx\} = \{(0,0)\}$ for all $x \in K$ such that $\|x\| \neq r$ because T is continuous at those points. For points $x = (r,\theta)$, with $\theta \in [0,\pi/2]$, we have three possibilities: if $\theta \in [0,\pi/4)$, then $\mathbb{T}x$ is the segment with endpoints $(0,0)$ and $(r,\pi/2)$; if $\theta \in (\pi/4,\pi/2]$, then $\mathbb{T}x$ is the segment with endpoints $(0,0)$ and $(r,0)$; finally, $\mathbb{T}(r,\pi/4)$ is the triangle with vertices $(0,0)$, $(r,0)$ and $(r,\pi/2)$. Therefore,

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \quad \text{for all } x \in K.$$

Moreover, conditions (a) and (b) in Theorem 2.2.5 are satisfied if we replace $\mathbb{T}$ by T (and we take $r_1 = R$, $r_2 = r$ and any $w \in \mathbb{R}^2 \setminus \{(0,0)\}$). However, T has no fixed point satisfying that $r < \|x\| < R$.

2.3 Krasnosel'skiĭ type fixed point theorems in cones for monotone operators

In the last years some authors studied new criteria for the existence of fixed points of completely continuous operators in cones, where the usual conditions regarding two different boundaries were relaxed, see [28–30, 43, 70, 72]. These new results only require assumptions about the behavior of the operator on one boundary instead of on two boundaries as in the classical fixed point theorems of compression–expansion type. The way to do that is by using some monotone iterative techniques. In this section we will generalize the mentioned results in order to allow their application to our class of possibly discontinuous operators.

First, we introduce a result on the existence of extremal fixed points for nondecreasing operators due to Heikkilä *et al.* [78, Theorem 1.2.2], which will be useful for us throughout this section.

Theorem 2.3.1. *Let Y be a subset of an ordered metric space X, $[a,b]$ a nonempty order interval in Y, and $G : [a,b] \longrightarrow [a,b]$ a nondecreasing mapping. If $(Gx_n)_{n=0}^{\infty}$ converges in Y whenever $(x_n)_{n=0}^{\infty}$ is a monotone sequence in $[a,b]$, then the well-ordered chain of G-iterations of a has the maximum x_*, and the inversely well-ordered chain of G-iterations of b has the minimum x^*, and*

$$x_* = \min\{y \,|\, Gy \leq y\}, \quad x^* = \max\{y \,|\, y \leq Gy\}.$$

In particular, x_* and x^* are the extremal fixed points of G.

Let $(X, \|\cdot\|)$ be a real Banach space and $K \subset X$ a solid cone. We state our results on the existence of non-trivial fixed points.

Theorem 2.3.2. *Let $T : K \longrightarrow K$ be an operator mapping bounded sets into relatively compact sets. Assume that*

1. *there exists $r > 0$ such that $\mathbb{T}x \subset \{Tx\} - K$ for all $x \in K$ with $\|x\| = r$,*
2. *there exist $\beta \in K$, with $T\,\beta \preceq \beta$, and $R \geq r$ such that $\overline{B}_R(\beta) \subset K$,*
3. *the mapping T is nondecreasing in $\mathcal{P} = \{x \in K : x \preceq \beta\}$ and $T\,\mathcal{P}$ is relatively compact,*
4. *there exists a bounded open set $V \subset K$ such that $i_K(T, V) = 0$ and $\overline{K}_r \subset V$ or $\overline{V} \subset K_r$.*

If T satisfies condition (2.1.1) *on $\mathcal{P} \cup \overline{V}$, then T has at least a non–trivial fixed point in K which either belongs to $\mathcal{P}$ or to* $\begin{cases} V \setminus \overline{K}_r, & \text{if } \overline{K}_r \subset V, \\ K_r \setminus \overline{V}, & \text{if } \overline{V} \subset K_r. \end{cases}$

Proof. Since $\overline{B}_R(\beta) \subset K$, if $x \in K$ with $\|x\| \leq R$, then $\beta - x \in K$ and so $x \preceq \beta$.

Now we have two possibilities. First, if we have that there exists $\alpha \in K$ with $\|\alpha\| = r$ and $T\alpha \succeq \alpha$, then $\alpha \preceq \beta$ and if $\alpha \preceq x \preceq \beta$, since T is a nondecreasing mapping and $T\beta \preceq \beta$, we obtain $\alpha \preceq T\alpha \preceq Tx \preceq T\beta \preceq \beta$. Thus T maps the order interval $[\alpha, \beta]$ into itself. Notice that $[\alpha, \beta]$ is a nonempty closed and convex set, $T\left([\alpha, \beta]\right) \subset T\,\mathcal{P}$ is relatively compact and condition (2.1.1) is fulfilled in $[\alpha, \beta]$, so Theorem 2.1.2 implies that T has at least a fixed point in $[\alpha, \beta]$.

On the other hand, if we have that $Tx \not\succeq x$ for all $x \in K$ with $\|x\| = r$, then $y \not\succeq x$ for all $y \in \mathbb{T}x$ with $x \in K$, $\|x\| = r$. Indeed, for $y \in \mathbb{T}x$ with $x \in K$ and $\|x\| = r$, we have $y - x = Tx - x - k$ for some $k \in K$, by hypothesis 1. Thus $Tx - x \notin K$ implies that $y - x \notin K$. Therefore, by Corollary 2.2.3, $i_K(T, K_r) = 1$. Since $i_K(T, V) = 0$, the properties of fixed point index (see Theorem 1.3.5) ensure that there exists a non-trivial fixed point of T in $V \setminus \overline{K}_r$ (if $\overline{K}_r \subset V$) or in $K_r \setminus \overline{V}$ (if $\overline{V} \subset K_r$). □

Remark 2.3.3. *Notice that hypothesis* 1 *in Theorem 2.3.2 is weaker than continuity of the operator T in a neighborhood of the origin.*

In fact, such a condition is satisfied if the operator T is "upper semicontinuous" for $x \in K$ with $\|x\| = r$, that is,

(1^*) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y - x\| < \delta$, $y \in K$, implies*

$$Ty \in \overline{B}_\varepsilon(Tx) - K.$$

Indeed, if (1^*) holds, then for each $\varepsilon > 0$,

$$\mathbb{T}x \subset \overline{\mathrm{co}}\left(\overline{B}_\varepsilon(Tx) - K\right) = \overline{B}_\varepsilon(Tx) - K,$$

so $\mathbb{T}x \subset \{Tx\} - K$.

Remark 2.3.4. *The previous result seems to be new even in the case of a completely continuous operator T. In [28–30, 43], the authors require the cone either to be normal or to satisfy a suitable condition which involves the partial ordering induced by the cone and the norm of the Banach space (see [29]). However, for some applications it can be useful to work in cones which do not satisfy those conditions, as we will show.*

Some assumptions in Theorem 2.3.2 can be weakened if we ask the cone K to be normal, as we prove in the following result.

Theorem 2.3.5. *Let K be a solid and normal cone with normal constant $d \geq 1$ and $T : K \longrightarrow K$ an operator which maps bounded sets into relatively compact ones. Assume that*

(a) *there exists $r > 0$ such that $\mathbb{T}x \subset \{Tx\} - K$ for all $x \in K$ with $\|x\| = r$,*

(b) *there exist $\beta \in K$, with $T\,\beta \preceq \beta$, and $R \geq r$ such that $\overline{B}_R(\beta) \subset K$,*

(c) *the mapping T is nondecreasing in $\mathcal{P} = \left\{x \in K : x \preceq \beta \text{ and } \frac{r}{d} \leq \|x\|\right\}$,*

(d) *there exists a bounded open set $V \subset K$ such that $i_K(T, V) = 0$ and $\overline{K}_r \subset V$ or $\overline{V} \subset K_r$.*

If T satisfies condition (2.1.1) *for all $x \in \overline{V}$, then T has at least a non–trivial fixed point in K which either belongs to $\mathcal{P}$ or to* $\begin{cases} V \setminus \overline{K}_r, & \text{if } \overline{K}_r \subset V, \\ K_r \setminus \overline{V}, & \text{if } \overline{V} \subset K_r. \end{cases}$

Proof. Now we consider two cases separately:

First, suppose that there exists $\alpha \in K$ with $\|\alpha\| = r$ and $T\alpha \succeq \alpha$. Then T maps the set $[\alpha, \beta]$ into itself. Consider a nondecreasing sequence $\{x_n\}_{n=0}^{\infty} \subset [\alpha, \beta]$. The set $\{Tx_n\}_{n=0}^{\infty} \subset [\alpha, \beta]$ is relatively compact and thus it has a convergent subsequence, say $\{Tx_{n_k}\} \to y$. Since T is nondecreasing, there exists a $N \in \mathbb{N}$ such that for all $n, m \geq N$ we have $Tx_{n_k} \preceq Tx_n \preceq Tx_{n_l}$ and $Tx_{n_k} \preceq Tx_m \preceq Tx_{n_l}$ for some $k, l \in \mathbb{N}$. Therefore, for all $n, m \geq N$ we have $Tx_n - Tx_m \preceq Tx_{n_l} - Tx_{n_k}$, so from the normality of the cone we obtain $\|Tx_n - Tx_m\| \leq d\,\|Tx_{n_l} - Tx_{n_k}\|$. It follows that $\{Tx_n\}_{n=0}^{\infty}$ is a Cauchy sequence and then the whole sequence $\{Tx_n\}_{n=0}^{\infty}$ converges to y. In a similar way we can show that $\{Tx_n\}_{n=0}^{\infty}$ converges whenever $\{x_n\}_{n=0}^{\infty}$ is a non-increasing sequence. Hence, Theorem 2.3.1 ensures that T has a fixed point in $[\alpha, \beta]$.

On the other hand, assume that $Tx \not\succeq x$ for all $x \in K$ with $\|x\| = r$, which implies $y \not\succeq x$ for all $y \in \mathbb{T}x$ with $x \in K$ and $\|x\| = r$. By Corollary 2.2.3, $i_K(T, K_r) = 1$, and since $i_K(T, V) = 0$ the conclusion follows from the additivity and existence properties of the fixed point index. □

Following the ideas of [29, 30] we obtain now a result which applies for non-increasing discontinuous operators.

Theorem 2.3.6. *Let $T : K \longrightarrow K$ be an operator which maps bounded sets into relatively compact sets. Assume that*

(a) *there exists $r > 0$ such that $\mathbb{T}x \subset \{Tx\} + K$ for all $x \in K$ with $\|x\| = r$,*

(b) *there exist $\alpha \in K$, with $\alpha \preceq T\alpha$, and $R \geq r$ such that $\overline{B}_R(\alpha) \subset K$,*

(c) *the mapping T is non-increasing in $\mathcal{P} = \{x \in K : r \leq \|x\| \leq \|\alpha\|\}$,*

(d) *there exists a bounded open set $V \subset K$ such that $i_K(T, V) = 1$ and $\overline{K}_r \subset V$ or $\overline{V} \subset K_r$.*

If T satisfies condition (2.1.1) *for all $x \in \overline{V}$, then T has at least a non–trivial fixed point in K which either belongs to $\mathcal{P}$ or to* $\begin{cases} V \setminus \overline{K}_r, & \text{if } \overline{K}_r \subset V, \\ K_r \setminus \overline{V}, & \text{if } \overline{V} \subset K_r. \end{cases}$

Proof. Let $x \in K$ be with $\|x\| = r$. Then assumption (b) implies $x \preceq \alpha$ and since $x, \alpha \in \mathcal{P}$, it follows from (c) that $Tx \succeq T\alpha \succeq \alpha \succeq x$.

If for some $x \in K$ with $\|x\| = r$ we have $Tx \preceq x$, then $Tx = x$. Otherwise, by condition (a), $y \npreceq x$ for all $y \in \mathbb{T}x$, with $x \in K$ and $\|x\| = r$. Hence Corollary 2.2.3 ensures that $i_K(T, K_r) = 0$. Therefore assumption (d) and the properties of the fixed point index imply the existence of the desired fixed point. $\square$

Remark 2.3.7. *Condition (a) holds if the operator T is "lower semicontinuous" at $x \in K$, $\|x\| = r$, that is,*

(a^*) *for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|y - x\| < \delta$, $y \in K$, implies*

$$Ty \in \overline{B}_\varepsilon(Tx) + K.$$

2.4 A generalization of Leggett–Williams' three–solutions theorem

Multiplicity of solutions of nonlinear differential equations is a very desirable property for some applications. Of course, this fact is closely related to the multiplicity of fixed points for the corresponding associated operator. This is the reason why many authors dealt with multiplicity of fixed points for nonlinear operators and a vast literature has been written in this direction. Classical examples of this are the *three–solutions theorems* due to Amann [4] or to Leggett and Williams [94]. Leggett–Williams' theorem was generalized in several directions in the last years, and among all these generalizations the result given by Avery [9] is one of the most relevant. In addition, Leggett–Williams' theorem was also extended to upper semicontinuous multivalued mappings in [116]. Our purpose in this section is to obtain a new generalization which applies for discontinuous operators satisfying condition (2.1.1).

Definition 2.4.1. *Given a cone K in a Banach space, we define a concave positive functional on K as a continuous mapping $\alpha : K \longrightarrow \mathbb{R}_+$ satisfying*

$$\alpha\left(\lambda x + (1 - \lambda)y\right) \geq \lambda\alpha(x) + (1 - \lambda)\alpha(y), \quad 0 \leq \lambda \leq 1,\ x, y \in K.$$

If α is a concave functional on K, then we denote

$$S(\alpha, a, b) = \{x \in K : \alpha(x) \geq a \ \text{ and } \ \|x\| \leq b\}.$$

These sets were introduced in [94] and they play an analogous role as order intervals, as far as they are also closed, convex and bounded subsets of K.

Moreover, for a given $\rho > 0$, we will use again the notation

$$K_\rho = \{x \in K : \|x\| < \rho\}.$$

We recall now the classical Leggett–Williams' theorem [94, Theorem 3.3].

Theorem 2.4.2. *Let $T : \overline{K}_c \longrightarrow \overline{K}_c$ be a completely continuous mapping and assume that there exist a concave positive functional α with $\alpha(x) \leq \|x\|$ ($x \in K$) and constants a, b, d, with $0 < d < a < b \leq c$, satisfying the following conditions:*

(1) *$\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(Tx) > a$ if $x \in S(\alpha, a, b)$;*

(2) *$\|Tx\| < d$ if $x \in \overline{K}_d$;*

(3) *$\alpha(Tx) > a$ for all $x \in S(\alpha, a, c)$ with $\|Tx\| > b$.*

Then T has at least three fixed points in $\overline{K}_c$.

As we said, our purpose is to replace the assumption of continuity by the weaker condition (2.1.1). To do so, we follow the steps given in [94] and we begin with a result that provides sufficient conditions on the existence of nontrivial fixed points.

Theorem 2.4.3. *Let $T : \overline{K}_c \longrightarrow K$ be a mapping such that $T\,\overline{K}_c$ is relatively compact and satisfies condition (2.1.1) in $\overline{K}_c$. Suppose there exist a concave positive functional α such that $\alpha(x) \leq \|x\|$, $x \in K$, and constants $b > a > 0$, $b \leq c$, satisfying the following:*

1. *$\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(z) > a$ if $z \in \mathbb{T}x$, $x \in S(\alpha, a, b)$;*
2. *$\mathbb{T}x \subset \overline{K}_c$ if $x \in S(\alpha, a, c)$;*
3. *$\alpha(z) > a$ for all $z \in \mathbb{T}x$, $x \in S(\alpha, a, c)$ with $\|z\| > b$.*

Then T has at least one fixed point $x \in S(\alpha, a, c)$.

Proof. Put $U = \{x \in S(\alpha, a, c) : \alpha(x) > a\}$ and take $x_0 \in S(\alpha, a, b)$ such that $\alpha(x_0) > a$. We define the mapping $H : \overline{U} \times [0, 1] \longrightarrow \overline{K}_c$ given by

$$H(x, t) = (1 - t)Tx + tx_0.$$

It is clear that $H\left(\overline{U} \times [0, 1]\right)$ is relatively compact and condition $\{x\} \cap \mathbb{H}_t x \subset \{H_t x\}$ holds for all $x \in \overline{U}$ when $t = 0, 1$.

Now we show that H is an admissible homotopy. Assume on the contrary that there exists $(x, t) \in \partial U \times [0, 1]$ such that $x \in \mathbb{H}(x, t)$. Then, $\alpha(x) = a$ and, in addition, there exists $z \in \mathbb{T}x$ such that $x = (1 - t)z + tx_0$. If $\|z\| > b$, then $\alpha(z) > a$, so

$$\alpha(x) = \alpha((1 - t)z + tx_0) \geq (1 - t)\alpha(z) + t\alpha(x_0) > a,$$

a contradiction. On the other hand, if $\|z\| \leq b$, then

$$\|x\| = \|(1 - t)z + tx_0\| \leq (1 - t)\,\|z\| + t\,\|x_0\| \leq b,$$

so $x \in S(\alpha, a, b)$. Hence, by hypothesis we have $\alpha(z) > a$ and we deduce again that $\alpha(x) > a$, a contradiction. Therefore, we obtain $x \notin \mathbb{H}(x, t)$ for all $(x, t) \in \partial U \times [0, 1]$.

Now homotopy invariance provides that

$$i_{\overline{K}_c}(T, U) = i_{\overline{K}_c}(x_0, U) = 1,$$

and therefore T has a fixed point in U. □

Remark 2.4.4. *Notice that condition $\alpha(x) \leq \|x\|$ guarantees that the fixed point provided by the previous theorem is, in fact, nontrivial.*

Now we introduce our main result in this section.

Theorem 2.4.5. *Let $T : \overline{K}_c \longrightarrow \overline{K}_c$ be a mapping such that $T\,\overline{K}_c$ is relatively compact and satisfies* (2.1.1) *in $\overline{K}_c$. Assume there exist a concave positive functional α with $\alpha(x) \leq \|x\|$, $x \in K$, and constants a, b, d with $0 < d < a < b \leq c$, such that the following conditions hold:*

(1) *$\{x \in S(\alpha, a, b) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(z) > a$ if $z \in \mathbb{T}x$, $x \in S(\alpha, a, b)$;*

(2) *$\|Tx\| < d$ if $x \in \overline{K}_d$;*

(3) *for all $x \in S(\alpha, a, c)$ and all $z \in \mathbb{T}x$ with $\|z\| > b$ we have $\alpha(z) > a$.*

Then T has at least three fixed points x_1, x_2 and x_3 in $\overline{K}_c$ such that $\|x_1\| < d$, $\alpha(x_2) > a$ and, $\|x_3\| > d$ and $\alpha(x_3) < a$.

Proof. Put $U_1 = \{x \in \overline{K}_c : \|x\| < d\}$ and $U_2 = \{x \in S(\alpha, a, c) : \alpha(x) > a\}$. The sets U_1 and U_2 are open convex subsets of $\overline{K}_c$ and T has no fixed point in $\partial U_1 \cup \partial U_2$. Indeed, assumption (2) guarantees that T has no fixed points in ∂U_1 and an analogous argument to that done in the proof of Theorem 2.4.3 ensures that T has no fixed points in ∂U_2.

By the additivity property of the index we have

$$i_{\overline{K}_c}(T, \overline{K}_c) = i_{\overline{K}_c}(T, U_1) + i_{\overline{K}_c}(T, U_2) + i_{\overline{K}_c}\big(T, \overline{K}_c \setminus \big(\overline{U_1 \cup U_2}\big)\big). \tag{2.4.1}$$

Now Theorem 2.1.1 and Proposition 1.3.9 ensure that $i_{\overline{K}_c}(T, \overline{K}_c) = i_{\overline{K}_c}(T, U_1) = 1$ and then T has a fixed point in U_1. Moreover, by application of Theorem 2.4.3, we obtain that $i_{\overline{K}_c}(T, U_2) = 1$ and thus T has another fixed point in U_2. Finally, one deduces from (2.4.1) that

$$i_{\overline{K}_c}\big(T, \overline{K}_c \setminus \big(\overline{U_1 \cup U_2}\big)\big) = 1 - 2 = -1,$$

and so T has a third fixed point in $\overline{K}_c \setminus \big(\overline{U_1 \cup U_2}\big)$. □

Remark 2.4.6. *Theorem 2.4.5 guarantees that two fixed points are nonzero, but the third one could be a trivial fixed point, as in the classical Leggett–Williams' theorem, see Figure 2.4.1.*

In the case $b = c$, the third assumption of the previous theorem is superfluous, so we can write the following simpler result.

Corollary 2.4.7. *Let $T : \overline{K}_c \longrightarrow \overline{K}_c$ be a mapping such that $T\,\overline{K}_c$ is relatively compact and satisfies* (2.1.1) *in $\overline{K}_c$. Assume there exist a concave positive functional α with $\alpha(x) \leq \|x\|$ ($x \in K$) and constants a and d with $0 < d < a < c$, satisfying the following conditions:*

1. *$\{x \in S(\alpha, a, c) : \alpha(x) > a\} \neq \emptyset$ and $\alpha(z) > a$ if $z \in \mathbb{T}x$, $x \in S(\alpha, a, c)$;*
2. *$\|Tx\| < d$ if $x \in \overline{K}_d$.*

Then T has at least three fixed points in $\overline{K}_c$.

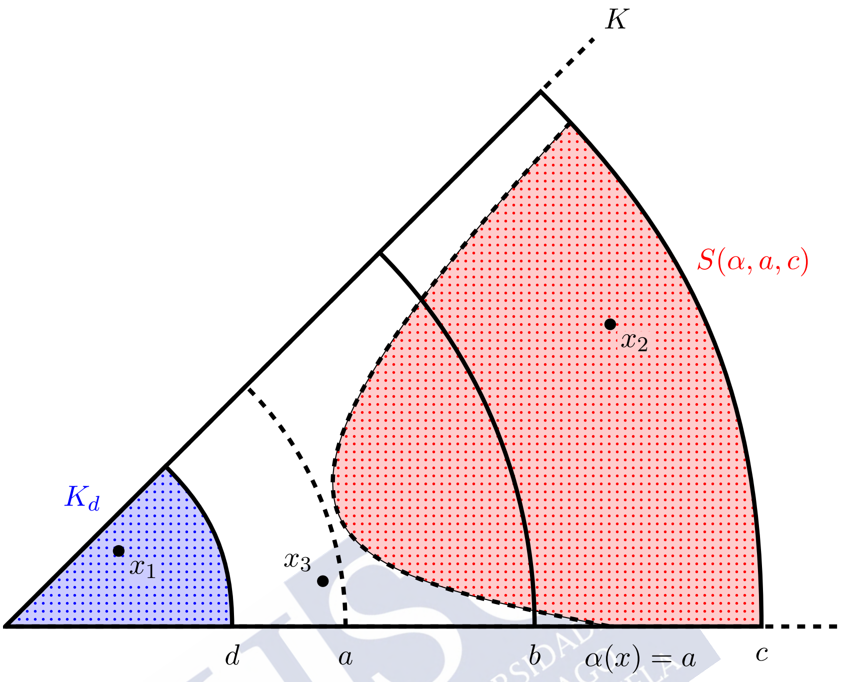

Figure 2.4.1: An illustration of the Leggett–Williams' type Theorem 2.4.5.

Some assumptions in Theorem 2.4.5 may be difficult to check in practice. By this reason we provide now another version where the required conditions about the operator T can be established in an easier way.

Remark 2.4.8. *Notice that assumption* (3) *in Theorem 2.4.5 is satisfied if one of the following two conditions holds:*

(i) $\alpha(z) \geq \dfrac{a}{b}\,\|z\|$ *if* $z \in \mathbb{T}x$, $x \in S(\alpha, a, c)$*;*

(ii) $\|z\| - \alpha(z) \leq b - a$ *if* $z \in \mathbb{T}x$, $x \in S(\alpha, a, c)$.

This remark is a modification of [94, Remark 1].

The following result is based on the previous inequalities and a small perturbation of the set $S(\alpha, a, b)$.

Theorem 2.4.9. *Let* $T : \overline{K}_c \longrightarrow \overline{K}_c$ *be a mapping such that* $T\,\overline{K}_c$ *is relatively compact and it satisfies* (2.1.1) *in* $\overline{K}_c$*. Assume there exist a concave positive functional* α *with* $\alpha(x) \leq \|x\|$, $x \in K$*, and constants* a, a_j, b, d *with* $0 < d < a < a_j < b \leq c$, $j = 1, 2$*, such that:*

1. $\{x \in S(\alpha, a, b) : \alpha(x) > a,\ \|x\| < b\} \neq \emptyset$ *and* $\alpha(Tx) > a_1 > a$ *if* $x \in S(\alpha, a, b)$*;*

2. $\|Tx\| < d$ *if* $x \in \overline{K}_d$;

3. *One of the following conditions holds:*

   $(3a)$ $\alpha(Tx) \geq \dfrac{a_2}{b}\|Tx\|$, *if* $x \in S(\alpha, a, c)$;

   $(3b)$ $\|Tx\| - \alpha(Tx) \leq b - a_2$, *if* $x \in S(\alpha, a, c)$.

Then T has at least three fixed points in $\overline{K}_c$.

Proof. Let $x_0 \in \{x \in S(\alpha, a, b) : \alpha(x) > a,\ \|x\| < b\}$ and $\varepsilon > 0$ be such that

$$\varepsilon < \min\left\{\alpha(x_0) - a, b - \|x_0\|, \frac{a_j - a}{2}, \frac{(a_2 - a)b}{a_2 + b}\right\}.$$

Then $x_0 \in \{x \in S(\alpha, a + \varepsilon, b - \varepsilon) : \alpha(x) > a + \varepsilon\}$, so this set is nonempty.

Take $x \in S(\alpha, a+\varepsilon, b-\varepsilon)$ and $z \in \mathbb{T}x$. Since α is continuous, there exists $\delta > 0$, $\delta \leq \varepsilon$, such that $\alpha(y) \geq a$ for all $y \in \overline{B}_\delta(x)$. First, suppose that $z = \sum \lambda_i T x_i$ with $x_i \in \overline{B}_\delta(x)$, $\lambda_i \in [0, 1]$ and $\sum \lambda_i = 1$. Then, by condition 1. and the concavity of α, we deduce

$$\alpha\left(\sum \lambda_i T x_i\right) \geq \sum \lambda_i \alpha(Tx_i) > \sum \lambda_i a_1 = a_1 > a + \varepsilon.$$

Note that if $z \in \mathbb{T}x$, then it is the limit of elements of the form $\sum \lambda_i T x_i$ (see Definition 1.1.1) and hence the continuity of α implies that

$$\alpha(z) \geq a_1 > a + \varepsilon.$$

Now assume that $(3a)$ holds. Then we have that

$$\alpha(z) \geq \frac{a + \varepsilon}{b - \varepsilon}\|z\|, \quad \text{for all } z \in \mathbb{T}x,\ x \in S(\alpha, a + \varepsilon, c). \tag{2.4.2}$$

Indeed, if z is a convex combination of the form $z = \sum \lambda_i T x_i$, $x_i \in \overline{B}_\delta(x)$, we obtain by condition $(3a)$ and concavity of α that

$$\alpha\left(\sum \lambda_i T x_i\right) \geq \sum \lambda_i \alpha(Tx_i) \geq \sum \lambda_i \frac{a_2}{b}\|Tx_i\| \geq \frac{a_2}{b}\left\|\sum \lambda_i T x_i\right\|,$$

and so

$$\alpha(z) \geq \frac{a_2}{b}\|z\|.$$

By the definition of $\mathbb{T}$, we have then

$$\alpha(z) \geq \frac{a_2}{b}\|z\|, \quad \text{for all } z \in \mathbb{T}x,\ x \in S(\alpha, a + \varepsilon, c),$$

and thus (2.4.2) follows from the choice of ε.

Assume, on the other hand, that $(3b)$ holds and then we will check that

$$\|z\| - \alpha(z) \leq b - a - 2\varepsilon, \quad \text{for all } z \in \mathbb{T}x,\ x \in S(\alpha, a + \varepsilon, c). \tag{2.4.3}$$

Observe that it is sufficient to show condition (2.4.3) for those z such that $z = \sum \lambda_i T x_i$ with $x_i \in \overline{B}_\delta(x)$, $\lambda_i \in [0,1]$ and $\sum \lambda_i = 1$. For such a z, we have

$$\begin{aligned}\alpha(z) = \alpha\left(\sum \lambda_i T x_i\right) &\geq \sum \lambda_i \alpha(T x_i) \geq \sum \lambda_i \|T x_i\| - (b - a_2) \\ &\geq \left\|\sum \lambda_i T x_i\right\| - (b - a_2) = \|z\| - (b - a_2),\end{aligned}$$

and so

$$\|z\| - \alpha(z) \leq b - a_2.$$

Hence the inequality (2.4.3) follows again from the choice of ε.

Therefore, Theorem 2.4.5 implies that T has at least three fixed points in $\overline{K}_c$. □

Remark 2.4.10. *In the case $b = c$, the third assumption in Theorem 2.4.9 can be again removed and so we obtain a result in the line of Corollary 2.4.7, where assumptions on $\mathbb{T}$ are now replaced by conditions on T.*

2.5 A vectorial version of Krasnosel'skiĭ's fixed point theorem

Consider the following system of operator equations

$$\begin{cases} x_1 = N_1(x_1, x_2), \\ x_2 = N_2(x_1, x_2), \end{cases} \tag{2.5.1}$$

where K_1 and K_2 are cones of the Banach space $(X, \|\cdot\|_X)$ and so $K_1 \times K_2$ is a cone of the Banach space X^2 (for instance, with the norm $\|(x_1, x_2)\| = \max\{\|x_1\|_X, \|x_2\|_X\}$), and $N = (N_1, N_2) : K_1 \times K_2 \longrightarrow X^2$ is an operator under the hypotheses of Krasnosel'skiĭ's fixed point theorem. As noticed by Precup [119], if we apply Krasnosel'skiĭ's fixed point theorem directly to (2.5.1), then we obtain a fixed point $x = (x_1, x_2)$ of this operator N such that

$$r \leq \|x\| = \max\{\|x_1\|_X, \|x_2\|_X\} \leq R,$$

which implies that $0 \leq \|x_1\|_X \leq R$ and $0 \leq \|x_2\|_X \leq R$, but it is not possible to obtain a lower bound for the norm of every component. This fact motivates the use of a vectorial version of Krasnosel'skiĭ's fixed point theorem, see Figure 2.5.1 for a comparison between the localization given by Krasnosel'skiĭ's fixed point theorem and that given by its vectorial version. Another advantage of the vectorial approach is that it allows different behaviors (that is, compression or expansion) in each component of the system.

We begin by recalling the result by Precup regarding single–valued completely continuous operators.

Let $(X, \|\cdot\|)$ be a Banach space, $K_1, K_2 \subset X$ two cones and $K := K_1 \times K_2$ the corresponding cone of $X^2 = X \times X$. For $r, R \in \mathbb{R}^2_+$, $r = (r_1, r_2)$, $R = (R_1, R_2)$, we denote

$$\begin{aligned}(K_i)_{r_i, R_i} &:= \{x \in K_i \,:\, r_i \leq \|x\| \leq R_i\} \quad (i = 1, 2), \\ K_{r,R} &:= \{x \in K \,:\, r_i \leq \|x_i\| \leq R_i \text{ for } i = 1, 2\}.\end{aligned}$$

Theorem 2.5.1 ([119, Theorem 2.1]). *Let $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $r_i = \min\{\alpha_i, \beta_i\}$ and $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $N : K_{r,R} \longrightarrow K$, $N = (N_1, N_2)$, is a compact map and that there exist $h_i \in K_i \setminus \{0\}$, $i = 1, 2$, such that for each $i \in \{1, 2\}$ the following conditions hold in $K_{r,R}$:*

$$\lambda x_i \neq N_i x \quad \text{for all } x \in K_{r,R} \text{ with } \|x_i\| = \alpha_i \text{ and all } \lambda > 1;$$
$$x_i \neq N_i x + \mu h_i \quad \text{for all } x \in K_{r,R} \text{ with } \|x_i\| = \beta_i \text{ and all } \mu > 0.$$

Then N has a fixed point $x = (x_1, x_2)$ in K with $r_i \leq \|x_i\| \leq R_i$ for $i = 1, 2$.

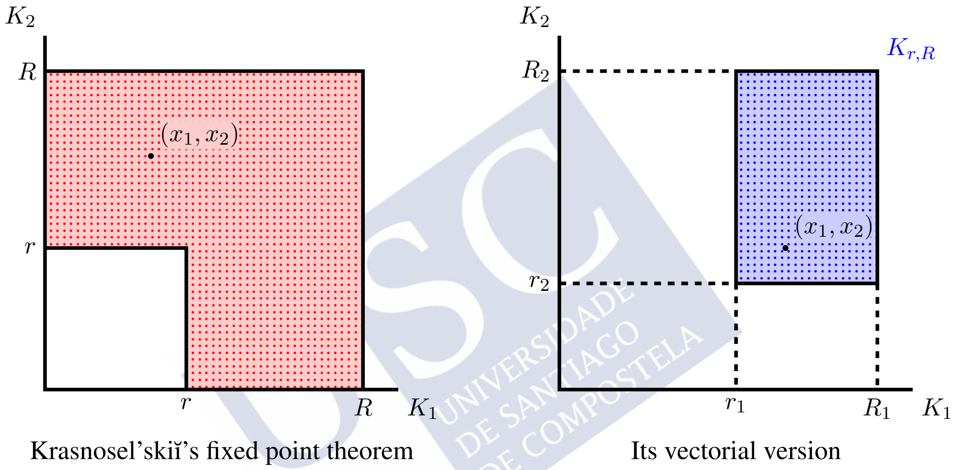

Figure 2.5.1: Comparison between localizations.

Remark 2.5.2. *Theorem 2.5.1 can be stated for n–dimensional systems, see [83], but for simplicity we only consider here two–dimensional systems.*

Now we generalize the previous theorem to usc multivalued mappings. Its proof is based on Bohnenblust–Karlin fixed point theorem.

Theorem 2.5.3. *Let $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $r_i = \min\{\alpha_i, \beta_i\}$ and $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $N : K_{r,R} \longrightarrow 2^K$, $N = (N_1, N_2)$, is an usc map with nonempty closed and convex values such that $N(K_{r,R})$ is relatively compact, and that there exist $h_i \in K_i \setminus \{0\}$, $i = 1, 2$, such that for each $i \in \{1, 2\}$ the following conditions hold in $K_{r,R}$:*

$$\lambda x_i \notin N_i x \quad \text{for all } x \in K_{r,R} \text{ with } \|x_i\| = \alpha_i \text{ and all } \lambda > 1; \tag{2.5.2}$$
$$x_i \notin N_i x + \mu h_i \quad \text{for all } x \in K_{r,R} \text{ with } \|x_i\| = \beta_i \text{ and all } \mu > 0. \tag{2.5.3}$$

Then N has a fixed point $x = (x_1, x_2)$ in K with $r_i \leq \|x_i\| \leq R_i$ for $i = 1, 2$.

Proof. We shall consider the four possible combinations of compression–expansion conditions for N_1 and N_2.

1. Assume first that $\beta_i < \alpha_i$ for both $i = 1, 2$ (compression for N_1 and N_2). Then $r_i = \beta_i$ and $R_i = \alpha_i$ for $i = 1, 2$. Denote $h = (h_1, h_2)$ and define the map $\tilde{N} : K \longrightarrow K$ given, for $x \in K$, by

$$\tilde{N}x = \eta(x) N \left(\delta_1(x_1) \frac{x_1}{\|x_1\|}, \delta_2(x_2) \frac{x_2}{\|x_2\|} \right) + (1 - \eta(x))\, h,$$

where $\delta_i(x_i) = \max\{\min\{x_i, R_i\}, r_i\}$ for $i = 1, 2$, and

$$\eta(x) = \min \left\{ \frac{\|x_1\|}{r_1}, \frac{\|x_2\|}{r_2}, 1 \right\}.$$

The map $\tilde{N}$ is usc because N is so and, as

$$\tilde{N}(K) \subset \overline{\mathrm{co}} \left(N(K_{r,R}) \cup \{h\} \right),$$

$\tilde{N}(K)$ is relatively compact. Then by application of Bohnenblust–Karlin fixed point theorem there exists $x \in K$ such that $x \in \tilde{N}x$.

Clearly $\|x_i\| > 0$ since $h_i \neq 0$ for $i = 1, 2$. Now we will show by contradiction that $x \in K_{r,R}$. To do so, we need to consider four cases separately, but we will study only one of them because the other ones are analogous.

Assume $0 < \|x_1\| < r_1$ and $0 < \|x_2\| < r_2$. If $\min \left\{ \frac{\|x_1\|}{r_1}, \frac{\|x_2\|}{r_2} \right\} = \frac{\|x_1\|}{r_1}$, then

$$x \in \frac{\|x_1\|}{r_1} N \left(\frac{r_1}{\|x_1\|} x_1, \frac{r_2}{\|x_2\|} x_2 \right) + \left(1 - \frac{\|x_1\|}{r_1} \right) h,$$

and so

$$\frac{r_1}{\|x_1\|} x_1 \in N_1 \left(\frac{r_1}{\|x_1\|} x_1, \frac{r_2}{\|x_2\|} x_2 \right) + \frac{r_1}{\|x_1\|} \left(1 - \frac{\|x_1\|}{r_1} \right) h_1,$$

which contradicts (2.5.3) for $i = 1$.

2. Assume that $\beta_1 < \alpha_1$ (compression for N_1) and $\beta_2 > \alpha_2$ (expansion for N_2). Let $N_i^* : K_{r,R} \longrightarrow K_i$ ($i = 1, 2$) be given by

$$\begin{aligned} N_1^* x &= N_1 \left(x_1, \left(\frac{R_2}{\|x_2\|} + \frac{r_2}{\|x_2\|} - 1 \right) x_2 \right), \\ N_2^* x &= \left(\frac{R_2}{\|x_2\|} + \frac{r_2}{\|x_2\|} - 1 \right)^{-1} N_2 \left(x_1, \left(\frac{R_2}{\|x_2\|} + \frac{r_2}{\|x_2\|} - 1 \right) x_2 \right). \end{aligned} \tag{2.5.4}$$

Notice that the map $N^* = (N_1^*, N_2^*)$ is in case 1, and thus N^* has a fixed point $v \in K_{r,R}$. Now, the point $x = (x_1, x_2)$ given by

$$x_1 = v_1 \quad \text{and} \quad x_2 = \left(\frac{R_2}{\|v_2\|} + \frac{r_2}{\|v_2\|} - 1 \right) v_2$$

is a fixed point of N.

3. The case $\beta_1 > \alpha_1$ (expansion for N_1) and $\beta_2 < \alpha_2$ (compression for N_2) is similar to the previous one. Consider the map $N^* = (N_1^*, N_2^*)$ defined by

$$N_1^* x = \left(\frac{R_1}{\|x_1\|} + \frac{r_1}{\|x_1\|} - 1 \right)^{-1} N_1 \left(\left(\frac{R_1}{\|x_1\|} + \frac{r_1}{\|x_1\|} - 1 \right) x_1, x_2 \right), \qquad (2.5.5)$$
$$N_2^* x = N_2 \left(\left(\frac{R_1}{\|x_1\|} + \frac{r_1}{\|x_1\|} - 1 \right) x_1, x_2 \right).$$

4. The case $\beta_i > \alpha_i$ for $i = 1, 2$ (expansion for N_1 and N_2) reduces to case 1, if we consider the map $N^* = (N_1^*, N_2^*)$ defined by (2.5.5) and (2.5.4), respectively.

This ends the proof. □

Observe that Theorem 2.5.1 can be generalized to a class of possibly discontinuous operators in the sense of the previous sections as a simple application of Theorem 2.5.3. In order to do that, given a single–valued (not necessarily continuous) operator $T : \overline{U} \subset K \longrightarrow K$, $T = (T_1, T_2)$, we associate to it the following multivalued map $\mathbb{T} : \overline{U} \longrightarrow 2^K$ given by

$$\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2), \quad \mathbb{T}_i x = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, T_i \left(\overline{B}_\varepsilon(x) \cap \overline{U} \right) \quad \text{for every } x \in \overline{U} \quad (i = 1, 2), \qquad (2.5.6)$$

and then we require the following condition to T:

$$\text{Fix}(\mathbb{T}) \subset \text{Fix}(T),$$

as usual. Therefore, under suitable conditions about T, Theorem 2.5.3 provides the existence of a fixed point for the operator $\mathbb{T}$ which, in fact, will be a fixed point of T.

Theorem 2.5.4. *Let $\alpha_i, \beta_i > 0$ with $\alpha_i \neq \beta_i$, $r_i = \min\{\alpha_i, \beta_i\}$ and $R_i = \max\{\alpha_i, \beta_i\}$ for $i = 1, 2$. Assume that $T : K_{r,R} \longrightarrow K$, $T = (T_1, T_2)$, is a mapping such that $T(K_{r,R})$ is relatively compact, it satisfies that*

$$\text{Fix}(\mathbb{T}) \subset \text{Fix}(T),$$

and that there exist $h_i \in K_i \setminus \{0\}$, $i = 1, 2$, such that for each $i \in \{1, 2\}$ the following conditions hold in $K_{r,R}$:

$$\lambda x_i \notin \mathbb{T}_i x \quad \text{for all } x \in K_{r,R} \text{ with } \|x_i\| = \alpha_i \text{ and all } \lambda > 1;$$
$$x_i \notin \mathbb{T}_i x + \mu h_i \quad \text{for all } x \in K_{r,R} \text{ with } \|x_i\| = \beta_i \text{ and all } \mu > 0.$$

Then T has a fixed point $x = (x_1, x_2)$ in K with $r_i \leq \|x_i\| \leq R_i$ for $i = 1, 2$.

Chapter 3
First order problems

It is well–known due to Peano's theorem [115] that the Cauchy problem

$$x'(t) = f(t, x(t)) \text{ for a.a. } t \in I = [0,1], \quad x(0) = x_0 \in \mathbb{R}^n, \tag{3.0.1}$$

has at least one solution whenever f is continuous and bounded and, even more, by Kneser–Hukuhara's theorem we know that the set of solutions for (3.0.1) is connected when f is continuous and bounded (see [51, 90]). However, if f is discontinuous, the behavior of the set of solutions for problem (3.0.1) is more complicated. For instance, for the function

$$f(x) = \begin{cases} 1, & \text{if } x \geq 0, \\ -1, & \text{if } x < 0, \end{cases}$$

and the initial condition $x(0) = 0$, the Cauchy problem (3.0.1) has exactly two different absolutely continuous solutions. Moreover, for

$$f(x) = \begin{cases} -1, & \text{if } x \geq 0, \\ 1, & \text{if } x < 0, \end{cases} \tag{3.0.2}$$

and the initial condition $x(0) = 0$, the Cauchy problem (3.0.1) has no solutions, so it is clear that some additional condition is necessary at the discontinuity points.

Several papers in the literature (see [13, 25, 45, 59, 60, 76, 84, 132, 136]) generalize the existence result due to Carathéodory which guarantees that problem (3.0.1) has at least one absolutely continuous solution provided that the right-hand side $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ is a L^1-Carathéodory function, that is,

$(C1)$ for all $x \in \mathbb{R}^n$, $f(\cdot, x)$ is measurable;

$(C2)$ for a.a. $t \in I$, $f(t, \cdot)$ is continuous; and

$(C3)$ there exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}^n$, $\|f(t,x)\| \leq M(t)$.

A function $x : I \to \mathbb{R}^n$ is a Carathéodory solution of (3.0.1) if $x(t)$ is absolutely continuous (shortly, $\mathcal{AC}(I)$), $x(0) = x_0$ and $x'(t) = f(t, x(t))$ a.e. on I. All the mentioned papers give conditions on f in order to replace condition $(C2)$ by a weaker condition which allows the nonlinearity to be discontinuous with respect to the spatial variable.

A very useful technique to study this kind of problems consists on considering a differential inclusion, where the original nonlinearity is replaced by a multivalued usc map. In this case, new notions of solutions to problem (3.0.1) appear, depending on the regularization chosen, and so we can talk about Filippov solutions, Krasovskij solutions, Hermes solutions and so on. Therefore, a comparison between such notions of solution is interesting and needed, and it has been widely treated in literature. For example, a comparison between Carathéodory and Filippov solutions can be seen in [134], between Carathéodory and Krasovskij solutions in [84] or between Hermes, Filippov and Krasovskij solutions in [75]. Moreover, an overview about different notions of solutions for discontinuous differential equations can be found in [41, Chapter 1].

In particular, in Section 3.2, we will use the Krasovskij regularization, where the nonlinearity f is replaced by its 'convexification', namely

$$\mathcal{K}f(t,x)=\bigcap_{\varepsilon>0}\overline{\text{co}}f(t,\overline{B}_\varepsilon(x)),$$

and then the Cauchy problem (3.0.1) turns into the differential inclusion

$$x'(t)\in\mathcal{K}f(t,x(t)) \text{ for a.a. } t\in I,\quad x(0)=x_0\in\mathbb{R}^n. \tag{3.0.3}$$

To do this, we follow the line of [45, 59, 84], where the authors provide some conditions on the existence of solutions for the inclusion and then they prove that such solutions are, in fact, solutions of the original Cauchy problem. On the other hand, in Section 3.1 we will use this idea from a different of view, and so we will convexify the associated fixed point operator instead of the nonlinearity itself. Then we apply the topological degree or the fixed point theorems of the previous chapters to obtain existence results to discontinuous differential equations.

This chapter is organized as follows: in Section 3.1 we prove the existence of solutions for a scalar first order problem with functional initial conditions by using the degree defined in Chapter 1; in Section 3.2 we improve an existence result for discontinuous autonomous systems due to Bressan and Shen, and in Section 3.3 we obtain the existence of extremal solutions for a first order functional differential equation under weak conditions in the nonlinear part and monotonicity with respect to the functional variable. Our results are illustrated by several examples and they can be found in the papers [64, 102, 105].

3.1 Existence result for first order scalar problems with functional initial conditions

We shall consider the functional initial value problem

$$x'(t)=f(t,x(t)) \text{ for a.a. } t\in I=[0,L],\quad x(0)=F(x), \tag{3.1.1}$$

where $L>0$ is given and $F:\mathcal{C}(I)\to\mathbb{R}$ is assumed to be continuous, but not necessarily linear or bounded.

Clearly, as a particular case, we have the initial value problem (3.0.1) when $n=1$. Unlike the classical situation, we do not assume that $f:I\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. Indeed, we shall allow f to be discontinuous over the graphs of countably many functions in the conditions of the following definition.

Definition 3.1.1. *An admissible discontinuity curve for the differential equation $x' = f(t,x)$ is an absolutely continuous function $\gamma : [c,d] \subset I \to \mathbb{R}$ satisfying one of the following conditions:*

either $\gamma'(t) = f(t,\gamma(t))$ for a.a. $t \in [c,d]$ (and we say that γ is viable for the differential equation),

or there exists $\varepsilon > 0$ and $\psi \in L^1(c,d)$, $\psi(t) > 0$ for a.a. $t \in [c,d]$, such that

either

$$\gamma'(t) + \psi(t) < f(t,y) \quad \textit{for a.a. } t \in [c,d] \textit{ and all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon]\,, \tag{3.1.2}$$

or

$$\gamma'(t) - \psi(t) > f(t,y) \quad \textit{for a.a. } t \in [c,d] \textit{ and all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon]\,. \tag{3.1.3}$$

We say that γ is inviable for the differential equation if it satisfies (3.1.2) *or* (3.1.3).

Observe that conditions (3.1.2) and (3.1.3) recall the notion of strict lower and upper solutions for the differential equation $x' = f(t,x)$.

Now, we state three technical results that we need in the proof of our main existence result for (3.1.1). Their proofs can be lookep up in [101].

In the sequel m denotes Lebesgue measure in $\mathbb{R}$.

Lemma 3.1.2. *Let $a, b \in \mathbb{R}$, $a < b$, and let $g, h \in L^1(a,b)$, $g \geq 0$ a.e., and $h > 0$ a.e. on (a,b).*

For every measurable set $J \subset (a,b)$ such that $m(J) > 0$ there is a measurable set $J_0 \subset J$ satisfying that $m(J \setminus J_0) = 0$ and for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\setminus J} g(s)\,ds}{\int_{\tau_0}^{t} h(s)\,ds} = 0 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\setminus J} g(s)\,ds}{\int_{t}^{\tau_0} h(s)\,ds}.$$

Corollary 3.1.3. *Let $a, b \in \mathbb{R}$, $a < b$, and let $h \in L^1(a,b)$ be such that $h > 0$ a.e. on (a,b).*

For every measurable set $J \subset (a,b)$ such that $m(J) > 0$ there is a measurable set $J_0 \subset J$ satisfying that $m(J \setminus J_0) = 0$ and for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\cap J} h(s)\,ds}{\int_{\tau_0}^{t} h(s)\,ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\cap J} h(s)\,ds}{\int_{t}^{\tau_0} h(s)\,ds}.$$

Corollary 3.1.4. *Let $a, b \in \mathbb{R}$, $a < b$, and let $f,\ f_n : [a,b] \to \mathbb{R}$ be absolutely continuous functions on $[a,b]$ ($n \in \mathbb{N}$), such that $f_n \to f$ uniformly on $[a,b]$ and for a measurable set $A \subset [a,b]$ with $m(A) > 0$ we have*

$$\lim_{n \to \infty} f_n'(t) = g(t) \quad \textit{for a.a. } t \in A.$$

If there exists $M \in L^1(a,b)$ such that $|f'(t)| \leq M(t)$ a.e. in $[a,b]$ and also $|f_n'(t)| \leq M(t)$ a.e. in $[a,b]$ ($n \in \mathbb{N}$), then $f'(t) = g(t)$ for a.a. $t \in A$.

In addition, the following well–known result is crucial throughout the text. It is a straightforward consequence of [16, Lemma 5.8.13], [111, Theorem 38.2] or [135, Lemma 6.92].

Lemma 3.1.5. *Let $a,\ b \in \mathbb{R}$, $a < b$.*

If $\varphi : [a,b] \longrightarrow \mathbb{R}$ is almost everywhere differentiable on $[a,b]$, then to any null measure set $A \subset \mathbb{R}$ there exists a null measure set $B \subset \varphi^{-1}(A)$ such that

$$\varphi'(t) = 0 \quad \text{for all } t \in \varphi^{-1}(A) \setminus B.$$

We are already in a position to prove an existence result for (3.1.1) by means of the degree theory introduced in Chapter 1. We note that it is an alternative proof to that given in [60] where the same result has been proven by means of Theorem 2.1.2, but we intend to show how to apply our degree theory. Observe that f can be discontinuous with respect to both of its variables.

We shall work in the Banach space $(\mathcal{C}(I), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the usual maximum norm. Given $R > 0$ fixed, we state the following assumptions regarding the nonlinearity f:

(H1) There exist $r < R$, $N \geq 0$ and $M \in L^1(I)$ such that $N + \|M\|_{L^1} < r$, $|F(x)| \leq N$ if $\|x\|_\infty \leq R$, and for a.a. $t \in I$ and all $x \in [-R, R]$ we have $|f(t,x)| \leq M(t)$.

(H2) Any composition $t \in I \mapsto f(t, x(t))$ is measurable if $x \in \mathcal{C}(I)$ and $\|x\|_\infty \leq R$.

(H3) There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \to \mathbb{R}$ $(n \in \mathbb{N})$ such that for a.a. $t \in I$ the function $x \mapsto f(t,x)$ is continuous on $[-R, R] \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$.

Theorem 3.1.6. *Problem* (3.1.1) *has at least an absolutely continuous solution $x : I \to \mathbb{R}$ such that $\|x\|_\infty < R$ provided that f satisfies conditions (H1)–(H3).*

Proof. Consider the Banach space $X = \mathcal{C}(I)$ with the norm $\|\cdot\|_\infty$. We define the integral operator $T : \overline{B}_R(0) \to X$ given by

$$Tx(t) = F(x) + \int_0^t f(s, x(s))\, ds \quad (t \in I,\ x \in X). \tag{3.1.4}$$

Clearly, finding fixed points of the operator T is equivalent to finding absolutely continuous solutions of problem (3.1.1). To prove that T has at least one fixed point we shall use Theorem 1.2.4 and the normalization and existence properties of the degree.

First, note that operator T is bounded. Indeed, by virtue of conditions (H1) and (H2), we have for all $x \in \overline{B}_R(0)$ and for all $t \in I$ that

$$|(Tx)(t)| \leq |F(x)| + \int_0^t |f(s, x(s))|\, ds \leq N + \|M\|_{L^1} < r,$$

so $T(\overline{B}_R(0)) \subset B_r(0) = \{x \in X : \|x\|_\infty < r\}$. In addition, T is well defined and maps $\overline{B}_R(0)$ into itself.

Second, $T\,\overline{B}_R(0)$ is equicontinuous. Since $(Tx)'(t) = f(t, x(t))$ for a.a. $t \in I$, we have

$$\begin{aligned} |(Tx)(t) - (Tx)(s)| &= \left| \int_s^t (Tx)'(r)\, dr \right| = \left| \int_s^t f(r, x(r))\, dr \right| \\ &\leq \int_s^t |f(r, x(r))|\, dr \leq \int_s^t M(r)\, dr \quad (s \leq t). \end{aligned} \tag{3.1.5}$$

Therefore, $T\,\overline{B}_R(0)$ is relatively compact in X.

On the other hand, we have that

$$\mathbb{T}(\overline{B}_R(0)) \subset \overline{\text{co}}\left(T(\overline{B}_R(0))\right) \subset \overline{B}_r(0) \subset B_R(0),$$

which implies that $x \notin \sigma\mathbb{T}x$ for all $(x,\sigma) \in \partial\, B_R(0) \times [0,1]$, where $\mathbb{T}$ is the cc–envelope of T defined as in (1.1.1). Thus, $\deg\,(Id - \sigma\mathbb{T}, B_R(0))$ is well defined as the degree for multivalued operators.

We consider the homotopy $H : \overline{B}_R(0) \times [0,1] \to \overline{B}_R(0)$ given by

$$H(x,\sigma) = \sigma\, Tx.$$

It is obvious that $H\left(\overline{B}_R(0) \times [0,1]\right)$ is relatively compact, because $T\,\overline{B}_R(0)$ is so.

If the operator T satisfies the condition $\{x\} \cap \mathbb{T}x \subset \{Tx\}$ for all $x \in \overline{B}_R(0) \cap \mathbb{T}\,\overline{B}_R(0)$, then Theorem 1.2.4 and the normalization property of Proposition 1.2.1 guarantee us that

$$\deg\,(Id - T, B_R(0)) = \deg\,(Id, B_R(0)) = 1,$$

so the operator T would have at least a fixed point as we want to see.

Therefore, to finish we only have to prove that

$$\{x\} \cap \mathbb{T}x \subset \{Tx\} \;\text{ for all } x \in \overline{B}_R(0) \cap \mathbb{T}\,\overline{B}_R(0).$$

To do so, fix $x \in \overline{B}_R(0) \cap \mathbb{T}\,\overline{B}_R(0)$ and consider the following three cases.

Case 1: $m\,(\{t \in I_n : x(t) = \gamma_n(t)\}) = 0$ *for all* $n \in \mathbb{N}$. Let us prove that then T is continuous at x.

By assumption, for a.a. $t \in I$ the mapping $f(t,\cdot)$ is continuous at $x(t)$. Thus, if $x_k \to x$ in $\overline{B}_R(0)$ then

$$f(t, x_k(t)) \to f(t, x(t)) \quad \text{for a.a. } t \in I,$$

which, along with (H1) and Lebesgue's dominated convergence theorem, yield $Tx_k \to Tx$ uniformly on I.

Case 2: $m\,(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$ *for some* $n \in \mathbb{N}$ *such that* γ_n *is inviable*. We suppose that $x \in \mathbb{T}x$ and we will prove, by *reductio ad absurdum*, that it is false.

We consider the set

$$K = \left\{ x \in \mathcal{C}(I) : |x(t) - x(s)| \le \int_s^t M(r)\,dr \quad (s \le t) \right\},$$

which is a convex and closed subset of X.

It is clear, by inequality (3.1.5), that $T\,\overline{B}_R(0) \subset K$, so $\mathbb{T}\,\overline{B}_R(0) \subset K$ because K is a convex and closed set. Hence, $x \in K$.

Now, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have

$$m\,(\{t \in I_n : x(t) = \gamma_n(t)\}) > 0$$

and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for a.a. $t \in I_n$, such that (3.1.3) holds with γ replaced by γ_n. (We can prove the result in a similar way if we assume (3.1.2) instead of (3.1.3), so we omit it).

We denote $J = \{t \in I_n : x(t) = \gamma_n(t)\}$, and we deduce from Lemma 3.1.2 that there is a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{2\int_{[\tau_0,t]\setminus J} M(s)\,ds}{(1/4)\int_{\tau_0}^{t} \psi(s)\,ds} = 0 = \lim_{t \to \tau_0^-} \frac{2\int_{[t,\tau_0]\setminus J} M(s)\,ds}{(1/4)\int_{t}^{\tau_0} \psi(s)\,ds}. \tag{3.1.6}$$

By Corollary 3.1.3 there exists $J_1 \subset J_0$ with $m(J_0) = m(J_1)$ such that for all $\tau_0 \in J_1$ we have

$$\begin{aligned}\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\cap J} \psi(s)\,ds}{\int_{\tau_0}^{t} \psi(s)\,ds} &= \lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\cap J_0} \psi(s)\,ds}{\int_{\tau_0}^{t} \psi(s)\,ds} = 1 \\ &= \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\cap J_0} \psi(s)\,ds}{\int_{t}^{\tau_0} \psi(s)\,ds} = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\cap J} \psi(s)\,ds}{\int_{t}^{\tau_0} \psi(s)\,ds},\end{aligned} \tag{3.1.7}$$

since $J_0 \subset J$ with $m(J_0) = m(J)$.

Let us fix a point $\tau_0 \in J_1$. From (3.1.6) and (3.1.7) we deduce that there exist $t_- < \tau_0$ and $t_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied:

$$2\int_{[\tau_0,t_+]\setminus J} M(s)\,ds < \frac{1}{4}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{3.1.8}$$

$$2\int_{[t_-,\tau_0]\setminus J} M(s)\,ds < \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds, \tag{3.1.9}$$

$$\int_{[\tau_0,t_+]\cap J} \psi(s)\,ds > \frac{1}{2}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{3.1.10}$$

$$\int_{[t_-,\tau_0]\cap J} \psi(s)\,ds > \frac{1}{2}\int_{t_-}^{\tau_0} \psi(s)\,ds. \tag{3.1.11}$$

Now, we define a positive number

$$\rho = \min\left\{\frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds, \frac{1}{4}\int_{\tau_0}^{t_+} \psi(s)\,ds\right\}. \tag{3.1.12}$$

We will prove that for $\varepsilon > 0$ given by our assumptions over γ_n and ρ as in (3.1.12), for every finite family $x_i \in \overline{B}_\varepsilon(x) \cap K$ and $\lambda_i \in [0,1]$ ($i = 1, 2, \dots, m$), with $\sum \lambda_i = 1$, we have $\|x - \sum \lambda_i T x_i\|_\infty \geq \rho$. Hence, we will get a contradiction with the hypothesis $x \in \mathbb{T}x$, so we can conclude that $x \notin \mathbb{T}x$.

Let us denote $y = \sum \lambda_i T x_i$. For a.a. $t \in I$ we have

$$y'(t) = \sum_{i=1}^{m} \lambda_i (Tx_i)'(t) = \sum_{i=1}^{m} \lambda_i f\left(t, x_i(t)\right) \leq M(t). \tag{3.1.13}$$

On the other hand, for every $t \in J = \{t \in I_n : x(t) = \gamma_n(t)\}$ we have

$$|x_i(t) - \gamma_n(t)| = |x_i(t) - x(t)| < \varepsilon,$$

and then the assumptions on γ_n ensure that for a.a. $t \in J$ we have

$$y'(t) = \sum_{i=1}^{m} \lambda_i f(t, x_i(t)) < \sum_{i=1}^{m} \lambda_i \left(\gamma_n'(t) - \psi(t)\right) = \gamma_n'(t) - \psi(t).$$

Lemma 3.1.5 guarantees that $\gamma_n'(t) = x'(t)$ for a.a. $t \in J$, hence

$$y'(t) < x'(t) - \psi(t) \quad \text{for a.a. } t \in J. \tag{3.1.14}$$

Now we use (3.1.13) and (3.1.14) first, and later (3.1.9) and (3.1.11), to deduce the following estimate:

$$\begin{aligned}
y(\tau_0) - y(t_-) &= \int_{t_-}^{\tau_0} y'(s)\,ds = \int_{[t_-,\tau_0]\cap J} y'(s)\,ds + \int_{[t_-,\tau_0]\setminus J} y'(s)\,ds \\
&< \int_{[t_-,\tau_0]\cap J} x'(s)\,ds - \int_{[t_-,\tau_0]\cap J} \psi(s)\,ds + \int_{[t_-,\tau_0]\setminus J} M(s)\,ds \\
&= x(\tau_0) - x(t_-) - \int_{[t_-,\tau_0]\setminus J} x'(s)\,ds - \int_{[t_-,\tau_0]\cap J} \psi(s)\,ds \\
&+ \int_{[t_-,\tau_0]\setminus J} M(s)\,ds \\
&\le x(\tau_0) - x(t_-) - \int_{[t_-,\tau_0]\cap J} \psi(s)\,ds + 2\int_{[t_-,\tau_0]\setminus J} M(s)\,ds \\
&< x(\tau_0) - x(t_-) - \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds.
\end{aligned}$$

Hence $\|x - y\|_\infty \ge y(t_-) - x(t_-) \ge \rho$ provided that $y(\tau_0) \ge x(\tau_0)$.

Similar computations with t_+ instead of t_- show that if $y(\tau_0) \le x(\tau_0)$ then we have $\|x - y\|_\infty \ge \rho$ too and we conclude that $x \notin \mathbb{T}x$.

Case 3: $m\left(\{t \in I_n : x(t) = \gamma_n(t)\}\right) > 0$ *only for some of those* $n \in \mathbb{N}$ *such that* γ_n *is viable*. We will assume that all admissible discontinuity curves are viable and $m(J_n) > 0$ for all $n \in \mathbb{N}$, where $J_n = \{t \in I_n : x(t) = \gamma_n(t)\}$. Hence, by Definition 3.1.1, for a.a. $t \in A = \bigcup_{n\in\mathbb{N}} J_n$ we have $x'(t) = f(t, x(t))$.

We will show that in this case condition $x \in \mathbb{T}x$ implies $x = Tx$.

First, since $x \in \mathbb{T}x$ then for each $k \in \mathbb{N}$ we can choose $\varepsilon = \rho = 1/k$ to guarantee that we can find functions $x_{k,i} \in \overline{B}_{1/k}(x) \cap K$ and coefficients $\lambda_{k,i} \in [0,1]$ $(i = 1, 2, \dots, m(k))$ such that $\sum_i \lambda_{k,i} = 1$ and

$$\left\| x - \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i} \right\|_\infty < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i}$, and notice that $y_k \to x$ uniformly in I and that $\|x_{k,i} - x\|_\infty \le 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \dots, m(k)\}$.

On the other hand, for a.a. $t \in I \setminus A$ we have that $f(t,\cdot)$ is continuous at $x(t)$ so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$, we have

$$|f(t,x_{k,i}(t)) - f(t,x(t))| < \varepsilon \quad \text{for all } i \in \{1,2,\ldots,m(k)\},$$

and hence

$$|y_k'(t) - f(t,x(t))| \leq \sum_{i=1}^{m(k)} \lambda_{k,i} \, |f(t,x_{k,i}(t)) - f(t,x(t))| < \varepsilon.$$

Therefore $y_k'(t) \to f(t,x(t))$ for a.a. $t \in I \setminus A$, and then we conclude from Corollary 3.1.4 that $x'(t) = f(t,x(t))$ for a.a. $t \in I \setminus A$.

To finish we only have to prove that $x(0) = F(x)$. To do so, we use essentially the continuity of F. Fix $\rho > 0$. Then there exists $\varepsilon > 0$ such that

$$|F(x) - F(y)| < \rho/2 \quad \text{for all } y \in \overline{B}_\varepsilon(x). \tag{3.1.15}$$

Since $x \in \mathbb{T}x$ we can find functions $x_i \in \overline{B}_\varepsilon(x) \cap K$ and coefficients $\lambda_i \in [0,1]$ ($i = 1,2,\ldots,m$) such that $\sum_{i=1}^m \lambda_i = 1$ and

$$\left\| x - \sum_{i=1}^m \lambda_i T x_i \right\|_\infty < \frac{\rho}{2}.$$

By the definition of T, we have $\sum_{i=1}^m \lambda_i T x_i(0) = \sum_{i=1}^m \lambda_i F(x_i)$ and, thus

$$\left| x(0) - \sum_{i=1}^m \lambda_i F(x_i) \right| < \frac{\rho}{2}. \tag{3.1.16}$$

By (3.1.15) and (3.1.16), we obtain that

$$\begin{aligned} |x(0) - F(x)| &\leq \left| x(0) - \sum_{i=1}^m \lambda_i F(x_i) \right| + \left| \sum_{i=1}^m \lambda_i F(x_i) - F(x) \right| \\ &\leq \frac{\rho}{2} + \sum_{i=1}^m \lambda_i \, |F(x_i) - F(x)| < \frac{\rho}{2} + \frac{\rho}{2} = \rho. \end{aligned}$$

Hence $x(0) = F(x)$, since $\rho > 0$ is arbitrary, and we conclude. □

Now we discuss briefly about Definition 3.1.1 and condition $(H3)$ in order to clarify the type of discontinuities we are considering.

First, notice that condition $(H3)$ does not imply that the set of discontinuity points of $f(t,\cdot)$ is a sequence of points (depending on t), but it only needs to be contained in such a sequence. Of course, a function f which is continuous with respect to the second variable clearly satisfies condition $(H3)$.

Moreover, we consider time-dependent discontinuity sets (see Example 3.1.7 below): the graphs of the curves γ_n, so we allow the function f to be discontinuous over the graphs of a countable number of curves satisfying the 'transversality' condition given by Definition 3.1.1. It is clear that such a condition is necessary, otherwise the existence result is not guaranteed. For example, function f given by (3.0.2) does not satisfy this condition and the associated Cauchy problem has no solution. Moreover, notice that conditions in Definition 3.1.1 are local, that is, only the behavior of f in a neighborhood of the curves is taking into account, in contrast with other results in the literature, see [25].

Example 3.1.7. *Consider the Cauchy problem*

$$x' = f(t,x) := \frac{1}{\sqrt{t}} + H(x+at) \quad \textit{for a.a. } t \in I = [0,1], \quad x(0) = 0,$$

where $a > 0$ and H is the Heaviside step function

$$H(x) = \begin{cases} 1, & \textit{if } x \geq 0, \\ 0, & \textit{if } x < 0. \end{cases} \tag{3.1.17}$$

Observe that for a.a. $t \in I$ the function $x \mapsto f(t,x)$ is continuous on $\mathbb{R} \setminus \{\gamma(t)\}$ with $\gamma : I \to \mathbb{R}$ given by $\gamma(t) = -at$. In this case, $\gamma'(t) = -a$ and then

$$\gamma'(t) + a < f(t,x) \quad \textit{for a.a. } t \in I \textit{ and all } x \in \mathbb{R},$$

so condition (3.1.2) *in Definition 3.1.1 holds for $\psi(t) \equiv a$. Hence, $(H3)$ is satisfied.*

Notice that the function f in Example 3.1.7 is nonnegative. For nonnegative nonlinearities any linear function with negative derivative is a simple example of an admissible curve, since it satisfies condition (3.1.2) in Definition 3.1.1. For this class of functions some existence results for (3.0.1) were established in [44] by means of different techniques to those employed here.

Another interesting particular case is that of constant discontinuity curves, that is, nonlinearities f which are discontinuous at fixed points, not depending on t. In this context, conditions in Definition 3.1.1 read as follows (since $\gamma(t) \equiv \gamma$ and so, $\gamma'(t) \equiv 0$):
either $f(t,\gamma) = 0$ for a.a. $t \in I$,
or there exists $\varepsilon > 0$ and $\psi \in L^1(I)$, $\psi(t) > 0$ for a.a. $t \in I$, such that
either

$$\psi(t) < f(t,y) \quad \text{for a.a. } t \in I \text{ and all } y \in [\gamma - \varepsilon, \gamma + \varepsilon]\,, \tag{3.1.18}$$

or

$$-\psi(t) > f(t,y) \quad \text{for a.a. } t \in I \text{ and all } y \in [\gamma - \varepsilon, \gamma + \varepsilon]\,. \tag{3.1.19}$$

In particular, we require to the function f to have constant sign in some neighborhood of the point γ or, otherwise, to vanish at this point. Similar conditions are frequently used in the literature, cf. [18, 19, 21, 84]. Observe that for autonomous problems time-dependent discontinuity sets have no sense, and thus the discontinuity points must satisfy conditions as above. Nevertheless, in the scalar autonomous case, necessary and sufficient conditions for the existence of Carathéodory solutions for Cauchy problems are known, see [15].

A usual assumption in the literature concerning discontinuous nonlinearities is the following, see [22, 84]:

$$\inf_{t \in I,\; x \in \mathbb{R}} f(t,x) > 0. \tag{3.1.20}$$

Then any constant curve is admissible for the differential equation since it is inviable, so the following result is immediately deduced.

Corollary 3.1.8. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies conditions $(H1)$, $(H2)$ and* (3.1.20)*, and the function $x \mapsto f(t,x)$ is continuous except at most in a countable number of points. Then problem* (3.1.1) *has at least one absolutely continuous solution.*

This is not true if we replace (3.1.20) by $f(t,x) > 0$ for all $t \in I$ and all $x \in \mathbb{R}$, as it can be seen by taking $f(x) = 1$ if $x \le 0$, $f(x) = x/2$ if $x > 0$ and $x_0 = 0$.

Theorem 3.1.6 provides existence results even for nonnegative functions f for which condition (3.1.20) fails.

Example 3.1.9. *Consider the Cauchy problem* (3.0.1) *with $n = 1$, $x_0 = 0$ and*

$$f(t,x) = \frac{1}{\sqrt{t}} - 1 + H(x - c)$$

where $c \in \mathbb{R}$ and H is the Heaviside step function.

The function f is discontinuous in the second variable for $x = c$ or, equivalently, it is discontinuous over the graph of the constant curve $\gamma \equiv c$. By taking $\psi(t) = (1/\sqrt{t} - 1)/2$, we have that

$$\psi(t) < f(t,x) \quad \text{for a.a. } t \in I \text{ and all } x \in \mathbb{R},$$

and $\psi \in L^1(I)$, $\psi(t) > 0$ for a.a. $t \in I$. Hence, γ is inviable.

However, $\inf_{t \in I,\, x \in \mathbb{R}} f(t,x) = 0$. Moreover, notice that condition (3.1.2) *in Definition 3.1.1 is not satisfied for any constant function ψ.*

Finally we illustrate the applicability of Theorem 3.1.6 with the following examples.

Example 3.1.10. *The initial value problem* (3.0.1) *has at least one absolutely continuous solution on $I = [0,1]$ for*

$$f(t,x) = \begin{cases} 1/\sqrt{t} + \sin\lfloor 1/x \rfloor & \text{for } x > 0, \\ 1/\sqrt{t} & \text{for } x \le 0, \end{cases}$$

and $x_0 = 0$ (where $\lfloor x \rfloor$ means integer part of x).

To check condition $(H2)$ in Theorem 3.1.6, note that for every continuous function x, we can write the composition $t \in I \mapsto f(t, x(t))$ as

$$t \in I \mapsto f(t,x(t)) = \frac{1}{\sqrt{t}} + \sum_{n=1}^{\infty} \sin(n) \chi_{E_n}(t), \tag{3.1.21}$$

where χ denotes the characteristic function and $E_n = x^{-1}((1/(n+1), 1/n])$, $n \in \mathbb{N}$, are measurable sets. Therefore, (3.1.21) *is a measurable function and so condition $(H2)$ holds.*

Moreover, in this case $f(t,\cdot)$ is discontinuous at $x = 0$ or $x = 1/n$, $n \in \mathbb{N}$, but it is not monotone. Notice that $\gamma_0 = 0$ and $\gamma_n = 1/n$ ($n \in \mathbb{N}$) are admissible discontinuity curves. Moreover, all of them are necessarily crossed by any solution because $x'(t) > 1/\sqrt{t} - 1$ for a.a. $t \in [0,1]$.

Example 3.1.11. *Consider the scalar initial value problem*

$$x'(t) = f(t,x) = \lfloor 1/(t+2|x|) \rfloor^{1/2} + 1 \text{ for a.a. } t \in [0,1], \quad x(0) = 0.$$

First, observe that $f(t,x) \le M(t) = 1/\sqrt{t} + 1$ for a.a. $t \in [0,1]$ and all $x \in \mathbb{R}$, so condition $(H1)$ in Theorem 3.1.6 holds.

Now, to check $(H2)$, let $x \in \mathcal{C}(I)$ be given. Then the function $\varphi(t) = t + 2|x(t)|$ is also continuous and, therefore, its composition with the monotone (and so, Borel–measurable) function $\psi(z) = (\lfloor 1/z \rfloor)^{1/2}$ is measurable. This allows us to conclude that the composition $t \in I \mapsto f(t,x(t))$ is measurable because it coincides almost everywhere with the function $t \in I \mapsto 1 + \psi(\varphi(t))$.

Finally, the function f is discontinuous over the graphs of the following two families of admissible curves ($n \in \mathbb{N}$):

$$\gamma_n^1(t) = -\frac{t}{2} + \frac{1}{2n}, \quad t \in I_n = \left[0, \frac{1}{n}\right],$$

and

$$\gamma_n^2(t) = \frac{t}{2} - \frac{1}{2n}, \quad t \in I_n = \left[0, \frac{1}{n}\right].$$

As $f(t,x) \ge 1$ for a.a. $t \in [0,1]$ and all $x \in \mathbb{R}$, we can conclude that such curves are in fact inviable, since for each $n \in \mathbb{N}$ we have ${\gamma_n^1}'(t) = -1/2$ and ${\gamma_n^2}'(t) = 1/2$.

Therefore, Theorem 3.1.6 implies the existence of at least one absolutely continuous solution for this problem.

In the following example we show the application of Theorem 3.1.6 to Cauchy problems with functional initial conditions and unbounded nonlinearities.

Example 3.1.12. *Consider the family of multipoint problems*

$$\begin{cases} x'(t) = f(t,x(t)) & \text{for a.a. } t \in I = [0,1], \\ x(0) = \sum_{i=1}^m c_i \, x(\eta_i), & \end{cases} \tag{3.1.22}$$

where $\eta_i \in [0,1]$ for $i = 1, \dots, m$, and

$$f(t,x) = \frac{a}{\sqrt{t}} + b\, x^\alpha \sin\lfloor 1/x \rfloor, \; x \neq 0, \quad f(t,0) = \frac{a}{\sqrt{t}},$$

with $a, b, c_i \in \mathbb{R}$ and $\alpha \in \mathbb{N}$ satisfying that $|a| \ge |b|$ and $4|a| + |b| + \sum_{i=1}^m |c_i| < 1$. Then problem (3.1.22) *has at least one solution x such that $\|x\|_\infty < 1$.*

Observe that the function $f(t,\cdot)$ is discontinuous over the countable family of constant curves $\{\pm 1/n \,:\, n \in \mathbb{N}\}$ which are inviable by condition $|a| \ge |b|$. Moreover, since $\int_{-1}^1 |s|^{-1/2}\, ds = 4$ and $4|a| + |b| + \sum_{i=1}^m |c_i| < 1$, condition $(H1)$ in Theorem 3.1.6 is satisfied for $M(t) = |a|\,|t|^{-1/2} + |b|$, $N = \sum_{i=1}^m |c_i|$ and $R = 1$. Finally, condition $(H2)$ can be verified as in Example 3.1.10.

3.2 Existence results for non–autonomous systems

In this section we study the existence of Carathéodory solutions for the system

$$x'(t) = f(t, x(t)) \quad \text{for a.a. } t \in I = [0, L], \;\; x(0) = x_0 \in \mathbb{R}^n, \tag{3.2.1}$$

where $L > 0$ and $n \in \mathbb{N}$ are fixed, and the nonlinear part $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ need not be continuous.

The following theorem, due to Bressan and Shen, was our motivation for this section.

Theorem 3.2.1. *[25, Theorem 1] Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ has the form*

$$f(x) = F\left(g_1(\tau_1(x), x), \dots, g_N(\tau_N(x), x)\right) \quad \textit{for some } N \in \mathbb{N},$$

where

(i) *Each map $\tau_i : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable. Each $g_i : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in t and continuous in x, that is, it is a Carathéodory function. Moreover, $F : \mathbb{R}^N \to \mathbb{R}^n$ is continuous.*

(ii) *For some compact set $K \subset \mathbb{R}^n$, at every point x there holds:*

$$f(x) \in K, \qquad \nabla\tau_i(x) \cdot z > 0 \quad \textit{for every } z \in K. \tag{3.2.2}$$

Then the Cauchy problem

$$x' = f(x), \quad x(0) = x_0 \in \mathbb{R}^n, \; t \in I = [0, L], \tag{3.2.3}$$

has at least one solution.

Theorem 3.2.1 was extended to discontinuous autonomous differential inclusions in [6] and to non-autonomous perturbations of autonomous discontinuous equations in [7].

The transversality condition (3.2.2) is the key assumption in the proof of Theorem 3.2.1 to deduce the existence of solutions as a consequence of the continuity of the Picard operator and the application of Schauder's fixed point theorem. Observe that (3.2.2) is a global condition, i.e. imposed at every point $x \in \mathbb{R}^n$, and this is an important drawback in Theorem 3.2.1. For instance, condition (3.2.2) is not satisfied if $f(x) = (0, 0, \dots, 0)$ for some $x \in \mathbb{R}^n$, thus leaving many nonlinearities outside its scope. The previous remark also shows that Theorem 3.2.1 does not contain as a particular case the classical situation when f is continuous everywhere.

Here we present an alternative version of Theorem 3.2.1 which

a) is valid for the general nonautonomous case;

b) relaxes condition (3.2.2) and imposes it where really needed (namely, discontinuity sets);

c) includes the classical Peano theorem as a particular case.

Our approach starts by replacing the differential equation in (3.2.1) by a differential inclusion

$$x'(t) \in \mathcal{K}f(t,x(t)) \quad \text{for a.a. } t \in I = [0,L], \;\; x(0) = x_0 \in \mathbb{R}^n. \tag{3.2.4}$$

We define the multivalued mapping $\mathcal{K}f : I \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ as

$$\mathcal{K}f(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}} f\left(t, \overline{B}_\varepsilon(x)\right) \qquad \text{for every } (t,x) \in I \times \mathbb{R}^n, \tag{3.2.5}$$

where $\overline{B}_\varepsilon(x)$ is the closed ball centered at x and radius $\varepsilon > 0$ (we consider the maximum norm in $\mathbb{R}^n$). Observe that, in the scalar case ($n = 1$) one can easily deduce for $(t,x) \in I \times \mathbb{R}$ that

$$\mathcal{K}f(t,x) = \left[\min\left\{f(t,x), \liminf_{y\to x} f(t,y)\right\}, \max\left\{f(t,x), \limsup_{y\to x} f(t,y)\right\}\right]. \tag{3.2.6}$$

Absolutely continuous solutions of (3.2.4) are usually called *Krasovskij solutions* of (3.2.1). It is clear that every Carathéodory solution of (3.2.1) is also a solution for the inclusion (3.2.4), but the converse is false in general. Following the spirit of [45], we will introduce some assumptions on f (similar to those in Bressan and Shen's paper [25]) so that, first, problem (3.2.1) has Krasovskij solutions and, second, every Krasovskij solution is a Carathéodory solution.

We need the following known result concerning the existence of Krasovskij solutions. It is just a part of [45, Proposition 2.1].

Proposition 3.2.2. *Let $f : I \times \mathbb{R}^n \to \mathbb{R}^n$ be such that*

(1) *There exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}^n$, we have*

$$\|f(t,x)\| \le M(t)(1 + \|x\|);$$

(2) *For all $x \in \mathbb{R}^n$, $f(\cdot, x)$ is measurable.*

Then the set of Krasovskij solutions to (3.2.1) *is a nonempty, compact and convex subset of* $\mathcal{C}(I;\mathbb{R}^n)$.

We are already in a position to state and prove our main existence result in this section.

Theorem 3.2.3. *Assume that $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the following conditions:*

(1) (Sublinear growth) There exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in \mathbb{R}^n$, we have

$$\|f(t,x)\| \le M(t)(1 + \|x\|);$$

(2) (Measurability) For all $x \in \mathbb{R}^n$, $f(\cdot, x)$ is measurable;

(3) (Inviable discontinuity sets) The function $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ can be expressed in the form

$$f(t,x) = F(t, g_1(\tau_1(t,x), x), g_2(\tau_2(t,x), x), \dots), \tag{3.2.7}$$

where for each $i \in N$ (N is a nonempty countable set) we have that

(i) *a) Each function $\tau_i : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable;*
b) Each function $g_i : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous in $(\mathbb{R} \setminus A_i) \times \mathbb{R}^n$, where A_i is a null–measure set;
c) For a.a. $t \in I$ and all $x \in \mathbb{R}^n$, the condition $\tau_i(t,x) \in \mathbb{R} \setminus A_i$ for all $i \in N$ implies that $f(t,\cdot)$ is continuous at x.

(ii) For each $(t,x) \in \tau_i^{-1}(A_i) \cap D$ we have

$$\nabla\tau_i(t,x) \cdot (1,z) \neq 0 \quad \textit{for all } z \in \mathcal{K}f(t,x), \tag{3.2.8}$$

where $D = \{(t,x) \in I \times \mathbb{R}^n \,:\, f$ is discontinuous at $(t,x)\}$ and $\mathcal{K}f(t,x)$ is as in (3.2.5).

Then the set of Carathéodory solutions to (3.2.1) *is a nonempty, compact and convex subset of $\mathcal{C}(I;\mathbb{R}^n)$.*

Proof. Let us consider the Banach space $X = \mathcal{C}(I;\mathbb{R}^n)$ with the maximum norm (also denoted by $\|\cdot\|$). Since for a.a. $t \in I$ the multivalued mapping $\mathcal{K}f(t,\cdot)$ is upper semicontinuous with nonempty, convex and compact values (see, for instance, Example 1.2 in [50] for details), we can ensure, by virtue of Proposition 3.2.2, that the set of Krasovskij solutions to (3.2.1) is a nonempty, compact and convex subset of $\mathcal{C}(I;\mathbb{R}^n)$.

To finish the proof, we will show that the assumptions imply that every Krasovskij solution to (3.2.1) is a Carathéodory solution. To do so, we need the following claim.

Claim: If u is a Krasovskij solution, then

$$m(\{t \in I \,:\, \tau_i(t,u(t)) \in A_i,\ (t,u(t)) \in D\}) = 0$$

for all $i \in N$ (here m denotes the Lebesgue measure).

Let us denote $J_i = \{t \in I \,:\, \tau_i(t,u(t)) \in A_i,\ (t,u(t)) \in D\}$ and $\varphi_i(t) = \tau_i(t,u(t))$ for all $t \in I$. Since $m(A_i) = 0$, Lemma 3.1.5 ensures the existence of some null set $B_i \subset \varphi_i^{-1}(A_i)$ such that for every $t \in \varphi_i^{-1}(A_i) \setminus B_i$ we have

$$\frac{d\tau_i(t,u(t))}{dt} = 0.$$

By the chain rule we deduce

$$\frac{d\tau_i(t,u(t))}{dt} = \nabla\tau_i(t,u(t)) \cdot (1,u'(t)),$$

so

$$\nabla\tau_i(t,u(t)) \cdot (1,u'(t)) = 0 \quad \text{ for all } t \in \varphi_i^{-1}(A_i) \setminus B_i. \tag{3.2.9}$$

The definition of Krasovskij solution implies the existence of a null measure set $C_i \subset \varphi^{-1}(A_i)$ such that $u'(t) \in \mathcal{K}f(t,u(t))$ for all $t \in \varphi_i^{-1}(A_i) \setminus C_i$, so condition (3.2.9) restricted to $\varphi_i^{-1}(A_i) \setminus (B_i \cup C_i)$ and the assumption (3.2.8) are contradictory unless $\varphi_i^{-1}(A_i) = B_i \cup C_i$. The claim is proven.

Therefore, if u is a Krasovskij solution, then condition (3) (i) c) implies that for a.a. $t \in I$ the mapping $f(t,\cdot)$ is continuous at $u(t)$, and thus $\mathcal{K}f(t,u(t)) = \{f(t,u(t))\}$ for a.a. $t \in I$. Since u is a Krasovskij solution, $u'(t) \in \mathcal{K}f(t,u(t)) = \{f(t,u(t))\}$ for a.a. $t \in I$, so u is a solution in the Carathéodory sense. □

Remark 3.2.4. *Theorem 3.2.3 includes the classical Peano theorem as a particular case because condition* (ii) *is only required at the points where the function* f *may present discontinuities.*

Remark 3.2.5. *Condition (3) (i) implies that the set of discontinuity points of the function* $f(t,\cdot)$ *is determined by those points satisfying that* $\tau_i(t,x) \in A_i$ *for some* $i \in N$*. Moreover, the behavior of* f *at the discontinuity points is controlled by the transversality condition (3) (ii).*

On the other hand, notice that in the case of a finite set N *condition (3) (i) c) holds trivially if for a.a.* $t \in I$*, the function* $F(t,\cdot)$ *is continuous.*

Observe that the specific form of f given by (3.2.7) allows to consider any Carathéodory function just by taking $N = \{1, 2, \dots, n\}$ and $g_i(t,x) = x_i$ with $i \in N$. Even though in the previous papers where f was considered as the composition (3.2.7) the set N was finite (see [7, 25]), the following example motivates to consider a countable set N.

Example 3.2.6. *Consider the problem* (3.2.1) *with* $n = 2$*,* $L = 1$*,* $x_0 = (0,0)$ *and the function* $f = (f_1, f_2)$ *where*

$$f_1(t,x,y) = \begin{cases} 2, & \text{if } \dfrac{1-t^{2m}}{2m} \le x + y < \dfrac{1-t^{2m-1}}{2m-1}, \ m \in \mathbb{N}, \\ 1, & \text{otherwise,} \end{cases}$$

and

$$f_2(t,x,y) = \begin{cases} |y|, & \text{if } \dfrac{1}{2m} \le x < \dfrac{1}{2m-1}, \ m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

It is not clear how to express $f = (f_1, f_2)$ *as the function composition* (3.2.7) *with a finite set* N*. Nevertheless, that difficulty can be easily overcome with a countable set* N*. Define the auxiliary functions* $H_1, H_2 : \mathbb{R} \to \mathbb{R}$ *given by*

$$H_1(y) = \begin{cases} 1, & \text{if } y \ge 0, \\ 0, & \text{if } y < 0, \end{cases} \quad \text{and} \quad H_2(y) = \begin{cases} 1, & \text{if } \dfrac{1}{2m} \le y < \dfrac{1}{2m-1}, \ m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Then the functions f_1 *and* f_2 *can be written as*

$$f_1(t,x,y) = 1 + \sum_{n \in \mathbb{N}} \left[H_1\left(x + y - \frac{1-t^{2n}}{2n}\right) - H_1\left(x + y - \frac{1-t^{2n-1}}{2n-1}\right) \right].$$

and

$$f_2(t,x,y) = |y|\, H_2(x).$$

Finally, taking $N = \mathbb{N} \cup \{0\}$*,* $\tau_0(t,x,y) = x$*,* $g_0(t,x,y) = |y|\, H_2(t)$ *and for each* $i \in \mathbb{N}$*,* $\tau_i(t,x,y) = x + y - (1-t^i)/i$*,* $g_i(t,x,y) = H_1(t)$ *and*

$$F(t, x_0, x_1, x_2, \dots) = \left(1 + \sum_{i \in \mathbb{N}} (x_{2i} - x_{2i-1}), x_0\right),$$

we obtain f in the form (3.2.7).

Now, it only remains to check the transversality condition (3.2.8). *Notice that*

$$f_1(t,x,y) \geq 1 \text{ and } f_2(t,x,y) \geq 0 \text{ for all } (t,x,y) \in I \times \mathbb{R}^2,$$

and so

$$z_1 \geq 1 \text{ and } z_2 \geq 0 \text{ for every } (z_1,z_2) \in \mathcal{K}f(t,x,y).$$

Then for every $(z_1,z_2) \in \mathcal{K}f(t,x,y)$ we have

$$\frac{\partial \tau_0}{\partial t}(t,x,y) + \frac{\partial \tau_0}{\partial x}(t,x,y)z_1 + \frac{\partial \tau_0}{\partial y}(t,x,y)z_2 = z_1 \geq 1$$

and for each $i \in \mathbb{N}$,

$$\frac{\partial \tau_i}{\partial t}(t,x,y) + \frac{\partial \tau_i}{\partial x}(t,x,y)z_1 + \frac{\partial \tau_i}{\partial y}(t,x,y)z_2 = t^{i-1} + z_1 + z_2 \geq 1.$$

Therefore, Theorem 3.2.3 ensures that the above problem has at least one solution.

Remark 3.2.7. *Condition* (3.2.8) *seems to be difficult to check in concrete examples when $n > 1$. The reason is that it could be hard to determine exactly what are the points which belong to the set $\mathcal{K}f(t,x)$. Therefore, we present two sufficient conditions for verifying* (3.2.8)*:*

(a) *There exist $\varepsilon, \delta > 0$ such that*

$$\nabla\tau_i(t,x)\cdot(1,z) > \delta \quad \text{for all } z \in f(t, \overline{B}_\varepsilon(x));$$

(b) *There exist $\varepsilon, \delta > 0$ such that*

$$\nabla\tau_i(t,x)\cdot(1,z) < -\delta \quad \text{for all } z \in f(t, \overline{B}_\varepsilon(x)).$$

Indeed, if condition (a) holds, then for $z_1, z_2 \in f(t, \overline{B}_\varepsilon(x))$ and $\lambda \in [0,1]$, we have

$$\nabla\tau_i(t,x)\cdot(1,\lambda z_1 + (1-\lambda)z_2) = \lambda\nabla\tau_i(t,x)\cdot(1,z_1) + (1-\lambda)\nabla\tau_i(t,x)\cdot(1,z_2) > \delta,$$

which implies

$$\nabla\tau_i(t,x)\cdot(1,z) > \delta \quad \text{for all } z \in \operatorname{co}f(t,\overline{B}_\varepsilon(x)),$$

and thus

$$\nabla\tau_i(t,x)\cdot(1,z) \geq \delta > 0 \quad \text{for all } z \in \mathcal{K}f(t,x) \subset \overline{\operatorname{co}}f(t,\overline{B}_\varepsilon(x)).$$

Similarly, if condition (b) holds, it is possible to show that (3.2.8) *is satisfied.*

Remark 3.2.8. *Carathéodory solutions of* (3.2.1) *can be localized a priori, which allows us to consider a weaker version of condition* (3.2.8)*. Indeed, in the conditions of Theorem 3.2.3, assume that u is a Carathéodory solution of the Cauchy problem* (3.2.1)*. Then*

$$\|u(t)\| = \left\| u_0 + \int_0^t f(s,u(s))\,ds \right\| \leq \|u_0\| + \|M\|_{L^1(I)} + \int_0^t M(s)\,\|u(s)\|\,ds.$$

By Gronwall's inequality,

$$\|u(t)\| \le \Big(\|u_0\| + \|M\|_{L^1(I)}\Big) \exp\left(\int_0^t M(s)\,ds\right),$$

and thus there exists $R > 0$ such that $\|u(t)\| \le R$ for all $t \in I$.

Define $\tilde{f}(t,x) = f(t,\rho(x))$ with $\rho(x) = \max\{\min\{R,x\},-R\}$. It is clear that

$$\left\|\tilde{f}(t,x)\right\| \le M(t)(1+R) =: \tilde{M}(t) \quad \textit{for a.a. } t \in I \textit{ and for all } x \in \mathbb{R}^n,$$

and $\tilde{M} \in L^1(I)$.

Observe that the solutions of (3.2.1) *belong to the set*

$$K = \left\{ u \in \mathcal{C}(I;\mathbb{R}^n) \,:\, u(0) = x_0,\ \|u(t)-u(s)\| \le \int_s^t \tilde{M}(r)\,dr \quad (s \le t) \right\}, \tag{3.2.10}$$

which is a compact and convex subset of $\mathcal{C}(I;\mathbb{R}^n)$. Note that solutions of (3.2.4) *belong to K too.*

Therefore, condition (3.2.8) *need not to be required for each $(t,x) \in \tau_i^{-1}(A_i) \cap D$, but only for those such that*

$$\|x\| \le \|x_0\| + \int_0^t \tilde{M}(s)\,ds.$$

In the case of the autonomous problem (3.2.3) we obtain the following existence result.

Corollary 3.2.9. *Assume that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a sublinear function which can be expressed in the form*

$$f(x) = F(g_1(\tau_1(x),x), g_2(\tau_2(x),x), \dots, g_N(\tau_N(x),x)),$$

where:

(i)
a) Each function $\tau_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is differentiable;
b) Each function $g_i : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous in $(\mathbb{R} \setminus A_i) \times \mathbb{R}^n$, where $A_i \subset \mathbb{R}$ is a null–measure set;
c) $F : \mathbb{R}^N \to \mathbb{R}^n$ is continuous.

(ii) For each $i \in \{1,2,\dots,N\}$ and every $x \in \tau_i^{-1}(A_i) \cap D$ the following condition is satisfied

$$\nabla\tau_i(x) \cdot z \neq 0 \quad \textit{for all } z \in \mathcal{K}f(x). \tag{3.2.11}$$

Then problem (3.2.3) *has at least one solution.*

Remark 3.2.10. *Corollary 3.2.9 replaces the global condition* (3.2.2) *in Theorem 3.2.1 by the local assumption* (3.2.11)*. To do so, we need to impose stronger conditions on functions g_i, namely, that they are continuous with respect to its first variable except on a null–measure set, instead of being only measurable, as in Theorem 3.2.1. Nevertheless, the assumptions on g_i in Theorem 3.2.1 imply that for a given $\varepsilon > 0$ there exists a closed set J_i such that $m(\mathbb{R} \setminus J_i) \le \varepsilon$ and g_i is continuous when restricted to the set $J_i \times \mathbb{R}^n$, by Scorza-Dragoni's theorem.*

Now we give an example which shows that our results are more applicable than Theorem 3.2.1.

Example 3.2.11. *Consider the Cauchy problem* (3.2.3) *with* $n = 2$, $x_0 = (0,0)$, *and* $f = (f_1, f_2)$ *given by*

$$f_1(x,y) = \begin{cases} \sin^2\lfloor 1/x \rfloor + x\cos y + 2 & \textit{if } x \neq 0, \\ 2 & \textit{if } x = 0, \end{cases} \tag{3.2.12}$$

and

$$f_2(x,y) = \begin{cases} (1+\sin\lfloor 1/y \rfloor)\,|y| + 1 - \cos x & \textit{if } y \neq 0, \\ 1-\cos x & \textit{if } y = 0, \end{cases} \tag{3.2.13}$$

where $\lfloor z \rfloor$ *denotes the integer part of* z.

Observe that f_1 *and* f_2 *have infinitely many discontinuity points in any neighborhood of the initial condition. However, we shall show that* $f = (f_1, f_2)$ *satisfies the conditions of Corollary 3.2.9, and therefore the corresponding initial value problem has at least one absolutely continuous solution. To do so, we start by rewriting*

$$f_1(x,y) = g_1(\tau_1(x,y),(x,y)), \tag{3.2.14}$$

$$f_2(x,y) = g_2(\tau_2(x,y),(x,y)), \tag{3.2.15}$$

so that

$$f(x,y) = F(g_1(\tau_1(x,y),(x,y)), g_2(\tau_2(x,y),(x,y))), \tag{3.2.16}$$

for $F(u,v) = (u,v)$.

Now in (3.2.14) *we define* $\tau_1(x,y) = x$ *and* $g_1(t,(x,y)) = \sin^2\lfloor 1/t \rfloor + x\cos y + 2$ *if* $t \neq 0$ *and* $g_1(0,(x,y)) = x\cos y + 2$. *The function* g_1 *is continuous in* $(\mathbb{R} \setminus A_1) \times \mathbb{R}^2$, *where* A_1 *is the countable set*

$$A_1 = \left\{ \pm\frac{1}{n} \,:\, n \in \mathbb{N} \right\} \cup \{0\}.$$

We have that for every $(x,y) \in \mathbb{R}^2$ *and for all* $(z_1, z_2) \in ([-3/2, 3/2] \setminus \{0\}) \times \mathbb{R}$,

$$\begin{aligned} \nabla\tau_1(x,y)\cdot f(z_1,z_2) &= (1,0)\cdot f(z_1,z_2) \\ &= \sin^2\lfloor 1/z_1 \rfloor + z_1\cos z_2 + 2 \\ &\geq 2 - 3/2 = 1/2 > 0 \end{aligned}$$

and

$$\nabla\tau_1(x,y)\cdot f(0,z_2) = f_1(0,z_2) = 2 > 0,$$

so, in virtue of Remark 3.2.7, condition (ii) *in Corollary 3.2.9 is satisfied for* $i = 1$.

On the other hand, in (3.2.15) *it suffices to define* $\tau_2(x,y) = y$ *and*

$$g_2(t,(x,y)) = (1+\sin\lfloor 1/t \rfloor)\,|t| + 1 - \cos x$$

if $t \neq 0$ *and* $g_2(0,(x,y)) = 1 - \cos x$. *The function* g_2 *is continuous in* $(\mathbb{R} \setminus A_2) \times \mathbb{R}^2$, *where* A_2 *is the countable set*

$$A_2 = \left\{ \pm\frac{1}{n} \,:\, n \in \mathbb{N} \right\}.$$

For each $n \in \mathbb{N}$*, every* $(x, y) \in \tau_2^{-1}(\pm 1/n)$ *and all* $(z_1, |z_2|) \in \mathbb{R} \times (1/(n+1), 1/(n-1))$*, we have*

$$\begin{aligned}\nabla\tau_2(x,y) \cdot f(z_1,z_2) &= (1+\sin\lfloor 1/z_2 \rfloor)\,|z_2| + 1 - \cos z_1 \\ &\geq (1 - \max\{|\sin n|\,, |\sin(n-1)|\})\,\frac{1}{n+1} > 0,\end{aligned}$$

and thus condition (ii) *holds.*

Therefore, Corollary 3.2.9 ensures the existence of at least one solution for this Cauchy problem. However, Theorem 3.2.1 is not applicable here, since $f_2(0,0) = 0$ *and* $\nabla\tau_2(x,y) = (0,1)$ *and thus condition* (3.2.2) *is not satisfied.*

The Cauchy problem considered in the previous example cannot be analyzed by means of the existence results in [45, 84], since in those results the set of discontinuity points of the nonlinearity f can be at most the Cartesian product of null–measure sets. The results in [132] are also not applicable because only differential equations with right-hand sides bounded from the origin are considered, whereas in Example 3.2.11 it is $f_2(0,0) = 0$. Moreover, the example falls outside the scope of the existence principles established in [38, 76, 78] because the right–hand side is neither monotone nor quasimonotone.

More information can be given for the scalar case of (3.2.1). Exactly as in the classical setting of continuous right–hand sides, we can deduce the existence of extremal solutions, that is, a least and a greatest one. The proof is based on the following result on existence of extremal solutions for problem (3.2.4), which is the second part of [45, Proposition 2.1].

Proposition 3.2.12. *Let* $n = 1$*. In the conditions of Proposition 3.2.2, the set of Krasovskij solutions to* (3.2.1) *has a pointwise maximum* x^* *and minimum* x_**, which are the extremal Krasovskij solutions to* (3.2.1)*.*

Moreover, for each $t \in I$ *we have*

$$\begin{aligned}x^*(t) &= \max\{v(t) : v \in \mathcal{AC}(I),\, v'(s) \in \mathcal{K}f(s, v(s)) - \mathbb{R}_+ \textit{ a.e.},\ v(0) \leq x_0\}, \\ x_*(t) &= \min\{v(t) : v \in \mathcal{AC}(I),\, v'(s) \in \mathcal{K}f(s, v(s)) + \mathbb{R}_+ \textit{ a.e.},\ v(0) \geq x_0\}.\end{aligned}$$

From Proposition 3.2.12 and the proof of Theorem 3.2.3, it is easily obtained the following result on extremal Carathéodory solutions for the scalar case of (3.2.1).

Theorem 3.2.13. *Assume that* $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies the following conditions:*

(1) (Sublinear growth) There exists $M \in L^1(I)$ *such that for a.a.* $t \in I$ *and all* $x \in \mathbb{R}$*, we have*

$$|f(t,x)| \leq M(t)(1 + |x|);$$

(2) (Measurability) For all $x \in \mathbb{R}$*,* $f(\cdot, x)$ *is measurable;*

(3) (Inviable discontinuity sets) The function $f : I \times \mathbb{R} \to \mathbb{R}$ *can be expressed in the form*

$$f(t,x) = F(t, g_1(\tau_1(t,x), x), g_2(\tau_2(t,x), x), \dots),$$

where for each $i \in N$ *(*N *is a nonempty countable set) we have that* *(i)* *a)* *Each function* $\tau_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ *is differentiable;*

b) *Each function* $g_i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous in* $(\mathbb{R} \setminus A_i) \times \mathbb{R}$*, where* $A_i \subset \mathbb{R}$ *is a null–measure set;*

c) *For a.a.* $t \in I$ *and all* $x \in \mathbb{R}$*, the condition* $\tau_i(t,x) \in \mathbb{R} \setminus A_i$ *for all* $i \in N$ *implies that* $f(t,\cdot)$ *is continuous at* x*.*

(ii) *For each* $(t,x) \in \tau_i^{-1}(A_i) \cap D$ *we have*

$$\frac{\partial \tau_i}{\partial t}(t,x) + \frac{\partial \tau_i}{\partial x}(t,x)z \neq 0 \quad \text{for all } z \in \mathcal{K}f(t,x). \tag{3.2.17}$$

Then problem (3.2.1) *has the least solution,* x_**, and the greatest solution,* x^**. Moreover, for each* $t \in I$ *we have*

$$x^*(t) = \max\{v(t) : v \in \mathcal{AC}(I),\ v'(s) \leq f(s,v(s))\ \text{a.e.},\ v(0) \leq x_0\},$$
$$x_*(t) = \min\{v(t) : v \in \mathcal{AC}(I),\ v'(s) \geq f(s,v(s))\ \text{a.e.},\ v(0) \geq x_0\}.$$

Remark 3.2.14. *We cannot expect to obtain extremal solutions to* (3.2.1) *when* $n \geq 2$*, even if* f *is continuous, see [45, Remark 1, p. 624].*

Next we present a general family of problems whose solvability can be deduced from Theorem 3.2.13. We base our examples on a function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ which is discontinuous at every point of Cantor's ternary set C. We define ψ as a 1-periodic function such that $\psi = 0$ on C, and if (a,b) is one of the open intervals removed from $[0,1]$ at some step of the construction of C (see Figure 3.2.1), then define

$$\psi(x) = \frac{2x - a - b}{b - a} \quad \text{for all } x \in (a,b).$$

Note that ψ is linear with slope $2/(b-a)$ on (a,b), $\lim_{x \to a^+} \psi(x) = -1$ and $\lim_{x \to b^-} \psi(x) = 1$. Therefore, ψ is continuous on $\mathbb{R} \setminus \tilde{C}$, where $\tilde{C} = C + \mathbb{Z}$, and discontinuous at every $x_0 \in \tilde{C}$. More precisely, for each $x_0 \in \tilde{C}$ we have

$$\liminf_{x \to x_0} \psi(x) = -1 \quad \text{and} \quad \limsup_{x \to x_0} \psi(x) = 1.$$

We emphasize that the set of discontinuities of ψ is uncountable.

Proposition 3.2.15. *For any continuously differentiable function* $p = p(x)$ *and any bounded and continuous function* $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$*, the problem*

$$x' = \psi(p(x) + \alpha\, t) + \varphi(x),\ t \in I = [0,1], \quad x(0) = 0, \tag{3.2.18}$$

has the extremal solutions provided that $|\alpha|$ *is sufficiently large.*

Proof. Problem (3.2.18) is the particular case of (3.2.1) corresponding to

$$f(t,x) = g(\tau(t,x),x)$$

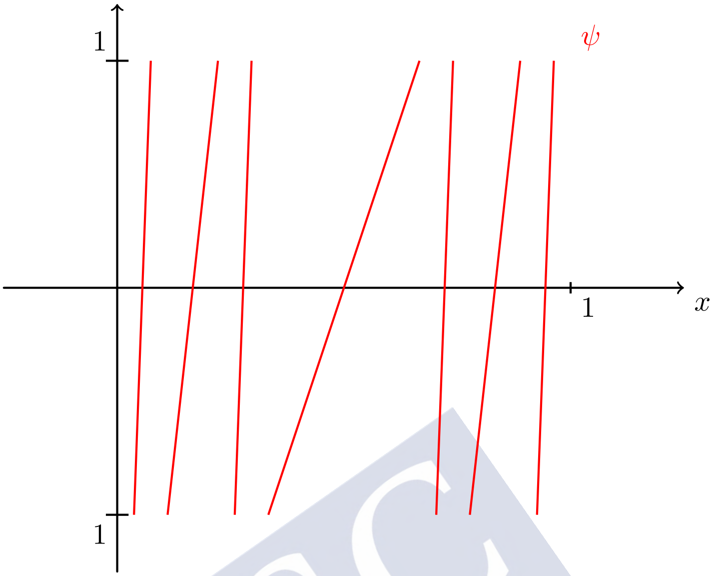

Figure 3.2.1: First three steps of the construction of the function ψ on $[0,1]$.

for $\tau(t,x) = p(x) + \alpha\, t$ and $g(t,x) = \psi(t) + \varphi(x)$. Observe that τ is continuously differentiable, g is continuous on $(\mathbb{R} \setminus \tilde{C}) \times \mathbb{R}$ and $\tilde{C}$ is a null–measure set.

Let $M > 0$ be such that $|\varphi(x)| \le M$ for all $x \in \mathbb{R}$; then we have $|f(t,x)| \le 1 + M$ for all $(t,x) \in I \times \mathbb{R}$, so one deduces by integration that any possible solution of (3.2.18) satisfies

$$|x(t)| \le 1 + M \quad \text{for all } t \in I.$$

This allows us to restrict the verification of condition (3.2.8) at points $(t,x) \in I \times \mathbb{R}$ such that $|x| \le 1 + M$. Observe that for every $(t,x) \in I \times [-(1+M), 1+M]$ we have

$$\mathcal{K}f(t,x) \subset [-(1+M), 1+M],$$

so if we take

$$\bar{M} = (1+M) \sup_{|x| \le 1+M} |p'(x)| \ge 0,$$

then we have

$$|zp'(x)| \le \bar{M} \quad \text{for all } x \in [-(1+M), 1+M] \text{ and all } z \in \mathcal{K}f(t,x).$$

Therefore, if $|\alpha| > \bar{M}$, then for every $(t,x) \in I \times [-(1+M), 1+M]$ such that $\tau(t,x) \in \tilde{C}$ we have

$$\nabla\tau(t,x) \cdot (1,z) = \alpha + zp'(x) \ne 0 \quad \text{for all } z \in \mathcal{K}f(t,x).$$

Then, Theorem 3.2.13 implies the existence of extremal solutions to problem (3.2.18) provided that $|\alpha| > \bar{M}$. □

Remark 3.2.16. *Proposition 3.2.15 cannot be deduced from Theorem 3.1.6. There, the nonlinearity f may be discontinuous over the graphs of a countable number of* admissible *curves, whereas here such a number is given by the set of discontinuities of ψ and so being uncountable. Even it is not clear in general how to write the set of points $(t,x) \in I \times \mathbb{R}$ such that $p(x) + \alpha\, t \in \tilde{C}$ as the union of several graphs of curves $\gamma(t)$.*

Remark 3.2.17. *If we try to apply Theorem 3.2.13 to the Cauchy problem in Example 3.1.11, then we need to write the function*

$$f(t,x) = \lfloor 1/(t + 2\,|x|) \rfloor^{1/2} + 1,$$

in the form (3.2.7)*. One first attempt would be to take $f(t,x) = g(\tau(t,x),x)$ with $g(t,x) = \lfloor 1/t \rfloor^{1/2} + 1$ if $t > 0$ and $g(t,x) = 1$, otherwise; and $\tau(t,x) = t + 2\,|x|$. Unfortunately, Theorem 3.2.13 cannot be directly applied since τ is not differentiable.*

Nevertheless, we can still expect to write f in the form (3.2.7)*. Consider the auxiliary functions*

$$H_1(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } y \geq 0, \end{cases} \quad , \quad H_2(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ 1, & \text{if } y > 0. \end{cases}$$

Observe that we can express the function $\lfloor 1/(t + 2\,|x|) \rfloor$ as

$$\lfloor 1/(t + 2\,|x|) \rfloor = \sum_{n \in \mathbb{N}} \left(H_1\left(\frac{1}{n} - 2x - t\right) - H_2\left(t - 2x - \frac{1}{n}\right)\right),$$

and thus f can be written as in (3.2.7) *in the obvious way.*

Even though the previous comments and examples suggest that Theorem 3.2.13 is more general than Theorem 3.1.6 when restricted to the scalar Cauchy problem (3.0.1), that is not true since Theorem 3.2.13 does not allow to consider the case of viable discontinuity curves.

To finish this section, we generalize Theorem 3.2.13 by assuming the existence of well–ordered lower and upper solutions for problem (3.2.1).

Definition 3.2.18. *A function $\alpha : I \to \mathbb{R}$, $\alpha \in \mathcal{AC}(I)$, is a lower solution for problem* (3.2.1) *if*

$$\alpha'(t) \leq f(t, \alpha(t)) \quad \text{for a.a. } t \in I, \quad \alpha(0) \leq x_0.$$

Similarly, a function $\beta \in \mathcal{AC}(I)$ is an upper solution for (3.2.1) *if it satisfies the inequalities in the reverse order.*

Theorem 3.2.19. *Suppose that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(H1) (Lower and upper solutions) There exist $\alpha, \beta \in \mathcal{AC}(I)$ lower and upper solutions for (3.2.1)*, respectively, such that $\alpha \leq \beta$ on I.*

Let us denote $\Delta = \{(t,x) \in I \times \mathbb{R} \,:\, \alpha(t) \leq x \leq \beta(t)\}$.

(H2) (L^1 bound) There exists $M \in L^1(I)$ such that for a.a. $t \in I$ and for all $x \in [\alpha(t), \beta(t)]$, we have $|f(t,x)| \leq M(t)$.

(H3) (Measurability) For all $x \in \mathbb{R}$, $f(\cdot, x)$ is measurable.

(H4) (Inviable discontinuity sets) The function $f : I \times \mathbb{R} \to \mathbb{R}$ can be expressed in the form

$$f(t,x) = F(t, g_1(\tau_1(t,x),x), g_2(\tau_2(t,x),x), \ldots),$$

where for each $i \in N$ (N is a nonempty countable set) we have that

(i) a) Each function $\tau_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable on Δ;
b) Each function $g_i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous in $(\mathbb{R} \setminus A_i) \times \mathbb{R}$, where A_i is a null–measure set;
c) For a.a. $t \in I$ and all $x \in [\alpha(t), \beta(t)]$, the condition $\tau_i(t,x) \in \mathbb{R} \setminus A_i$ for all $i \in N$ implies that $f(t,\cdot)$ is continuous at x.

(ii) For each $(t,x) \in \tau_i^{-1}(A_i) \cap D$, $x \in [\alpha(t), \beta(t)]$, condition (3.2.17) *holds.*

Then the set of Carathéodory solutions between α and β on I to the problem (3.2.1) *is a nonempty, compact and convex subset of $\mathcal{C}(I)$. Moreover, such a set has pointwise maximum, x^*, and minimum, x_*, which are extremal solutions to* (3.2.1) *between α and β, and satisfy*

$$x^*(t) = \max\{v(t) : v \in \mathcal{AC}(I),\ v'(s) \le f(s,v(s))\ \text{a.e.},\ v(0) \le x_0,\ \alpha \le v \le \beta \text{ on } I\},$$
$$x_*(t) = \min\{v(t) : v \in \mathcal{AC}(I),\ v'(s) \ge f(s,v(s))\ \text{a.e.},\ v(0) \ge x_0,\ \alpha \le v \le \beta \text{ on } I\}.$$

Proof. Consider the modified problem

$$x'(t) \in \mathcal{K}\tilde{f}(t,x(t)) \quad \text{for a.a. } t \in I, \quad x(0) = x_0, \tag{3.2.19}$$

where

$$\tilde{f}(t,x) = \begin{cases} \beta'(t) & \text{if } x > \beta(t), \\ f(t,x) & \text{if } \alpha(t) \le x \le \beta(t), \\ \alpha'(t) & \text{if } x < \alpha(t). \end{cases}$$

Let us divide the proof in several steps.

Step 1: Problem (3.2.19) *has extremal solutions.*

Clearly, for a.a. $t \in I$ and all $x \in \mathbb{R}$, we have $\left|\tilde{f}(t,x)\right| \le \tilde{M}(t)$ with $\tilde{M}(t) := \max\{|\alpha'(t)|, |\beta'(t)|, M(t)\}$. Then Proposition 3.2.12 implies that (3.2.19) has extremal solutions.

Step 2: All solutions of (3.2.19) *belong to* $[\alpha, \beta]$.

Let $u \in \mathcal{AC}(I)$ be such that $u'(t) \in \mathcal{K}\tilde{f}(t,u(t))$ for a.a. $t \in I$ and $u(0) = x_0$. Let us show that $\alpha(t) \le u(t) \le \beta(t)$ for all $t \in I$. Suppose that there exists $t \in I$ such that $u(t) > \beta(t)$. Since u and β are continuous functions and $u(0) = x_0 \le \beta(0)$, there exist $\bar{t} \in I$ and $\varepsilon > 0$ such that $u(\bar{t}) = \beta(\bar{t})$ and $u(t) > \beta(t)$ for all $t \in (\bar{t}, \bar{t}+\varepsilon) \subset I$. Hence, $\tilde{f}(t,u(t)) = \beta'(t)$ for all $t \in (\bar{t}, \bar{t}+\varepsilon)$ and by the definition of $\mathcal{K}\tilde{f}$ we deduce that $\mathcal{K}\tilde{f}(t,u(t)) = \{\beta'(t)\}$ for all $t \in (\bar{t}, \bar{t}+\varepsilon)$. Since u is a solution of the modified problem (3.2.19), we have that $u'(t) = \beta'(t)$ for a.a. $t \in (\bar{t}, \bar{t}+\varepsilon)$. Now, $u(\bar{t}) = \beta(\bar{t})$ implies that $u(t) = \beta(t)$ in $(\bar{t}, \bar{t}+\varepsilon)$, a contradiction. Similarly, $\alpha \le u$ for all $t \in I$.

Therefore, the set of solutions to (3.2.19) coincides with the set of Krasovskij solutions between α and β to the problem (3.2.1).

Step 3: If u is a Krasovskij solution between α and β to (3.2.1)*, then u is a Carathéodory solution to* (3.2.1).

It can be shown exactly as in the proof of Theorem 3.2.3. □

Remark 3.2.20. *Theorem 3.2.19 allows to obtain a new result for infinite quasimonotone systems of functional equations following the technique described in [42].*

3.3 Discontinuous first–order functional boundary value problems

Now we prove the existence of extremal absolutely continuous solutions to the functional problem

$$\begin{cases} x'(t) = f(t, x(t), x) & \text{for a.a. } t \in I = [0, L], \\ B(x(0), x) = 0, \end{cases} \tag{3.3.1}$$

where $L > 0$, and $f : I \times \mathbb{R} \times \mathcal{AC}(I) \to \mathbb{R}$ and $B : \mathbb{R} \times \mathcal{AC}(I) \to \mathbb{R}$ may be discontinuous with respect to all of their arguments.

The functional boundary condition in (3.3.1) includes initial and periodic conditions, but also some more sophisticated types as multipoint, integral and other conditions with, for example, maximum or minimum arguments.

We need the following version of Bolzano's theorem whose proof can be looked up in [71, Lemma 2.3].

Lemma 3.3.1. *Let $a, b \in \mathbb{R}$, $a \le b$, and let $h : \mathbb{R} \to \mathbb{R}$ be such that $h(a) \le 0 \le h(b)$ and*

$$\liminf_{z \to x^-} h(z) \ge h(x) \ge \limsup_{z \to x^+} h(z) \quad \text{for all } x \in [a, b].$$

Then there exist $c_1, c_2 \in [a, b]$ such that $h(c_1) = 0 = h(c_2)$ and if $h(c) = 0$ for some $c \in [a, b]$ then $c_1 \le c \le c_2$, i.e., c_1 and c_2 are, respectively, the least and the greatest of the zeros of h in $[a, b]$.

The following fixed point theorem is an immediate consequence of [78, Proposition 1.4.4].

Proposition 3.3.2. *Given a nonempty order interval $[\alpha, \beta]$ in $\mathcal{AC}(J)$, where J is a compact real interval, and a nondecreasing mapping $G : [\alpha, \beta] \to [\alpha, \beta]$, assume that there is $M \in L^1(J)$ such that*

$$|(Gx)'(t)| \le M(t) \quad \text{for all } x \in [\alpha, \beta] \text{ and for a.a. } t \in J.$$

Then G has in $[\alpha, \beta]$ a maximum, x^*, and the minimum, x_*, fixed points, and

$$x^* = \max\{x \in [\alpha, \beta] \, : \, x \le Gx\}, \quad x_* = \min\{x \in [\alpha, \beta] \, : \, Gx \le x\}.$$

We also need the notion of lower and upper solutions for (3.3.1), see [42, 59].

Definition 3.3.3. *A function $\alpha : I \to \mathbb{R}$, $\alpha \in \mathcal{AC}(I)$, is a lower solution for problem* (3.3.1) *if*

$$\alpha'(t) \le f(t, \alpha(t), \alpha) \quad \text{for a.a. } t \in I, \quad B(\alpha(0), \alpha) \le 0.$$

Similarly, a function $\beta \in \mathcal{AC}(I)$ is an upper solution for (3.3.1) *if it satisfies the inequalities in the reverse order.*

Next we proceed to establish the main existence result concerning problem (3.3.1).

Theorem 3.3.4. *Suppose that $f : I \times \mathbb{R} \times \mathcal{AC}(I) \to \mathbb{R}$ satisfies the following conditions:*

(C1) (Lower and upper solutions) There exist $\alpha, \beta \in \mathcal{AC}(I)$ lower and upper solutions for (3.3.1)*, respectively, such that $\alpha \le \beta$ on I.*

(C2) (L^1 bound) There exists $M \in L^1(I)$ such that for a.a. $t \in I$, all $x \in [\alpha(t), \beta(t)]$ and all $\gamma \in [\alpha, \beta] := \{x \in \mathcal{AC}(I) \,:\, \alpha \le x \le \beta \text{ on } I\}$, we have $|f(t, x, \gamma)| \le M(t)$.

(C3) (Measurability and inviable discontinuity sets) For each $\gamma \in [\alpha, \beta]$ the mapping

$$(t, x) \in I \times \mathbb{R} \mapsto f_\gamma(t, x) := f(t, x, \gamma)$$

satisfies the conditions (H3) and (H4) in Theorem 3.2.19.

(C4) (Functional dependence) For a.a. $t \in I$ and all $x \in [\alpha(t), \beta(t)]$, $f(t, x, \cdot)$ is nondecreasing in $[\alpha, \beta]$.

(C5) (Functional BCs) For all $\gamma \in [\alpha, \beta]$ and all $x \in \mathbb{R}$ we have

$$\liminf_{y \to x^-} B(y, \gamma) \ge B(x, \gamma) \ge \limsup_{y \to x^+} B(y, \gamma),$$

and $B(x, \cdot)$ is nonincreasing in $[\alpha, \beta]$.

Then problem (3.3.1) *has extremal solutions between α and β.*

Proof. Consider the mapping $G : [\alpha, \beta] \to [\alpha, \beta]$ defined as follows: for each $\gamma \in [\alpha, \beta]$, $G\gamma$ is the greatest solution between α and β to the Cauchy problem

$$x'(t) = f(t, x(t), \gamma) \quad \text{for a.a. } t \in I, \quad x(0) = x_\gamma, \tag{3.3.2}$$

where x_γ is the greatest solution in $[\alpha(0), \beta(0)]$ of the algebraic equation $B(x, \gamma) = 0$.

Note that, since $B(x, \cdot)$ is nonincreasing in $[\alpha, \beta]$, we have

$$B(\alpha(0), \gamma) \le B(\alpha(0), \alpha) \le 0 \le B(\beta(0), \beta) \le B(\beta(0), \gamma),$$

and thus, by assumption (C5) and Lemma 3.3.1, we obtain that x_γ is well–defined. The existence of the greatest solution between α and β to (3.3.2) is guaranteed by Theorem 3.2.19, so the mapping G is also well–defined.

Observe that if $x \in [\alpha, \beta]$ is a fixed point of G, then x is a solution to (3.3.1) between α and β.

Let us prove that G is nondecreasing in $[\alpha, \beta]$. Let $\gamma_1, \gamma_2 \in [\alpha, \beta]$ be such that $\gamma_1 \le \gamma_2$. By definition, x_{γ_i} is the greatest solution in $[\alpha(0), \beta(0)]$ of $B(x, \gamma_i) = 0$. Hence, since for all $x \in (x_{\gamma_2}, \beta(0)]$, we have $0 < B(x, \gamma_2) \le B(x, \gamma_1)$, then $x_{\gamma_1} \le x_{\gamma_2}$. Let us denote $y_1 = G\gamma_1$, which is a solution to

$$x'(t) = f(t, x(t), \gamma_1) \quad \text{for a.a. } t \in I, \quad x(0) = x_{\gamma_1},$$

so, by (C4),

$$y_1'(t) \le f(t, y_1(t), \gamma_2) \quad \text{for a.a. } t \in I, \quad y_1(0) \le x_{\gamma_2},$$

and thus y_1 is a lower solution for problem

$$x'(t) = f(t, x(t), \gamma_2) \quad \text{for a.a. } t \in I, \quad x(0) = x_{\gamma_2}. \tag{3.3.3}$$

Hence, Theorem 3.2.19 implies that problem (3.3.3) has at least one solution between y_1 and β. On the other hand, $y_2 = G\gamma_2$ is the greatest solution between α and β to (3.3.3). Therefore, $y_1 \le y_2$ and so G is nondecreasing in $[\alpha, \beta]$.

Moreover, by (C2), there exists $M \in L^1(I)$ such that for every $\gamma \in [\alpha, \beta]$ we have $|(G\gamma)'(t)| \le M(t)$ for a.a. $t \in I$. Hence, Proposition 3.3.2 guarantees that G has in $[\alpha, \beta]$ a maximum fixed point, x^*. Furthermore, it satisfies

$$x^* = \max\{x \in [\alpha, \beta] \,:\, x \le Gx\}. \tag{3.3.4}$$

Let us show that x^* is the greatest solution to (3.3.1) in $[\alpha, \beta]$. If $\bar{x} \in [\alpha, \beta]$ is another solution to (3.3.1), then we have

$$\bar{x}'(t) = f(t, \bar{x}(t), \bar{x}) \quad \text{for a.a. } t \in I, \quad B(\bar{x}(0), \bar{x}) = 0.$$

Then $B(\bar{x}(0), \bar{x}) = 0$ and $\bar{x}(0) \in [\alpha(0), \beta(0)]$ imply that $\bar{x}(0) \le x_{\bar{x}}$, so $\bar{x}$ is a lower solution for the problem

$$x'(t) = f(t, x(t), \bar{x}) \quad \text{for a.a. } t \in I, \quad x(0) = x_{\bar{x}}. \tag{3.3.5}$$

Since $G\bar{x}$ is the greatest solution in $[\alpha, \beta]$ to (3.3.5), then $\bar{x} \le G\bar{x}$, and by (3.3.4), $\bar{x} \le x^*$.

The existence of the least solution to (3.3.1) in $[\alpha, \beta]$ can be deduced by redefining the operator G in the obvious way. □

We illustrate the previous result with a couple of examples. The first one is devoted to a periodic problem and in the second one we deal with the existence of solutions to a generalized logistic equation.

Example 3.3.5. *Consider the periodic problem*

$$\begin{cases} x' = \psi(x + 5t) + x\cos(\pi x) & \textit{for a.a. } t \in [0, 1], \\ x(0) = x(1), \end{cases} \tag{3.3.6}$$

where ψ is defined as in Proposition 3.2.15.

First, note that the periodic problem can be seen as a particular case of (3.3.1) *by defining $B(x, \gamma) = x - \gamma(1)$. Moreover, the constant functions $\alpha_1 \equiv -3$, $\alpha_2 \equiv -1$, $\alpha_3 = 2$, $\beta_1 \equiv -2$, $\beta_2 \equiv 1$ and $\beta_2 \equiv 3$ are three pairs of well–ordered lower and upper solutions for* (3.3.6).

On the other hand, the function $f(t, x) = \psi(x + 5t) + x\cos(\pi x)$ can be written as the composition $f(t, x) = g(\tau(t, x), x)$ where $\tau(t, x) = x + 5t$ and $g(t, x) = \psi(t) + x\cos(\pi x)$. In order to prove that the transversality condition (3.2.17) *is satisfied for all $x \in [-3, -2] \cup [-1, 1] \cup [2, 3]$, observe that if $x \in [-3, 3]$, then $f(t, x) \in [-4, 4]$ and thus $z \in [-4, 4]$ for all $z \in \mathcal{K}f(t, x)$. Therefore, for a.a. $t \in [0, 1]$ and $x \in [-3, 3]$,*

$$\frac{\partial \tau}{\partial t}(t, x) + \frac{\partial \tau}{\partial x}(t, x) z = 5 + z \ge 1 \quad \textit{for all } z \in \mathcal{K}f(t, x).$$

In conclusion, problem (3.3.6) *has at least three solutions located in the intervals* $[-3,-2]$, $[-1,1]$ *and* $[2,3]$, *respectively.*

We highlight that this example is not covered by [59, Corollary 4.3] since ψ *has an uncountable number of discontinuity points.*

Example 3.3.6. *The logistic equation*

$$x'(t) = rx(t)(N - x(t)), \qquad x(0) = x_0,$$

where $r, N > 0$ *and* $x_0 \in (0, N)$*, is a well–known model of population growth. The constant* r *represents the growth rate and* N *is the carrying capacity.*

We propose a generalized logistic equation which includes a functional dependence (e.g. a delay or an integral term) to control not only the present state of the population, but also the previous ones. Also the carrying capacity N *is replaced by a function* $N : \mathbb{R} \to \mathbb{R}$ *which may be discontinuous. It symbolizes how the carrying capacity may be modified due to external factors depending on the number of individuals. For instance, we may think of a population of bacteria where we introduce (or remove) resources when the size of the population exceed some fixed value.*

We consider the following generalized logistic equation

$$x'(t) = r(t,x)(N(x(t)) - x(t)) \quad t \in I, \qquad x(0) = x_0, \tag{3.3.7}$$

where

- $N : \mathbb{R} \to \mathbb{R}$ *satisfies that*
  - *there exist* $N_1, N_2 > 0$ *such that* $N_1 \le N(x) \le N_2$ *for all* $x \in [N_1, N_2]$;
  - N *is continuous on* $[N_1, N_2] \setminus A$ *with* $m(A) = 0$*, and for each* $x \in A$ *either*

$$\min\{N(x), \liminf_{y \to x} N(y)\} > x \tag{3.3.8}$$

    or

$$\max\{N(x), \limsup_{y \to x} N(y)\} < x. \tag{3.3.9}$$

- $r : I \times \mathcal{AC}(I) \to \mathbb{R}$ *is a positive and* L^1 *bounded function on* $I \times [N_1, N_2]$ *such that for a.a.* $t \in I$*, we have that* $r(t, \cdot)$ *is nondecreasing; and*
- $x_0 \in [N_1, N_2]$.

Problem (3.3.7) *is a particular case of the functional problems of type* (3.3.1) *with*

$$f(t, x, \gamma) = r(t, \gamma)(N(x) - x).$$

Observe that the constant functions $\alpha = N_1$ *and* $\beta = N_2$ *are, respectively, lower and upper solutions for* (3.3.7)*. Moreover, we note that for each* $\gamma \in [N_1, N_2]$ *the mapping* $x \mapsto f_\gamma(t, x)$ *can be expressed in the form* (3.2.7) *as*

$$f_\gamma(t, x) = F(t, g(\tau(t, x), x)),$$

where $F(t,y) = r(t,\gamma)y$, $g(t,x) = N(t) - x$ *and* $\tau(t,x) = x$. *Then the transversality condition* (3.2.17) *is satisfied if for each* $t \in I$ *and* $x \in A$,

$$z \neq 0 \quad \textit{for all } z \in \mathcal{K}f_\gamma(t,x),$$

what is an immediate consequence of assumptions (3.3.8) *and* (3.3.9) *together with* (3.2.6). *Therefore, Theorem 3.3.4 implies that problem* (3.3.7) *has extremal solutions between* α *and* β.

We remark that, when modeling population growth, condition (3.3.8) *may just mean that we give additional resources to the population (for instance, food) before the limiting capacity for* x *individuals is attained.*

In particular, we can take $N(x) = 1 + \lfloor 1/(2x) \rfloor x$ *if* $x \neq 0$ *and* $N(0) = 3/2$, *and* $r(t,x) = c\int_0^t x(s)\,ds$, *where* $c > 0$ *and* $\lfloor y \rfloor$ *denotes the integer part of* y. *Then the assumptions on* N *hold for* $N_1 \in (0,1]$, $N_2 = 3/2$ *and* $A = \{1/2n\}_{n\in\mathbb{N}}$ *since* $N(x) \geq 1$ *on* $[0,3/2]$ *implies that condition* (3.3.8) *is satisfied for all* $x \in A$.

3.3.1 An application to second–order problems with functional boundary conditions

As an application, we consider the existence of extremal solutions to the second–order problem

$$\begin{cases} x''(t) = f(t,x(t),x'(t)) & \text{for a.a. } t \in I = [0,L], \\ x(0) = A(x), \quad B(x'(0),x') = 0, \end{cases} \tag{3.3.10}$$

where $L > 0$, $f : I \times \mathbb{R}^2 \to \mathbb{R}$ and $B : \mathbb{R} \times \mathcal{AC}(I) \to \mathbb{R}$ may be discontinuous with respect to all of their arguments and $A : \mathcal{AC}(I) \to \mathbb{R}$ is linear and nondecreasing.

Note that the function A is suitable to cover some usual conditions, such as initial, multipoint or integral conditions.

Problem (3.3.10) will be reduced to a functional problem of type (3.3.1) by order reduction, following similar reasonings to those from [32, 106].

First, we recall a result for the following first–order linear problem

$$\begin{cases} x'(t) = q(t) & \text{for a.a. } t \in I, \\ x(0) = A(x), \end{cases} \tag{3.3.11}$$

where $q \in L^1(I)$ and $A : \mathcal{AC}(I) \to \mathbb{R}$ is linear.

Lemma 3.3.7 ([71, Proposition 3.1]). *Let* $q \in L^1(I)$ *and a linear mapping* $A : \mathcal{AC}(I) \to \mathbb{R}$ *be fixed.*

(a) *If* $A(1) = 1$, *then* (3.3.11) *is solvable if and only if*

$$A\left(\int_0^{\cdot} q(s)\,ds\right) = 0,$$

and, in such a case, the problem has infinitely many solutions which are given by

$$x(t) = C + \int_0^t q(s)\,ds, \qquad C \in \mathbb{R}.$$

(b) *If $A(1) \neq 1$, then* (3.3.11) *has a unique solution given by*

$$x(t) = \frac{A\left(\int_0^{\cdot} q(s)\,ds\right)}{1 - A(1)} + \int_0^t q(s)\,ds.$$

Moreover, if $q(t) \geq 0$ for a.a. $t \in I$ and A is a nondecreasing linear mapping such that $A(1) < 1$, then the unique solution to (3.3.11) *is nonnegative.*

Note that if x is a solution to (3.3.10) and $A(1) < 1$, then $y = x'$ solves the first–order functional boundary value problem

$$\begin{cases} y'(t) = f\,(t, \mathcal{F}(y)(t), y(t)) & \text{for a.a. } t \in I, \\ B(y(0), y) = 0, \end{cases} \tag{3.3.12}$$

where $\mathcal{F}(y)$ is the unique solution of the linear problem

$$\begin{cases} x'(t) = y(t) & \text{for a.a. } t \in I, \\ x(0) = A(x). \end{cases}$$

Observe that $\mathcal{F}(y)$ is well–defined and the operator $\mathcal{F} : \mathcal{AC}(I) \to \mathcal{AC}(I)$ is nondecreasing. Indeed, if $y_1 \leq y_2$, then $x = \mathcal{F}(y_2) - \mathcal{F}(y_1)$ is a solution of the problem

$$\begin{cases} x'(t) = (y_2 - y_1)(t) & \text{for a.a. } t \in I, \\ x(0) = A(x), \end{cases}$$

and, by Lemma 3.3.7 (b), we have that $x = \mathcal{F}(y_2) - \mathcal{F}(y_1) \geq 0$, so $\mathcal{F}(y_1) \leq \mathcal{F}(y_2)$.

In addition, problem (3.3.12) is a particular case of the functional problems of type (3.3.1). Hence, we shall apply Theorem 3.3.4 to it in order to obtain a new existence result for (3.3.10), which allows the function f to be discontinuous with respect to all its variables.

Theorem 3.3.8. *Suppose that $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions:*

$(H1^*)$ *(Lower and upper solutions) There exist $\alpha, \beta \in W^{2,1}(I)$ such that $\alpha' \leq \beta'$ on I and*

$$\begin{array}{l} \alpha''(t) \leq f(t, \alpha(t), \alpha'(t)) \quad \textit{for a.a. } t \in I, \\ B(\alpha'(0), \alpha') \leq 0, \quad \alpha(0) \leq A(\alpha), \end{array}$$

$$\begin{array}{l} \beta''(t) \geq f(t, \beta(t), \beta'(t)) \quad \textit{for a.a. } t \in I, \\ B(\beta'(0), \beta') \geq 0, \quad \beta(0) \geq A(\beta). \end{array}$$

$(H2^*)$ *(L^1 bound) There exists $M \in L^1(I)$ such that for a.a. $t \in I$, for all $x \in [\alpha(t) - \alpha(0), \beta(t) - \beta(0)]$ and all $y \in [\alpha'(t), \beta'(t)]$, we have $|f(t, x, y)| \leq M(t)$.*

$(H3^*)$ *(Inviable discontinuity sets) For each $x \in [\min_{t \in I} \alpha(t) - \alpha(0), \max_{t \in I} \beta(t) - \beta(0)]$ the mapping*

$$(t, y) \in I \times \mathbb{R} \to f_x(t, y) := f(t, x, y)$$

satisfies the conditions (H3) and (H4) in Theorem 3.2.19.

$(H4^*)$ *(Monotonicity) For a.a. $t \in I$ and all $y \in [\alpha'(t), \beta'(t)]$, $f(t, \cdot, y)$ is nondecreasing in $[\alpha(t) - \alpha(0), \beta(t) - \beta(0)]$.*

$(H5^*)$ *(Functional BCs) For all $\gamma \in [\alpha', \beta']$ and all $x \in \mathbb{R}$ we have*

$$\liminf_{y \to x^-} B(y, \gamma) \geq B(x, \gamma) \geq \limsup_{y \to x^+} B(y, \gamma),$$

and $B(x, \cdot)$ is nonincreasing in $[\alpha', \beta']$.

$(H6^*)$ *(Initial BCs) The mapping $A : \mathcal{AC}(I) \to \mathbb{R}$ is linear nondecreasing and such that $A(1) < 1$.*

Then the problem (3.3.10) *has extremal solutions between $\alpha - \alpha(0)$ and $\beta - \beta(0)$.*

Proof. As mentioned above it is possible to reduce the second–order problem (3.3.10) to the first–order functional problem (3.3.12).

Let us show that $\tilde{\alpha} = \alpha'$ and $\tilde{\beta} = \beta'$ are lower and upper solutions for (3.3.12), respectively, where α and β are given by condition $(H1^*)$. To prove that $\tilde{\alpha}$ is a lower solution for (3.3.12) we have to verify that

$$\tilde{\alpha}'(t) \leq f\left(t, \mathcal{F}(\tilde{\alpha})(t), \tilde{\alpha}(t)\right) \quad \text{for a.a. } t \in I, \quad B(\tilde{\alpha}(0), \tilde{\alpha}) \leq 0.$$

Since $\tilde{\alpha} = \alpha'$ and $\alpha''(t) \leq f(t, \alpha(t), \alpha'(t))$ for a.a. $t \in I$, we have

$$\tilde{\alpha}'(t) \leq f\left(t, \alpha(0) + \int_0^t \tilde{\alpha}(s)\, ds, \tilde{\alpha}(t)\right) \quad \text{for a.a. } t \in I.$$

By $\alpha(0) \leq A(\alpha)$, we obtain that $\alpha(0) + \int_0^t \tilde{\alpha}(s)\, ds \leq \mathcal{F}(\tilde{\alpha})(t)$ for a.a. $t \in I$ (recall that, by Lemma 3.3.7, $\mathcal{F}(\tilde{\alpha})(t) = A(t \mapsto \int_0^t \tilde{\alpha}(s)\, ds)/(1 - A(1)) + \int_0^t \tilde{\alpha}(s)\, ds$) and thus condition $(H4^*)$ implies that

$$\tilde{\alpha}'(t) \leq f\left(t, \mathcal{F}(\tilde{\alpha})(t), \tilde{\alpha}(t)\right) \quad \text{for a.a. } t \in I.$$

Similarly $\tilde{\beta}$ is an upper solution for (3.3.12).

Therefore, the conclusion is obtained from applying Theorem 3.3.4 to problem (3.3.12). □

Remark 3.3.9. *Observe that the definition of the lower and upper solutions for the second–order problem given in condition $(H1^*)$ is not the standard one in the literature, cf. [27, 46].*

As an example, we study the existence of solutions to the following multipoint second–order problem

$$\begin{cases} x'' = \mu x + h(x') + p(t)\operatorname{sgn}(x') + c(t) & \text{for a.a. } t \in I = [0,1], \\ x'(0) = x'(1), \\ x(0) = \sum_{k=1}^{n} c_k\, x(\eta_k), \end{cases} \tag{3.3.13}$$

where $n \in \mathbb{N}$, $c_k \geq 0$, $\sum_{k=1}^n c_k < 1$, $\eta_k \in I$ and $\mu \geq 0$. Moreover, $c, p \in \mathcal{C}(I)$, $h : \mathbb{R} \to \mathbb{R}$ is a continuous function and sgn denotes the sign function which is defined as

$$\operatorname{sgn}(y) = \begin{cases} 1, & \text{if } y > 0, \\ 0, & \text{if } y = 0, \\ -1, & \text{if } y < 0. \end{cases}$$

Observe that the discontinuities on the right-hand side of (3.3.13) are precisely given by this sign function. This kind of discontinuous perturbation appears in some physical models such as those treating with a mechanical system subjected to dry friction, see e.g. [24, 108].

Proposition 3.3.10. *Assume that c, p, h are continuous functions and that there exist constants $r, R \in \mathbb{R}$, $r \leq R$, such that for a.a. $t \in I$, we have*

$$0 \leq h(r) + p(t)\operatorname{sgn}(r) + c(t) + \mu r \left(t + \frac{\sum_{k=1}^n c_k \eta_k}{1 - \sum_{k=1}^n c_k} \right), \tag{3.3.14}$$

$$0 \geq h(R) + p(t)\operatorname{sgn}(R) + c(t) + \mu R \left(t + \frac{\sum_{k=1}^n c_k \eta_k}{1 - \sum_{k=1}^n c_k} \right). \tag{3.3.15}$$

Then $\alpha(t) = r\left(t + \sum c_k \eta_k / (1 - \sum c_k)\right)$ and $\beta(t) = R\left(t + \sum c_k \eta_k / (1 - \sum c_k)\right)$ are, respectively, a lower and an upper solutions for problem (3.3.13).

Moreover, if

$$h(0) + \mu \min\{0, r\} + \inf_{t \in I}\{c(t) - |p(t)|\} > 0 \tag{3.3.16}$$

or

$$h(0) + \mu \max\{0, R\} + \sup_{t \in I}\{c(t) + |p(t)|\} < 0, \tag{3.3.17}$$

then problem (3.3.13) *has extremal solutions in $[\alpha^*, \beta^*]$ with $\alpha^*(t) = rt$ and $\beta^*(t) = Rt$.*

Proof. Note that problem (3.3.13) is a particular case of (3.3.10) with f defined as

$$f(t, x, y) = \mu x + h(y) + p(t)\operatorname{sgn}(y) + c(t),$$

$A(x) = \sum_{k=1}^n c_k x(\eta_k)$ and $B(x, \gamma) = x - \gamma(1)$. Hence, the first statement can be directly verified by applying the definition of the lower and upper solutions to α and β, respectively.

Indeed, α satisfies that

$$\begin{aligned} \alpha''(t) = 0 &\leq h(r) + p(t)\operatorname{sgn}(r) + c(t) + \mu r \left(t + \frac{\sum_{k=1}^n c_k \eta_k}{1 - \sum_{k=1}^n c_k} \right) \\ &= h(\alpha'(t)) + p(t)\operatorname{sgn}(\alpha'(t)) + c(t) + \mu \alpha(t) \\ &= f(t, \alpha(t), \alpha'(t)). \end{aligned}$$

Moreover, we have

$$B(\alpha'(0), \alpha') = \alpha'(0) - \alpha'(1) = r - r = 0$$

and

$$A(\alpha)=\sum_{k=1}^{n}c_k\alpha(\eta_k)=r\sum_{k=1}^{n}c_k\left(\eta_k+\frac{\sum_{j=1}^{n}c_j\eta_j}{1-\sum_{j=1}^{n}c_j}\right)$$
$$=r\frac{\sum_{k=1}^{n}c_k\eta_k-\sum_{k=1}^{n}c_k\eta_k\sum_{j=1}^{n}c_j+\sum_{k=1}^{n}c_k\sum_{j=1}^{n}c_j\eta_j}{1-\sum_{j=1}^{n}c_j}$$
$$=r\frac{\sum_{k=1}^{n}c_k\eta_k}{1-\sum_{j=1}^{n}c_j}=\alpha(0).$$

In a similar way, one may check that β is an upper solution for problem (3.3.13).

As conditions $(H2^*)$, $(H4^*)$, $(H5^*)$ and $(H6^*)$ are trivially satisfied, it only remains to check condition $(H3^*)$. For each $x\in\mathbb{R}$, the function $(t,y)\mapsto f_x(t,y)$ can be expressed in the form (3.2.7) with $g_1(t,y)=\operatorname{sgn}(t)$, $\tau_1(t,y)=y$, $g_2(t,y)=h(y)$ and $F(t,x_1,x_2)=\mu x+x_2+p(t)x_1+c(t)$. Since

$$\frac{\partial\tau_1}{\partial t}(t,y)=0\quad\text{and}\quad\frac{\partial\tau_1}{\partial y}(t,y)=1,$$

the transversality condition (3.2.17) holds if for each $t\in[0,1]$ and $x\in[\min\{r,0\},\max\{0,R\}]$ we have that

$$z\neq 0\qquad\text{for all } z\in\mathcal{K}f_x(t,0).$$

Notice that

$$\mathcal{K}f_x(t,0)=\mu x+h(0)+c(t)+[-|p(t)|\,,|p(t)|],$$

by (3.2.6). Therefore, condition $(H3^*)$ is fulfilled if one of the conditions (3.3.16) or (3.3.17) holds. □

Remark 3.3.11 (Asymptotic conditions)**.** *A sufficient condition for the existence of a constant r satisfying* (3.3.14) *is that:*

$$\liminf_{y\to-\infty}\frac{h(y)}{y}<-\mu\left(L+\frac{\sum_{k=1}^{n}c_k\eta_k}{1-\sum_{j=1}^{n}c_j}\right).\tag{3.3.18}$$

Similarly, if

$$\liminf_{y\to+\infty}\frac{h(y)}{y}<-\mu\left(L+\frac{\sum_{k=1}^{n}c_k\eta_k}{1-\sum_{j=1}^{n}c_j}\right)\tag{3.3.19}$$

holds, then condition (3.3.15) *is satisfied for $R>0$ large enough.*

Therefore, if the function h satisfies the asymptotic conditions (3.3.18) *and* (3.3.19)*, then there exist $r,R\in\mathbb{R}$, $r<0<R$, such that the functions α and β, defined as in Proposition 3.3.10, are a lower and an upper solutions for* (3.3.13)*, respectively.*

Chapter 4
Second order problems and lower and upper solutions

The method of lower and upper solutions was extensively employed to deal with second-order boundary value problems under Carathéodory assumptions. The aim of this technique, as shown in the previous chapter, is to replace the difficult problem of finding a solution by that of encountering lower and upper solutions for the problem. In addition, in the classical case of well–ordered lower and upper solutions, it provides not only existence results, but also localization for the solutions. Readers interested in the method of lower and upper solutions are referred to the monograph by Coster and Habets [46] and the expository paper by Cabada [27].

Our main goal in this chapter is to use this method to study the existence and multiplicity of solutions for the differential equation

$$x''(t) = f(t, x(t), x'(t)), \quad t \in I = [a, b], \tag{4.0.1}$$

where the nonlinear term f may be discontinuous in all the arguments. More specifically, we shall prove existence of extremal solutions and multiplicity results to (4.0.1) coupled with nonlinear functional boundary conditions of the form

$$\begin{aligned} 0 &= L_1(x(a), x(b), x'(a), x'(b), x), \\ 0 &= L_2(x(a), x(b)), \end{aligned} \tag{4.0.2}$$

where $L_1 : \mathbb{R}^4 \times \mathcal{C}(I) \to \mathbb{R}$ is continuous and it is nonincreasing with respect to its third and fifth variables, and nondecreasing with respect to the fourth one; and $L_2 : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and it is nondecreasing with respect to its first argument.

In particular, the nonlinear boundary conditions (4.0.2) contain Dirichlet boundary conditions

$$x(a) = x(b) = 0,$$

and periodic conditions

$$x(a) = x(b), \quad x'(a) = x'(b).$$

Since f may be discontinuous in all the arguments, we will employ the degree theory and the Schauder type fixed point theorems developed in Chapters 1 and 2, respectively, combined with the lower and upper solution method.

As a starting point for our work we consider the papers by Pouso [101] and Figueroa and Infante [58], where they study sufficient conditions for the existence of solutions for the differential equation

$$x''(t) = f(t, x(t)), \quad t \in I = [a, b],$$

coupled with Dirichlet and Sturm–Liouville boundary conditions. Here we will generalize their results in several directions: we consider derivative dependence, the boundary conditions are more general and we combine their technique with lower and upper solutions.

This chapter is organized as follows: in Section 4.1 we prove existence of Carathéodory solutions to (4.0.1)–(4.0.2) between a pair of well–ordered lower and upper solutions and, moreover, we give additional conditions in order to guarantee the existence of extremal solutions; in Section 4.2, inspired by the techniques used in [124–127], we get existence results when the lower and upper solutions are not well–ordered; in Section 4.3 we present multiplicity results in presence of more than a pair of lower and upper solutions; and, finally, in Section 4.4, we study a second–order problem on unbounded domains.

The main results in this chapter can be seen in the papers [62, 63, 65, 104].

4.1 Existence results via well–ordered lower and upper solutions

In the sequel we will work in the Banach space $X = \mathcal{C}^1(I)$ endowed with its usual norm

$$\|x\|_{C^1} = \|x\|_\infty + \|x'\|_\infty = \max_{t\in I} |x(t)| + \max_{t\in I} |x'(t)|.$$

Following [46] and the review article [27] we will use lower and upper solutions in order to obtain an existence result for problem (4.0.1)–(4.0.2). In the proof of the main result we consider a modified problem in the line of [34]. The notion of lower and upper solutions that we use is based on that introduced by De Coster and Habets in [46].

Definition 4.1.1. *We say that $\alpha \in \mathcal{C}(I)$ is a lower solution for the differential problem* (4.0.1)–(4.0.2) *if it satisfies the following conditions:*

(i) *For any $t_0 \in (a,b)$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$, where D_- and D^+ refers, respectively, to the lower–left and upper–right Dini derivatives,*
or there exists an open interval I_0 such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \textit{for a.a. } t \in I_0;$$

(ii) $D^+\alpha(a),\ D_-\alpha(b) \in \mathbb{R}$ *and* $L_1(\alpha(a), \alpha(b), D^+\alpha(a), D_-\alpha(b), \alpha) \leq 0$*;*

(iii) $L_2(\alpha(a), \alpha(b)) = 0$ *and* $L_2(\alpha(a), \cdot)$ *is injective.*

Similarly $\beta \in \mathcal{C}(I)$ is an upper solution for (4.0.1)–(4.0.2) *if it satisfies:*

(i) *For any $t_0 \in (a,b)$, either $D^-\beta(t_0) > D_+\beta(t_0)$, where D^- and D_+ refers, respectively, to the upper–left and lower–right Dini derivatives,*
or there exists an open interval I_0 such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and

$$\beta''(t) \leq f(t, \beta(t), \beta'(t)) \quad \textit{for a.a. } t \in I_0;$$

(ii) $D_+\beta(a),\ D^-\alpha(b) \in \mathbb{R}$ *and* $L_1(\beta(a), \beta(b), D_+\beta(a), D^-\beta(b), \beta) \geq 0$*;*

(iii) $L_2(\beta(a), \beta(b)) = 0$ *and* $L_2(\beta(a), \cdot)$ *is injective.*

If α and β are, respectively, a lower and an upper solution for problem (4.0.1)–(4.0.2), we say that they are well–ordered if $\alpha(t) \le \beta(t)$ for all $t \in I$.

When working with differential equations with derivative dependence it is usually needed to obtain *a priori* bounds on the derivative of the solutions. The usual way to do so is to impose the so–called Nagumo condition, see [46, 112], which in our case reads as follows.

Proposition 4.1.2. *Let* $\bar\alpha, \bar\beta \in \mathcal{C}(I)$ *be such that* $\bar\alpha \le \bar\beta$ *and define*

$$r = \max\left\{\bar\beta(b) - \bar\alpha(a), \bar\beta(a) - \bar\alpha(b)\right\}/(b-a).$$

Assume there exist a continuous function $\bar N : [0,\infty) \to (0,\infty)$*,* $\bar M \in L^1(I)$ *and* $R > r$ *such that*

$$\int_r^R \frac{1}{\bar N(s)} ds > \left\|\bar M\right\|_{L^1}.$$

Define $E := \left\{(t,x,y) \in I \times \mathbb{R}^2 : \bar\alpha(t) \le x \le \bar\beta(t)\right\}$*. If* $f : E \to \mathbb{R}$ *is a function satisfying*

$$|f(t,x,y)| \le \bar M(t)\bar N(|y|)$$

for a.a. $t \in I$ *and all* $(x,y) \in \mathbb{R}^2$ *with* $(t,x,y) \in E$*, then for every solution* x *of* (4.0.1) *such that* $\bar\alpha \le x \le \bar\beta$*, we have*

$$\|x'\|_\infty < R.$$

Proof. Let x be a solution of (4.0.1) and assume that $t \in I$ is such that $x'(t) > R$. Notice that

$$-r \le \frac{\bar\alpha(b) - \bar\beta(a)}{b-a} \le \frac{x(b) - x(a)}{b-a} \le \frac{\bar\beta(b) - \bar\alpha(a)}{b-a} \le r,$$

and then by virtue of Mean Value Theorem there exists $\tau \in I$ such that

$$|x'(\tau)| = \left|\frac{x(b)-x(a)}{b-a}\right| \le r.$$

Thus we can choose $t_0 < t_1$ (or $t_1 < t_0$) such that $x'(t_0) = r$, $x'(t_1) = R$ and $r \le x'(s) \le R$ in $[t_0, t_1]$ (or $[t_1, t_0]$).

Therefore we have

$$\begin{aligned}\int_r^R \frac{1}{\bar N(s)} ds &= \int_{t_0}^{t_1} \frac{x''(s)}{\bar N(x'(s))} ds = \int_{t_0}^{t_1} \frac{f(s,x(s),x'(s))}{\bar N(x'(s))} ds \\ &\le \left|\int_{t_0}^{t_1} \bar M(s)\, ds\right| \le \left\|\bar M\right\|_{L^1},\end{aligned}$$

a contradiction, so we deduce that $x'(t) < R$ for all $t \in I$. In a similar way we prove that $x'(t) > -R$ for all $t \in I$. □

Now we will introduce our result on the existence of $W^{2,1}$–solutions for problem (4.0.1)–(4.0.2). There, we will allow the nonlinearity to be discontinuous in the second argument over countably many curves satisfying again a type of 'transversality' condition, whose geometrical idea recalls that of the discontinuity surfaces described in [68], and their definition is similar to Definition 3.1.1 in the context of first–order problems.

Definition 4.1.3. *An admissible discontinuity curve for the differential equation* (4.0.1) *is a* $W^{2,1}$ *function* $\gamma : [c, d] \subset I \longrightarrow \mathbb{R}$ *satisfying one of the following conditions:*

either $\gamma''(t) = f(t, \gamma(t), \gamma'(t))$ *for a.a.* $t \in [c, d]$ *(and we then say that* γ *is viable for the differential equation),*

or there exist $\varepsilon > 0$ *and* $\psi \in L^1(c, d)$*,* $\psi(t) > 0$ *for a.a.* $t \in [c, d]$*, such that*
either

$$\gamma''(t) + \psi(t) < f(t, y, z) \quad \text{for a.a. } t \in [c, d], \text{ all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon] \text{ and all } z \in [\gamma'(t) - \varepsilon, \gamma'(t) + \varepsilon], \tag{4.1.1}$$

or

$$\gamma''(t) - \psi(t) > f(t, y, z) \quad \text{for a.a. } t \in [c, d], \text{ all } y \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon] \text{ and all } z \in [\gamma'(t) - \varepsilon, \gamma'(t) + \varepsilon]. \tag{4.1.2}$$

We say that the admissible discontinuity curve γ *is inviable for the differential equation if it satisfies* (4.1.1) *or* (4.1.2)*.*

Moreover, we shall allow f to be discontinuous in the third argument over some curves satisfying the conditions of the following definition, slightly different from the previous one. As far as we are aware, this is the first time that such discontinuity sets are considered.

Definition 4.1.4. *Given* α *and* β *lower and upper solutions for problem* (4.0.1)–(4.0.2) *such that* $\alpha \le \beta$ *on* I*, an inviable discontinuity curve for the derivative is an absolutely continuous function* $\Gamma : [c, d] \subset I \longrightarrow \mathbb{R}$ *satisfying that there exist* $\varepsilon > 0$ *and* $\psi \in L^1(c, d)$*,* $\psi(t) > 0$ *for a.a.* $t \in [c, d]$*, such that*
either

$$\Gamma'(t) + \psi(t) < f(t, y, z) \quad \text{for a.a. } t \in [c, d], \text{ all } y \in [\alpha(t), \beta(t)] \text{ and all } z \in [\Gamma(t) - \varepsilon, \Gamma(t) + \varepsilon] \cup \{\alpha'(t), \beta'(t)\}, \tag{4.1.3}$$

or

$$\Gamma'(t) - \psi(t) > f(t, y, z) \quad \text{for a.a. } t \in [c, d], \text{ all } y \in [\alpha(t), \beta(t)] \text{ and all } z \in [\Gamma(t) - \varepsilon, \Gamma(t) + \varepsilon] \cup \{\alpha'(t), \beta'(t)\}. \tag{4.1.4}$$

Observe that the conditions in Definition 4.1.3 are local (they only depend on the behavior of f in a neighborhood of γ). However, in the case of an inviable discontinuity curve for the derivative it is necessary to verify the inequalities not only in a neighborhood of the curve, but in the whole interval between the lower and upper solutions.

Now we present the main existence result in this section.

Theorem 4.1.5. *Suppose that there exist* $\alpha, \beta \in W^{1,\infty}(I)$ *lower and upper solutions to* (4.0.1)–(4.0.2)*, respectively, such that* $\alpha \le \beta$ *on* I*. Let*

$$r = \max\{\beta(b) - \alpha(a), \beta(a) - \alpha(b)\}/(b - a).$$

Assume that for $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *the following conditions hold:*

$(C1)$ *Compositions* $t \in I \mapsto f(t, x(t), y(t))$ *are measurable whenever* $x(t)$ *is continuous and* $y(t)$ *is measurable;*

$(C2)$ *There exist a continuous function* $N : [0, \infty) \to (0, \infty)$ *and* $M \in L^1(I)$ *such that:*

(a) *For a.a.* $t \in I$*, all* $x \in [\alpha(t), \beta(t)]$ *and all* $y \in \mathbb{R}$*, we have*

$$|f(t, x, y)| \le M(t)N(|y|);$$

(b) *There exists* $R > r$ *such that*

$$\int_r^R \frac{1}{N(s)} ds > \|M\|_{L^1} .$$

$(C3)$ *For a.a.* $t \in I$*, the mapping* $(x, y) \mapsto f(t, x, y)$ *is continuous on*

$$\left([\alpha(t), \beta(t)] \setminus \bigcup_{\{n \,:\, t \in I_n\}} \{\gamma_n(t)\} \right) \times \left([-R, R] \setminus \bigcup_{\{n \,:\, t \in \tilde{I}_n\}} \{\Gamma_n(t)\} \right),$$

where $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ *with* $\alpha \le \gamma_n \le \beta$ *on* I_n*,* $n \in \mathbb{N}$*, are admissible discontinuity curves with uniformly bounded derivatives and the functions* $\Gamma_n : \tilde{I}_n = [c_n, d_n] \longrightarrow \mathbb{R}$*,* $n \in \mathbb{N}$*, are inviable discontinuity curves for the derivative which are uniformly bounded.*

Then problem (4.0.1)–(4.0.2) *has at least a solution* $x \in W^{2,1}(I)$ *between* α *and* β *such that* $\|x'\|_\infty < R$*.*

Proof. Without loss of generality, suppose that

$$R > \max_{t \in I} \{|\alpha'(t)|, |\beta'(t)|, |\gamma_n'(t)|, |\Gamma_n(t)|\}$$

for all $n \in \mathbb{N}$ and define an integrable function

$$\tilde{M}(t) := \max_{s \in [0,R]} \{N(s)\} M(t).$$

Let us also define $\delta_R(z) = \max\{\min\{z, R\}, -R\}$ for all $z \in \mathbb{R}$ and

$$f^*(t, x, y) = f(t, x, \delta_R(y)) \quad \text{for all } (t, x, y) \in I \times \mathbb{R}^2. \tag{4.1.5}$$

Consider the modified problem

$$\left\{ \begin{array}{rcl} x''(t) & = & f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))') \quad \text{for a.a. } t \in I, \\ x(a) & = & L_1^*(x(a), x(b), x'(a), x'(b), x), \\ x(b) & = & L_2^*(x(a), x(b)), \end{array} \right. \tag{4.1.6}$$

where

$$\varphi(t, x) = \max\{\min\{x, \beta(t)\}, \alpha(t)\} \quad \text{for } (t, x) \in I \times \mathbb{R}, \tag{4.1.7}$$

and $L_1^*(x, y, z, w, \xi) = \varphi(a, x - L_1(x, y, z, w, \xi))$ for all $(x, y, z, w, \xi) \in \mathbb{R}^4 \times \mathcal{C}(I)$ and $L_2^*(x, y) = \varphi(b, y + L_2(x, y))$ for all $(x, y) \in \mathbb{R}^2$.

We know from [138, Lemma 2] that if $v,\ v_n \in \mathcal{C}^1(I)$ are such that $v_n \to v$ in $\mathcal{C}^1(I)$, then

1. $(\varphi(t,v(t)))'$ exists for a.a. $t \in I$;
2. $(\varphi(t,v_n(t)))' \to (\varphi(t,v(t)))'$ for a.a. $t \in I$.

Now we consider the compact and convex subset of $X = \mathcal{C}^1(I)$,

$$K = \left\{ x \in X : \begin{array}{c} \alpha(a) \le x(a) \le \beta(a),\ \alpha(b) \le x(b) \le \beta(b), \\ |x'(t) - x'(s)| \le \displaystyle\int_s^t \tilde{M}(r)\,dr\ (a \le s \le t \le b) \end{array} \right\} \tag{4.1.8}$$

and for each $x \in K$ we define

$$Tx(t) = L_1^*(x) + \frac{t-a}{b-a}\left(L_2^*(x) - L_1^*(x) - \int_a^b \int_a^s f^*(r,\varphi(r,x(r)),(\varphi(r,x(r)))')\,dr\,ds \right)$$
$$+ \int_a^t \int_a^s f^*(r,\varphi(r,x(r)),(\varphi(r,x(r)))')\,dr\,ds, \tag{4.1.9}$$

where, for simplicity, we use the following notation: $L_1^*(x) = L_1^*\,(x(a),x(b),x'(a),x'(b),x)$ and $L_2^*(x) = L_2^*\,(x(a),x(b))$. Observe that $y = Tx$ is just the solution of

$$\left\{ \begin{array}{rcl} y''(t) &=& f^*(t,\varphi(t,x(t)),(\varphi(t,x(t)))') \quad \text{for a.a. } t \in I, \\ y(a) &=& L_1^*(x), \quad y(b) = L_2^*(x), \end{array} \right. \tag{4.1.10}$$

and so fixed points of T, if they exist, correspond with solutions of problem (4.1.6).

Conditions $(C1)$ and $(C2)$ guarantee that the operator T is well defined. Moreover, T maps K into itself. Indeed, for any $x \in K$ and $y = Tx$ we have, thanks to $(C2)$ (a), that

$$|y''(t)| = |f^*(t,\varphi(t,x(t)),(\varphi(t,x(t)))')| \le M(t)N(|\delta_R((\varphi(t,x(t)))')|) \le \tilde{M}(t),$$

which, along with $y(a) = L_1^*(x)$ and $y(b) = L_2^*(x)$, imply that $y \in K$.

Next we prove that the operator T satisfies condition (2.1.1) for all $x \in K$ and then Theorem 2.1.3 ensures the existence of a fixed point or, equivalently, a solution to the modified problem (4.1.6). This part of the proof follows the steps of that in [101, Theorem 4.4] or Theorem 3.1.6 in Chapter 3, but here some changes are necessary due to the use of lower and upper solutions and the derivative dependence in the differential equation.

We fix an arbitrary function $x \in K$ and we consider four different cases.

Case 1: $m(\{t \in I_n : x(t) = \gamma_n(t)\} \cup \{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}) = 0$ *for all* $n \in \mathbb{N}$. Let us prove that then T is continuous at x.

The assumption implies that for a.a. $t \in I$ the mapping $f(t,\cdot,\cdot)$ is continuous at the point $(\varphi(t,x(t)),(\varphi(t,x(t))'))$. Hence if $x_k \to x$ in K, then

$$f^*(t,\varphi(t,x_k(t)),(\varphi(t,x_k(t)))') \to f^*(t,\varphi(t,x(t)),(\varphi(t,x(t)))') \quad \text{for a.a. } t \in I,$$

as one can easily check by considering all possible combinations of the cases $x(t) \in [\alpha(t),\beta(t)]$, $x(t) > \beta(t)$ or $x(t) < \alpha(t)$, and $|x'(t)| \le R$ or $|x'(t)| > R$.

Moreover,

$$|f^*(t,\varphi(t,x(t)),(\varphi(t,x(t)))')| \le \tilde{M}(t) \tag{4.1.11}$$

for a.a. $t \in I$, hence $Tx_k \to Tx$ in $\mathcal{C}^1(I)$.

Case 2: $m(\{t \in I_n \,:\, x(t) = \gamma_n(t)\}) > 0$ *for some* $n \in \mathbb{N}$ *such that* γ_n *is inviable.* In this case we can prove that $x \notin \mathbb{T}x$.

First, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have

$$m(\{t \in I_n \,:\, x(t) = \gamma_n(t)\}) > 0$$

and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for a.a. $t \in I_n$, such that (4.1.2) holds with γ replaced by γ_n. (The proof is similar if we assume (4.1.1) instead of (4.1.2), so we omit it.)

We denote $J = \{t \in I_n \,:\, x(t) = \gamma_n(t)\}$, and we observe that

$$m(\{t \in J \,:\, \gamma_n(t) = \beta(t)\}) = 0.$$

Indeed, if $m(\{t \in J \,:\, \gamma_n(t) = \beta(t)\}) > 0$, then from (4.1.2) it follows that

$$\beta''(t) - \psi(t) > f(t, \beta(t), \beta'(t))$$

on a set of positive measure, which is a contradiction with the definition of upper solution.

Now we distinguish between two sub-cases.

Case 2.1: $m(\{t \in J \,:\, x(t) = \gamma_n(t) = \alpha(t)\}) > 0$.

Since $m(\{t \in J \,:\, \gamma_n(t) = \beta(t)\}) = 0$, we deduce that

$$m(\{t \in J : x(t) = \alpha(t) \neq \beta(t)\}) > 0,$$

so there exists $n_0 \in \mathbb{N}$ such that

$$m\left(\left\{t \in J : x(t) = \alpha(t),\; x(t) < \beta(t) - \frac{1}{n_0}\right\}\right) > 0.$$

We denote $A = \{t \in J : x(t) = \alpha(t),\; x(t) < \beta(t) - 1/n_0\}$ and we deduce from Lemma 3.1.2 that there is a measurable set $J_0 \subset A$ with $m(J_0) = m(A) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{2\int_{[\tau_0,t]\setminus A} \tilde{M}(s)\,ds}{(1/4)\int_{\tau_0}^{t} \psi(s)\,ds} = 0 = \lim_{t \to \tau_0^-} \frac{2\int_{[t,\tau_0]\setminus A} \tilde{M}(s)\,ds}{(1/4)\int_{t}^{\tau_0} \psi(s)\,ds}. \tag{4.1.12}$$

By Corollary 3.1.3 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\cap J_0} \psi(s)\,ds}{\int_{\tau_0}^{t} \psi(s)\,ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\cap J_0} \psi(s)\,ds}{\int_{t}^{\tau_0} \psi(s)\,ds}. \tag{4.1.13}$$

Let us now fix a point $\tau_0 \in J_1$. From (4.1.12) and (4.1.13) we deduce that there exist

$t_- < \tau_0$ and $t_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied:

$$2\int_{[\tau_0,t_+]\setminus A} \tilde{M}(s)\,ds < \frac{1}{4}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{4.1.14}$$

$$\int_{[\tau_0,t_+]\cap A} \psi(s)\,ds \geq \int_{[\tau_0,t_+]\cap J_0} \psi(s)\,ds > \frac{1}{2}\int_{\tau_0}^{t_+} \psi(s)\,ds, \tag{4.1.15}$$

$$2\int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\,ds < \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds, \tag{4.1.16}$$

$$\int_{[t_-,\tau_0]\cap A} \psi(s)\,ds > \frac{1}{2}\int_{t_-}^{\tau_0} \psi(s)\,ds. \tag{4.1.17}$$

Finally, we define a positive number

$$\rho = \min\left\{\frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\,ds, \frac{1}{4}\int_{\tau_0}^{t_+} \psi(s)\,ds\right\}, \tag{4.1.18}$$

and we are now in a position to prove that $x \notin \mathbb{T}x$. It is sufficient to prove the following claim:

Claim: Let $\tilde{\varepsilon} > 0$ be defined as $\tilde{\varepsilon} = \min\{\varepsilon, 1/n_0\}$, where ε is given by our assumptions over γ_n and n_0 by the definition of the set A, and let ρ be as in (4.1.18)*. For every finite family $x_i \in \overline{B}_{\tilde{\varepsilon}}(x) \cap K$ and $\lambda_i \in [0,1]$ ($i = 1, 2, \ldots, m$), with $\sum \lambda_i = 1$, we have*

$$\left\| x - \sum \lambda_i T x_i \right\|_{\mathcal{C}^1} \geq \rho.$$

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T x_i$. For a.a. $t \in J = \{t \in I_n \,:\, x(t) = \gamma_n(t)\}$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i (Tx_i)''(t) = \sum_{i=1}^{m} \lambda_i\, f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))'). \tag{4.1.19}$$

On the other hand, for every $i \in \{1, 2, \ldots, m\}$ and for a.a. $t \in J$ we have

$$|x_i(t) - \gamma_n(t)| + |x_i'(t) - \gamma_n'(t)| = |x_i(t) - x(t)| + |x_i'(t) - x'(t)| < \varepsilon. \tag{4.1.20}$$

Since $\gamma_n(t) \in [\alpha(t), \beta(t)]$, for a.a. $t \in A$ we have

$$|\varphi(t, x_i(t)) - \gamma_n(t)| \leq |x_i(t) - \gamma_n(t)| < \varepsilon$$

and

$$|(\varphi(t, x_i(t)))' - \gamma_n'(t)| \leq |x_i'(t) - \gamma_n'(t)| < \varepsilon,$$

taking into account that if $x_i(t) < \alpha(t)$, then $(\varphi(t, x_i(t)))' = \alpha'(t) = \gamma_n'(t)$.

Hence, from (4.1.2) it follows that

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$ and for all $x_i(t)$ satisfying (4.1.20).

Moreover, since for a.a. $t \in A$ we have $|\gamma_n'(t)| < R$ and $|x_i'(t) - \gamma_n'(t)| < \varepsilon$, without loss of generality we can suppose $|(\varphi(t, x_i(t)))'| \leq R$ and thus

$$\gamma_n''(t) - \psi(t) > f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$.

Therefore the assumptions on γ_n ensure that for a.a. $t \in A$ we have

$$\begin{aligned} y''(t) &= \sum_{i=1}^{m} \lambda_i \, f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))') \\ &< \sum_{i=1}^{m} \lambda_i \, (\gamma_n''(t) - \psi(t)) = x''(t) - \psi(t). \end{aligned} \tag{4.1.21}$$

Now we compute

$$\begin{aligned} y'(\tau_0) - y'(t_-) &= \int_{t_-}^{\tau_0} y''(s)\, ds = \int_{[t_-,\tau_0]\cap A} y''(s)\, ds + \int_{[t_-,\tau_0]\setminus A} y''(s)\, ds \\ &< \int_{[t_-,\tau_0]\cap A} x''(s)\, ds - \int_{[t_-,\tau_0]\cap A} \psi(s)\, ds \\ &\quad + \int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\, ds \qquad \text{(by (4.1.21), (4.1.19) and (4.1.11))} \\ &= x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0]\setminus A} x''(s)\, ds - \int_{[t_-,\tau_0]\cap A} \psi(s)\, ds \\ &\quad + \int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\, ds \\ &\leq x'(\tau_0) - x'(t_-) - \int_{[t_-,\tau_0]\cap A} \psi(s)\, ds + 2\int_{[t_-,\tau_0]\setminus A} \tilde{M}(s)\, ds \\ &< x'(\tau_0) - x'(t_-) - \frac{1}{4}\int_{t_-}^{\tau_0} \psi(s)\, ds \quad \text{(by (4.1.16) and (4.1.17))}, \end{aligned}$$

hence $\|x - y\|_{\mathcal{C}^1} \geq y'(t_-) - x'(t_-) \geq \rho$ provided that $y'(\tau_0) \geq x'(\tau_0)$.

Similar computations with t_+ instead of t_- show that if $y'(\tau_0) \leq x'(\tau_0)$ then we also have $\|x - y\|_{\mathcal{C}^1} \geq \rho$. The claim is proven.

Case 2.2: $m(\{t \in J \,:\, \gamma_n(t) \in (\alpha(t), \beta(t))\}) > 0$.

The set $\{t \in J \,:\, \gamma_n(t) \in (\alpha(t), \beta(t))\}$ can be written as the following countable union

$$\bigcup_{n \in \mathbb{N}} \left\{ t \in J : \alpha(t) + \frac{1}{n} < x(t) < \beta(t) - \frac{1}{n} \right\},$$

so there exists some $n_0 \in \mathbb{N}$ such that

$$m(\{t \in J : \alpha(t) + 1/n_0 < x(t) < \beta(t) - 1/n_0\}) > 0.$$

Now we denote $A = \{t \in J : \alpha(t) + 1/n_0 < x(t) < \beta(t) - 1/n_0\}$. Since A is a set of positive measure we can argue as in Case 2.1 for obtaining inequalities (4.1.14)–(4.1.17) and we are in a position to prove the Claim again.

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T x_i$. Then for every $i \in \{1, 2, \ldots, m\}$ and all $t \in A$ we have $x_i(t) \in (\alpha(t), \beta(t))$, and so $\varphi(t, x_i(t)) = x_i(t)$ and $(\varphi(t, x_i(t)))' = x_i'(t)$ and thus

$$|\varphi(t, x_i(t)) - \gamma_n(t)| + |(\varphi(t, x_i(t)))' - \gamma_n'(t)| = |x_i(t) - x(t)| + |x_i'(t) - x'(t)| < \varepsilon,$$

for a.a. $t \in A$.

Hence, from (4.1.2) it follows that

$$\gamma_n''(t) - \psi(t) > f(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in A$ and all $x_i \in \overline{B}_{\tilde{\varepsilon}}(x)$.

Now the proof of the Claim follows exactly as in Case 2.1.

Case 3: $m(\{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}) > 0$ *for some* $n \in \mathbb{N}$ *such that* Γ_n *is an inviable discontinuity curve for the derivative.* In this case, we can prove again that $x \notin \mathbb{T}x$.

As done before, let us assume that for some $n \in \mathbb{N}$ we have

$$m(\{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}) > 0$$

and there exist $\varepsilon > 0$ and $\psi \in L^1(\tilde{I}_n)$, $\psi(t) > 0$ for a.a. $t \in \tilde{I}_n$, such that (4.1.4) holds with Γ replaced by Γ_n. Similarly, we can define ρ as in (4.1.18) and we shall prove the Claim.

Let x_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T x_i$. For a.a. $t \in J = \{t \in \tilde{I}_n : x'(t) = \Gamma_n(t)\}$ we have (4.1.19). On the other hand, for every $i \in \{1, 2, \ldots, m\}$ and for every $t \in J$ we have

$$|x_i'(t) - \Gamma_n(t)| = |x_i'(t) - x'(t)| < \varepsilon.$$

Moreover, from (4.1.4) it follows that

$$\Gamma_n'(t) - \psi(t) > f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))')$$

for a.a. $t \in I_n$ and for all $x_i(t)$ taking into account that $\varphi(t, x_i(t)) \in [\alpha(t), \beta(t)]$ and $(\varphi(t, x_i(t)))' \in \{x_i'(t), \alpha'(t), \beta'(t)\}$.

Therefore the assumptions on Γ_n ensure that for a.a. $t \in J$ we have

$$\begin{aligned} y''(t) &= \sum_{i=1}^{m} \lambda_i \, f^*(t, \varphi(t, x_i(t)), (\varphi(t, x_i(t)))') \\ &< \sum_{i=1}^{m} \lambda_i \, (\Gamma_n'(t) - \psi(t)) = x''(t) - \psi(t), \end{aligned}$$

and the proof of Case 3 follows as in Case 2.1, but now the set J plays the role of the set A there.

Case 4: $m(\{t \in I_n \,:\, x(t) = \gamma_n(t)\}) > 0$ *only for some of those* $n \in \mathbb{N}$ *such that* γ_n *is viable and* $m(\{t \in \tilde{I}_n \,:\, x'(t) = \Gamma_n(t)\}) = 0$ *for all* $n \in \mathbb{N}$. Let us prove that in this case the relation $x \in \mathbb{T}x$ implies $x = Tx$.

Note first that $x \in \mathbb{T}x$ implies that x satisfies the boundary conditions in (4.1.6), because every element in $\mathbb{T}x$ is, roughly speaking, a limit of convex combinations of functions y satisfying (4.1.10) and L_1 and L_2 are continuous functions.

Now it only remains to show that $x \in \mathbb{T}x$ implies that x satisfies the ODE in (4.1.6).

Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 4, which we denote again by $\{\gamma_n\}_{n\in\mathbb{N}}$ to avoid overloading notation. We have $m(J_n) > 0$ for all $n \in \mathbb{N}$, where

$$J_n = \{t \in I_n \,:\, x(t) = \gamma_n(t)\}.$$

For each $n \in \mathbb{N}$ and for a.a. $t \in J_n$ we have $\gamma_n''(t) = f\,(t, \gamma_n(t), \gamma_n'(t))$ and from $\alpha \le \gamma_n \le \beta$ and $|\gamma_n'(t)| < R$ it follows that $\gamma_n''(t) = f^*(t, \varphi(t, \gamma_n(t)), (\varphi(t, \gamma_n(t)))')$, so γ_n is viable for (4.1.6). Then for a.a. $t \in J_n$ we have

$$x''(t) = \gamma_n''(t) = f^*(t, \varphi(t, \gamma_n(t)), (\varphi(t, \gamma_n(t)))') = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))'),$$

and therefore

$$x''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))') \quad \text{a.e. in } J = \textstyle\bigcup_{n\in\mathbb{N}} J_n. \tag{4.1.22}$$

Now we assume that $x \in \mathbb{T}x$ and we prove that it implies that

$$x''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')$$

a.e. in $I \setminus J$, thus showing that $x = Tx$.

Since $x \in \mathbb{T}x$ then for each $k \in \mathbb{N}$ we can choose $\varepsilon = \rho = 1/k$ to guarantee that we can find functions $x_{k,i} \in B_{1/k}(x) \cap K$ and coefficients $\lambda_{k,i} \in [0,1]$ $(i = 1, 2, \ldots, m(k))$ such that $\sum \lambda_{k,i} = 1$ and

$$\left\| x - \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i} \right\|_{\mathcal{C}^1} < \frac{1}{k}.$$

Let us denote $y_k = \sum_{i=1}^{m(k)} \lambda_{k,i} T x_{k,i}$, and notice that $y_k' \to x'$ uniformly in I and, moreover, $\|x_{k,i} - x\|_{\mathcal{C}^1} \le 1/k$ for all $k \in \mathbb{N}$ and all $i \in \{1, 2, \ldots, m(k)\}$. Note also that

$$y_k''(t) = \sum_{i=1}^{m(k)} \lambda_{k,i} f^*(t, \varphi(t, x_{k,i}(t)), (\varphi(t, x_{k,i}(t)))') \quad \text{for a.a. } t \in I. \tag{4.1.23}$$

For a.a. $t \in I \setminus J$ we have that either $x(t) \in [\alpha(t), \beta(t)]$, and then $f^*(t, \varphi(t, \cdot), (\varphi(t, \cdot))')$ is continuous at $x(t)$, so for any $\varepsilon > 0$ there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \ge k_0$, we have

$$|f^*(t, \varphi(t, x_{k,i}(t)), (\varphi(t, x_{k,i}(t)))') - f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')| < \varepsilon$$

for all $i \in \{1, 2, \ldots, m(k)\}$, or $x(t) < \alpha(t)$ (analogously if $x(t) > \beta(t)$), so there is some $k_0 = k_0(t) \in \mathbb{N}$ such that for all $k \in \mathbb{N}$, $k \geq k_0$ we have $x_{k,i}(t) < \alpha(t)$ for all $i \in \{1, 2, \ldots, m(k)\}$ and then $\varphi(t, x(t)) = \alpha(t) = \varphi(t, x_{k,i}(t))$, which implies

$$|f^*(t, \varphi(t, x_{k,i}(t)), (\varphi(t, x_{k,i}(t)))') - f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')| = 0$$

for all $i \in \{1, 2, \ldots, m(k)\}$.

Now we deduce from (4.1.23) that $y_k''(t) \to f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))')$ for a.a. $t \in I \setminus J$, and then Corollary 3.1.4 guarantees that

$$x''(t) = f^*(t, \varphi(t, x(t)), (\varphi(t, x(t)))') \text{ for a.a. } t \in I \setminus J.$$

Combining this result with (4.1.22), we see that x solves (4.1.6), which implies that x is a fixed point of T.

So far, we have proven that the operator T satisfies condition (2.1.1) for all $x \in K$ and then Theorem 2.1.3 ensures the existence of a fixed point of T or, equivalently, a solution to the modified problem (4.1.6). It remains to prove that every solution of (4.1.6) is also a solution of the former problem (4.0.1)–(4.0.2).

First we will see that if x is a solution for (4.1.6), then $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I$. Assume on the contrary that there exists $t_0 \in I$ such that

$$x(t_0) - \alpha(t_0) = \min_{t \in I} (x(t) - \alpha(t)) < 0.$$

By the boundary conditions we have $\alpha(a) \leq x(a) \leq \beta(a)$ and $\alpha(b) \leq x(b) \leq \beta(b)$, and so $t_0 \in (a, b)$. Suppose that $x(t_0) - \alpha(t_0) < x(t) - \alpha(t)$ for all $t \in (t_0, b]$. Then we have

$$x'(t_0) - D_-\alpha(t_0) \leq x'(t_0) - D^+\alpha(t_0)$$

so, by the definition of lower solution, there exists an open interval I_0 such that $t_0 \in I_0$ and

$$\alpha''(t) \geq f(t, \alpha(t), \alpha'(t)) \quad \text{for a.a. } t \in I_0.$$

Further $x'(t_0) = \alpha'(t_0)$ and

$$\forall r > 0 \, \exists \, t_r \in (t_0, t_0 + r) \text{ such that } \alpha'(t_r) < x'(t_r). \tag{4.1.24}$$

On the other hand, the continuity of $x - \alpha$ implies that there exists $\varepsilon > 0$ such that for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have $x(t) - \alpha(t) < 0$. Then by definition of solution for (4.1.6), we obtain that

$$x''(t) = f(t, \alpha(t), \alpha'(t)) \quad \text{for a.a. } t \in [t_0, t_0 + \varepsilon],$$

and for $t \in [t_0, t_0 + \varepsilon]$,

$$x'(t) - \alpha'(t) = \int_{t_0}^{t} (x''(s) - \alpha''(s)) \, ds = \int_{t_0}^{t} (f(s, \alpha(s), \alpha'(s)) - \alpha''(s)) \, ds \leq 0,$$

a contradiction with (4.1.24). In a similar way we can see that $x \leq \beta$, so $\varphi(t, x(t)) = x(t)$.

In addition, by the Nagumo condition given in Proposition 4.1.2 it is immediate that $\|x'\|_\infty < R$.

To finish we will see that if x is a solution of (4.1.6) then x satisfies the boundary conditions (4.0.2). To do so, we follow the steps of [34, Lemma 3.5].

If $x(b) + L_2(x(a), x(b)) < \alpha(b)$ the definition of L_2^* gives us that $x(b) = \alpha(b)$. Since L_2 is nondecreasing with respect to its first variable we get a contradiction:

$$\alpha(b) > x(b) + L_2(x(a), x(b)) \geq \alpha(b) + L_2(\alpha(a), \alpha(b)) = \alpha(b).$$

Similarly if $x(b) + L_2(x(a), x(b)) > \beta(b)$ we have $x(b) = \beta(b)$ and we get a contradiction as above. Then $\alpha(b) \leq x(b) + L_2(x(a), x(b)) \leq \beta(b)$, so $L_2^*(x(a), x(b)) = x(b) + L_2(x(a), x(b))$ and $L_2^*(x(a), x(b)) = x(b)$ imply $L_2(x(a), x(b)) = 0$.

In a similar way, to prove that $L_1(x(a), x(b), x'(a), x'(b), x) = 0$ it is enough to show that

$$\alpha(a) \leq x(a) - L_1(x(a), x(b), x'(a), x'(b), x) \leq \beta(a).$$

If $x(a) - L_1(x(a), x(b), x'(a), x'(b), x) < \alpha(a)$ then $x(a) = \alpha(a)$ and thus

$$0 = L_2(x(a), x(b)) = L_2(\alpha(a), x(b)).$$

Now, since $L_2(\alpha(a), \cdot)$ is injective and $L_2(\alpha(a), \alpha(b)) = 0$, we obtain that $x(b) = \alpha(b)$. Previously, we saw that $x - \alpha$ is nonnegative in I and thus it attains its minimum at a and b, so $x'(a) \geq D^+\alpha(a)$ and $x'(b) \leq D_-\alpha(b)$. Using the definition of lower solution and the properties of L_1 we obtain a contradiction:

$$\begin{aligned}\alpha(a) &> x(a) - L_1(x(a), x(b), x'(a), x'(b), x) \\ &\geq \alpha(a) - L_1(\alpha(a), \alpha(b), D^+\alpha(a), D_-\alpha(b), \alpha) \geq \alpha(a).\end{aligned}$$

In an analogous way we can prove that $x(a) - L_1(x(a), x(b), x'(a), x'(b), x) \leq \beta(a)$.

Hence every solution for the modified problem (4.1.6) is a solution for problem (4.0.1)–(4.0.2). $\square$

Now we present some simple examples which illustrate the applicability of our existence result.

Example 4.1.6. *We will study the existence of solutions to the following second–order problem*

$$x''(t) = \begin{cases} \dfrac{1}{\sqrt{t}} \cos^2\left(\lfloor 1/(x+at) \rfloor\right) & \text{if } x > 0, \quad \text{for a.a. } t \in [0,1], \\ 0 & \text{if } x \leq 0, \quad \text{for a.a. } t \in [0,1], \end{cases} \tag{4.1.25}$$

where $a \geq 0$, coupled with the following nonlinear boundary conditions

$$\begin{aligned} &x(1) - x(0) = 1, \\ &x(0)\,x(1) - \int_0^1 x(t)\,dt = 0, \end{aligned} \tag{4.1.26}$$

where $\lfloor x \rfloor$ *denotes the integer part of* x.

The nonlinearity in equation (4.1.25) *is discontinuous over the curves*

$$\gamma_n(t) = -at + \frac{1}{n} \quad \textit{for } t \in \left[0, \min\left\{1, \frac{1}{an}\right\}\right], \; n \in \mathbb{N}, \; \textit{and } \gamma_0(t) \equiv 0,$$

so it falls outside the scope of the usual existence results and the applicability of the standard topological methods. Moreover, it is not monotone and thus the monotone iterative techniques for discontinuous differential equations are neither applicable here (see [78]).

Notice that problem (4.1.25)–(4.1.26) *is a particular case of problem* (4.0.1)–(4.0.2) *where*

$$f(t,x) = \begin{cases} \dfrac{1}{\sqrt{t}} \cos^2\left(\lfloor 1/(x+at) \rfloor\right) & \textit{if } x > 0, \\ 0 & \textit{if } x \leq 0, \end{cases}$$

and we have the following boundary conditions

$$\begin{aligned} &L_1(x(0), x(1), x'(0), x'(1), x) = x(0)\, x(1) - \int_0^1 x(t)\, dt = 0, \\ &L_2(x(0), x(1)) = x(0) - x(1) + 1 = 0. \end{aligned}$$

Observe that L_1 *and* L_2 *are continuous functions satisfying the required monotonicity conditions and* $L_2(x, \cdot)$ *is an injective function for each* $x \in \mathbb{R}$.

First, we will show that $\alpha(t) = (4\,t\sqrt{t} - t)/3$ *and* $\beta(t) = t+1$ *are, respectively, lower and upper solutions for problem* (4.1.25)–(4.1.26) *such that* $\alpha(t) \leq \beta(t)$ *for all* $t \in I$. *Indeed,*

$$\begin{aligned} &\alpha''(t) = \frac{1}{\sqrt{t}} \geq \frac{1}{\sqrt{t}} \cos^2\left(\lfloor 1/(\alpha(t)+at) \rfloor\right) \geq f(t, \alpha(t)), \\ &L_1(\alpha(0), \alpha(1), \alpha'(0), \alpha'(1), \alpha) = \alpha(0)\, \alpha(1) - \int_0^1 \alpha(t)\, dt = -11/30 \leq 0, \\ &L_2(\alpha(0), \alpha(1)) = \alpha(0) - \alpha(1) + 1 = 0 - 1 + 1 = 0, \end{aligned}$$

and

$$\begin{aligned} &\beta''(t) = 0 \leq \frac{1}{\sqrt{t}} \cos^2\left(\lfloor 1/(\beta(t)+at) \rfloor\right) = f(t, \beta(t)), \\ &L_1(\beta(0), \beta(1), \beta'(0), \beta'(1), \beta) = \beta(0)\, \beta(1) - \int_0^1 \beta(t)\, dt = 2 - 3/2 \geq 0, \\ &L_2(\beta(0), \beta(1)) = \beta(0) - \beta(1) + 1 = 1 - 2 + 1 = 0. \end{aligned}$$

Moreover, condition $(C1)$ *in Theorem 4.1.5 holds with a similar argument to that done in Example 3.1.10, and condition* $(C2)$ *is also satisfied with* $M(t) = 1/\sqrt{t}$ *and* $N(s) = 1$.

On the other hand, for a.a. $t \in I$ *the function* $x \mapsto f(t,x)$ *is continuous on*

$$[\alpha(t), \beta(t)] \setminus \bigcup_{n \in \mathbb{N}} \{\gamma_n(t), \gamma_0(t)\},$$

where the curves γ_n, $n \in \mathbb{N}$, *are inviable admissible discontinuity curves. Indeed, for each* $n \in \mathbb{N}$, *we have*

$$\gamma_n''(t) = 0 < \min\{\cos^2(n-1), \cos^2(n)\} \frac{1}{\sqrt{t}} \leq f(t,x)$$

for a.a. $t \in I$ *and for all* $x \in \left[\gamma_n(t) - \dfrac{1}{2n(n+1)}, \gamma_n(t) + \dfrac{1}{2n(n+1)}\right]$, *so condition* (4.1.1) *in Definition 4.1.3 is satisfied by taking*

$$\psi_n(t) = \min\{\cos^2(n-1), \cos^2(n)\}\frac{1}{2\sqrt{t}} \quad \text{and} \quad \varepsilon_n = \frac{1}{2n(n+1)}.$$

The curve γ_0 *is clearly a viable admissible discontinuity curve.*

Therefore, Theorem 4.1.5 implies that problem (4.1.25)–(4.1.26) *has at least one solution between* α *and* β.

Example 4.1.7. *Consider the following second-order equation*

$$x''(t) = \begin{cases} \dfrac{1}{\sqrt{t}}\cos^2(\lfloor 1/|1-x|\rfloor) & \text{if } x \neq 1, \\ 0 & \text{if } x = 1, \end{cases}$$

coupled with the boundary conditions in (4.1.26).

Now the admissible discontinuity curves are $\gamma_n \equiv 1 \pm 1/n$, *which are inviable for each* $n \in \mathbb{N}$, *and* $\gamma_0 \equiv 1$, *which is a viable admissible discontinuity curve. It is possible to prove, as in Example 4.1.6, that this problem has at least a solution between the curves* α *and* β *defined above. Notice that* $x \equiv 1$ *is not a solution for the problem and then any solution between* α *and* β *must cross an infinity number of admissible discontinuity curves, see Figure 4.1.1.*

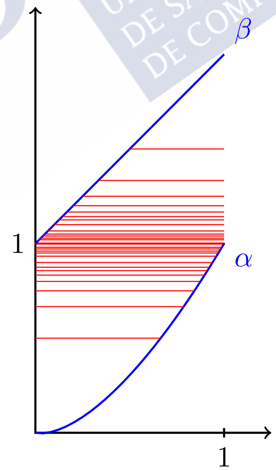

Figure 4.1.1: Lower and upper solutions in blue and some discontinuity curves in red.

Theorem 4.1.5 establishes sufficient conditions for the existence of a solution between well–ordered lower and upper solutions for problem (4.0.1)–(4.0.2) under general conditions by applying our extension of Schauder's fixed point theorem, namely, Theorem 2.1.3. Nevertheless, in Section 4.2, it will be useful for us to have a reformulation of that result in terms of degree, for example to obtain results to (4.0.1)–(4.0.2) when the lower and upper solutions are non well ordered. For this reason, we include now two versions of Theorem 4.1.5 which are written in terms of degree. For simplicity we will assume here that f is continuous in the last variable.

Assume that there exist $\alpha, \beta \in W^{1,\infty}(I)$ lower and upper solutions to (4.0.1)–(4.0.2) such that $\alpha < \beta$ on I, and that $f : I \times \mathbb{R}^2 \to \mathbb{R}$ satisfies conditions $(C1)$, $(C2)$ and

$(\widetilde{C3})$ There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ such that $\alpha \le \gamma_n \le \beta$ on I_n and their derivatives are uniformly bounded. For a.a. $t \in I$, the mapping $(x, y) \mapsto f(t, x, y)$ is continuous on

$$\left([\alpha(t), \beta(t)] \setminus \bigcup_{\{n \,:\, t \in I_n\}} \{\gamma_n(t)\} \right) \times [-R, R].$$

As done above, the idea is to transform the problem (4.0.1)–(4.0.2) with nonlinear boundary conditions into a Dirichlet BVP in the line of [34] or [127], where the periodic problem was studied. Hence we consider the following equivalent BVP

$$\begin{cases} x'' = f(t, x, x'), \qquad t \in I, \\ x(a) = x(a) - L_1(x(a), x(b), x'(a), x'(b), x), \\ x(b) = x(b) + L_2(x(a), x(b)), \end{cases} \tag{4.1.27}$$

whose associated fixed point operator, $T : \mathcal{C}^1(I) \to \mathcal{C}^1(I)$, is given by

$$\begin{aligned} Tx(t) =& x(a) - L_1(x) \\ &+ \frac{t-a}{b-a}\left[x(b) - x(a) + L_1(x) + L_2(x) - \int_a^b \int_a^s f(r, x(r), x'(r))\, drds \right] \\ &+ \int_a^t \int_a^s f(r, x(r), x'(r))\, drds, \end{aligned} \tag{4.1.28}$$

where $L_1(x)$ and $L_2(x)$ denote, respectively, $L_1(x(a), x(b), x'(a), x'(b), x)$ and $L_2(x(a), x(b))$.

For each $l > 0$ we define the open set

$$\Omega_l = \left\{ x \in \mathcal{C}^1(I) : \alpha < x < \beta \text{ on } I \text{ and } \|x'\| < l \right\}.$$

Theorem 4.1.8. *Assume that there exist $\alpha, \beta \in W^{1,\infty}(I)$ lower and upper solutions to* (4.0.1)–(4.0.2) *such that $\alpha < \beta$ on I, and that for $f : I \times \mathbb{R}^2 \to \mathbb{R}$ conditions $(C1)$, $(C2)$ and $(\widetilde{C3})$ hold.*

Let R be as in $(C2)$ and such that $R \ge \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty, \|\gamma_n'\|_\infty\}$ for all $n \in \mathbb{N}$.

Then for all values of $l \ge R$ we have

$$\deg(Id - T, \Omega_l) = 1 \quad \textit{provided that } Tx \ne x \textit{ for } x \in \partial\Omega_l.$$

In particular, problem (4.0.1)–(4.0.2) *has at least one solution x such that $\alpha \le x \le \beta$.*

Proof. Let $l \geq R$, define $\Omega := \Omega_l$, and assume that

$$Tx \neq x \quad \text{for every } x \in \partial\Omega. \tag{4.1.29}$$

Now consider the modified problem (4.1.6) and denote by $\tilde{T}$ the integral operator associated to it. For all $x \in \mathcal{C}^1(I)$ and a.a. $t \in I$, we have

$$|f(t, \varphi(t, x(t)), \delta((\varphi(t, x(t)))'))| \leq \tilde{M}(t) := \max_{s\in[0,l]} \{N(s)\} \, M(t),$$

so there exists $R_0 > 0$ such that $\Omega \subset B_{R_0/2}(0)$ and $\tilde{T}x \in B_{R_0/2}(0)$ for all $x \in \mathcal{C}^1(I)$. In particular, $\|x\| < R_0/2$ for every $\lambda \in [0,1]$ and $x \in \mathcal{C}^1(I)$ such that $x = \lambda\tilde{T}x$. Hence $x \notin \lambda\tilde{\mathbb{T}}x$ if $\|x\| = R_0$.

Now we define the homotopy $H : \overline{B}_{R_0}(0) \times [0,1] \to \overline{B}_{R_0}(0)$ given by $H(x,\lambda) = \lambda\tilde{T}x$. By virtue of Theorem 1.2.4 we have that

$$\deg\left(Id - \tilde{T}, B_{R_0}(0)\right) = \deg\left(Id, B_{R_0}(0)\right) = 1, \tag{4.1.30}$$

provided that $\{x\} \cap \tilde{\mathbb{T}}x \subset \left\{\tilde{T}x\right\}$ for all $x \in \overline{B}_{R_0}(0) \cap \tilde{\mathbb{T}}\overline{B}_{R_0}(0)$. Observe that

$$\overline{B}_{R_0}(0) \cap \tilde{\mathbb{T}}\overline{B}_{R_0}(0) \subset K,$$

where K is defined as in (4.1.8), and for every function $x \in K$ we can prove that $x \in \tilde{\mathbb{T}}x$ implies $x = \tilde{T}x$ just by following the steps in the proof of Theorem 4.1.5. Thus (4.1.30) is justified.

Let

$$\tilde{\Omega} = \{x \in \Omega : \alpha(a) < x(a) - L_1(x) < \beta(a),\ \alpha(b) < x(b) + L_2(x) < \beta(b)\}.$$

We shall see that $\tilde{T}x = x$ implies that $x \in \tilde{\Omega}$.

Indeed, if x is a fixed point of $\tilde{T}$, then we claim the following, which can be proven as in Theorem 4.1.5:

(a) $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in I$;

(b) $\|x'\|_\infty < R \leq l$, as a consequence of Proposition 4.1.2;

(c) $\alpha(a) \leq x(a) - L_1(x) \leq \beta(a)$ and $\alpha(b) \leq x(b) + L_2(x) \leq \beta(b)$.

Therefore $x \in \overline{\tilde{\Omega}}$. Notice that $\tilde{T} = T$ on $\overline{\tilde{\Omega}}$, so from (4.1.29) it follows that $x \in \tilde{\Omega}$. Finally, by the excision property of the degree (see Proposition 1.2.1), we can conclude that

$$\deg(Id - T, \Omega) = \deg\left(Id - T, \tilde{\Omega}\right) = \deg\left(Id - \tilde{T}, \tilde{\Omega}\right) = \deg\left(Id - \tilde{T}, B_{R_0}(0)\right) = 1,$$

which ends the proof. □

If we check the proofs of Theorem 4.1.5 or Theorem 4.1.8, we observe that the derivatives of the admissible discontinuity curves need not be uniformly bounded if the nonlinearity f is globally bounded with respect to its third variable. That is the reason why we finish this subsection with a result which concretes this fact.

Theorem 4.1.9. *Assume that there exist* $\alpha, \beta \in W^{1,\infty}(I)$ *lower and upper solutions to* (4.0.1)–(4.0.2) *such that* $\alpha < \beta$ *and that* $f : I \times \mathbb{R}^2 \to \mathbb{R}$ *satisfies conditions* $(C1)$ *and*

$(\overline{C2})$ *There exists* $M \in L^1(I)$ *such that for a.a.* $t \in I$*, all* $x \in [\alpha(t), \beta(t)]$ *and* $y \in \mathbb{R}$*, we have* $|f(t,x,y)| \le M(t)$*;*

$(\overline{C3})$ *There exist admissible discontinuity curves* $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ *such that* $\alpha \le \gamma_n \le \beta$ *on* I_n*, and for a.a.* $t \in I$ *the function* $(x,y) \mapsto f(t,x,y)$ *is continuous on* $\left([\alpha(t), \beta(t)] \setminus \bigcup_{\{n\,:\,t \in I_n\}} \{\gamma_n(t)\}\right) \times \mathbb{R}$.

Let R *be as given by Proposition 4.1.2 and such that* $R \ge \max\{\|\alpha'\|_\infty, \|\beta'\|_\infty\}$.
Then for all values of $l \ge R$ *we have*

$$\deg(Id - T, \Omega_l) = 1 \quad \textit{provided that } Tx \ne x \textit{ for } x \in \partial\,\Omega_l.$$

In particular, problem (4.0.1)–(4.0.2) *has at least one solution* x *such that* $\alpha \le x \le \beta$.

4.1.1 Existence of extremal solutions between the lower and upper solutions

Now sufficient conditions for the existence of extremal solutions for problem (4.0.1)–(4.0.2) are given.

Theorem 4.1.10. *Assume that the hypotheses of Theorem 4.1.5 hold and* $L_2(x, \cdot)$ *is injective for all* $x \in [\alpha(a), \beta(a)]$. *Then problem* (4.0.1)–(4.0.2) *has extremal solutions between* α *and* β.

Proof. Let $S = \{x \in [\alpha, \beta] : x \text{ is a solution for } (4.0.1) - (4.0.2)\}$, which is a nonempty subset of $[\alpha, \beta]$ by virtue of Theorem 4.1.5. Moreover, if now $K \subset \mathcal{C}^1$ and $T : K \to K$ are given, respectively, by (4.1.8) and (4.1.9), then we have

$$S = \left\{x \in \mathcal{C}^1(I) : x \text{ is a solution for } (4.1.6)\right\} = \{x \in K : x = Tx\}.$$

Since condition $\{x\} \cap \mathbb{T}x \subset \{Tx\}$ is satisfied for every $x \in K$ we obtain

$$S = \{x \in K : x \in \mathbb{T}x\} = (Id - \mathbb{T})^{-1}(\{0\})$$

which is a closed set because $\mathbb{T}$ is an upper semicontinuous mapping and $\{0\}$ is a closed subset of the Banach space. Now the fact that $S \subset K$ implies that S is compact.

Define $x_{\min}(t) = \inf\{x(t) : x \in S\}$ for $t \in I$. By the compactness of S in $\mathcal{C}^1(I)$ there exists, for each $t_0 \in I$, a function $x_0 \in S$ such that $x_0(t_0) = x_{\min}(t_0)$ and $x_{\min}$ is continuous in I. Indeed, given $\varepsilon > 0$, by the equicontinuity of S, there exists $\delta > 0$ such that $t, s \in I$ with $|t - s| < \delta$ implies

$$|x(t) - x(s)| < \varepsilon/2 \quad \text{ for all } x \in S.$$

Now, for $t, s \in I$ with $|t - s| < \delta$, there exist $x_t, x_s \in S$ such that $x_t(t) = x_{\min}(t)$ and $x_s(s) = x_{\min}(s)$. Notice that Bolzano's theorem guarantees the existence of a point $r \in [t, s]$ (or $r \in [s, t]$) such that $x_t(r) = x_s(r)$. Therefore,

$$|x_{\min}(t) - x_{\min}(s)| = |x_t(t) - x_s(s)| \le |x_t(t) - x_t(r)| + |x_s(r) - x_s(s)| < \varepsilon,$$

so $x_{\min}$ is continuous in I.

Let us show that $x_{\min}$ is the least solution. To do so, we follow the steps in [34, Theorem 4.1].

First, observe that if $y \in S$ and $y(a) = x_{\min}(a)$, then $y(b) = x_{\min}(b)$. Indeed, if we have that $y(b) > x_{\min}(b)$, we can find $x \in S$ such that $x(b) = x_{\min}(b)$. Note that, due to the definitions of lower and upper solutions, condition $L_2(x, \cdot)$ injective is equivalent to $L_2(x, \cdot)$ decreasing. Hence, we have that $L_2(y(a), \cdot)$ is decreasing, $L_2(\cdot, x_{\min}(b))$ is nondecreasing and $y(a) \le x(a)$, and so

$$0 = L_2(y(a), y(b)) < L_2(y(a), x_{\min}(b)) \le L_2(x(a), x_{\min}(b)) = 0,$$

a contradiction.

Let us see that $x_{\min}$ is a solution to problem (4.1.6). To do so, we shall prove that $x_{\min}$ is a limit in $\mathcal{C}^1(I)$ of a sequence of elements of S. Clearly, it will be the least solution.

Given $\varepsilon > 0$, we shall show that there exists $v \in S$ such that

$$\|v - x_{\min}\|_\infty < \varepsilon.$$

Hence, there exists a sequence of elements in S which converges pointwise to $x_{\min}$ and by the compactness of S, up to a subsequence, it converges in S.

By the equicontinuity of S and the continuity of $x_{\min}$ on I, there exists $\delta > 0$ such that $t, s \in I$ with $|t - s| < \delta$ implies

$$|x(t) - x(s)| < \varepsilon/2 \quad \text{for all } x \in S \cup \{x_{\min}\}.$$

Let $\{t_0, t_1, \ldots, t_n\} \subset I$ such that $t_0 = a$, $t_n = b$ and $t_{i+1} - t_i < \delta$ for $i = 0, 1, \ldots, n-1$. Choose a function $x_0 \in S$ such that $x_0(a) = x_{\min}(a)$, $x_0(b) = x_{\min}(b)$ and denote $\beta_0 \equiv x_0$.

For each $i \in \{1, 2, \ldots, n-1\}$, define recursively $\beta_i \equiv \beta_{i-1}$ if $\beta_{i-1}(t_i) = x_{\min}(t_i)$ and otherwise, take $x_i \in S$ such that $x_i(t_i) = x_{\min}(t_i)$, define

$$\begin{aligned} s_i &= \inf\{t \in [t_{i-1}, t_i] \,:\, x_i(s) < \beta_{i-1}(s) \text{ for all } s \in [t, t_i]\}, \\ s_{i+1} &= \sup\{t \in [t_i, b] \,:\, x_i(s) < \beta_{i-1}(s) \text{ for all } s \in [t_i, t]\} \end{aligned}$$

and the function

$$\beta_i(t) = \begin{cases} \beta_{i-1}(t) & \text{if } t \in [a, s_i] \cup [s_{i+1}, b], \\ x_i(t) & \text{if } t \in (s_i, s_{i+1}). \end{cases}$$

Then $\beta_{n-1}(a) = x_0(a)$, $\beta_{n-1}(b) = x_0(b)$, $\beta'_{n-1}(a) \le x'_0(a)$, $\beta'_{n-1}(b) \ge x'_0(b)$ and $\beta_{n-1}(t) \le x_0(t)$ for all $t \in I$, so from the monotonicity hypotheses on L_1 and the fact that $x_0 \in S$, we have

$$\begin{aligned} &L_1(\beta_{n-1}(a), \beta_{n-1}(b), \beta'_{n-1}(a), \beta'_{n-1}(b), \beta_{n-1}) \\ &\qquad \ge L_1(x_0(a), x_0(b), x'_0(a), x'_0(b), x_0) = 0, \end{aligned}$$

and now it is immediate to check that β_{n-1} is an upper solution for problem (4.0.1)–(4.0.2).

By Theorem 4.1.5, we obtain that there exists $v \in S$ such that $\alpha(t) \le v(t) \le \beta_{n-1}(t)$ for $t \in I$ and, by the construction of β_{n-1} and the definition of $x_{\min}$, we have $v(t_i) = x_{\min}(t_i)$

for $i = 0, 1, \dots, n-1$. Hence, for each $t \in I$ there is $i \in \{0, 1, \dots, n-1\}$ such that $t \in [t_i, t_{i+1}]$, so

$$|v(t) - x_{\min}(t)| \le |v(t) - v(t_i)| + |x_{\min}(t_i) - x_{\min}(t)| < \varepsilon.$$

A similar reasoning shows that problem (4.1.6) has the greatest solution between α and β. $\square$

Remark 4.1.11. *Observe that $L_2(x, \cdot)$ is injective for all $x \in \mathbb{R}$, for example, when $L_2(x, y) = x - y$.*

Remark 4.1.12. *Notice that both problems in Example 4.1.6 and 4.1.7 are under the hypotheses of Theorem 4.1.10, and thus they have extremal solutions between their lower and upper solutions.*

Finally, we illustrate the previous results with an example which includes discontinuities in the third variable.

Example 4.1.13. *Consider in $I = [0, 1]$ the differential equation*

$$x'' = t^2 \lfloor 1/(t^2 + |x|) \rfloor \cos(x') + \frac{(x-1)^2\,|x'|}{54} \sin^2(x') \left[1 + H\left(\sin\left(\frac{1}{x' + at} \right) \right) H(x') \right]$$

where $a \in (1, \pi/2)$, $\lfloor x \rfloor$ denotes the integer part of x and H is the Heaviside step function (3.1.17)*, joint with the boundary conditions*

$$\max_{t \in [0,1]} x(t) = 0, \quad x(1) = 0.$$

Note that this problem falls inside the scope of Theorem 4.1.5 and Theorem 4.1.10 with the differential equation given by the nonlinearity

$$f(t, x, y) = t^2 \lfloor 1/(t^2 + |x|) \rfloor \cos(y) + \frac{(x-1)^2\,|y|}{54} \sin^2(y) \left[1 + H\left(\sin\left(\frac{1}{y + at} \right) \right) H(y) \right]$$

and the boundary conditions given by

$$L_1(x, y, z, w, \gamma) = -\max_{t \in [0,1]} \gamma(t), \quad L_2(x, y) = y.$$

Observe that f is unbounded and discontinuous with respect to the second and third arguments.

First, to check condition $(C1)$ in Theorem 4.1.5, note that it suffices to show that the function

$$t \in I \mapsto H\left(\sin\left(\frac{1}{y(t) + at} \right) \right) H(y(t))$$

is measurable for any measurable function y, since it can be proven as in Example 3.1.11 that the function $t \mapsto \lfloor 1/(t^2 + |x(t)|) \rfloor$ is measurable for every continuous function x and the sum and product of measurable functions is so. Now observe that we can write

$$t \in I \mapsto H\left(\sin\left(\frac{1}{y(t) + at} \right) \right) H(y(t)) = \chi_{\varphi^{-1}(E)}(t)\chi_{y^{-1}([0,+\infty))}(t), \tag{4.1.31}$$

where E and φ are, respectively, the following measurable set and function

$$E = \bigcup_{n=1}^{\infty} \left[\frac{1}{(2n+1)\pi}, \frac{1}{2n\pi} \right] \bigcup \left[\frac{1}{\pi}, +\infty \right) \quad \textit{and} \quad \varphi(t) = y(t) + at.$$

Then (4.1.31) *is a measurable function and thus condition* $(C1)$ *holds.*

Functions $\alpha(t) = \pi t - \pi$ *and* $\beta(t) = 0$ *for* $t \in [0,1]$ *are, respectively, a lower and an upper solutions for our problem. Indeed, they satisfy the boundary conditions and for a.a.* $t \in I$ *we have*

$$f(t, \alpha(t), \alpha'(t)) = -t^2 \lfloor 1/(t^2 + \pi(1-t)) \rfloor \le 0 = \alpha''(t)$$

and

$$f(t, \beta(t), \beta'(t)) = t^2 \lfloor 1/t^2 \rfloor \ge 0 = \beta''(t).$$

On the other hand, for a.a. $t \in [0,1]$*, the function* $f(t,\cdot,\cdot)$ *is continuous on*

$$\left([\alpha(t), \beta(t)] \setminus \bigcup_{\{n : t \in I_n^i,\, i=1,2\}} \{\gamma_n^i(t)\} \right) \times \left(\mathbb{R} \setminus \bigcup_{\{n : t \in \tilde{I}_n\}} \{\Gamma_n(t)\} \right)$$

where for each $n \in \mathbb{N}$,

$$\begin{aligned} \gamma_n^1(t) &= t^2 - n^{-1} \quad \textit{for all } t \in I_n^1 = [0, n^{-1/2}], \\ \gamma_n^2(t) &= -t^2 + n^{-1} \quad \textit{for all } t \in I_n^2 = [n^{-1/2}, 1], \end{aligned}$$

and

$$\Gamma_n(t) = -at + \frac{1}{n\pi} \quad \textit{for all } t \in \tilde{I}_n = [0, (an\pi)^{-1}].$$

The curves γ_n^1 *and* γ_n^2 *are inviable for the differential equation. Indeed, we can choose* $\varepsilon_n^1 = \dfrac{1}{2n(n+1)}$ *and* $\psi_n^1 \equiv 1/4$ *and then for all* $u \in [\gamma_n^1(t) - \varepsilon_n^1, \gamma_n^1(t) + \varepsilon_n^1]$ *and all* $v \in [{\gamma_n^1}'(t) - \varepsilon_n^1, {\gamma_n^1}'(t) + \varepsilon_n^1]$ *we have*

$$\begin{aligned} f(t,u,v) &\le 1 + \frac{1}{9} \max \left\{ t^2 - \frac{1}{n} + \varepsilon_n^1 - 1, t^2 - \frac{1}{n} - \varepsilon_n^1 - 1 \right\}^2 \\ &\le 1 + \frac{1}{9} \left(\frac{9}{4} \right)^2 = 1 + \frac{9}{16}, \end{aligned}$$

so (4.1.2) *holds. In a similar way, we may show that condition* (4.1.1) *holds for* γ_n^2, $n \in \mathbb{N}$.

Moreover, for all $t \in [0,1]$, $x \in [\alpha(t), \beta(t)]$ *and* $y \in [-a + (n\pi)^{-1} - \varepsilon, (n\pi)^{-1} + \varepsilon]$*, for* $\varepsilon > 0$ *small enough, we have* $|y| \le \pi/2$ *what implies that* $f(t,x,y) \ge 0$*; and for all* $t \in [0,1]$, $x \in [\alpha(t), \beta(t)]$ *and* $y = \alpha'(t) = \pi$ *we have*

$$f(t,x,\pi) = -t^2 \lfloor 1/(t^2 + |x|) \rfloor \ge -1.$$

Therefore the curves Γ_n satisfy (4.1.3)*, so they are inviable discontinuity curves for the derivative, and condition $(C3)$ in Theorem 4.1.5 is satisfied.*

Hence, this problem has extremal solutions between α and β as a consequence of Theorem 4.1.10.

4.2 Existence results via non–ordered lower and upper solutions

Our purpose in this section is to consider the case where the lower and upper solutions are not well–ordered. To do so, we follow the ideas of [127], where the authors consider the particular case of (4.0.1)–(4.0.2) with a Carathéodory nonlinearity and periodic boundary conditions. The main idea is the following: given a pair of not well–ordered lower and upper solutions, we will be able to construct a pair of constant well–ordered lower and upper solutions and then the existence of solutions will be a consequence of the results proven in Section 4.1.

Here and henceforth, we shall be concerned with the following problem:

$$\begin{aligned} &x''(t) = f(t, x(t), x'(t)), \quad t \in I = [a, b], \\ &0 = L_1(x(a), x(b), x'(a), x'(b), x), \\ &x(a) = x(b), \end{aligned} \tag{4.2.1}$$

where L_1 is as in the previous sections. Notice that (4.2.1) is the particular case of (4.0.1)–(4.0.2) with $L_2(x(a), x(b)) = x(a) - x(b)$.

Let us start with a technical result. The reader is referred to [127, Lemma 3.1] for its proof.

Lemma 4.2.1. *Let $\overline{\alpha}, \overline{\beta} \in \mathcal{C}(I)$ and $x \in \mathcal{C}(I)$ be such that*

$$x(t_x) < \overline{\alpha}(t_x) \text{ and } x(s_x) > \overline{\beta}(s_x) \quad \text{for some } t_x, s_x \in I. \tag{4.2.2}$$

Then there exist $\tau_x \in I$ such that

$$\min\left\{\overline{\alpha}(\tau_x), \overline{\beta}(\tau_x)\right\} \le x(\tau_x) \le \max\left\{\overline{\alpha}(\tau_x), \overline{\beta}(\tau_x)\right\}.$$

We are ready for our main result in this section.

Theorem 4.2.2. *Assume that there exist $\alpha, \beta \in W^{1,\infty}(I)$ lower and upper solutions to* (4.2.1)*, respectively, such that*

$$\alpha(\tau) > \beta(\tau) \quad \text{for some } \tau \in I,$$

and that f satisfies the following conditions:

$(C1^*)$ *Compositions $t \in I \mapsto f(t, x(t), y(t))$ are measurable whenever $x(t)$ and $y(t)$ are continuous;*

$(C2^*)$ *There exists $M \in L^1(I)$ such that for a.a. $t \in I$ and all $x, y \in \mathbb{R}$, we have $|f(t,x,y)| \le M(t)$;*

$(C3^*)$ *There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ such that they are uniformly bounded, and for a.a. $t \in I$ the function $(x,y) \mapsto f(t,x,y)$ is continuous on $\left(\mathbb{R} \setminus \bigcup_{\{n\,:\,t\in I_n\}}\{\gamma_n(t)\}\right) \times \mathbb{R}$.*

Then problem (4.2.1) *has a solution x such that for some $\tau_x \in I$*

$$\min\{\alpha(\tau_x), \beta(\tau_x)\} \le x(\tau_x) \le \max\{\alpha(\tau_x), \beta(\tau_x)\}\,. \tag{4.2.3}$$

Proof. Define $\tilde{M}(t) := 2M(t) + 1$ and take $R > 0$ such that $\|x'\|_\infty < R$ for all $x \in X = \mathcal{C}^1(I)$ satisfying

$$|x''(t)| \le \tilde{M}(t), \quad x(a) = x(b). \tag{4.2.4}$$

Now fix some value $r \ge \|\alpha\|_\infty + \|\beta\|_\infty + (b-a)R$ and $r \ge \|\gamma_n\|_\infty$ for all $n \in \mathbb{N}$, and define

$$\tilde{f}(t,x,y) = \begin{cases} f(t,x,y) - M(t) - 1 & \text{if } x \le -(r+1), \\ f(t,x,y) + (x+r)(M(t)+1) & \text{if } -(r+1) < x < -r, \\ f(t,x,y) & \text{if } -r \le x \le r, \\ f(t,x,y) + (x-r)(M(t)+1) & \text{if } r < x < r+1, \\ f(t,x,y) + M(t) + 1 & \text{if } x \ge r+1, \end{cases} \tag{4.2.5}$$

and

$$\tilde{L_1}(x,y,z,w,\xi) = \begin{cases} w - z & \text{if } x \le -(r+1), \\ (r+1+x)L_1(x,y,z,w,\xi) - (x+r)(w-z) & \text{if } r < -x < r+1, \\ L_1(x,y,z,w,\xi) & \text{if } -r \le x \le r, \\ (r+1-x)L_1(x,y,z,w,\xi) + (x-r)(w-z) & \text{if } r < x < r+1, \\ w - z & \text{if } x \ge r+1. \end{cases} \tag{4.2.6}$$

Consider the auxiliary problem

$$\begin{cases} x'' = \tilde{f}(t,x,x') \qquad t \in I, \\ 0 = \tilde{L_1}(x(a), x(b), x'(a), x'(b), x), \\ x(a) = x(b). \end{cases} \tag{4.2.7}$$

Notice that α and β are, respectively, lower and upper solutions of (4.2.7).

In addition

$$\left|\tilde{f}(t,x,y)\right| \le \tilde{M}(t) := 2M(t) + 1 \quad \text{ for a.a. } t \in I \text{ and all } x, y \in \mathbb{R}, \tag{4.2.8}$$

and

$$\begin{matrix} \tilde{f}(t,x,y) < 0 & \text{for a.a. } t \in I \text{ and all } x \in (-\infty, -r-1],\ y \in \mathbb{R}, \\ \tilde{f}(t,x,y) > 0 & \text{for a.a. } t \in I \text{ and all } x \in [r+1, \infty),\ y \in \mathbb{R}. \end{matrix} \tag{4.2.9}$$

In particular, $\tilde{\alpha}(t) \equiv -\rho$ and $\tilde{\beta}(t) \equiv \rho$ (where $\rho = r + 2$) are, respectively, lower and upper solutions for the auxiliary problem (4.2.7).

Denote

$$\Omega_0 = \left\{ x \in \mathcal{C}^1(I) : \tilde{\alpha} < x < \tilde{\beta} \text{ on } I,\ \|x'\|_\infty < R \right\},$$

$$\Omega_1 = \{ x \in \Omega_0 : \tilde{\alpha} < x < \beta \text{ on } I \},$$

$$\Omega_2 = \left\{ x \in \Omega_0 : \alpha < x < \tilde{\beta} \text{ on } I \right\}$$

and

$$\Omega = \Omega_0 \setminus \left(\overline{\Omega_1 \cup \Omega_2} \right), \tag{4.2.10}$$

which is the set of points $x \in \Omega_0$ such that (4.2.2) holds with $\overline{\alpha} = \alpha$ and $\overline{\beta} = \beta$.

Problem (4.2.7) is equivalent to the equation $\tilde{T}x = x$ where $\tilde{T} : \mathcal{C}^1(I) \to \mathcal{C}^1(I)$ is defined as in (4.1.28) but replacing f by $\tilde{f}$ and $L_1(x)$ by $\tilde{L}_1(x)$. Observe that $\tilde{T}$ maps bounded sets into relatively compact sets.

Now we shall show that if $\tilde{T}x = x$ and $x \in \overline{\Omega}_0$, then $x \in \Omega_0$. Assume on the contrary that $x \in \partial\,\Omega_0$ is such that $\tilde{T}x = x$. As $\|x'\|_\infty < R$, by virtue of (4.2.4) and (4.2.8), then the only possibility is

$$x(\sigma_1) = \max_{t \in I} x(t) = \rho \quad \text{or} \quad x(\sigma_1) = \min_{t \in I} x(t) = -\rho$$

for some $\sigma_1 \in [a, b)$. In the first case, if $\sigma_1 \in (a, b)$, then we have that $x'(\sigma_1) = 0$ and $x(t) > r + 1$ on $[\sigma_1, \sigma_2]$ for some $\sigma_2 \in (\sigma_1, b]$. By (4.2.9), $x''(t) = \tilde{f}(t, x(t), x'(t)) > 0$ for a.a. $t \in [\sigma_1, \sigma_2]$, which implies $x'(t) > 0$ on $(\sigma_1, \sigma_2]$, a contradiction with the fact that there is a maximum for x at σ_1. If $\sigma_1 = a$, since $x'(a) \le 0$, $x'(b) \ge 0$ and

$$0 = \tilde{L}_1(x(a), x(b), x'(a), x'(b), x) = x'(b) - x'(a),$$

then $x'(a) = 0$ and we can argue as before. A similar reasoning shows the impossibility of the case $x(\sigma_1) = \min_{t \in I} x(t) = -\rho$.

Moreover if $\tilde{T}x = x$ with $x \in \overline{\Omega}$, then $\|x\|_\infty < r$, because by (4.2.4) and (4.2.8) we obtain $\|x'\|_\infty < R$ and by Lemma 4.2.1 we have

$$\|x\|_\infty < \|\alpha\|_\infty + \|\beta\|_\infty + (b - a)R = r.$$

Then there are two possible cases:

(i) $\tilde{T}x = x$ for some $x \in \partial\,\Omega_0 \cup \partial\,\Omega_1 \cup \partial\,\Omega_2$. Then $\|x\|_\infty < r$, so $Tx = \tilde{T}x = x$ and x is a solution for (4.2.1).

(ii) $\tilde{T}x \neq x$ on $\partial\,\Omega_0 \cup \partial\,\Omega_1 \cup \partial\,\Omega_2$. By Theorem 4.1.9,

$$\deg\left(Id - \tilde{T}, \Omega_0\right) = \deg\left(Id - \tilde{T}, \Omega_1\right) = \deg\left(Id - \tilde{T}, \Omega_2\right) = 1.$$

Since $\alpha(\tau) > \beta(\tau)$ for some $\tau \in I$, $\Omega_1 \cap \Omega_2 = \emptyset$. Therefore, by the additivity property of the degree

$$\begin{aligned}\deg\left(Id - \tilde{T}, \Omega\right) &= \deg\left(Id - \tilde{T}, \Omega_0\right) - \deg\left(Id - \tilde{T}, \Omega_1\right) - \deg\left(Id - \tilde{T}, \Omega_2\right) \\ &= -1.\end{aligned}$$

Hence there exists $x \in \Omega$ such that $\tilde{T}x = x$. Then $\|x\|_\infty < r$ which implies $\tilde{f} = f$ and $\tilde{L}_1(x) = L_1(x)$, so $Tx = \tilde{T}x = x$, that is, x is a solution for problem (4.2.1).

To finish we have to show that

$$\{x\} \cap \tilde{\mathbb{T}}x \subset \left\{\tilde{T}x\right\} \quad \text{for every } x \in \overline{\Omega_0} \cap \tilde{\mathbb{T}}\overline{\Omega_0} \tag{4.2.11}$$

in order to guarantee that the degree is well–defined. The function $(x, y) \mapsto \tilde{f}(t, x, y)$ is continuous except over the graphs of the curves γ_n and these curves satisfy that $-r \leq \gamma_n \leq r$, so at these points $\tilde{f} = f$ and condition (4.2.11) can be proven as in Theorem 4.1.5. □

Remark 4.2.3. *Observe that Theorem 4.2.2 allows to obtain solutions via non-ordered lower and upper solutions to BVP with functional nonlinear boundary conditions, such as multi-point or maximum conditions, which fall outside the scope of the papers [123–127] where the periodic conditions were considered. In this sense, as far as the authors are aware, Theorem 4.2.2 gives new existence results even in the case of Carathéodory nonlinearities.*

The following example illustrates the existence of solutions for (4.2.1) with non-ordered lower and upper solutions.

Example 4.2.4. *Consider the problem* (4.0.1)–(4.0.2) *along with the following functional boundary conditions*

$$\begin{aligned} 0 &= L_1(x(0), x(1), x'(0), x'(1), x) = -x(1/2) - \max\nolimits_{t\in[0,1]} x(t), \\ 0 &= L_2(x(0), x(1)) = x(0) - x(1), \end{aligned}$$

and

$$f(t, x, y) = \lfloor 1/(t + |x|) \rfloor^{1/2} \cos(y) + 1/2,$$

for all $x \in \mathbb{R}$, $t \in [0, 1]$, $t > 0$ and where $\lfloor x \rfloor$ denotes the integer part of x.

First, standard arguments similar to those done in Example 3.1.11 show that condition $(C1^*)$ is satisfied. Moreover, $(C2^*)$ holds by taking $M(t) = t^{-1/2} + 1/2$.

Now we consider the functions $\alpha(t) = \pi\,(t - 1/2)^2$ and $\beta(t) = 0$ for $t \in [0, 1]$, which are a lower and an upper solutions for our problem, respectively. Indeed,

$$\begin{aligned} f(t, \alpha(t), \alpha'(t)) &= \lfloor 1/(t + \pi(t - 1/2)^2) \rfloor^{1/2} \cos(2\pi(t - 1/2)) + 1/2 \\ &\leq \lfloor 4\pi/(2\pi - 1) \rfloor^{1/2} + 1/2 \\ &= \sqrt{2} + 1/2 < 2\pi = \alpha''(t), \end{aligned}$$

and

$$\begin{aligned} L_1(\alpha(0), \alpha(1), \alpha'(0), \alpha'(1), \alpha) &= -\alpha(1/2) - \max_{t\in[0,1]} \alpha(t) = 0 - \pi/4 \leq 0, \\ L_2(\alpha(0), \alpha(1)) &= \alpha(0) - \alpha(1) = \pi/4 - \pi/4 = 0. \end{aligned}$$

Notice that α and β are not well–ordered.

On the other hand, for a.a. $t \in [0, 1]$, the function $f(t, \cdot, \cdot)$ is continuous on

$$\left(\mathbb{R} \setminus \bigcup_{\{n : t\in I_n^i,\ i=1,2\}} \{\gamma_n^i(t)\} \right) \times \mathbb{R},$$

where for each $n \in \mathbb{N}$,

$$\gamma_{n,1}(t) = -t + n^{-1} \quad \text{for all } t \in I_n^1 = [0, n^{-1}],$$

and

$$\gamma_{n,2}(t) = t - n^{-1} \quad \text{for all } t \in I_n^2 = [n^{-1}, 1].$$

These curves are inviable for the differential equation (see Definition 4.1.3). Indeed, there exists $\varepsilon > 0$ small enough such that $\cos(1+\varepsilon) \geq 0$ and thus for all $n \in \mathbb{N}$ and $i = 1, 2$,

$$f(t, x, y) \geq 1/2 > 1/4 + \gamma_{n,i}''(t)$$

for a.a. $t \in I_n^i$, all $x \in [\gamma_{n,i}(t) - \varepsilon, \gamma_{n,i}(t) + \varepsilon]$ and all $y \in [(-1)^i - \varepsilon, (-1)^i + \varepsilon]$.

Hence Theorem 4.2.2 guarantees the existence of a solution $x \in W^{2,1}(0,1)$ such that

$$\min\{\alpha(\tau_x), \beta(\tau_x)\} \leq x(\tau_x) \leq \max\{\alpha(\tau_x), \beta(\tau_x)\}$$

for some $\tau_x \in [0, 1]$.

4.3 Multiplicity results

The existence of multiple solutions for second–order boundary value problems was largely studied in the literature. For instance, in the papers [47, 123, 131, 143] the authors obtain results based on degree theory and the lower and upper solutions technique. Here, by combining the results we got in Sections 4.1 and 4.2, we obtain new multiplicity results in this line.

To this end we need to present the notion of strict lower and upper solutions, in order to guarantee that the boundary of the set where we compute the degree does not contain fixed points of the fixed point operator associated to our problem.

Definition 4.3.1. *We say that $\alpha \in \mathcal{C}(I)$ is a strict lower solution for the differential problem* (4.0.1)–(4.0.2) *if it satisfies the following conditions:*

(i) *For any $t_0 \in (a, b)$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$,*
or there exist an open interval I_0 and $\varepsilon_0 > 0$ such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and for a.a. $t \in I_0$, all $u \in [\alpha(t), \alpha(t) + \varepsilon_0]$ and all $v \in [\alpha'(t) - \varepsilon_0, \alpha'(t) + \varepsilon_0]$,

$$\alpha''(t) \geq f(t, u, v).$$

(ii) $D^+\alpha(a),\ D_-\alpha(b) \in \mathbb{R}$ *and* $L_1(\alpha(a), \alpha(b), D^+\alpha(a), D_-\alpha(b), \alpha) < 0$.

(iii) $L_2(\alpha(a), \alpha(b)) = 0$, $L_2(\alpha(a), \cdot)$ *and* $L_2(\cdot, \alpha(b))$ *are injective.*

Similarly $\beta \in \mathcal{C}(I)$ is a strict upper solution for (4.0.1)–(4.0.2) *if it satisfies:*

(i) *For any $t_0 \in (a, b)$, either $D^-\beta(t_0) > D_+\beta(t_0)$,*
or there exist an open interval I_0 and $\varepsilon_0 > 0$ such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and for a.a. $t \in I_0$, all $u \in [\beta(t) - \varepsilon_0, \beta(t)]$ and all $v \in [\beta'(t) - \varepsilon_0, \beta'(t) + \varepsilon_0]$,

$$\beta''(t) \leq f(t, u, v).$$

(*ii*) $D_+\beta(a),\ D^-\beta(b) \in \mathbb{R}$ *and* $L_1\left(\beta(a), \beta(b), D_+\beta(a), D^-\beta(b), \beta\right) > 0$.

(*iii*) $L_2\left(\beta(a), \beta(b)\right) = 0$*,* $L_2\left(\beta(a), \cdot\right)$ *and* $L_2\left(\cdot, \beta(b)\right)$ *are injective.*

Lemma 4.3.2. *Let* α, β *be strict lower and upper solutions and let* x *be a solution for problem* (4.0.1)–(4.0.2)*. Then* $\alpha \le x$ *implies* $\alpha < x$ *and* $x \le \beta$ *implies* $x < \beta$*.*

Proof. Let $\alpha \le x$. If $x(a) = \alpha(a)$, since $L_2(x(a), x(b)) = 0 = L_2(\alpha(a), \alpha(b))$ and $L_2(\alpha(a), \cdot)$ is injective, then $x(b) = \alpha(b)$, so $x - \alpha$ attains its minimum at a and b, and thus $x'(a) \ge \alpha'(a)$ and $x'(b) \le \alpha'(b)$. Therefore, by using the monotonicity conditions on L_1, we obtain the contradiction

$$0 = L_1(x(a), x(b), x'(a), x'(b), x) \le L_1(\alpha(a), \alpha(b), \alpha'(a), \alpha'(b), \alpha) < 0.$$

If $x(b) = \alpha(b)$, the fact that $L_2\left(\cdot, \alpha(b)\right)$ is injective implies that $x(a) = \alpha(a)$. Then we reach another contradiction as above.

Now assume that there exists $t_0 \in (a, b)$ such that $0 = x(t_0) - \alpha(t_0)$ and $x(t) > \alpha(t)$ for all $t \in (t_0, b]$. As $x - \alpha$ attains its minimum at t_0, we have that

$$x'(t_0) - D_-\alpha(t_0) \le x'(t_0) - D^+\alpha(t_0)$$

so, by the definition of strict lower solution, $x'(t_0) = \alpha'(t_0)$ and there exist an open interval I_0 and $\varepsilon_0 > 0$ such that $t_0 \in I_0$ and for a.a. $t \in I_0$, all $u \in [\alpha(t), \alpha(t) + \varepsilon_0]$ and all $v \in [\alpha'(t) - \varepsilon_0, \alpha'(t) + \varepsilon_0]$,

$$\alpha''(t) \ge f(t, u, v).$$

On the other hand,

$$\forall r > 0\, \exists\, t_r \in (t_0, t_0 + r) \text{ such that } \alpha'(t_r) < x'(t_r).$$

Hence, we can choose $t_r \in I_0$, $t_r > t_0$ such that $\alpha'(t_r) < x'(t_r)$ and for every $t \in (t_0, t_r)$,

$$x(t) \le \alpha(t) + \varepsilon_0, \qquad |x'(t) - \alpha'(t)| < \varepsilon_0.$$

Then for a.a. $t \in (t_0, t_r)$ we have that

$$\alpha''(t) \ge f(t, x(t), x'(t)),$$

and thus

$$x'(t_r) - \alpha'(t_r) = \int_{t_0}^{t_r} \left(x''(s) - \alpha''(s)\right) ds = \int_{t_0}^{t_r} \left(f(s, x(s), x'(s)) - \alpha''(s)\right) ds \le 0,$$

a contradiction. □

Notice that even in the case of Carathéodory nonlinearities Lemma 4.3.2 does not hold if the strict lower and upper solutions are defined just by taking strict inequalities in Definition 4.1.1, see [46, Chapter III-1].

Remark 4.3.3. *Notice that the proof of Lemma 4.3.2 remains valid for the following statement:*

Let α be a strict lower solution for problem (4.0.1)–(4.0.2) *and let $\beta \in W^{2,1}(I)$ be an upper solution. Then $\alpha \leq \beta$ implies $\alpha < \beta$.*

Theorem 4.3.4. *Assume there exist $\alpha_1, \alpha_2 \in W^{1,\infty}(I)$ lower solutions and $\beta \in W^{1,\infty}(I)$ an upper solution for* (4.2.1) *such that $\alpha_1 \leq \beta$, $\alpha_2(\tau) > \beta(\tau)$ for some $\tau \in I$ and α_2, β are strict lower and upper solutions, respectively.*

Suppose that the function f satisfies the hypotheses $(C1)$, $(C2^*)$ and $(C3^*)$.

Then the problem (4.2.1) *has at least two solutions $x_1, x_2 \in W^{2,1}(I)$ satisfying that $\alpha_1 \leq x_1 < \beta$ and there exist $t_1, t_2 \in I$ such that $x_2(t_1) < \alpha_2(t_1)$ and $x_2(t_2) > \beta(t_2)$.*

Proof. By Theorem 4.1.9 there exists a solution $x_1 \in W^{2,1}(I)$ for problem (4.2.1) such that $\alpha_1 \leq x_1 \leq \beta$. On the other hand, the fact that α_2 and β are strict lower and upper solutions, respectively, implies that the case (i) in the proof of Theorem 4.2.2 is not possible (due to Lemma 4.3.2), so (4.2.1) has a solution $x_2 \in \Omega$ where Ω is defined as in (4.2.10) taking $\alpha = \alpha_2$. Observe that Ω is the set of functions satisfying (4.2.2) with $\alpha = \alpha_2$. $\square$

Remark 4.3.5. *A similar result can be obtained by interchanging the role of lower and upper solutions.*

Example 4.3.6. *Consider the problem* (4.0.1) *along with the boundary conditions*

$$\begin{aligned} 0 &= L_1(x(0), x(1), x'(0), x'(1), x) = -x(0)^2 - x(1/4), \\ 0 &= L_2(x(0), x(1)) = x(0) - x(1), \end{aligned}$$

and the nonlinearity

$$f(t, x, y) = \lfloor 1/(t + |x|) \rfloor^{1/2} \cos(y) + 1/2,$$

for all $x \in \mathbb{R}$, $t \in [0, 1]$, $t > 0$.

This function f satisfies conditions $(C1)$, $(C2^*)$ and $(C3^*)$ (see Example 4.2.4). On the other hand, one can easily verify that $\alpha_1(t) = \pi(t - 1/2)^2 - \pi$ and $\alpha_2(t) = \pi(t - 1/2)^2$ are strict lower solutions for the previous problem and that $\beta(t) = -(t - 1/2)^2/2$ is a strict upper solution. Moreover, they satisfy the following order conditions: $\alpha_1 \leq \beta \leq \alpha_2$. Hence, Theorem 4.3.4 ensures that the considered problem has at least two different solutions.

Following the ideas of Amann [5], in [47] a three solution theorem is given in presence of two pairs of lower and upper solutions with order relations, see Figure 4.3.1. Here an analogous result is possible for our problem (4.2.1) as an immediate consequence of Theorems 4.1.9 and 4.2.2.

Theorem 4.3.7. *Assume there exist $\alpha_1, \alpha_2 \in W^{1,\infty}(I)$ lower solutions and $\beta_1, \beta_2 \in W^{1,\infty}(I)$ upper solutions for* (4.2.1) *such that $\alpha_1 \leq \beta_1$, $\alpha_2 \leq \beta_2$, $\alpha_2(\tau) > \beta_1(\tau)$ for some $\tau \in I$ and α_2, β_1 are strict lower and upper solutions, respectively.*

Suppose that the function f satisfies the hypotheses $(C1)$, $(C2^*)$ and $(C3^*)$.

Then problem (4.2.1) *has at least three solutions $x_1, x_2, x_3 \in W^{2,1}(I)$ satisfying that $\alpha_1 \leq x_1 < \beta_1$, $\alpha_2 < x_2 \leq \beta_2$ and there exist $t_1, t_2 \in I$ such that $x_3(t_1) < \alpha_2(t_1)$ and $x_3(t_2) > \beta_1(t_2)$.*

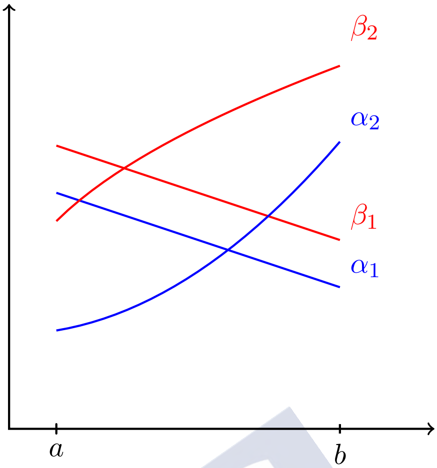

Figure 4.3.1: Order relations for the lower and upper solutions in Theorem 4.3.7.

If the lower and upper solutions are strict, then by means of the topological degree it is possible to obtain a three-solution result, see [143], where f need not be bounded with respect to the spatial variables and, moreover, the more general boundary conditions (4.0.2) can be considered. However, to do so, more restrictive conditions about the order relations for the lower and upper solutions are required in this case, see Figure 4.3.2. This is the idea of the following result.

Theorem 4.3.8. *Assume there exist $\alpha_1, \alpha_2 \in W^{1,\infty}(I)$ strict lower solutions and $\beta_1, \beta_2 \in W^{1,\infty}(I)$ strict upper solutions for problem* (4.0.1)–(4.0.2) *with the following order relations: $\alpha_1 < \beta_1 \leq \beta_2$, $\alpha_1 \leq \alpha_2 < \beta_2$, $\alpha_2(\tau) > \beta_1(\tau)$ for some $\tau \in I$.*

Suppose that the function f satisfies the hypotheses $(C1)$, $(C2)$ and $(\widetilde{C3})$ taking $\alpha = \alpha_1$ and $\beta = \beta_2$.

Then problem (4.0.1)–(4.0.2) *has at least three solutions $x_1, x_2, x_3 \in W^{2,1}(I)$ such that $\alpha_1 < x_1 < \beta_1$, $\alpha_2 < x_2 < \beta_2$ and there exist $t_1, t_2 \in I$ such that $x_3(t_1) < \alpha_2(t_1)$ and $x_3(t_2) > \beta_1(t_2)$.*

Proof. Consider the sets

$$\Omega_0 = \left\{x \in \mathcal{C}^1(I) : \alpha_1 < x < \beta_2 \text{ on } I,\ \|x'\|_\infty < R\right\},$$
$$\Omega_1 = \{x \in \Omega_0 : \alpha_1 < x < \beta_1 \text{ on } I\},$$
$$\Omega_2 = \{x \in \Omega_0 : \alpha_2 < x < \beta_2 \text{ on } I\},$$

where R is given by condition $(C2)$. Note that since the lower and upper solutions are strict, then the operator T, defined as in (4.1.28), has no fixed points on the boundary of Ω_i, $i = 0, 1, 2$. Hence, by Theorem 4.1.8,

$$\deg(Id - T, \Omega_0) = \deg(Id - T, \Omega_1) = \deg(Id - T, \Omega_2) = 1.$$

Then T has fixed points $x_1 \in \Omega_1$ and $x_2 \in \Omega_2$ which are different fixed points because $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \emptyset$ due to the order hypotheses about the lower and upper solutions. Now the properties of the degree ensure that

$$\begin{aligned}\deg\left(Id - T, \Omega_0 \setminus \left(\overline{\Omega}_1 \cup \overline{\Omega}_2\right)\right) &= \deg\left(Id - T, \Omega_0\right) - \deg\left(Id - T, \Omega_1\right) - \deg\left(Id - T, \Omega_2\right) \\ &= -1,\end{aligned}$$

so T has a third fixed point $x_3 \in \Omega_0 \setminus \left(\overline{\Omega}_1 \cup \overline{\Omega}_2\right)$. Since $x_3 \notin \overline{\Omega}_1 \cup \overline{\Omega}_2$, there exist $t_1, t_2 \in I$ such that $x_3(t_1) < \alpha_2(t_1)$ and $x_3(t_2) > \beta_1(t_2)$. □

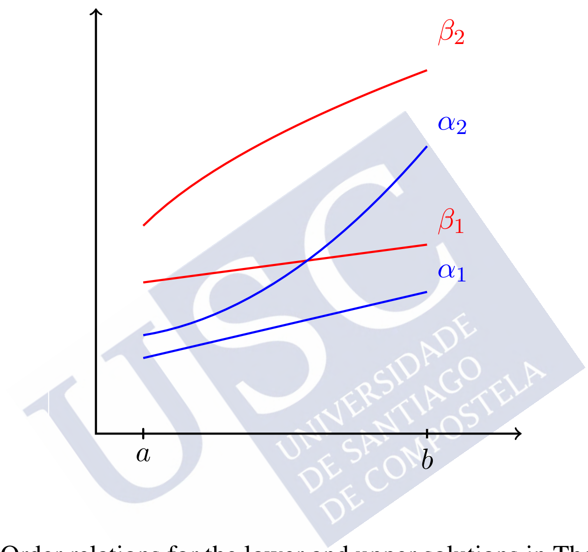

Figure 4.3.2: Order relations for the lower and upper solutions in Theorem 4.3.8.

To finish we illustrate our multiplicity results by an example.

Example 4.3.9. *Consider the problem* (4.0.1) *along with the following nonlinear boundary conditions*

$$\begin{aligned}0 &= L_1(x(0), x(3\pi), x'(0), x'(3\pi), x) = \cos(x(0)),\\ 0 &= L_2(x(0), x(3\pi)) = x(0) - x(3\pi) - \sin(x(3\pi)),\end{aligned}$$

and the function

$$f(t, x, y) = \phi(x - t^2)\cos(x) - |y|\,,$$

for all $x, y \in \mathbb{R}$ and $t \in [0, 3\pi]$, where the function ϕ is defined below. Consider a bijection between the rational numbers and the positive integers and denote as $\{q_n\}_{n\in\mathbb{N}}$ the sequence of rational numbers. The function $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(u) = \sum_{n:\, q_n < u} 2^{-n}.$$

Notice that ϕ is continuous at the irrational points and discontinuous at the rational numbers, see [130, Prop. 2, p. 108-109]. Moreover, $\phi(u) \in (0, 1)$ for each $u \in \mathbb{R}$ as far as it is $\phi(u) < \sum_{n=1}^{\infty} 2^{-n} = 1$.

Observe that $\alpha_1 \equiv -\pi$ *and* $\alpha_2 \equiv \pi$ *are strict lower solutions for the problem and* $\beta_1 \equiv 0$ *and* $\beta_2 \equiv 2\pi$ *are strict upper solutions.*

On the other hand, for a.a. $t \in [0, 3\pi]$ *the function* $(x, y) \mapsto f(t, x, y)$ *is continuous on* $\left([-\pi, 2\pi] \setminus \bigcup_{\{n\,:\,t \in I_n\}} \{\gamma_n(t)\}\right) \times \mathbb{R}$*, where for each* $n \in \mathbb{N}$*,*

$$\gamma_n(t) = t^2 + q_n \quad \textit{for all } t \in I_n = [0, 3\pi].$$

In addition, the curves γ_n *are inviable admissible discontinuity curves. Indeed, for every* $n \in \mathbb{N}$ *and for all* $x, y \in \mathbb{R}$ *and* $t \in [0, 3\pi]$*, we have*

$$f(t, x, y) = \phi(x - t^2)\cos(x) - |y| \leq 1 < 2 = \gamma_n''(t).$$

Therefore, Theorem 4.3.8 guarantees the existence of at least three solutions for the considered problem satisfying some suitable localization conditions.

4.4 Second order problems on unbounded domains

We study the existence of solutions of the nonlinear equation on the half–line

$$x''(t) = f(t, x(t), x'(t)), \quad t \in \mathbb{R}_+, \tag{4.4.1}$$

coupled with the functional boundary conditions

$$L(x(0), x'(0), x) = 0, \quad x'(+\infty) := \lim_{t \to +\infty} x'(t) = B, \tag{4.4.2}$$

where $B \in \mathbb{R}$ and $L : \mathbb{R}^2 \times \mathcal{C}(\mathbb{R}_+) \to \mathbb{R}$ is a continuous function and it is nonincreasing in the second and third variables.

The functional boundary conditions considered here are quite general and were recently studied in [57]. They include the extensively studied Sturm–Liouville conditions [96, 97], integral boundary conditions [2], multipoint [145] and other boundary conditions with, for example, maximum or minimum arguments. Observe that the boundary condition at infinity implies that solutions, if they exist, are unbounded if $B \neq 0$.

The main novelty here is that we allow the function f to satisfy weaker conditions than the usual ones in the literature. In particular, it may be discontinuous in the second variable over a countable number of *admissible* curves.

We shall assume the existence of well ordered upper and lower solutions on unbounded domains and a Nagumo condition to control the first derivative in order to obtain existence results for (4.4.1)–(4.4.2). Another interesting point is that we are able to relax the usual definition of lower and upper solutions, cf. [2, 57, 96–98], and therefore our main existence result is new even in the classical case of continuous right–hand sides in (4.4.1). See Remark 4.4.13 for details.

Furthermore, we also prove the existence of extremal solutions for (4.4.1)–(4.4.2) by adapting the arguments in [34] to unbounded domains. This is also a new result even for a continuous nonlinearity.

4.4.1 Existence results on the half line

First we present some definitions and results concerning the problem (4.4.1)–(4.4.2). Consider the space

$$X = \left\{ x \in \mathcal{C}^1(\mathbb{R}_+) : \lim_{t\to\infty} \frac{x(t)}{1+t} \in \mathbb{R} \text{ and } \lim_{t\to\infty} x'(t) \in \mathbb{R} \right\}$$

endowed with a Bielecki–type norm in $\mathcal{C}^1(\mathbb{R}_+)$,

$$\|x\| := \max\left\{\|x\|_0, \|x\|_1\right\},$$

where

$$\|x\|_0 = \sup_{0\le t<\infty} \frac{|x(t)|}{1+t} \quad \text{and} \quad \|x\|_1 = \sup_{0\le t<\infty} |x'(t)|.$$

It is clear that $(X, \|\cdot\|)$ is a Banach space. For convenience we denote

$$Y = \left\{ x \in \mathcal{C}(\mathbb{R}_+) : \lim_{t\to\infty} \frac{x(t)}{1+t} \in \mathbb{R} \right\}.$$

Our approach is based on lower and upper solutions method and fixed point theory. Thus we define the lower and upper solutions for problem (4.4.1)–(4.4.2) and we present a Nagumo condition which gives some *a priori* bound on the first derivative for all possible solutions of the differential equation (4.4.1) between the lower and the upper solutions.

Definition 4.4.1. *A function $\alpha \in Y$ is said to be a lower solution for the problem* (4.4.1)–(4.4.2) *if the following conditions are satisfied:*

(i) *For any $t_0 \in (0,\infty)$, either $D_-\alpha(t_0) < D^+\alpha(t_0)$, or there exists an open interval I_0 such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and*

$$\alpha''(t) \ge f(t, \alpha(t), \alpha'(t)) \quad \textit{for a.a. } t \in I_0.$$

(ii) *$D^+\alpha(0) \in \mathbb{R}$ and $L(\alpha(0), D^+\alpha(0), \alpha) \le 0$.*

(iii) *There exists $N \in \mathbb{N}$ such that $\alpha \in W^{2,1}((N,\infty))$ and $\alpha'(+\infty) \le B$.*

Similarly $\beta \in Y$ is an upper solution for (4.4.1)–(4.4.2) *if it satisfies:*

(i) *For any $t_0 \in (0,\infty)$, either $D^-\beta(t_0) > D_+\beta(t_0)$, or there exists an open interval I_0 such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and*

$$\beta''(t) \le f(t, \beta(t), \beta'(t)) \quad \textit{for a.a. } t \in I_0.$$

(ii) *$D_+\beta(0) \in \mathbb{R}$ and $L(\beta(0), D_+\beta(0), \beta) \ge 0$.*

(iii) *There exists $N \in \mathbb{N}$ such that $\beta \in W^{2,1}((N,\infty))$ and $\beta'(+\infty) \ge B$.*

Proposition 4.4.2. *Let $\bar\alpha, \bar\beta \in Y$ be such that $\bar\alpha \le \bar\beta$. Assume there exist a continuous function $\bar N : [0,\infty) \to (0,\infty)$ and $\bar M \in L^1(\mathbb{R}_+)$ such that for each $r > 0$ we have*

$$\int_r^\infty \frac{1}{\bar N(s)} ds = +\infty. \tag{4.4.3}$$

Define $E := \left\{(t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^2 : \bar\alpha(t) \le x \le \bar\beta(t)\right\}$. Then, there exists $R > 0$ such that for every function $f : E \to \mathbb{R}$ satisfying for a.a. $t \in \mathbb{R}_+$ and all $(x,y) \in \mathbb{R}^2$ with $(t,x,y) \in E$,

$$|f(t,x,y)| \le \bar M(t)\bar N(|y|),$$

and for every solution x of (4.4.1) *such that $\bar\alpha \le x \le \bar\beta$, we have*

$$\|x\|_1 < R.$$

Proof. Choose $\delta > 0$ and let $r > 0$ such that

$$r > \max\left\{\sup_{\delta \le t < \infty} \frac{\bar\beta(t) - \bar\alpha(0)}{t}, \sup_{\delta \le t < \infty} \frac{\bar\beta(0) - \bar\alpha(t)}{t}\right\},$$

which is well–defined since $\bar\alpha, \bar\beta \in Y$. Let $R > r$ be big enough such that

$$\int_r^R \frac{1}{\bar N(s)}\, ds > \int_0^\infty \bar M(s)\, ds. \tag{4.4.4}$$

Let x be a solution of (4.4.1) and $t \in \mathbb{R}_+$ such that $x'(t) > R$.

If $x'(t) > r$ for all $t \in \mathbb{R}_+$, then for any $T > \delta$ we have

$$\frac{\bar\beta(T) - \bar\alpha(0)}{T} \ge \frac{x(T) - x(0)}{T} = \frac{\int_0^T x'(s)\, ds}{T} > r \ge \frac{\bar\beta(T) - \bar\alpha(0)}{T},$$

a contradiction.

Therefore, there exist $t_0 < t_1$ (or $t_1 < t_0$) such that $x'(t_0) = r$, $x'(t_1) = R$ and, moreover, $r \le x'(s) \le R$ in $[t_0, t_1]$ (or $[t_1, t_0]$). Then we have

$$\begin{aligned}\int_r^R \frac{1}{\bar N(s)} ds &= \int_{t_0}^{t_1} \frac{x''(s)}{\bar N(x'(s))} ds = \int_{t_0}^{t_1} \frac{f(s,x(s),x'(s))}{\bar N(x'(s))} ds \\ &\le \int_{t_0}^{t_1} \bar M(s)\, ds \le \int_0^\infty \bar M(s)\, ds,\end{aligned}$$

a contradiction, so we deduce that $x'(t) < R$.

In the same way we prove that $x'(t) > -R$. □

Remark 4.4.3. *Observe that condition* (4.4.3) *in Proposition 4.4.2 could just be replaced by condition* (4.4.4)*. However, the first one is easier to check in practice.*

Remark 4.4.4. *Notice that a better condition about $\bar{N}$, which allows a quadratic growth with respect to the third variable of the nonlinear term f for the differential equation* (4.4.1)*, is commonly employed in the literature (see, e.g. [2, 96, 97, 99]), namely,*

$$\int_r^{\infty} \frac{s}{\bar{N}(s)} ds = +\infty.$$

Unfortunately, this type of conditions require harder assumptions about $\bar{M}$ such as

$$\sup_{0 \le t < \infty} (1+t)^k \bar{M}(t) < +\infty \quad \text{for some } k > 1.$$

In particular, the previous hypothesis avoids that $\bar{M}$ could be singular at $t = 0$.

Lemma 4.4.5. *Let $h \in L^1(\mathbb{R}_+)$. Then $x \in X$ is the unique solution of the problem*

$$\begin{aligned} x''(t) &= h(t) \quad t \in \mathbb{R}_+, \\ x(0) &= A, \\ x'(+\infty) &= B, \end{aligned}$$

with $A, B \in \mathbb{R}$, if and only if,

$$x(t) = A + \int_0^t \left(B - \int_s^{\infty} h(r)\, dr \right) ds.$$

Proof. It is immediate, see [57, Lemma 2.3]. □

In order to apply fixed point theorems of Chapter 2 it is necessary to guarantee that certain sets are relatively compact. Nevertheless, the classical Ascoli–Arzelà's theorem fails due to the non–compactness of the infinite interval $\mathbb{R}_+$, so this difficulty is overcome by the following result, see [1].

Lemma 4.4.6. *Let $A \subset X$. The set A is relatively compact if the following conditions hold:*

(a) *A is uniformly bounded in X;*

(b) *the functions belonging to A are equicontinuous on any compact interval of $\mathbb{R}_+$;*

(c) *the functions f from A are equiconvergent at $+\infty$, i.e., given $\varepsilon > 0$ there exists $T(\varepsilon) > 0$ such that $\|f(t) - f(+\infty)\| < \varepsilon$ for any $t > T(\varepsilon)$ and $f \in A$.*

Now we shall construct a modified problem for proving the existence of solutions for (4.4.1)–(4.4.2) under well–ordered lower and upper solutions.

Suppose that there exist $\alpha, \beta \in W^{1,\infty}((0,\infty))$ a lower and an upper solutions for problem (4.4.1)–(4.4.2), respectively, such that $\alpha(t) \le \beta(t)$ for all $t \in \mathbb{R}_+$.

Assume that for $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ the following conditions hold:

$(H1)$ Compositions $t \in \mathbb{R}_+ \mapsto f(t, x(t), y(t))$ are measurable whenever $x(t)$ is continuous and $y(t)$ is measurable;

$(H2)$ There exist a continuous function $N : [0,\infty) \to (0,\infty)$ and $M \in L^1(\mathbb{R}_+)$ such that for each $r > 0$,

$$\int_r^\infty \frac{1}{N(s)} ds = +\infty,$$

and for a.a. $t \in \mathbb{R}_+$, all $x \in [\alpha(t), \beta(t)]$ and all $y \in \mathbb{R}$, we have

$$|f(t,x,y)| \le M(t)N(|y|).$$

Consider the modified problem

$$\begin{cases} x''(t) = f(t, \varphi(t,x(t)), \delta_R((\varphi(t,x(t)))')), \\ x(0) = \varphi(0, x(0) - L(x(0), x'(0), x)), \\ x'(+\infty) = B, \end{cases} \tag{4.4.5}$$

where

$$\varphi(t,x) = \max\{\min\{x, \beta(t)\}, \alpha(t)\} \quad \text{for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{4.4.6}$$

and

$$\delta_R(y) = \max\{\min\{y, R\}, -R\} \quad \text{for all } y \in \mathbb{R}, \tag{4.4.7}$$

with R given by Proposition 4.4.2.

Notice that for $t \in \mathbb{R}_+$,

$$\varphi(t, x(t)) = x(t) + (\alpha - x)^+(t) - (x - \beta)^+(t),$$

where $(u)^+(t) = \max\{u(t), 0\}$. Hence, we have $\varphi(\cdot, x) \in W^{1,\infty}((0,\infty))$ and

$$(\varphi(t,x(t)))' = \frac{d}{dt}\varphi(t,x(t)) = \begin{cases} \alpha'(t), & \text{if } x(t) < \alpha(t), \\ x'(t), & \text{if } \alpha(t) \le x(t) \le \beta(t), \\ \beta'(t), & \text{if } x(t) > \beta(t). \end{cases}$$

Furthermore, if $\{x_n\} \subset X$ is such that $x_n \to x$ in X, then

$$\lim_{n\to\infty} (\varphi(t, x_n(t)))' = (\varphi(t,x(t)))',$$

see [46, 138].

The operator $T : X \to X$ associated to the modified problem (4.4.5) is defined as

$$Tx(t) = L^*(x) + \int_0^t \left(B - \int_s^\infty f(r, \varphi(r, x(r)), \delta_R((\varphi(r, x(r)))'))\, dr \right) ds, \tag{4.4.8}$$

where $L^*(x) = \varphi(0, x(0) - L(x(0), x'(0), x))$.

In order to achieve an existence result for problem (4.4.1)–(4.4.2) we shall prove that the operator T has a fixed point by applying Theorem 2.1.2. In this direction we present some previous lemmas.

Lemma 4.4.7. *Assume that conditions $(H1)$ and $(H2)$ hold. Then the operator T is well defined.*

Proof. Given $x \in X$, we shall show that $Tx \in X$. First, we have

$$\begin{aligned}\lim_{t\to\infty} \frac{(Tx)(t)}{1+t} &= \lim_{t\to\infty} \frac{\int_0^t \left(B - \int_s^\infty f(r,\varphi(r,x(r)),\delta_R((\varphi(r,x(r)))'))\,dr\right) ds}{1+t} \\ &= \lim_{t\to\infty} \frac{Bt - \int_0^t \left(\int_s^\infty f(r,\varphi(r,x(r)),\delta_R((\varphi(r,x(r)))'))\,dr\right) ds}{1+t}.\end{aligned}$$

Now, from $(H2)$, (4.4.6) and (4.4.7) we obtain that

$$\lim_{t\to\infty} \frac{\left|\int_0^t \left(\int_s^\infty f(r,\varphi(r,x(r)),\delta_R((\varphi(r,x(r)))'))\,dr\right) ds\right|}{1+t} \leq \lim_{t\to\infty} \frac{\int_0^t \left(\int_s^\infty \tilde{M}(r)\,dr\right) ds}{1+t},$$

where $\tilde{M}(t) = \max_{s\in[0,R]} N(s)M(t)$.

Let us show that

$$\lim_{t\to\infty} \frac{\int_0^t \left(\int_s^\infty \tilde{M}(r)\,dr\right) ds}{1+t} = 0.$$

Observe that the function $F : \mathbb{R}_+ \to \mathbb{R}_+$ defined as $F(s) = \int_s^\infty \tilde{M}(r)\,dr$ is continuous and nonincreasing, so the function $t \mapsto \int_0^t F(s)\,ds$ is nondecreasing and thus there exists the limit

$$l := \lim_{t\to\infty} \int_0^t F(s)\,ds.$$

We consider two different cases. If $l < +\infty$, then we have

$$\lim_{t\to\infty} \frac{\int_0^t \left(\int_s^\infty \tilde{M}(r)\,dr\right) ds}{1+t} = \lim_{t\to\infty} \frac{\int_0^t F(s)\,ds}{1+t} = 0. \tag{4.4.9}$$

On the other hand, if $l = +\infty$, then L'Hôpital's rule ensures again that (4.4.9) holds. Therefore, we conclude that

$$\lim_{t\to\infty} \frac{(Tx)(t)}{1+t} = B < +\infty. \tag{4.4.10}$$

Moreover,

$$\lim_{t\to\infty} (Tx)'(t) = \lim_{t\to\infty} \left(B - \int_t^\infty f(s,\varphi(s,x(s)),\delta_R((\varphi(s,x(s)))'))\,ds\right) = B. \tag{4.4.11}$$

Therefore T is well defined. □

Lemma 4.4.8. *Assume that conditions $(H1)$ and $(H2)$ hold. Then TX is relatively compact.*

Proof. Let us apply Lemma 4.4.6. First, let us show that TX is uniformly bounded in X. For $x \in X$ we have

$$\|Tx\|_0 = \sup_{t\in\mathbb{R}_+} \frac{|(Tx)(t)|}{1+t} \leq \sup_{t\in\mathbb{R}_+} \left(\max\{|\alpha(0)|, |\beta(0)|\} + \frac{(|B|+k_1)t}{1+t}\right) \leq \rho$$

and

$$\begin{aligned}\|Tx\|_1 &= \sup_{t\in\mathbb{R}_+} |(Tx)'(t)| \\ &\le \sup_{t\in\mathbb{R}_+} \left| B - \int_t^\infty f(s,\varphi(s,x),\delta_R((\varphi(s,x))'))\,ds \right| \le |B| + k_1 \le \rho,\end{aligned}$$

where $\rho := \max\{|\alpha(0)|, |\beta(0)|\} + |B| + k_1$ and $k_1 := \int_0^\infty \tilde{M}(r)\,dr$.

Next, given $T_0 > 0$ let us prove that the functions in TX are equicontinuous on $[0, T_0]$. Let us consider $t_1, t_2 \in [0, T_0]$, then we have

$$\begin{aligned}\left|\frac{Tx(t_1)}{1+t_1} - \frac{Tx(t_2)}{1+t_2}\right| &\le \frac{|(t_2-t_1)L^*(x)|}{(1+t_1)(1+t_2)} \\ &\quad + \frac{\left|\int_{t_1}^{t_2} \left(B - \int_s^\infty f(r,\varphi(r,x(r)),\delta_R((\varphi(r,x(r)))'))\,dr\right) ds\right|}{(1+t_1)(1+t_2)} \\ &\le \frac{1}{(1+t_1)(1+t_2)} \left(\max\{|\alpha(0)|, |\beta(0)|\} + |B| + k_1\right)|t_2 - t_1| \\ &\le \left(\max\{|\alpha(0)|, |\beta(0)|\} + |B| + k_1\right)|t_2 - t_1|,\end{aligned}$$

which tends to zero as $|t_2 - t_1| \to 0$. Moreover,

$$\begin{aligned}|(Tx)'(t_1) - (Tx)'(t_2)| &= \left|\int_{t_1}^{t_2} f(s,\varphi(s,x(s)),\delta_R((\varphi(s,x(s)))'))\,ds\right| \\ &\le \left|\int_{t_1}^{t_2} \tilde{M}(s)\,ds\right| \to 0,\end{aligned}$$

as $|t_2 - t_1| \to 0$.

Finally, let us prove that TX is equiconvergent at $+\infty$. Let $x \in X$, then by (4.4.9) and (4.4.10) we have

$$\begin{aligned}\left|\frac{Tx(t)}{1+t} - \lim_{t\to\infty}\frac{Tx(t)}{1+t}\right| &= \left|\frac{Tx(t)}{1+t} - B\right| \\ &\le \frac{|L^*(x)| + |B|}{1+t} + \frac{\int_0^t \int_s^\infty f(r,\varphi(r,x(r)),\delta_R((\varphi(r,x(r)))'))\,dr ds}{1+t} \\ &\le \frac{|L^*(x)| + |B|}{1+t} + \frac{\int_0^t \int_s^\infty \tilde{M}(r)\,dr ds}{1+t} \to 0,\end{aligned}$$

as $t \to +\infty$. In addition, by (4.4.11),

$$\left|(Tx)'(t) - \lim_{t\to\infty}(Tx)'(t)\right| = |(Tx)'(t) - B| \le \int_t^\infty \tilde{M}(s)\,ds \to 0,$$

as $t \to +\infty$. □

We shall allow the nonlinearity f to be discontinuous in the second variable over some curves as in Definition 4.1.3. As far as we know, this is the first time that such type of discontinuity conditions were presented for boundary value problems on infinity intervals.

Definition 4.4.9. *An admissible discontinuity curve for the differential equation* (4.4.1) *is a* $W^{2,1}$ *function* $\gamma : [a,b] \subset \mathbb{R}_+ \longrightarrow \mathbb{R}$ *satisfying one of the following conditions:*

either $\gamma''(t) = f(t, \gamma(t), \gamma'(t))$ *for a.a.* $t \in [a,b]$ *(and we then say that* γ *is viable for the differential equation),*

or there exist $\varepsilon > 0$ *and* $\psi \in L^1(a,b)$, $\psi(t) > 0$ *for a.a.* $t \in [a,b]$, *such that*
either

$$\gamma''(t) + \psi(t) < f(t,y,z) \quad \text{for a.a. } t \in [a,b], \text{ all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon] \text{ and all } z \in [\gamma'(t)-\varepsilon, \gamma'(t)+\varepsilon], \tag{4.4.12}$$

or

$$\gamma''(t) - \psi(t) > f(t,y,z) \quad \text{for a.a. } t \in [a,b], \text{ all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon] \text{ and all } z \in [\gamma'(t)-\varepsilon, \gamma'(t)+\varepsilon]. \tag{4.4.13}$$

We say that the admissible discontinuity curve γ *is inviable for the differential equation if it satisfies* (4.4.12) *or* (4.4.13).

Now we present the result which gives the main difference between our existence results and the classical ones. It guarantees that condition (2.1.1) holds what allows to avoid the continuity of the operator T and thus the continuity of f.

Lemma 4.4.10. *Assume that conditions* $(H1)$, $(H2)$ *and*

$(H3)$ *There exist admissible discontinuity curves* $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ *such that* $\alpha \le \gamma_n \le \beta$ *on* $\mathbb{R}_+$, *their derivatives are uniformly bounded and for a.a.* $t \in \mathbb{R}_+$, *the function* $(x,y) \mapsto f(t,x,y)$ *is continuous on*

$$\left([\alpha(t), \beta(t)] \setminus \bigcup_{\{n \,:\, t \in I_n\}} \{\gamma_n(t)\} \right) \times \mathbb{R};$$

hold.

Then the operator T *satisfies condition* (2.1.1) *for all* $x \in X$, *i.e.,* $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$ *where* $\mathbb{T}$ *is the cc-envelope of* T.

Its proof is similar to that done in Theorem 4.1.5, so we omit it here.

Remark 4.4.11. *Admissible discontinuity curves may be defined in infinite intervals as the union of those in Definition 4.4.9.*

Now we establish an existence and localization result for (4.4.1)–(4.4.2).

Theorem 4.4.12. *Suppose that there exist* α *and* β *lower and upper solutions to* (4.4.1)–(4.4.2), *respectively, such that* $\alpha \le \beta$ *on* $\mathbb{R}_+$ *and* $\alpha, \beta \in W^{1,\infty}((0,\infty))$. *Assume that conditions* $(H1)$–$(H3)$ *hold. Then problem* (4.4.1)–(4.4.2) *has at least one solution* $x \in X$ *such that* $\alpha(t) \le x(t) \le \beta(t)$.

Proof. For simplicity, we divide the proof in several steps. First, we will prove that the modified problem (4.4.5) has at least one solution, that is, we will ensure that the operator T defined as in (4.4.8) has a fixed point. Then, we will show that this fixed point is a solution for problem (4.4.1)–(4.4.2).

Step 1. Problem (4.4.5) *has at least a solution* $x \in X$.

By Lemma 4.4.7, the operator T is well defined. Consider the closed and convex set D defined as

$$D = \{x \in X : \|x\| \leq \rho\},$$

where

$$\rho := \max\{|\alpha(0)|, |\beta(0)|\} + |B| + k_1,$$

and $k_1 = \int_0^\infty \tilde{M}(r)\, dr$.

For $x \in D$ we have

$$\|Tx\|_0 = \sup_{t\in\mathbb{R}_+} \frac{|(Tx)(t)|}{1+t} \leq \sup_{t\in\mathbb{R}_+} \left(\max\{|\alpha(0)|, |\beta(0)|\} + \frac{(|B| + k_1)t}{1+t} \right) \leq \rho$$

and

$$\begin{aligned} \|Tx\|_1 &= \sup_{t\in\mathbb{R}_+} |(Tx)'(t)| \\ &\leq \sup_{t\in\mathbb{R}_+} \left| B - \int_t^\infty f(s, \varphi(s,x), \delta_R((\varphi(s,x))'))\, ds \right| \leq |B| + k_1 \leq \rho. \end{aligned}$$

Therefore, $TD \subset D$. In addition, TD is relatively compact by virtue of Lemma 4.4.8 and T satisfies condition (2.1.1), by Lemma 4.4.10. Then Theorem 2.1.2 implies that the operator T has at least one fixed point $x \in D$.

Step 2. Every solution of (4.4.5) *satisfies* $\alpha(t) \leq x(t) \leq \beta(t)$ *for all* $t \in \mathbb{R}_+$.

Let $x \in X$ be a solution of (4.4.5). Suppose that there exists $t \in \mathbb{R}_+$ such that $\alpha(t) > x(t)$. Then

$$\inf_{t\in\mathbb{R}_+} \{x(t) - \alpha(t)\} < 0.$$

First, this cannot happen at $t = 0$ since

$$x(0) = \varphi\left(0, x(0) - L(x(0), x'(0), x)\right) \geq \alpha(0).$$

On the other hand, if the infimum is attained as t tends to infinity, then there exists $T > 0$ such that

$$x(t) - \alpha(t) < 0 \quad \text{for all } t \in [T, \infty),$$

and $\alpha \in W^{2,1}((T,\infty))$. Hence, we have

$$x''(t) = f(t, \varphi(t,x), \delta_R((\varphi(t,x))')) = f(t, \alpha(t), \alpha'(t)) \leq \alpha''(t) \quad \text{for a.a. } t \in [T,\infty), \tag{4.4.14}$$

and so $x - \alpha$ is a concave function on $[T, \infty)$.

Then there are two options: either there exists $t_0 > T$ such that t_0 is a relative minimum (in this case the reasoning is analogous to that we do below when the infimum is attained at $t_0 \in (0,\infty)$) or there exists $\tilde{T} > T$ such that $(x-\alpha)'(\tilde{T}) < 0$ and, by (4.4.14),

$$(x-\alpha)'(t) \le (x-\alpha)'(\tilde{T}) \quad \text{for all } t \ge \tilde{T},$$

which implies

$$\lim_{t\to\infty} (x'(t) - \alpha'(t)) = x'(+\infty) - \alpha'(+\infty) < 0.$$

However, by the definition of α,

$$0 > x'(+\infty) - \alpha'(+\infty) = B - \alpha'(+\infty) \ge 0,$$

a contradiction.

Hence there exist $t_0 \in (0,\infty)$ such that

$$\min_{t\in\mathbb{R}_+} (x(t) - \alpha(t)) = x(t_0) - \alpha(t_0) < 0.$$

Then we have

$$x'(t_0) - D_-\alpha(t_0) \le x'(t_0) - D^+\alpha(t_0)$$

so, by the definition of lower solution, there exists an open interval I_0 such that $t_0 \in I_0$ and

$$\alpha''(t) \ge f(t,\alpha(t),\alpha'(t)) \quad \text{for a.a. } t \in I_0.$$

Further $x'(t_0) = \alpha'(t_0)$ and for all $r > 0$ there exists $t_r \in (t_0 - r, t_0)$ such that $x'(t_r) < \alpha'(t_r)$.

On the other hand, by the continuity of $x - \alpha$, there exists $\varepsilon > 0$ such that for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ we have $x(t) - \alpha(t) < 0$. Then,

$$x''(t) = f(t,\varphi(t,x),\delta_R((\varphi(t,x))')) = f(t,\alpha(t),\alpha'(t)) \le \alpha''(t) \text{ for a.a. } t \in [t_0-\varepsilon,t_0]\cap I_0.$$

Thus, the function $x'(t)-\alpha'(t)$ is nonincreasing on $(t_0-\varepsilon,t_0)\cap I_0$, so for $t \in (t_0-\varepsilon,t_0)\cap I_0$ it is

$$x'(t) - \alpha'(t) \ge x'(t_0) - \alpha'(t_0) = 0,$$

a contradiction.

Step 3. Every solution x of problem (4.4.5) *satisfies* $|x'(t)| < R$ *for all* $t \in \mathbb{R}_+$.

It is an immediate consequence of the Nagumo condition, see Proposition 4.4.2.

Step 4. Every solution x of problem (4.4.5) *is a solution to* (4.4.1)–(4.4.2).

Let x be a solution of the modified problem (4.4.5). It is enough to show that

$$\alpha(0) \le x(0) - L(x(0),x'(0),x) \le \beta(0).$$

Assume on the contrary that

$$\alpha(0) > x(0) - L(x(0),x'(0),x).$$

Then, $x(0) = \varphi(0, x(0) - L(x(0),x'(0),x)) = \alpha(0)$. Therefore, since $\alpha \le x$ and $x'(0) \ge D^+\alpha(0)$, from the monotonicity properties of L, we get the contradiction

$$0 > x(0) - L(x(0),x'(0),x) - \alpha(0) = -L(x(0),x'(0),x) \ge -L(\alpha(0),D^+\alpha(0),\alpha) \ge 0.$$

In an analogous way it can be proven that $x(0) - L(x(0),x'(0),x) \le \beta(0)$. □

Remark 4.4.13. *It is usual in the literature (see [57, 96–98]) to require the upper and lower solutions to satisfy strict inequalities at infinity, that is, $\alpha'(+\infty) < B$ and $\beta'(+\infty) > B$. Moreover, in the recent paper [98, Remark 3.3], the authors observe that it is unknown if this strict inequality can be weakened. Notice that this difficulty was overcome in our previous theorem.*

Now we illustrate our existence result with an example which shows its applicability.

Example 4.4.14. *Define $\phi : \mathbb{R} \to \mathbb{R}$ as in Example 4.3.9. Notice that ϕ is continuous at the irrational points and discontinuous at the rational numbers. Moreover, $\phi(u) \in (0,1)$ for each $u \in \mathbb{R}$.*

Consider the problem (4.4.1)–(4.4.2) *with the following functional boundary conditions*

$$L(x(0), x'(0), x) = 3(x(0))^3 - x'(0) - \int_0^\eta x(t)\,dt = 0,$$
$$x'(+\infty) = 0,$$

where $0 \le \eta \le 1$, and nonlinearity

$$f(t,x,y) = \frac{1}{1+t^2}\phi(t-x) + \min\left\{\frac{1}{\sqrt{t}}, \frac{1}{t^2}\right\} y\cos(2\pi y),$$

for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}_+$.

First, the functions $M(t) = \min\left\{1/\sqrt{t}, 1/t^2\right\}$ and $N(y) = 1+y$ satisfy condition $(H2)$.

For a.a. $t \in \mathbb{R}_+$, the function $(x,y) \mapsto f(t,x,y)$ is continuous on

$$\left(\mathbb{R} \setminus \bigcup_{\{n\,:\,t\in I_n\}} \{\gamma_n(t)\}\right) \times \mathbb{R},$$

where for each $n \in \mathbb{N}$,

$$\gamma_n(t) = t + q_n \quad \text{for all } t \in I_n = \mathbb{R}_+.$$

Notice that these curves can be defined in compact domains as in Definition 4.4.9 by writing the infinite interval $\mathbb{R}_+$ as a countable union of compact intervals.

The curves γ_n are inviable admissible discontinuity curves. Indeed, for $\varepsilon > 0$ small enough we have

$$\gamma''(t) = 0 < \frac{1}{2}\min\left\{\frac{1}{\sqrt{t}}, \frac{1}{t^2}\right\} < f(t,y,z)$$

for a.a. $t \in I_n$, all $y \in [\gamma_n(t) - \varepsilon, \gamma_n(t) + \varepsilon]$ and $z \in [\gamma_n'(t) - \varepsilon, \gamma_n'(t) + \varepsilon]$.

It is easy to check that the functions $\alpha(t) = -t - 1$ and $\beta(t) = 0$ are, respectively, a lower and an upper solutions for this problem.

Therefore, Theorem 4.4.12 ensures that it has a solution between α and β.

Observe that the function f is not monotone in the third argument, changes sign and it is discontinuous over a set of curves which is dense in $\mathbb{R}_+ \times \mathbb{R}$. Moreover, it is singular at $t = 0$, and even in the case of being continuous, it would fall outside the scope of other results in the literature, as for example those in [57], because the function M is not a possible bound for the nonlinearities considered there, see Remark 4.4.4.

Multiplicity results can be also derived by means of degree theory and the existence of two pairs of lower and upper solutions as done, for example, in [2, 96, 98].

4.4.2 Extremal solutions between the lower and upper solutions

To finish, we provide sufficient conditions for the existence of extremal solutions for the following problem

$$\left\{\begin{array}{l} x''(t) = f(t, x(t), x'(t)), \quad t \in \mathbb{R}_+, \\ L(x(0), x'(0)) = 0, \quad x'(+\infty) = B, \end{array}\right. \tag{4.4.15}$$

where $B \in \mathbb{R}$ and L is continuous and nonincreasing in the second argument.

Since problem (4.4.15) is a particular case of (4.4.1)–(4.4.2) (by removing the functional dependence in the boundary conditions), the existence of solutions is guaranteed by Theorem 4.4.12 when well ordered lower and upper solutions exist. Now we establish the existence of extremal solutions between them.

Theorem 4.4.15. *In the conditions of Theorem 4.4.12, problem* (4.4.15) *has extremal solutions between α and β.*

Proof. Consider the set of solutions for problem (4.4.15) located between the lower and upper solution

$$\begin{aligned} S &= \{x \in [\alpha, \beta] : x \text{ solution of } (4.4.15)\} \\ &= \{x \in X : x \text{ solution of } (4.4.5)\} \\ &= \{x \in X : x = Tx\}. \end{aligned}$$

By Lemma 4.4.10,

$$S = \{x \in X : x \in \mathbb{T}x\} = (Id - \mathbb{T})^{-1}(\{0\}),$$

which implies that S is a closed subset of X due to $\mathbb{T}$ is upper semicontinuous and $\{0\}$ is a closed set. Hence, since TX is relatively compact and $S \subset TX$, S is a compact set.

Define $x_{\min}(t) = \inf\{x(t) : x \in S\}$ for $t \in \mathbb{R}_+$. The evaluation map $\delta_t : X \to \mathbb{R}$ given by $\delta_t(x) = x(t)$ is a continuous map and then $\delta_t(S) = \{x(t) : x \in S\}$ is compact. Thus, for each $t_0 \in \mathbb{R}_+$, there exists a function $x_0 \in S$ such that $x_0(t_0) = x_{\min}(t_0)$ and $x_{\min}$ is a continuous function on $\mathbb{R}_+$.

Let us see that $x_{\min}$ is a solution of (4.4.5), and in such a case it will be the least one between α and β.

By the upper semicontinuity of the operator $\mathbb{T}$ and the condition $\mathrm{Fix}(\mathbb{T}) = \mathrm{Fix}(T)$, the limit in X of a sequence of elements in S must be a solution of (4.4.5). Indeed, given $\tau > 0$ and $\varepsilon > 0$, if we prove that there exists $v \in S$ such that

$$|v(t) - x_{\min}(t)| \le \varepsilon \quad \text{for all } t \in [0, \tau], \tag{4.4.16}$$

then we obtain a sequence of elements in S which converges pointwise to $x_{\min}$ and by the compactness of S, up to a subsequence, it converges in S.

To do so, and following the proof of Theorem 4.1.10, the idea is to construct an upper solution for problem (4.4.15) and to apply Theorem 4.4.12 in order to obtain a function $v \in S$ satisfying (4.4.16).

So, fix $\tau, \varepsilon > 0$. By the equicontinuity of S and the continuity of $x_{\min}$ on $[0,\tau]$, there exists $\delta > 0$ such that $t, s \in [0,\tau]$ with $|t-s| < \delta$ implies

$$|x(t) - x(s)| < \varepsilon/2 \quad \text{for all } x \in S \cup \{x_{\min}\}.$$

Let $\{t_0, t_1, \dots, t_n\} \subset [0,\tau]$ such that $t_0 = 0$, $t_n = \tau$ and $t_{i+1} - t_i < \delta$ for $i = 0, 1, \dots, n-1$. Choose a function $x_0 \in S$ such that $x_0(0) = x_{\min}(0)$ and denote $\beta_0 \equiv x_0$.

For each $i \in \{1, 2, \dots, n-1\}$, define recursively $\beta_i \equiv \beta_{i-1}$ if $\beta_{i-1}(t_i) = x_{\min}(t_i)$ and, otherwise, take $x_i \in S$ such that $x_i(t_i) = x_{\min}(t_i)$, define

$$s_i = \inf\{t \in [t_{i-1}, t_i] : x_i(s) < \beta_{i-1}(s) \quad \text{for all } s \in [t, t_i]\}$$

and the function

$$\beta_i(t) = \begin{cases} \beta_{i-1}(t) & \text{if } t \in [0, s_i], \\ x_i(t) & \text{if } t \in (s_i, \infty). \end{cases}$$

Then $\beta_{n-1}(0) = x_0(0)$ and $\beta'_{n-1}(0) \le x'_0(0)$, so from the monotonicity hypotheses about L and the fact that $x_0 \in S$, we have

$$L(\beta_{n-1}(0), \beta'_{n-1}(0)) \ge L(x_0(0), x'_0(0)) = 0,$$

and it is immediate to check that β_{n-1} is an upper solution for problem (4.4.15).

From Theorem 4.4.12 it follows that there exists $v \in S$ such that $\alpha(t) \le v(t) \le \beta_{n-1}(t)$ for $t \in \mathbb{R}_+$ and, by the construction of β_{n-1} and the definition of $x_{\min}$, we have that $v(t_i) = x_{\min}(t_i)$ for $i = 0, 1, \dots, n-1$. Hence, for each $t \in [0,\tau]$ there is $i \in \{0, 1, \dots, n-1\}$ such that $t \in [t_i, t_{i+1}]$, and so

$$|v(t) - x_{\min}(t)| \le |v(t) - v(t_i)| + |x_{\min}(t_i) - x_{\min}(t)| < \varepsilon.$$

A similar reasoning shows that problem (4.4.15) has the greatest solution between α and β. □

Remark 4.4.16. *We note that the existence of extremal solutions in addition to some information about the set of solutions for problem* (4.4.15) *provides a method to achieve new existence results for problems where the nonlinearity has a functional dependence, see [35, 42] and Section 3.3.*

Chapter 5

Positive solutions for second and higher order problems

The existence of *positive* solutions to integral and differential equations has been widely studied in the literature, see for example [30, 43, 48, 53, 55, 67, 70, 86–88, 92–94, 107, 118, 120, 141, 142] and the references therein. The importance of this property is due to the fact that in most applications differential equations model physical or biological processes where the magnitudes cannot attain negative values. That is the case, for instance, of the number of individuals in a population or the temperature measured in Kelvin. The Krasnosel'skiĭ's fixed point theorem and, more in general, the fixed point index in cones are the main tools to study the existence of positive solutions for different boundary value problems in most of the papers mentioned above.

A classical problem [92, 93, 141] is that of the existence of positive solutions for the differential equation

$$u''(t) + g(t)f(u(t)) = 0, \quad (0 < t < 1), \tag{5.0.1}$$

along with suitable boundary conditions. This problem arises in the study of radial solutions in $\mathbb{R}^n,\ n \geq 2$, for the partial differential equation

$$\Delta v + h(\|x\|)f(v) = 0, \quad x \in \mathbb{R}^n,\ \|x\| \in [R_1, R_2]\,,$$

with the appropriate boundary conditions, see [55, 92, 93].

Recently, in the paper [87], the authors studied the existence of non trivial radial solutions for a system of PDEs of the previous type. The key on that work was to transform the former problem into a system of ordinary differential equations similar to (5.0.1).

Here we will consider both scalar and systems of second order differential equations with discontinuous nonlinearities and Sturm–Liouville BCs. Going from scalar discontinuous problems to systems of discontinuous equations is not a trivial matter and two different notions of discontinuity sets can be considered at this point: the first of them is related to consider discontinuities in a, roughly speaking, *componentwise* form and the second one considers discontinuous time–dependent curves in $\mathbb{R}^2_+$. For example, the first approach (see Definition 5.2.4 and Theorem 5.2.5) is used to guarantee the existence of one positive solution for a system like

$$\begin{cases} -x''(t) = x^2 + x^2y^2H(1-x)H(1-y), \\ -y''(t) = \sqrt{x} + \sqrt{y} + H(1-x)H(1-y), \end{cases}$$

subject to the Sturm–Liouville BCs, meanwhile the second approach, see Theorem 5.2.11, allows us to establish the existence of a positive solution for a system like

$$\begin{cases} -x''(t) = (xy)^{1/3}, \\ -y''(t) = \left(1 + (xy)^{1/3}\right)\left(1 - H(-x^2 - y^2)\right), \end{cases}$$

subject to the Sturm–Liouville boundary conditions. In both examples H denotes the Heaviside step function.

In the first two sections of this chapter we will study the existence and localization of positive solutions for second order scalar and multidimensional equations coupled with Sturm–Liouville BCs. Our results will rely on our generalization of Krasnosel'skiĭ's fixed point theorem and its multidimensional version for multivalued maps, respectively. Section 5.3 is devoted to obtain multiplicity results for a kind of three point boundary value problem via our extension of Leggett–Williams' fixed point theorem. Finally, in Section 5.4, we will focus on the existence and multiplicity of positive solutions for a kind of two point fourth–order problems.

The results of this chapter can be found in the papers [61, 66, 103, 128, 129].

5.1 Second order problems with Sturm–Liouville boundary conditions

We consider the following generalization of equation (5.0.1) with separated BCs:

$$\begin{cases} u''(t) + g(t)f(t,u(t)) = 0, \quad t \in I = [0,1], \\ \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{5.1.1}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$ and $\Gamma := \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

The usual approach to this problem consists in turning it into a fixed point problem with the integral operator

$$Tu(t) := \int_0^1 G(t,s)g(s)f(s,u(s))\,ds,$$

where G is the Green's function associated to the differential problem.

Motivated by this situation, we study existence of fixed points of Hammerstein integral operators

$$Tu(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,ds, \tag{5.1.2}$$

defined in a suitable space. Here we consider $\mathcal{C}(I)$, endowed with the usual supremum norm $\|u\|_\infty = \max_{t\in I} |u(t)|$.

Fixed points of T will be looked for in the cone

$$K = \left\{ u \in \mathcal{C}(I) : \; u \geq 0, \; \min_{t\in[a,b]} u(t) \geq c\,\|u\|_\infty \right\},$$

where $[a,b] \subset I$ and $c \in (0,1]$. This cone was introduced by Guo and it was intensively employed in recent years, see for example [86, 92, 141].

In the sequel we will assume the following hypotheses:

(H1) $f : I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is such that:

(a) Compositions $f(\cdot, u(\cdot))$ are measurable whenever $u \in \mathcal{C}(I)$; and

(b) For each $r > 0$ there exists $R > 0$ such that $f(t,u) \le R$ for a.a. $t \in I$ and all $u \in [0, r]$.

(H2) g is measurable and $g(s) \ge 0$ almost everywhere.

(H3) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in I$, $|t_1 - t_2| < \delta$ implies

$$|k(t_1,s) - k(t_2,s)| < \varepsilon \quad \text{for all } s \in I.$$

(H4) There exists a measurable function $\Phi : I \to \mathbb{R}_+$ satisfying

$$\Phi g \in L^1(I) \text{ and } \int_a^b \Phi(s) g(s)\, ds > 0,$$

and a constant $c \in (0,1]$ such that

$$\begin{array}{ll} k(t,s) \le \Phi(s) & \text{for all } t, s \in I, \\ c\,\Phi(s) \le k(t,s) & \text{for all } t \in [a,b],\ s \in I. \end{array}$$

Remark 5.1.1. *Conditions $(H1) - (H4)$ are similar to those requested in [92] with the exception that we do not require f to be continuous. In addition, our assumptions are more general than those in [93] or [141], where the authors require $g \in L^1(I)$ and $\Phi \in \mathcal{C}(I)$.*

Lemma 5.1.2. *If conditions $(H1) - (H4)$ are satisfied, then the operator $T : K \to K$ introduced in* (5.1.2) *is well–defined and maps bounded sets into relatively compact sets.*

Proof. The operator T maps K into K. Indeed, we have

$$\|Tu\|_\infty = \max_{t \in I} \left\{ \int_0^1 k(t,s) g(s) f(s,u(s))\, ds \right\} \le \int_0^1 \Phi(s) g(s) f(s,u(s))\, ds,$$

and so (H4) implies

$$\min_{t \in [a,b]} \{Tu(t)\} \ge c \int_0^1 \Phi(s) g(s) f(s,u(s))\, ds.$$

Hence, $Tu \in K$ for every $u \in K$.

Now we prove that if $B \subset K$ is an arbitrary nonempty bounded set, then TB is relatively compact. By assumption (H1) (b), there exists $R > 0$ such that $f(t,u) \le R$ for a.a. $t \in I$ and all $u \in B$. Then for $u \in B$ we have

$$\int_0^1 k(t,s) g(s) f(s,u(s))\, ds \le R \int_0^1 \Phi(s) g(s)\, ds < \infty,$$

so TB is uniformly bounded. To see that TB is equicontinuous, it suffices to show that for every $\tau \in I$ and $t_n \to \tau$, we have

$$\lim_{t_n \to \tau} \int_0^1 |k(t_n,s) g(s) f(s,u(s)) - k(\tau,s) g(s) f(s,u(s))|\, ds = 0 \quad \text{uniformly in } u \in B. \tag{5.1.3}$$

To prove it, we note that for every $u \in B$ we have

$$|k(t_n, s)g(s)f(s, u(s)) - k(\tau, s)g(s)f(s, u(s))| \leq Rg(s)\,|k(t_n, s) - k(\tau, s)|\,, \tag{5.1.4}$$

which tends to zero uniformly on s by virtue of condition (H3). Moreover,

$$R\,g(s)\,|k(t_n, s) - k(\tau, s)| \leq 2R\Phi(s)g(s) \quad \text{for all } n \in \mathbb{N},$$

and $2R\Phi\, g \in L^1(I)$, by (H4), so the dominated convergence theorem and (5.1.4) yield (5.1.3). □

In addition, assume that the discontinuities of f allow the operator T to satisfy the condition

$$\{u\} \cap \mathbb{T}\,u \subset \{Tu\} \qquad \text{for all } u \in K \cap \mathbb{T}\,K, \tag{5.1.5}$$

where $\mathbb{T}$ is the multivalued mapping associated to T defined as in (1.1.1). Examples of this type of nonlinearities f can be seen in the previous chapters.

Lemma 5.1.3. *Suppose that condition* (5.1.5) *holds and that*

(I_ρ^1) *There exist $\rho > 0$ and $\varepsilon > 0$ such that $f^{\rho,\varepsilon} < m$, where*

$$f^{\rho,\varepsilon} := \sup_{0 \leq t \leq 1,\, 0 \leq u \leq \rho+\varepsilon} \left\{ \frac{f(t,u)}{\rho} \right\} \quad \textit{and} \quad \frac{1}{m} := \sup_{t \in [0,1]} \int_0^1 k(t,s)g(s)\,ds.$$

Then $\lambda u \notin \mathbb{T}u$ for all $u \in \partial_K B_\rho(0)$ and all $\lambda \geq 1$.

Proof. Suppose that there exist $\lambda \geq 1$ and $u \in \partial_K B_\rho(0)$ such that $\lambda u = Tv$ for some $v \in \overline{B}_\varepsilon(u) \cap K$, i.e.,

$$\lambda u(t) = \int_0^1 k(t,s)g(s)f(s, v(s))\,ds.$$

Taking the supremum for $t \in [0, 1]$,

$$\begin{aligned}
\lambda\rho &\leq \sup_{t \in [0,1]} \int_0^1 k(t,s)g(s)f(s, v(s))\,ds \\
&\leq \rho f^{\rho,\varepsilon} \sup_{t \in [0,1]} \int_0^1 k(t,s)g(s)\,ds \\
&\leq \rho f^{\rho,\varepsilon} \frac{1}{m} < \rho,
\end{aligned} \tag{5.1.6}$$

a contradiction.

Now, given $p \in \mathbb{N}$, we prove similarly that $\lambda u \neq \sum_{i=1}^p \lambda_i T v_i$ for any $v_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0, 1]$ with $\sum_{i=1}^p \lambda_i = 1$. Hence, $\lambda u \notin \operatorname{co}\left(T\left(\overline{B}_\varepsilon(u) \cap K\right)\right)$.

Finally, to see that $\lambda u \notin \overline{\operatorname{co}}\left(T\left(\overline{B}_\varepsilon(u) \cap K\right)\right)$ we consider two cases: $\lambda = 1$ and $\lambda > 1$.

If $\lambda = 1$, we obtain by the reasonings done above that $u \neq Tu$ and so condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ implies $u \notin \mathbb{T}u$.

If $\lambda > 1$, by inequality (5.1.6), we obtain that $\lambda\rho \leq \rho$, a contradiction. □

In the sequel, given $\rho > 0$ we will denote

$$V_\rho = \left\{ u \in K : \min_{a \le t \le b} u(t) < \rho \right\}.$$

In addition, it is trivial to see that $B_\rho(0) \cap K \subset V_\rho \subset B_{\rho/c}(0) \cap K$, and V_ρ is a relatively open subset of K (since minimum function is continuous).

Lemma 5.1.4. *Suppose that condition* (5.1.5) *holds and that*

(I_ρ^0) *There exist $\rho > 0$ and $\varepsilon > 0$ such that $f_{\rho,\varepsilon} > M(a,b)$, where*

$$f_{\rho,\varepsilon} := \inf_{a \le t \le b,\, \rho-\varepsilon \le u \le \frac{\rho}{c}+\varepsilon} \left\{ \frac{f(t,u)}{\rho} \right\} \quad \text{and} \quad \frac{1}{M(a,b)} := \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds.$$

Then $u \notin \mathbb{T}u + \lambda e$ for all $u \in \partial_K V_\rho$, all $\lambda \ge 0$ and $e(t) \equiv 1$.

Proof. Suppose that there exist $u \in \partial_K V_\rho$ and $\lambda \ge 0$ such that $u = Tv + \lambda e$ for some $v \in \overline{B}_\varepsilon(u) \cap K$. Then

$$u(t) = \int_0^1 k(t,s)g(s)f(s,v(s))\,ds + \lambda.$$

Notice that $\|v\|_\infty \le \|u\|_\infty + \varepsilon \le \rho/c + \varepsilon$ and

$$\min_{t\in[a,b]} v(t) \ge \min_{t\in[a,b]} u(t) - \varepsilon = \rho - \varepsilon.$$

Therefore, for $t \in [a,b]$

$$\begin{aligned} u(t) &= \int_0^1 k(t,s)g(s)f(s,v(s))\,ds + \lambda \\ &\ge \int_a^b k(t,s)g(s)f(s,v(s))\,ds + \lambda \\ &\ge \rho f_{\rho,\varepsilon} \int_a^b k(t,s)g(s)\,ds + \lambda. \end{aligned}$$

Taking the infimum in $[a,b]$ we have

$$\rho \ge \rho f_{\rho,\varepsilon} \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,ds + \lambda > \rho + \lambda, \tag{5.1.7}$$

a contradiction because $\lambda \ge 0$.

Now, given $p \in \mathbb{N}$, it is similar to check that $u \ne \sum_{i=1}^p \lambda_i T v_i + \lambda e$ for any $v_i \in \overline{B}_\varepsilon(u)$ and $\lambda_i \in [0,1]$ $(i = 1,\dots,p)$ with $\sum_{i=1}^p \lambda_i = 1$. Hence,

$$u \notin \operatorname{co}\left(T\left(\overline{B}_\varepsilon(u) \cap K\right)\right) + \lambda e.$$

Finally to see that $u \notin \mathbb{T}u + \lambda e$, we argue in a similar way that in the previous lemma: if $\lambda = 0$, then the conclusion follows from condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ and the fact that $Tu \ne u$; if $\lambda > 0$, then we obtain from (5.1.7) that $\rho \ge \rho + \lambda$, a contradiction. $\square$

Theorem 5.1.5. *Under the hypotheses* $(H1)$–$(H4)$ *and* (5.1.5)*, the Hammerstein integral operator* (5.1.2) *has at least a nonzero fixed point in* K *if either of the following conditions holds:*

(a) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1/c < \rho_2$ *such that* $(I^0_{\rho_1})$ *and* $(I^1_{\rho_2})$ *hold.*

(b) *There exist* $\rho_1, \rho_2 \in (0, \infty)$ *with* $\rho_1 < \rho_2$ *such that* $(I^1_{\rho_1})$ *and* $(I^0_{\rho_2})$ *hold.*

Proof. The conclusion will be a consequence of our Krasnoselskii's–type theorem, Theorem 2.2.4. Indeed, if we take $R > \rho_2$, then we obtain by Lemma 5.1.2 that $T(\overline{K}_R)$ is relatively compact.

Now, condition (a) implies by virtue of Theorem 2.2.1 that

$$i_K(T, V_{\rho_1}) = 0, \qquad i_K(T, B_{\rho_2}(0) \cap K) = 1,$$

and therefore the operator T has a fixed point in $(B_{\rho_2}(0) \cap K) \setminus \overline{V}_{\rho_1}$. On the other hand, if condition (b) holds, then we obtain again by application of Theorem 2.2.1 that

$$i_K(T, V_{\rho_2}) = 0, \qquad i_K(T, B_{\rho_1}(0) \cap K) = 1,$$

which implies that $i_K(T, V_{\rho_2} \setminus (B_{\rho_1}(0) \cap K)) = -1$ and hence T has a fixed point in $V_{\rho_2} \setminus (B_{\rho_1}(0) \cap K)$. $\square$

Remark 5.1.6. *Multiplicity results can be obtained by combining the previous conditions (see [92]).*

Now we return to the differential BVP (5.1.1). We will say that u is a solution of that problem if $u \in W^{2,1}(I)$ and satisfies both the boundary conditions and the differential equation almost everywhere on I.

We can write the differential problem (5.1.1) in terms of the integral equation

$$u(t) = \int_0^1 G(t,s) g(s) f(s, u(s))\, ds =: Tu(t),$$

where G is the associated Green function, that in this case (see [92]) is given by

$$G(t,s) = \frac{1}{\Gamma} \begin{cases} (\gamma + \delta - \gamma\, t)(\beta + \alpha\, s), & \text{if } 0 \le s \le t \le 1, \\ (\beta + \alpha\, t)(\gamma + \delta - \gamma\, s), & \text{if } 0 \le t < s \le 1, \end{cases} \tag{5.1.8}$$

and it is nonnegative.

Now we define

$$\Phi(s) = G(s,s) = \frac{1}{\Gamma}(\gamma + \delta - \gamma\, s)(\beta + \alpha\, s),$$

and we choose a, b and c in the following way [92]:

$(C1)$ $a, b \in [0,1]$ such that $-\beta/\alpha < a < b < 1 + \delta/\gamma$, where we consider $\beta/\alpha = \infty$ if $\alpha = 0$ and $\delta/\gamma = \infty$ if $\gamma = 0$;

$(C2)$ $c = \min\{(\gamma+\delta-\gamma\, b)/(\gamma+\delta), (\beta+\alpha\, a)/(\alpha+\beta)\}$.

These choices guarantee that $c\,\Phi(s) \le G(t,s)$ for $t \in [a,b]$ and $s \in [0,1]$.

We shall work, as done before, in the cone

$$K = \left\{ u \in \mathcal{C}(I) :\ u \ge 0,\ \min_{t\in[a,b]} u(t) \ge c\,\|u\|_\infty \right\}.$$

As in Chapter 4, we allow $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ to have discontinuities over the graphs of some time–dependent *admissible* curves.

Definition 5.1.7. *We say that $\gamma : [r,s] \subset I = [0,1] \to \mathbb{R}_+$, $\gamma \in W^{2,1}([r,s])$, is an admissible discontinuity curve for the differential equation $u'' = -g(t)f(t,u)$ if one of the following conditions holds:*

(a) *$\gamma''(t) = -g(t)f(t,\gamma(t))$ for a.a. $t \in [r,s]$ (then we say γ is viable for the differential equation),*

(b) *There exist $\varepsilon > 0$ and $\psi \in L^1(r,s)$, $\psi(t) > 0$ for a.a. $t \in [r,s]$ such that either*

$$\gamma''(t) + \psi(t) < -g(t)f(t,y) \text{ for a.a. } t \in [r,s] \text{ and all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon]\,, \tag{5.1.9}$$

or

$$\gamma''(t) - \psi(t) > -g(t)f(t,y) \text{ for a.a. } t \in [r,s] \text{ and all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon]\,. \tag{5.1.10}$$

In this case we say that γ is inviable.

In the proof of our main result in this section we shall need the following technical lemma.

Lemma 5.1.8. *If $M \in L^1(I)$, $M \ge 0$ almost everywhere, then the set*

$$Q = \left\{ u \in \mathcal{C}^1(I) : |u'(t) - u'(s)| \le \int_s^t M(r)\,dr \quad \text{whenever } 0 \le s \le t \le 1 \right\},$$

is closed in $\mathcal{C}(I)$ with the maximum norm topology.

Moreover, if $u_n \in Q$ for all $n \in \mathbb{N}$ and $u_n \to u$ uniformly in I, then there exists a subsequence $\{u_{n_k}\}$ which tends to u in the $\mathcal{C}^1$ norm.

Proof. Let $\{u_n\}$ be a sequence of elements of Q which converges uniformly on $[0,1]$ to some function $u \in \mathcal{C}(I)$; we have to show that $u \in Q$ and a subsequence $\{u_{n_k}\}$ tends to u in the $\mathcal{C}^1$ norm.

Since each u_n is continuously differentiable, the Mean Value Theorem guarantees the existence of some $t_n \in (0,1)$ such that

$$u_n'(t_n) = u_n(1) - u_n(0).$$

This implies the existence of some $K > 0$ such that $|u_n'(t_n)| \le K$ for all $n \in \mathbb{N}$, because $\{u_n\}$ is uniformly bounded in $[0,1]$. Hence, for every $n \in \mathbb{N}$ and every $t \in [0,1]$, we have

$$|u_n'(t)| \le |u_n'(t) - u_n'(t_n)| + |u_n'(t_n)| \le \int_0^1 M(s)\,ds + K,$$

so $\{u_n\}$ is bounded in the $\mathcal{C}^1$ norm. Moreover, the definition of Q implies that the sequence $\{u_n'\}$ is equicontinuous in $[0,1]$, so the Ascoli–Arzelá Theorem ensures that there exists some subsequence of $\{u_n\}$, say $\{u_{n_k}\}$, which converges in the $\mathcal{C}^1$ norm to some $v \in \mathcal{C}^1(I)$. As a result, $u = v$, so u is continuously differentiable in $[0,1]$ and $\{u_{n_k}\}$ tends to u in the $\mathcal{C}^1$ norm. In particular, $\{u_{n_k}'\}$ tends to u' uniformly in $[0,1]$.

Moreover, for $s, t \in [0,1]$, $s \le t$, and all $k \in \mathbb{N}$, we have

$$|u_{n_k}'(t) - u_{n_k}'(s)| \le \int_s^t M(r)\, dr,$$

and going to the limit as k tends to infinity we deduce that $|u'(t) - u'(s)| \le \int_s^t M(r)\, dr$. □

We are now ready for the proof of the main result in this section.

Theorem 5.1.9. *Suppose that f and g satisfy the following hypotheses:*

i. *$f : I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is such that:*

- *Compositions $f(\cdot, u(\cdot))$ are measurable whenever $u \in \mathcal{C}(I)$; and*
- *For each $r > 0$ there exists $R > 0$ such that $f(t,u) \le R$ for a.a. $t \in I$ and all $u \in [0,r]$.*

ii. *There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \to \mathbb{R}_+$, $n \in \mathbb{N}$, such that for a.a. $t \in I$ the function $u \mapsto f(t,u)$ is continuous on $\mathbb{R}_+ \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$.*

iii. *$g \in L^1(I)$ and $g(s) \ge 0$ almost everywhere with $\int_a^b g(s)\, ds > 0$, where a and b are given in $(C1)$.*

Moreover, assume that one of the following two conditions hold:

(a) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1/c < \rho_2$ such that $(I^0_{\rho_1})$ and $(I^1_{\rho_2})$ hold.*

(b) *There exist $\rho_1, \rho_2 \in (0,\infty)$ with $\rho_1 < \rho_2$ such that $(I^1_{\rho_1})$ and $(I^0_{\rho_2})$ hold.*

Then the differential problem with separated BCs (5.1.1) *has at least one positive solution $u \in W^{2,1}(I)$.*

Proof. The operator $T : K \to K$ given by

$$Tu(t) = \int_0^1 G(t,s) g(s) f(s, u(s))\, ds$$

is well–defined and maps bounded sets into relatively compact ones, as a consequence of Lemma 5.1.2. In addition, as G is the Green's function associated to a second–order homogeneous differential problem, $Tu \in W^{2,1}(I)$ for all $u \in K$. On the other hand, given $u \in B_{\rho_2/c}(0) \cap K = K_2$, conditions on f guarantee that there exists $R > 0$ such that $f(t,u) \le R$ for a.a. $t \in I$ and all $u \in K_2$. Therefore, we have $t \mapsto g(t) f(t, u(t)) \in L^1(I)$, and there exists $M(t) \in L^1(I)$ such that

$$h(t,u) := g(t) f(t,u) \le M(t) \text{ for a.a. } t \in I \text{ and all } u \in K_2. \tag{5.1.11}$$

We consider the set

$$Q = \left\{ u \in \mathcal{C}^1(I) : |u'(t) - u'(s)| \le \int_s^t M(r)\,dr \quad (s \le t) \right\}, \tag{5.1.12}$$

which is closed in $(\mathcal{C}(I), \|\cdot\|_\infty)$ by virtue of Lemma 5.1.8.

Hence, since $T\,K_2 \subset Q$ and Q is a closed and convex subset of $\mathcal{C}(I)$, we have $\mathbb{T}\,K_2 \subset Q$.

Now we will prove that

$$\{u\} \cap \mathbb{T}u \subset \{Tu\} \qquad \text{for all } u \in K_2 \cap \mathbb{T}K_2. \tag{5.1.13}$$

To do so, we fix an arbitrary function $u \in K_2 \cap Q$ and we consider three different cases.

Case 1: $m(\{t \in I_n \,:\, u(t) = \gamma_n(t)\}) = 0$ *for all* $n \in \mathbb{N}$. Let us prove that then T is continuous at u.

The assumption implies that for a.a. $t \in I$ the mapping $h(t,\cdot)$ is continuous at $u(t)$. Hence if $u_k \to u$ in $K_2 \cap Q$ then

$$h(t, u_k(t)) \to h(t, u(t)) \quad \text{for a.a. } t \in I,$$

which, along with (5.1.11), yield $Tu_k \to Tu$ in $\mathcal{C}(I)$.

Case 2: $m(\{t \in I_n \,:\, u(t) = \gamma_n(t)\}) > 0$ *for some* $n \in \mathbb{N}$ *such that* γ_n *is inviable.* In this case we can prove that $u \notin \mathbb{T}u$.

First, we fix some notation. Let us assume that for some $n \in \mathbb{N}$ we have

$$m(\{t \in I_n \,:\, u(t) = \gamma_n(t)\}) > 0$$

and there exist $\varepsilon > 0$ and $\psi \in L^1(I_n)$, $\psi(t) > 0$ for a.a. $t \in I_n$, such that (5.1.10) holds with γ replaced by γ_n. (The proof is similar if we assume (5.1.9) instead of (5.1.10), so we omit it.)

We denote $J = \{t \in I_n \,:\, u(t) = \gamma_n(t)\}$, and we deduce from Lemma 3.1.2 that there is a measurable set $J_0 \subset J$ with $m(J_0) = m(J) > 0$ such that for all $\tau_0 \in J_0$ we have

$$\lim_{t \to \tau_0^+} \frac{2\int_{[\tau_0,t]\setminus J} M(s)\,ds}{(1/4)\int_{\tau_0}^t \psi(s)\,ds} = 0 = \lim_{t \to \tau_0^-} \frac{2\int_{[t,\tau_0]\setminus J} M(s)\,ds}{(1/4)\int_t^{\tau_0} \psi(s)\,ds}. \tag{5.1.14}$$

By Corollary 3.1.3 there exists $J_1 \subset J_0$ with $m(J_0 \setminus J_1) = 0$ such that for all $\tau_0 \in J_1$ we have

$$\lim_{t \to \tau_0^+} \frac{\int_{[\tau_0,t]\cap J_0} \psi(s)\,ds}{\int_{\tau_0}^t \psi(s)\,ds} = 1 = \lim_{t \to \tau_0^-} \frac{\int_{[t,\tau_0]\cap J_0} \psi(s)\,ds}{\int_t^{\tau_0} \psi(s)\,ds}. \tag{5.1.15}$$

Let us now fix a point $\tau_0 \in J_1$. From (5.1.14) and (5.1.15) we deduce that there exist $t_- < \tilde{t}_- < \tau_0$ and $t_+ > \tilde{t}_+ > \tau_0$, $t_\pm$ sufficiently close to τ_0 so that the following inequalities are satisfied for all $t \in [\tilde{t}_+, t_+]$:

$$2\int_{[\tau_0,t]\setminus J} M(s)\,ds < \frac{1}{4}\int_{\tau_0}^t \psi(s)\,ds, \tag{5.1.16}$$

$$\int_{[\tau_0,t]\cap J} \psi(s)\,ds \ge \int_{[\tau_0,t]\cap J_0} \psi(s)\,ds > \frac{1}{2}\int_{\tau_0}^t \psi(s)\,ds, \tag{5.1.17}$$

and for all $t \in [t_-, \tilde{t}_-]$:

$$2 \int_{[t,\tau_0]\setminus J} M(s)\, ds < \frac{1}{4} \int_t^{\tau_0} \psi(s)\, ds, \tag{5.1.18}$$

$$\int_{[t,\tau_0]\cap J} \psi(s)\, ds > \frac{1}{2} \int_t^{\tau_0} \psi(s)\, ds. \tag{5.1.19}$$

Finally, we define a positive number

$$\tilde{\rho} = \min \left\{ \frac{1}{4} \int_{\tilde{t}_-}^{\tau_0} \psi(s)\, ds, \frac{1}{4} \int_{\tau_0}^{\tilde{t}_+} \psi(s)\, ds \right\}, \tag{5.1.20}$$

and we are now in a position to prove that $u \notin \mathbb{T}u$. It suffices to prove the following claim:
Claim: Let $\varepsilon > 0$ be given by our assumptions over γ_n and let

$$\rho = \frac{\tilde{\rho}}{2} \min \left\{ \tilde{t}_- - t_-, t_+ - \tilde{t}_+ \right\},$$

where $\tilde{\rho}$ is as in (5.1.20). *For every finite family $u_i \in \overline{B}_\varepsilon(x) \cap K$ and $\lambda_i \in [0,1]$ ($i = 1, 2, \dots, m$), with $\sum \lambda_i = 1$, we have $\|u - \sum \lambda_i T u_i\|_\infty \geq \rho$.*

Let u_i and λ_i be as in the Claim and, for simplicity, denote $y = \sum \lambda_i T u_i$. For a.a. $t \in J = \{t \in I_n \,:\, u(t) = \gamma_n(t)\}$ we have

$$y''(t) = \sum_{i=1}^{m} \lambda_i (Tu_i)''(t) = -\sum_{i=1}^{m} \lambda_i\, h(t, u_i(t)). \tag{5.1.21}$$

On the other hand, for every $i \in \{1, 2, \dots, m\}$ and every $t \in J$ we have

$$|u_i(t) - \gamma_n(t)| = |u_i(t) - u(t)| < \varepsilon,$$

and then the assumptions on γ_n ensure that for a.a. $t \in J$ we have

$$y''(t) = -\sum_{i=1}^{m} \lambda_i\, h(t, u_i(t)) < \sum_{i=1}^{m} \lambda_i \left(\gamma_n''(t) - \psi(t)\right) = u''(t) - \psi(t). \tag{5.1.22}$$

Now for $t \in [t_-, \tilde{t}_-]$ we compute

$$\begin{aligned}
y'(\tau_0) - y'(t) &= \int_t^{\tau_0} y''(s)\, ds = \int_{[t,\tau_0]\cap J} y''(s)\, ds + \int_{[t,\tau_0]\setminus J} y''(s)\, ds \\
&< \int_{[t,\tau_0]\cap J} u''(s)\, ds - \int_{[t,\tau_0]\cap J} \psi(s)\, ds \\
&\qquad + \int_{[t,\tau_0]\setminus J} M(s)\, ds \qquad \text{(by (5.1.22), (5.1.21) and (5.1.11))} \\
&= u'(\tau_0) - u'(t) - \int_{[t,\tau_0]\setminus J} u''(s)\, ds - \int_{[t,\tau_0]\cap J} \psi(s)\, ds + \int_{[t,\tau_0]\setminus J} M(s)\, ds \\
&\leq u'(\tau_0) - u'(t) - \int_{[t,\tau_0]\cap J} \psi(s)\, ds + 2 \int_{[t,\tau_0]\setminus J} M(s)\, ds \\
&< u'(\tau_0) - u'(t) - \frac{1}{4} \int_t^{\tau_0} \psi(s)\, ds \quad \text{(by (5.1.18) and (5.1.19))},
\end{aligned}$$

hence $y'(t)-u'(t)\ge\tilde\rho$ provided that $y'(\tau_0)\ge u'(\tau_0)$. Therefore, by integration we obtain

$$\begin{aligned} y(\tilde t_-)-u(\tilde t_-) &= y(t_-)-u(t_-)+\int_{t_-}^{\tilde t_-}(y'(t)-u'(t))\,dt \\ &\ge y(t_-)-u(t_-)+\tilde\rho(\tilde t_- - t_-), \end{aligned}$$

so if $y(t_-)-u(t_-)\le-\rho$, then $\|y-u\|_\infty\ge\rho$. Otherwise, if $y(t_-)-u(t_-)>-\rho$, then we have $y(\tilde t_-)-u(\tilde t_-)>\rho$ and thus $\|y-u\|_\infty\ge\rho$ too.

Similar computations in the interval $[\tilde t_+,t_+]$ show that if $y'(\tau_0)\le u'(\tau_0)$, then we have $u'(t)-y'(t)\ge\tilde\rho$ for all $t\in[\tilde t_+,t_+]$ and this also implies $\|y-u\|_\infty\ge\rho$. The claim is proven.

Case 3: $m(\{t\in I_n\,:\,u(t)=\gamma_n(t)\})>0$ only for some of those $n\in\mathbb{N}$ such that γ_n is viable. Let us prove that in this case the relation $u\in\mathbb{T}u$ implies $u=Tu$.

Let us consider the subsequence of all viable admissible discontinuity curves in the conditions of Case 3, which we denote again by $\{\gamma_n\}_{n\in\mathbb{N}}$ to avoid overloading notation. We have $m(J_n)>0$ for all $n\in\mathbb{N}$, where

$$J_n=\{t\in I_n\,:\,u(t)=\gamma_n(t)\}.$$

For each $n\in\mathbb{N}$ and for a.a. $t\in J_n$ we have

$$u''(t)=\gamma_n''(t)=-h(t,\gamma_n(t))=-h(t,u(t)),$$

and therefore $u''(t)=-h(t,u(t))$ a.e. in $J=\cup_{n\in\mathbb{N}}J_n$.

Now we assume that $u\in\mathbb{T}u$ and we prove that it implies that $u''(t)=-h(t,u(t))$ a.e. in $I\setminus J$, thus showing that $u=Tu$.

Since $u\in\mathbb{T}u$, then for each $k\in\mathbb{N}$ we can find functions $u_{k,i}\in\overline{B}_{1/k}(u)\cap K_2$ and coefficients $\lambda_{k,i}\in[0,1]$ $(i=1,2,\dots,m(k))$ such that $\sum\lambda_{k,i}=1$ and

$$\left\|u-\sum_{i=1}^{m(k)}\lambda_{k,i}Tu_{k,i}\right\|_\infty<\frac{1}{k}.$$

Let us denote $y_k=\sum_{i=1}^{m(k)}\lambda_{k,i}Tu_{k,i}$, and notice that $y_k\to u$ uniformly in I and that $\|u_{k,i}-u\|\le1/k$ for all $k\in\mathbb{N}$ and all $i\in\{1,2,\dots,m(k)\}$.

For every $k\in\mathbb{N}$ we have $y_k\in Q$ as defined in (5.1.12), and therefore Lemma 5.1.8 guarantees that $u\in Q$ and, up to a subsequence, $y_k\to u$ in the $\mathcal{C}^1$ – topology.

For a.a. $t\in I\setminus J$ we have that $h(t,\cdot)$ is continuous at $u(t)$, so for any $\varepsilon>0$ there is some $k_0=k_0(t)\in\mathbb{N}$ such that for all $k\in\mathbb{N}$, $k\ge k_0$, we have

$$|h(t,u_{k,i}(t))-h(t,u(t))|<\varepsilon\quad\text{for all } i\in\{1,2,\dots,m(k)\},$$

and therefore

$$|y_k''(t)+h(t,u(t))|\le\sum_{i=1}^{m(k)}\lambda_{k,i}|h(t,u_{k,i}(t))-h(t,u(t))|<\varepsilon.$$

Hence $y_k''(t) \to -h(t,u(t))$ for a.a. $t \in I \setminus J$, and then Corollary 3.1.4 guarantees that $u''(t) = -h(t,u(t))$ for a.a. $t \in I \setminus J$.

Therefore the proof of condition (5.1.13) is over and the conclusion is obtained by means of Theorem 5.1.5. □

Remark 5.1.10. *The differential problem* (5.1.1) *contains Dirichlet and Robin problems as particular cases, so the previous result generalizes the existence results given in [93].*

Finally, we illustrate the applicability of our result with a discontinuous modification of an example due to Infante [86].

Example 5.1.11. *Let $\lambda > 0$ and consider the following second–order Dirichlet problem*

$$\begin{cases} u''(t) + g(t)f(t,u(t)) = 0 \quad (0 < t < 1), \\ u(0) = u(1) = 0, \end{cases} \tag{5.1.23}$$

where $g(t) = t^{-\alpha}$ with $\alpha \in (0,1)$ and f is given by

$$f(t,u) = \lambda u^2 \left(\cos^2\left(\lfloor 1/(t^2-u)\rfloor\right) + 1\right) \quad \text{if } u \neq t^2, \quad f(t,t^2) = 2\lambda t^4.$$

First, standard arguments show that f satisfies condition i. *in Theorem 5.1.9.*

On the other hand, for a.a. $t \in I$, the function $u \mapsto f(t,u)$ is continuous on

$$\mathbb{R}_+ \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\},$$

where for each $n \in \mathbb{Z} \setminus \{0\}$ the curves

$$\gamma_n(t) = t^2 - n^{-1} \qquad \text{and} \qquad \gamma_0(t) = t^2,$$

are admissible discontinuity curves as far as

$$2 > \gamma_n''(t) - 1 > 0 \geq -g(t)f(t,u(t)) \quad \text{for a.a. } t \in I \text{ and all } u \in \mathbb{R}_+,$$

which shows that condition (5.1.10) *holds with $\psi \equiv 1$.*

Finally, we can take $a = 1/4$ and $b = 3/4$ in $(C1)$, so $c = 1/4$. In addition it is easy to check that

$$m = (2-\alpha)^{\frac{3-2\alpha}{1-\alpha}} \quad \text{and} \quad M(1/4,3/4) = \frac{4^{3-\alpha}(2-\alpha)}{3^{2-\alpha}-1},$$

as defined in Lemma 5.1.3 and Lemma 5.1.4, respectively. Moreover, we have the following bounds for f

$$\lambda u^2 \leq f(t,u) \leq 2\lambda u^2 \qquad \text{for a.a. } t \in I \text{ and all } u \in \mathbb{R}_+.$$

Since

$$f^{\rho_1,\varepsilon} := \sup_{0 \leq t \leq 1,\, 0 \leq u \leq \rho_1+\varepsilon} \left\{\frac{f(t,u)}{\rho_1}\right\} \leq \frac{2\lambda(\rho_1+\varepsilon)^2}{\rho_1},$$

it is sufficient to choose

$$\rho_1 < \frac{(2-\alpha)^{\frac{3-2\alpha}{1-\alpha}}}{2\lambda}$$

for ensuring that $f^{\rho_1,\varepsilon} < m$ *and so* $(I^1_{\rho_1})$ *is satisfied. Now if we take* $\rho_0 = 1$*, then*

$$f_{\rho_0,\varepsilon} := \inf_{1/4 \le t \le 3/4,\, 1-\varepsilon \le u \le 4+\varepsilon} \{f(t,u)\} \ge \lambda\,(1-\varepsilon)^2 > \lambda/2$$

if ε *is small enough and so* $f_{\rho_0,\varepsilon} > M(1/4,3/4)$ *provided that*

$$\lambda > \frac{4^{3-\alpha}(4-2\alpha)}{3^{2-\alpha}-1}.$$

Hence Theorem 5.1.9 guarantees that the differential problem (5.1.23) *has at least a positive solution for every* $\lambda > 4^{3-\alpha}(4-2\alpha)/(3^{2-\alpha}-1)$.

5.2 Second order systems

In this section we will extend the previous results concerning positive solutions to the following two–dimensional Sturm–Liouville problem

$$\begin{cases} u_1''(t) + g_1(t) f_1(t, u_1(t), u_2(t)) = 0, \\ u_2''(t) + g_2(t) f_2(t, u_1(t), u_2(t)) = 0, \end{cases} \tag{5.2.1}$$

for $t \in I = [0,1]$, coupled with the following boundary conditions

$$\alpha_i u_i(0) - \beta_i u_i'(0) = 0, \quad \gamma_i u_i(1) + \delta_i u_i'(1) = 0, \tag{5.2.2}$$

for $i = 1,2$, where $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}_+$ and $\Gamma_i := \beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$ for $i = 1,2$. Assume that, for $i = 1,2$,

(H_1) $g_i \in L^1(I)$, $g_i(t) \ge 0$ for a.a. $t \in I$ and $\int_{a_i}^{b_i} g_i(s)\,ds > 0$;

(H_2) $f_i : I \times \mathbb{R}_+^2 \to \mathbb{R}_+$ satisfies that

- (i) $f_i(\cdot, u_1(\cdot), u_2(\cdot))$ are measurable whenever $(u_1,u_2) \in \mathcal{C}(I)^2$;
- (ii) for each $r > 0$ there exists $R_{i,r} > 0$ such that

$$f_i(t,u_1,u_2) \le R_{i,r} \quad \text{for } u_1,u_2 \in [0,r] \text{ and a.a. } t \in I.$$

Let $X = \mathcal{C}(I)$ be endowed with the usual norm $\|v\|_\infty = \max_{t\in I}|v(t)|$ and let P be the cone of all nonnegative functions of X. The existence of positive solutions to problem (5.2.1)-(5.2.2) is equivalent to the existence of fixed points of the integral operator $T : P^2 \to P^2$, $T = (T_1,T_2)$, given by

$$(T_i u)(t) = \int_0^1 G_i(t,s) g_i(s) f_i(s, u_1(s), u_2(s))\,ds, \qquad i = 1,2, \tag{5.2.3}$$

where $G_i(t,s)$ are the corresponding Green's functions which are explicitly given by

$$G_i(t,s) = \frac{1}{\Gamma_i} \begin{cases} (\gamma_i + \delta_i - \gamma_i t)(\beta_i + \alpha_i s), & \text{if } 0 \le s \le t \le 1, \\ (\beta_i + \alpha_i t)(\gamma_i + \delta_i - \gamma_i s), & \text{if } 0 \le t \le s \le 1. \end{cases}$$

Now we can choose the constants a_i, b_i and c_i in the following way:

$(C1)$ $a_i, b_i \in [0,1]$ such that $-\beta_i/\alpha_i < a_i < b_i < 1+\delta_i/\gamma_i$, where we consider $\beta_i/\alpha_i = \infty$ if $\alpha_i = 0$ and $\delta_i/\gamma_i = \infty$ if $\gamma_i = 0$;

$(C2)$ $c_i = \min\{(\gamma_i + \delta_i - \gamma_i\, b_i)/(\gamma_i + \delta_i), (\beta_i + \alpha_i\, a_i)/(\alpha_i + \beta_i)\}$.

Then it is possible to check the following inequalities:

$$\begin{aligned} G_i(t,s) &\le G_i(s,s) \quad \text{for } t, s \in I, \\ c_i\, G_i(s,s) &\le G_i(t,s) \quad \text{for } t \in [a_i, b_i],\ s \in I. \end{aligned}$$

Consider in X the cones K_1 and K_2 defined as

$$K_i = \{v \in P : v(t) \ge c_i \|v\|_\infty \text{ for all } t \in [a_i, b_i]\},$$

and the corresponding cone $K := K_1 \times K_2$ in X^2. Then, $TK \subset K$. Indeed, for $u \in K$ and $i = 1, 2$,

$$\begin{aligned} c_i \|T_i u\|_\infty &= c_i \max_{t\in[0,1]} \int_0^1 G_i(t,s) g_i(s) f_i(s, u_1(s), u_2(s))\, ds \\ &\le c_i \int_0^1 G_i(s,s) g_i(s) f_i(s, u_1(s), u_2(s))\, ds \le \min_{t\in[a_i,b_i]} T_i u(t). \end{aligned}$$

Hence, $T_i u \in K_i$ for every $u \in K$ and $i = 1, 2$.

Our purpose is to apply the vectorial version of Krasnosel'skiĭ's fixed point theorem to a multivalued regularization of the discontinuous system of single-valued operators T_i. To do so, we will consider the closed–convex envelope of T componentwise, in the following way:

$$\mathbb{T}_i u = \bigcap_{\varepsilon>0} \overline{\text{co}}\, T_i \left(\overline{B}_\varepsilon(u) \cap K_i\right) \quad \text{for every } u \in K_i \quad (i = 1, 2). \tag{5.2.4}$$

Therefore, we proceed in the following way: first, we intend to apply Theorem 2.5.3 to the multivalued operator $\mathbb{T} = (\mathbb{T}_1, \mathbb{T}_2)$ associated to the discontinuous operator $T = (T_1, T_2)$. Later, we shall provide conditions on the nonlinearities f_i $(i = 1, 2)$ to guarantee that $\text{Fix}(\mathbb{T}) \subset \text{Fix}(T)$ and, as a consequence, we will obtain some results concerning the existence of positive solutions for system (5.2.1)-(5.2.2).

Let us introduce some notations. For $\rho_i^0, \rho_i^1 > 0$ with $\rho_i^0 \ne \rho_i^1$ and $\varepsilon > 0$, we denote $r_i = \min\{\rho_i^0, \rho_i^1\}$, $R_i = \max\{\rho_i^0, \rho_i^1\}$ and we consider the following subset of K:

$$K_{r,R} = \{u = (u_1, u_2) \in K \,:\, r_i \le \|u_i\|_\infty \le R_i,\ i = 1, 2\}.$$

Moreover, we denote

$$\begin{aligned}
f_1^{0,\varepsilon} &:= \inf\{f_1(t,u_1,u_2)\,:\, t\in[a_1,b_1],\, c_1(\rho_1^0-\varepsilon)\le u_1\le \rho_1^0+\varepsilon,\, c_2r_2\le u_2\le R_2\},\\
f_2^{0,\varepsilon} &:= \inf\{f_2(t,u_1,u_2)\,:\, t\in[a_2,b_2],\, c_1r_1\le u_1\le R_1,\, c_2(\rho_2^0-\varepsilon)\le u_2\le \rho_2^0+\varepsilon\},\\
f_1^{1,\varepsilon} &:= \sup\{f_1(t,u_1,u_2)\,:\, t\in[0,1],\, 0\le u_1\le \rho_1^1+\varepsilon,\, 0\le u_2\le R_2\},\\
f_2^{1,\varepsilon} &:= \sup\{f_2(t,u_1,u_2)\,:\, t\in[0,1],\, 0\le u_1\le R_1,\, 0\le u_2\le \rho_2^1+\varepsilon\}.
\end{aligned}$$

and

$$\frac{1}{M_i(a_i,b_i)} := \inf_{t\in[a_i,b_i]}\int_{a_i}^{b_i} G_i(t,s)g_i(s)\,ds, \qquad \frac{1}{m_i} := \sup_{t\in[0,1]}\int_0^1 G_i(t,s)g_i(s)\,ds$$

for $i=1,2$.

Lemma 5.2.1. *Assume that there exist* $\rho_i^0,\rho_i^1>0$ *with* $\rho_i^0\neq\rho_i^1$, $i=1,2$, *and* $\varepsilon>0$ *such that*

$$f_i^{1,\varepsilon} < m_i\rho_i^1, \quad f_i^{0,\varepsilon} > M_i(a_i,b_i)\rho_i^0 \quad \textit{for } i=1,2. \tag{5.2.5}$$

Then, for each $i\in\{1,2\}$, *the following conditions are satisfied in* $K_{r,R}$*:*

$$\lambda u_i \notin \mathbb{T}_i u \quad \textit{for } \|u_i\|_\infty=\rho_i^1 \textit{ and } \lambda>1, \tag{5.2.6}$$

$$u_i \notin \mathbb{T}_i u+\mu h_i \quad \textit{for } \|u_i\|_\infty=\rho_i^0 \textit{ and } \mu>0, \tag{5.2.7}$$

where $h_1=h_2\equiv 1$.

Moreover, the map $\mathbb{T}$ *defined as in* (5.2.4) *has at least one fixed point in* $K_{r,R}$.

Proof. First, observe that if $v\in K_{r,R}$, then

$$c_ir_i\le v_i(t)\le R_i \quad \text{for all } t\in[a_i,b_i] \quad (i=1,2),$$

and if $v\in \overline{B}_\varepsilon(u)\cap K_{r,R}$ for some $u\in K_{r,R}$, and $\|u_1\|_\infty=\rho_1^1$, then $v_1(t)\le\rho_1^1+\varepsilon$ for all $t\in[0,1]$ and

$$c_1(\rho_1^1-\varepsilon)\le v_1(t)\le \rho_1^1+\varepsilon \quad \text{for all } t\in[a_1,b_1]\,.$$

Now we prove (5.2.6) for $i=1$. Assume that $\|u_1\|_\infty=\rho_1^1$ and let us see that $\lambda u_1\notin\mathbb{T}_1u$ for $\lambda>1$. First, we shall show that given a family of vectors $v_k\in\overline{B}_\varepsilon(u)\cap K_{r,R}$ and numbers $\lambda_k\in[0,1]$ such that $\sum\lambda_k=1$ ($k=1,\dots,m$), then

$$\lambda u_1 \neq \sum_{k=1}^m \lambda_k\, T_1 v_k,$$

what implies that $\lambda u_1\notin \operatorname{co}\left(T_1\left(\overline{B}_\varepsilon(u)\cap K_{r,R}\right)\right)$. Indeed, assume on the contrary that the previous equation holds, then taking the supremum for $t \in [0,1]$,

$$\begin{aligned}
\lambda \rho_1^1 &= \sup_{t\in[0,1]} \sum_{k=1}^m \lambda_k \int_0^1 G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds \\
&\le \sum_{k=1}^m \lambda_k \sup_{t\in[0,1]} \int_0^1 G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds \\
&\le \sum_{k=1}^m \lambda_k f_1^{1,\varepsilon}/m_1 = f_1^{1,\varepsilon}/m_1 < \rho_1^1,
\end{aligned}$$

a contradiction. Notice that if $\lambda u_1 \in \overline{\text{co}}\left(T_1\left(\overline{B}_\varepsilon(u) \cap K_{r,R}\right)\right)$, then λu_1 is the limit of a sequence of functions satisfying the previous inequality and thus it satisfies $\lambda\, \rho_1^1 \le \rho_1^1$, which is also a contradiction since $\lambda > 1$. Therefore, $\lambda u_1 \notin \mathbb{T}_1 u$ for $\lambda > 1$.

In order to prove (5.2.7), assume that $\|u_1\|_\infty = \rho_1^0$ where $u_1 = \sum_{k=1}^m \lambda_k\, T_1 v_k + \mu$ for some family of vectors $v_k \in \overline{B}_\varepsilon(u) \cap K_{r,R}$ and numbers $\lambda_k \in [0,1]$ such that $\sum \lambda_k = 1$ ($k = 1, \dots, m$) and some $\mu > 0$. Then for $t \in [a_1, b_1]$,

$$\begin{aligned}
u_1(t) &= \sum_{k=1}^m \lambda_k \int_0^1 G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds + \mu \\
&\ge \sum_{k=1}^m \lambda_k \int_{a_1}^{b_1} G_1(t,s) g_1(s) f_1(s, v_{k,1}(s), v_{k,2}(s))\, ds + \mu \\
&\ge \sum_{k=1}^m \lambda_k\, f_1^{0,\varepsilon} \int_{a_1}^{b_1} G_1(t,s) g_1(s)\, ds + \mu \\
&\ge f_1^{0,\varepsilon}/M_1(a_1,b_1) + \mu > \rho_1^0 + \mu,
\end{aligned}$$

so $\rho_1^0 > \rho_1^0 + \mu$, a contradiction. Hence, $u_1 \notin \text{co}\left(T_1\left(\overline{B}_\varepsilon(u) \cap K_{r,R}\right)\right) + \mu h_1$. As before,

$$u_1 \notin \overline{\text{co}}\left(T_1\left(\overline{B}_\varepsilon(u) \cap K_{r,R}\right)\right) + \mu h_1$$

because in that case we arrive to the inequality $\rho_1^0 \ge \rho_1^0 + \mu$ for $\mu > 0$. Therefore, we obtain $u_1 \notin \mathbb{T}_1(u) + \mu h_1$.

In a similar way we show that conditions (5.2.6) and (5.2.7) hold for $i = 2$.

Then we obtain by application of Theorem 2.5.3 that operator $\mathbb{T}$ has at least one fixed point in $K_{r,R}$. □

Remark 5.2.2 (Asymptotic conditions)**.** *The existence of $\rho_i^0, \rho_i^1 > 0$ with $\rho_i^0 \neq \rho_i^1$, $i = 1, 2$, and $\varepsilon > 0$ satisfying* (5.2.5) *is guaranteed, in the autonomous case, by any of the following sufficient conditions:*

(a) *$f_1(\cdot, y)$ has a superlinear behavior and $f_2(x, \cdot)$, a sublinear one, that is,*

$$\lim_{x\to\infty} \frac{f_1(x,y)}{x} = +\infty \quad \textit{for all } y > 0, \qquad \lim_{x\to 0} \frac{f_1(x,y)}{x} = 0 \quad \textit{for all } y \ge 0;$$
$$\lim_{y\to\infty} \frac{f_2(x,y)}{y} = 0 \quad \textit{for all } x \ge 0, \qquad \lim_{y\to 0} \frac{f_2(x,y)}{y} = +\infty \quad \textit{for all } x > 0.$$

(b) *Both $f_1(\cdot,y)$ and $f_2(x,\cdot)$ have a superlinear behavior, that is,*

$$\lim_{x\to\infty}\frac{f_1(x,y)}{x}=+\infty \quad \textit{for all } y>0, \quad \lim_{x\to 0}\frac{f_1(x,y)}{x}=0 \quad \textit{for all } y\geq 0;$$
$$\lim_{y\to\infty}\frac{f_2(x,y)}{y}=+\infty \quad \textit{for all } x>0, \quad \lim_{y\to 0}\frac{f_2(x,y)}{y}=0 \quad \textit{for all } x\geq 0.$$

(c) *Both $f_1(\cdot,y)$ and $f_2(x,\cdot)$ have a sublinear behavior, that is,*

$$\lim_{x\to\infty}\frac{f_1(x,y)}{x}=0 \quad \textit{for all } y\geq 0, \quad \lim_{x\to 0}\frac{f_1(x,y)}{x}=+\infty \quad \textit{for all } y>0;$$
$$\lim_{y\to\infty}\frac{f_2(x,y)}{y}=0 \quad \textit{for all } x\geq 0, \quad \lim_{y\to 0}\frac{f_2(x,y)}{y}=+\infty \quad \textit{for all } x>0.$$

Remark 5.2.3. *If f_1 and f_2 are monotone in both variables, then it is possible to compute the numbers $f_i^{0,\varepsilon}$ and $f_i^{1,\varepsilon}$ $(i=1,2)$, so in this case, conditions* (5.2.5) *only depend on the behavior of the functions at four points in $\mathbb{R}^2_+$, see [118, 119].*

Note that Lemma 5.2.1 gives us sufficient conditions for the existence of a fixed point in $K_{r,R}$ of the multivalued operator $\mathbb{T}$. Hence, it remains to provide some hypotheses on the functions f_i $(i=1,2)$ which imply $\mathrm{Fix}(\mathbb{T})\subset\mathrm{Fix}(T)$ in order to obtain a solution for the system (5.2.1)–(5.2.2). Observe moreover that no continuity hypotheses were required to the functions f_i until now. Related to this, we introduce now a definition which extends to the present problem the notion of inviable discontinuity time–dependent curves that we are using throughout this work.

Definition 5.2.4. *We say that $\Gamma_1:[a,b]\subset I=[0,1]\to\mathbb{R}_+$, $\Gamma_1\in W^{2,1}(a,b)$, is an inviable discontinuity curve with respect to the first variable u_1 if there exist $\varepsilon>0$ and $\psi_1\in L^1(a,b)$, $\psi_1(t)>0$ for a.a. $t\in[a,b]$ such that either*

$$\Gamma_1''(t)+\psi_1(t)<-g_1(t)f_1(t,y,z) \;\; \textit{for a.a. } t\in[a,b], \textit{ all } y\in[\Gamma_1(t)-\varepsilon,\Gamma_1(t)+\varepsilon] \textit{ and all } z\in\mathbb{R}_+, \tag{5.2.8}$$

or

$$\Gamma_1''(t)-\psi_1(t)>-g_1(t)f_1(t,y,z) \;\; \textit{for a.a. } t\in[a,b], \textit{ all } y\in[\Gamma_1(t)-\varepsilon,\Gamma_1(t)+\varepsilon] \textit{ and all } z\in\mathbb{R}_+. \tag{5.2.9}$$

Similarly, we say that $\Gamma_2:[a,b]\subset I=[0,1]\to\mathbb{R}_+$, $\Gamma_2\in W^{2,1}(a,b)$, is an inviable discontinuity curve with respect to the second variable u_2 if there exist $\varepsilon>0$ and $\psi_2\in L^1(a,b)$, $\psi_2(t)>0$ for a.a. $t\in[a,b]$ such that either

$$\Gamma_2''(t)+\psi_2(t)<-g_2(t)f_2(t,y,z) \;\; \textit{for a.a. } t\in[a,b], \textit{ all } y\in\mathbb{R}_+ \textit{ and all } z\in\left[\Gamma_2(t)-\varepsilon,\Gamma_2(t)+\varepsilon\right],$$

or

$$\Gamma_2''(t)-\psi_2(t)>-g_2(t)f_2(t,y,z) \;\; \textit{for a.a. } t\in[a,b], \textit{ all } y\in\mathbb{R}_+ \textit{ and all } z\in\left[\Gamma_2(t)-\varepsilon,\Gamma_2(t)+\varepsilon\right].$$

Now we are ready to present the following existence and localization result for the differential system (5.2.1)–(5.2.2).

Theorem 5.2.5. *Suppose that the functions f_i and g_i ($i = 1, 2$) satisfy conditions (H_1), (H_2) and*

(H_3) *There exist inviable discontinuity curves $\Gamma_{1,n} : I_{1,n} := [a_{1,n}, b_{1,n}] \subset I \to \mathbb{R}_+$ with respect to the first variable, $n \in \mathbb{N}$, and inviable discontinuity curves $\Gamma_{2,n} : I_{2,n} := [a_{2,n}, b_{2,n}] \subset I \to \mathbb{R}_+$ with respect to the second variable, $n \in \mathbb{N}$, such that for each $i \in \{1, 2\}$ and for a.a. $t \in I$ the function $(u_1, u_2) \mapsto f_i(t, u_1, u_2)$ is continuous on*

$$\left(\mathbb{R}_+ \setminus \bigcup_{\{n:t\in I_{1,n}\}} \{\Gamma_{1,n}(t)\}\right) \times \left(\mathbb{R}_+ \setminus \bigcup_{\{n:t\in I_{2,n}\}} \{\Gamma_{2,n}(t)\}\right).$$

Moreover, assume that there exist $\rho_i^0, \rho_i^1 > 0$ with $\rho_i^0 \neq \rho_i^1$, $i = 1, 2$, and $\varepsilon > 0$ such that

$$f_i^{1,\varepsilon} < m_i\rho_i^1, \quad f_i^{0,\varepsilon} > M_i(a_i, b_i)\rho_i^0 \quad \text{for } i = 1, 2.$$

Then system (5.2.1)–(5.2.2) *has at least one solution in $K_{r,R}$.*

Proof. The operator $T : K_{r,R} \to K$, $T = (T_1, T_2)$, given by (5.2.3), is well–defined and the hypotheses (H_1) and (H_2) imply that $TK_{r,R}$ is relatively compact as an immediate consequence of the Ascoli–Arzelá theorem. Moreover, by (H_1) and (H_2), there exist functions $M_i \in L^1(I)$ ($i = 1, 2$) such that

$$g_i(t)f_i(t, u_1, u_2) \leq M_i(t) \quad \text{for a.a. } t \in I \text{ and all } u_1 \in [0, R_1],\ u_2 \in [0, R_2]. \tag{5.2.10}$$

Therefore, $TK_{r,R} \subset Q_1 \times Q_2$, where

$$Q_i = \left\{u \in \mathcal{C}^1([0, 1]) : |u'(t) - u'(s)| \leq \int_s^t M_i(r)\, dr \quad \text{whenever } 0 \leq s \leq t \leq 1\right\},$$

for $i = 1, 2$, which by virtue of Lemma 5.1.8 is a closed and convex subset of $X = \mathcal{C}(I)$. Then, by ‘convexification’, $\mathbb{T}K_{r,R} \subset Q_1 \times Q_2$, where $\mathbb{T}$ is the multivalued map associated to T defined as in (5.2.4).

By Lemma 5.2.1, the multivalued map $\mathbb{T}$ has a fixed point in $K_{r,R}$. Hence, if we show that all the fixed points of the operator $\mathbb{T}$ are fixed points of T, the conclusion is obtained. To do so, we fix an arbitrary function $u \in K_{r,R} \cap (Q_1 \times Q_2)$ and we consider three different cases.

Case 1: $m(\{t \in I_{1,n} : u_1(t) = \Gamma_{1,n}(t)\} \cup \{t \in I_{2,n} : u_2(t) = \Gamma_{2,n}(t)\}) = 0$ for all $n \in \mathbb{N}$. Let us prove that T is continuous at u, which implies that $\mathbb{T}u = \{Tu\}$, and therefore the relation $u \in \mathbb{T}u$ gives that $u = Tu$.

The assumption implies that for a.a. $t \in I$ the mappings $f_1(t, \cdot)$ and $f_2(t, \cdot)$ are continuous at $u(t) = (u_1(t), u_2(t))$. Hence if $u_k \to u$ in $K_{r,R}$ then

$$f_i(t, u_k(t)) \to f_i(t, u(t)) \quad \text{for a.a. } t \in I \text{ and for } i = 1, 2,$$

which, along with (5.2.10), yield $Tu_k \to Tu$ in $\mathcal{C}(I)^2$, so T is continuous at u.

Case 2: $m(\{t \in I_{1,n} \,:\, u_1(t) = \Gamma_{1,n}(t)\}) > 0$ *for some* $n \in \mathbb{N}$. In this case we can prove that $u_1 \notin \mathbb{T}_1 u$, and thus $u \notin \mathbb{T}u$. The details are similar to those in Case 2 from Theorem 5.1.9, with obvious changes, so we omit them.

Case 3: $m(\{t \in I_{2,n} \,:\, u_2(t) = \Gamma_{2,n}(t)\}) > 0$ *for some* $n \in \mathbb{N}$. This case is analogous to the previous one, and then we can show that $u_2 \notin \mathbb{T}_2 u$. □

Remark 5.2.6. *Observe that Definition 5.2.4 allows us to study the discontinuities of the functions f_i componentwise, as shown in condition (H_3).*

In addition, a continuum set of discontinuity points is possible: for instance, the function f_1 may be discontinuous at the point $x = 1$ for all $y \in \mathbb{R}_+$ provided that the constant function $\Gamma_1 \equiv 1$ is an inviable discontinuity curve with respect to the first variable. This fact improves the ideas given in [84] for first–order autonomous systems where "only" a countable set of discontinuity points are allowed.

Remark 5.2.7. *Notice that conditions* (5.2.8) *and* (5.2.9) *are not local in the last variable. However, the condition*

$$\inf_{t \in I, x, y \in \mathbb{R}_+} f_1(t, x, y) > 0$$

implies that any constant function stands for an inviable discontinuity curve with respect to the first variable (since condition (5.2.9) *holds). Moreover, any function with strictly positive second derivative is always an inviable discontinuity curve with respect to the first variable without any additional condition on f_1.*

Now we illustrate our existence result with some examples.

Example 5.2.8. *Consider the coupled autonomous system*

$$\left\{ \begin{array}{l} -x''(t) = x^2 + x^2 y^2 H(a - x) H(b - y), \\ -y''(t) = \sqrt{x} + \sqrt{y} + H(c - x) H(d - y), \end{array} \right. \tag{5.2.11}$$

subject to the BCs (5.2.2)*, where $a, b, c, d > 0$ and H denotes the Heaviside function.*

First it is clear that conditions $(H1)$ and $(H2)$ hold with $g_1 = g_2 \equiv 1$, $f_1(x, y) = x^2 + x^2 y^2 H(a - x) H(b - y)$ and $f_2(x, y) = \sqrt{x} + \sqrt{y} + H(c - x) H(d - y)$. On the other hand, the existence of numbers ρ_i^0 and ρ_i^1 in the conditions of (5.2.5) *is guaranteed by Remark 5.2.2 (a) since $f_1(\cdot, y)$ is a superlinear function and $f_2(x, \cdot)$ is a sublinear one.*

Finally, the function $(x, y) \mapsto f_1(x, y)$ is continuous on $(\mathbb{R}_+ \setminus \{a\}) \times (\mathbb{R}_+ \setminus \{b\})$ and the constant function $\Gamma_1 \equiv a$ stands for an inviable curve with respect to the first variable. Indeed,

$$-\Gamma_1''(t) + \frac{a^2}{8} = \frac{a^2}{8} < f_1(y, z) \quad \textit{for a.a. } t \in [0, 1] \textit{ and for all } y \in \left[\frac{a}{2}, \frac{3a}{2}\right] \textit{ and } z \in \mathbb{R}_+,$$

hence (5.2.9) *holds with $\psi_1 \equiv a^2/8$.*

Moreover, the constant function $\Gamma_2 \equiv b$ is an inviable curve with respect to the second variable, according to Remark 5.2.7 since

$$\inf_{x, y \in \mathbb{R}_+} f_2(x, y) > 0.$$

Similarly, the function $f_2(x,y) = \sqrt{x} + \sqrt{y} + H(c-x)H(d-y)$ *satisfies the hypothesis* (H_3) *too.*

Then we conclude by application of Theorem 5.2.5 that the system (5.2.11)–(5.2.2) *has at least one positive solution.*

Example 5.2.9. *Consider now the nonautonomous system*

$$\begin{cases} -x''(t) = x^2 + x^2y^2H(a+t^2-x)H(b+pt-y), \\ -y''(t) = \sqrt{x} + \sqrt{y} + H(c-x)H(d-y), \end{cases} \tag{5.2.12}$$

subject to the BCs (5.2.2)*, where* $a,b,c,d > 0$ *and* $p \in \mathbb{R}$*.*

Now, for a.a. $t \in I$*, the function* $(x,y) \mapsto f_1(t,x,y)$*, where*

$$f_1(t,x,y) = x^2 + x^2y^2H(a+t^2-x)H(b+pt-y),$$

is continuous on $\left(\mathbb{R}_+ \setminus \{a+t^2\}\right) \times (\mathbb{R}_+ \setminus \{b+pt\})$ *and the curve* $\Gamma_1(t) = a+t^2$ *is inviable with respect to the first variable. Indeed,* (5.2.9) *is satisfied with* $\psi_1 \equiv 1$ *since*

$$-\Gamma_1''(t) + 1 = -1 < f_1(t,y,z) \quad \textit{for a.a. } t \in [0,1] \textit{ and for all } y,z \in \mathbb{R}_+.$$

On the other hand, the curve $\Gamma_2(t) = b + pt$ *is inviable with respect to the variable* y*, according to Remark 5.2.7, since* $\Gamma_2''(t) \equiv 0$ *and* $\inf_{x,y\in\mathbb{R}_+} f_2(x,y) > 0$*.*

As conditions $(H1)$ *and* $(H2)$ *in Theorem 5.2.5 also hold by virtue of standard arguments, we can conclude the existence of one positive solution for problem* (5.2.12)–(5.2.2)*.*

In some applications conditions of Definition 5.2.4 are too strong, even for functions which have discontinuities only at single isolated points. This is the motivation of introducing another definition of the notion of discontinuity curves, which is a generalization of the admissible curves presented in Section 5.1.

Definition 5.2.10. *We say that* $\gamma = (\gamma_1,\gamma_2) : [a,b] \subset I = [0,1] \to \mathbb{R}^2_+$*,* $\gamma_i \in W^{2,1}(a,b)$ *(*$i = 1,2$*), is an admissible discontinuity curve for the equation* $u_1'' = -g_1(t)f_1(t,u_1(t),u_2(t))$ *if one of the following conditions holds:*

(a) $\gamma_1''(t) = -g_1(t)f_1(t,\gamma_1(t),\gamma_2(t))$ *for a.a.* $t \in [a,b]$ *(then we say* γ *is viable for the differential equation),*

(b) There exist $\varepsilon > 0$ *and* $\psi \in L^1(a,b),\ \psi(t) > 0$ *for a.a.* $t \in [a,b]$ *such that either*

$$\gamma_1''(t) + \psi(t) < -g_1(t)f_1(t,y,z) \quad \begin{array}{l}\textit{for a.a. } t \in [a,b],\ \textit{all } y \in [\gamma_1(t)-\varepsilon,\gamma_1(t)+\varepsilon] \\ \textit{and all } z \in [\gamma_2(t)-\varepsilon,\gamma_2(t)+\varepsilon]\,,\end{array}$$

or

$$\gamma_1''(t) - \psi(t) > -g_1(t)f_1(t,y,z) \quad \begin{array}{l}\textit{for a.a. } t \in [a,b],\ \textit{all } y \in [\gamma_1(t)-\varepsilon,\gamma_1(t)+\varepsilon] \\ \textit{and all } z \in [\gamma_2(t)-\varepsilon,\gamma_2(t)+\varepsilon]\,.\end{array}$$

In this case we say that γ *is inviable.*

We define admissible discontinuity curves for $u_2'' = -g_2(t)f_2(t,u_1(t),u_2(t))$ *in a similar way.*

Theorem 5.2.11. *Suppose that the functions* f_i *and* g_i *(*$i = 1,2$*) satisfy conditions* (H_1), (H_2) *and*

(H_3^*) *There exist admissible discontinuity curves for the first differential equation* $\gamma_n : I_n := [a_n, b_n] \to \mathbb{R}_+^2$, $n \in \mathbb{N}$, *such that for a.a.* $t \in I$ *the function* $(u_1,u_2) \mapsto f_1(t,u_1,u_2)$ *is continuous on* $\mathbb{R}_+^2 \setminus \bigcup_{\{n:t\in I_n\}} \{(\gamma_{n,1}(t), \gamma_{n,2}(t))\}$*;*

(H_4^*) *There exist admissible discontinuity curves for the second differential equation* $\tilde{\gamma}_n : \tilde{I}_n \to \mathbb{R}_+^2$, $n \in \mathbb{N}$, *such that for a.a.* $t \in I$ *the function* $(u_1,u_2) \mapsto f_2(t,u_1,u_2)$ *is continuous on* $\mathbb{R}_+^2 \setminus \bigcup_{\{n:t\in \tilde{I}_n\}} \{(\tilde{\gamma}_{n,1}(t), \tilde{\gamma}_{n,2}(t))\}$.

Moreover, assume that there exist $\rho_i^0, \rho_i^1 > 0$ *with* $\rho_i^0 \neq \rho_i^1$, $i = 1,2$, *and* $\varepsilon > 0$ *such that*

$$f_i^{1,\varepsilon} < m_i\rho_i^1, \quad f_i^{0,\varepsilon} > M_i(a_i,b_i)\rho_i^0 \quad \textit{for } i = 1,2.$$

Then the differential system (5.2.1)-(5.2.2) *has at least one solution in* $K_{r,R}$.

Proof. Notice that by virtue of Lemma 5.2.1 it is sufficient to show $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$. Reasoning as in the proof of Theorem 5.2.5, if we fix a function $u \in K_{r,R} \cap (Q_1 \times Q_2)$, we have to consider three different cases.

Case 1: $m(\{t \in I_n : u(t) = \gamma_n(t)\} \cup \{t \in \tilde{I}_n : u(t) = \tilde{\gamma}_n(t)\}) = 0$ *for all* $n \in \mathbb{N}$. Then T is continuous at u.

Case 2: $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ *or* $m(\{t \in \tilde{I}_n : u(t) = \tilde{\gamma}_n(t)\}) > 0$ *for some* γ_n *or* $\tilde{\gamma}_n$ *inviable*. Then $u \notin \mathbb{T}u$. The proof follows the reasonings from Case 2 in Theorem 5.1.9.

Case 3: $m(\{t \in I_n : u(t) = \gamma_n(t)\}) > 0$ *or* $m(\{t \in \tilde{I}_n : u(t) = \tilde{\gamma}_n(t)\}) > 0$ *only for viable curves.* Then the relation $u \in \mathbb{T}u$ implies $u = Tu$. In this case the idea is to show that u is a solution of the differential system. The proof is analogous to that of the Case 3 in Theorem 5.1.9, so we omit it here. □

Remark 5.2.12. *Notice that, in the case of a function* $(u_1,u_2) \mapsto f_1(t,u_1,u_2)$ *which is discontinuous at a single point* (x_0,y_0)*, Definition 5.2.10 requires that one of the following two conditions holds:*

(i) $f_1(t,x_0,y_0) = 0$ *for a.a.* $t \in [0,1]$*;*

(ii) *there exist* $\varepsilon > 0$ *and* $\psi \in L^1(0,1)$, $\psi(t) > 0$ *for a.a.* $t \in I$ *such that*

$$0 < \psi(t) < g_1(t)f_1(t,x,y) \quad \textit{for a.a. } t \in I, \textit{ all } x \in [x_0-\varepsilon, x_0+\varepsilon] \textit{ and all } y \in [y_0-\varepsilon, y_0+\varepsilon].$$

In particular, condition (ii) *hold if there exist* $\varepsilon, \delta > 0$ *such that*

$$0 < \delta < f_1(t,x,y) \textit{ for a.a. } t \in I, \textit{ all } x \in [x_0-\varepsilon, x_0+\varepsilon] \textit{ and all } y \in [y_0-\varepsilon, y_0+\varepsilon].$$

Remark 5.2.13. *It could be interesting to give an unified notion of the inviable curves from Definitions 5.2.4 and 5.2.10. An analogous approach to that in Section 3.2 in the case of first order systems may be helpful, but that research exceeds the aims of the present chapter.*

To finish, we present two simple examples which fall outside of the applicability of Theorem 5.2.5, but which can be studied by means of Theorem 5.2.11.

Example 5.2.14. *Consider the problem*

$$\left\{ \begin{array}{l} -x''(t) = f_1(x,y), \\ -y''(t) = f_2(x,y), \end{array} \right. \tag{5.2.13}$$

subject to the boundary conditions (5.2.2)*, where*

$$\begin{array}{l} f_1(x,y) = (xy)^{1/3}\left[2-\cos\left(1/((x-1)^2+(y-1)^2)\right)\left(1-H\left(-(x-1)^2-(y-1)^2\right)\right)\right], \\ f_2(x,y) = (xy)^{1/3}. \end{array}$$

It is clear that f_1 and f_2 satisfy conditions $(H1)$ and $(H2)$. Moreover, they have a sublinear behavior, see Remark 5.2.2.

On the other hand, the function $(x,y) \mapsto f_1(x,y)$ is continuous on $\mathbb{R}^2_+ \setminus \{(1,1)\}$ and the constant function $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \equiv (1,1)$ is an inviable admissible discontinuity curve for the differential equation $-x''(t) = f_1(x,y)$ since $0 < 1/\sqrt[3]{4} \le f_1(x,y)$ for all $x \in [1/2, 3/2]$ and all $y \in [1/2, 3/2]$; and $\gamma_1''(t) = 0$.

Therefore, Theorem 5.2.11 guarantees the existence of a positive solution for problem (5.2.13)–(5.2.2)*.*

Example 5.2.15. *Consider the following system*

$$\left\{ \begin{array}{l} -x''(t) = f_1(x,y) = (xy)^{1/3}, \\ -y''(t) = f_2(x,y) = \left(1+(xy)^{1/3}\right)(1-H(-x^2-y^2)), \end{array} \right. \tag{5.2.14}$$

subject to the boundary conditions (5.2.2)*.*

The nonlinearities in this system satisfy $(H1)$ and $(H2)$ and have a sublinear behavior. Now, the function $(x,y) \mapsto f_2(x,y)$ is continuous on $\mathbb{R}^2_+ \setminus \{(0,0)\}$ and the constant function

$$\gamma(t) = (\gamma_1(t), \gamma_2(t)) \equiv (0,0)$$

is a viable admissible discontinuity curve for the differential equation.

Hence, the system (5.2.14)–(5.2.2) *has at least one positive solution as a consequence of Theorem 5.2.11.*

5.3 Multiplicity result to a three–point problem

Now we consider the following second order three–point boundary value problem:

$$\left\{ \begin{array}{l} u''(t) + f(t,u(t)) = 0, \quad t \in I = [0,1], \\ u(0) = 0, \ \alpha\, u(\eta) = u(1), \end{array} \right. \tag{5.3.1}$$

where $0 < \eta < 1$, $0 < \alpha\,\eta < 1$.

The existence of solutions for problem (5.3.1) for continuous or Carathéodory nonlinearities was intensively studied in the literature (see [77, 107, 141] and references therein) by using fixed point techniques. For instance, Krasnoselskii's fixed point theorem was applied in [107, 141] to obtain the existence of positive solutions and Leggett–Williams' fixed point theorem was used in [77, 89] for multiplicity results. Moreover, Leggett–Williams' theorem and its generalizations are useful in order to establish existence and multiplicity results for a wide assortment of BVPs, as we can see for example in the recent paper [37].

Our previous generalizations of some well–known fixed point theorems, in particular that of Leggett–Williams' fixed point theorem (see Section 2.4), now allow us to obtain new results for this problem in the case that f has some discontinuities in both arguments. Indeed, $f : I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ may be discontinuous over the graphs of countable many functions in the conditions of the following definition, which is analogous to Definition 5.1.7.

Definition 5.3.1. *An admissible discontinuity curve for the equation* $-u'' = f(t,u)$ *is a* $W^{2,1}$*– function* $\gamma : [a,b] \subset I \longrightarrow \mathbb{R}_+$ *satisfying one of the following conditions:*

either $-\gamma''(t) = f(t,\gamma(t))$ *for a.a.* $t \in [a,b]$ *(and we say that* γ *is viable for the differential equation),*

or there exists $\varepsilon > 0$ *and* $\psi \in L^1(a,b)$*,* $\psi(t) > 0$ *for a.a.* $t \in [a,b]$*, such that*

either

$$-\gamma''(t) + \psi(t) < f(t,y) \quad \textit{for a.a. } t \in [a,b] \textit{ and all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon]\,, \tag{5.3.2}$$

or

$$-\gamma''(t) - \psi(t) > f(t,y) \quad \textit{for a.a. } t \in [a,b] \textit{ and all } y \in [\gamma(t)-\varepsilon, \gamma(t)+\varepsilon]\,. \tag{5.3.3}$$

We say that γ *is inviable for the differential equation if it satisfies* (5.3.2) *or* (5.3.3).

In this section we will work in the cone $K = \{u \in \mathcal{C}(I) : u \geq 0\}$ contained in the Banach space $\mathcal{C}(I)$ with the maximum norm $\|\cdot\|_\infty$. It is easy to check, by routine integration, that the Green's function related to the boundary value problem (5.3.1) is given by

$$G(t,s) = \frac{1}{1-\alpha\,\eta}t(1-s) - \begin{cases} \dfrac{\alpha}{1-\alpha\,\eta}t(\eta - s), & s \leq \eta \\ 0, & s > \eta \end{cases} - \begin{cases} t-s, & s \leq t, \\ 0, & s > t. \end{cases}$$

Now we look for upper and lower bounds for G, and following [141] we have

$$G(t,s) \leq \Phi(s) := \max\{1,\alpha\}\,\frac{s(1-s)}{1-\alpha\,\eta} \quad \text{for all } t,s \in [0,1].$$

Then for all $s,t \in [0,1]$,

$$G(t,s) \leq \max\{1,\alpha\}\,\frac{1}{4(1-\alpha\eta)} =: m.$$

Taking arbitrary $a_1 > 0$ and $a_2 \leq 1$, we can choose

$$r = \min\{a_1, \alpha\,\eta, 4a_1(1-\eta), \alpha(1-\eta)\} \quad \text{if } \alpha < 1, \tag{5.3.4}$$

$$r = \min\{a_1\eta, 4a_1(1-\alpha\eta)\eta, \eta(1-\alpha\eta)\} \qquad \text{if } \alpha \geq 1, \tag{5.3.5}$$

such that $r\,\Phi(s) \leq G(t,s)$ for all $t \in [a_1, a_2]$, $s \in [0,1]$.

Now we assume that the nonlinearity f satisfies the following conditions:

(H1) There exist $c > 0$ and $M \in L^1(I)$, $M \geq 0$ a.e., such that $f(t,u) \leq M(t)$ for a.a. $t \in I$ and all $u \in [0,c]$. Moreover, $c > m\,\|M\|_{L^1}$.

(H2) Any composition $t \in I \longmapsto f(t, u(t))$ is measurable if $u \in \mathcal{C}(I)$.

(H3) There exist admissible discontinuity curves $\gamma_n : I_n = [a_n, b_n] \longrightarrow \mathbb{R}_+,\ n \in \mathbb{N}$, such that for a.a. $t \in I$ the function $u \longmapsto f(t,u)$ is continuous on $\mathbb{R}_+ \setminus \bigcup_{\{n:t\in I_n\}} \{\gamma_n(t)\}$.

To prove the existence of positive solutions for the differential problem (5.3.1) we will look for the fixed points of the equivalent integral operator $T : \overline{K}_c \longrightarrow \overline{K}_c$ given by

$$Tu(t) := \int_0^1 G(t,s) f(s, u(s))\,ds,$$

where $K_c = \{x \in K : \|x\|_\infty < c\}$ and $\overline{K}_c$ denotes its closure, following the notation in Chapter 2.

Proposition 5.3.2. *Under the assumptions (H1)–(H2), the operator $T : \overline{K}_c \longrightarrow \overline{K}_c$ is well–defined and $T\,\overline{K}_c$ is relatively compact.*

Proof. Since the Green's function G is nonnegative and condition $(H1)$ holds, it is clear that $T\overline{K}_c \subset \overline{K}_c$. Now, given $u \in \overline{K}_c$, we have

$$\|Tu\|_\infty \leq m \int_0^1 f(s, u(s))\,ds \leq m\,\|M\|_{L^1},$$

so $T\,\overline{K}_c$ is uniformly bounded. To see that $T\,\overline{K}_c$ is equicontinuous, notice that for a.a. $t \in I$ and all $u \in \overline{K}_c$, we have

$$|(Tu)'(t)| \leq \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| M(s)\,ds \leq \frac{1}{1-\alpha\eta}\,\|M\|_{L^1},$$

since a simple computation shows that

$$\left|\frac{\partial G}{\partial t}(t,s)\right| \leq \frac{1}{1-\alpha\eta} \qquad \text{a.e. on } [0,1]\times[0,1].$$

This implies that

$$|Tu(t) - Tu(s)| \leq \int_s^t |(Tu)'(r)|\,dr \leq \frac{1}{1-\alpha\eta}\,\|M\|_{L^1}\,|t-s|.$$

Hence $T\,\overline{K}_c$ is a relatively compact subset of $\mathcal{C}(I)$, by Áscoli–Arzela's theorem. □

Proposition 5.3.3. *If (H1)–(H3) holds, then the operator $T : \overline{K}_c \longrightarrow \overline{K}_c$ satisfies condition* (2.1.1) *in $\overline{K}_c$.*

Proof. It follows exactly as in the proof of Theorem 5.1.9. □

Now we are ready to use our new version of Leggett–William's theorem to obtain multiplicity of solutions for problem (5.3.1). To do that, we consider the concave positive functional $\alpha : K \longrightarrow \mathbb{R}_+$ given by

$$\alpha(u) = \min_{t\in[a_1,a_2]} u(t).$$

Lemma 5.3.4. *For each $u \in K$ we have $\alpha(Tu) \geq r\,\|Tu\|_\infty$, where r is defined as in* (5.3.4)–(5.3.5).

Proof. On the one hand, we have

$$\|Tu\|_\infty = \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s)f(s,u(s))\,ds \right\} \leq \int_0^1 \Phi(s)f(s,u(s))\,ds.$$

On the other hand,

$$\alpha(Tu) = \min_{t\in[a_1,a_2]} \{Tu(t)\} \geq r \int_0^1 \Phi(s)f(s,u(s))\,ds.$$

Hence, $\alpha(Tu) \geq r\,\|Tu\|_\infty$. □

Denote

$$\delta = \frac{1}{2(1-\alpha\,\eta)} \min_{i=1,2} \left\{ a_i\,(1-\max\{a_i,\eta\})^2 + (1-a_i+\alpha a_i)a_i^2 \right\}.$$

A new result regarding the existence of positive solutions for the differential problem (5.3.1) is obtained by applying our Leggett–Williams' type Theorem 2.4.9.

Theorem 5.3.5. *Assume $(H1)$–$(H3)$ and that there exist constants $0 < a < b < b\,d/r \leq c$ with $d > 1$ such that*

1. *There exists $M_a \in L^1(I)$ such that $f(t,u) \leq M_a(t)$ for a.a. $t \in I$ and all $u \in [0,a]$ where $a > m\,\|M_a\|_{L^1}$.*
2. *$f(t,u) > b\,\delta^{-1}$ for all $a_1 \leq t \leq a_2$, $b \leq u \leq b\,d/r$.*

Then the boundary value problem (5.3.1) *has at least three nonnegative solutions.*

Proof. We will see that under these conditions we can apply Theorem 2.4.9. First, it is clear that $\alpha(u) \leq \|u\|_\infty$ for all $u \in K$. Given $u \in \overline{K}_c$, by assumption $(H1)$ we have

$$\|Tu\|_\infty = \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))\,ds \leq m \int_0^1 f(s,u(s))\,ds \leq c,$$

and hence $T : \overline{K}_c \longrightarrow \overline{K}_c$. By condition 1 we deduce in a similar way that $\|Tu\|_\infty < a$ if $u \in \overline{K}_a$.

On the other hand, the set $\{S(\alpha,b,b\,d/r) : \alpha(u) > b\}$ is nonempty. If $u \in S(\alpha,b,b\,d/r)$ then $b \leq u(t) \leq b\,d/r$ for all $t \in [a_1,a_2]$, so hypothesis 2 implies that $f(t,u(t)) > b\,\delta^{-1}$.

Moreover, since $(Tu)''(t) = -f(t, u(t)) \leq 0$ for all $0 \leq t \leq 1$, we deduce that Tu is concave and by the definition of α we obtain $\alpha(Tu) = Tu(a_1)$ or $\alpha(Tu) = Tu(a_2)$. Suppose that $\alpha(Tu) = Tu(a_1)$. Then if $\eta \geq a_1$ we have

$$\begin{aligned}
Tu(a_1) &= -\int_0^{a_1} (a_1 - s) f(s,u)\, ds - \frac{\alpha a_1}{1-\alpha\eta}\int_0^{\eta} (\eta - s) f(s,u)\, ds \\
&\quad + \frac{a_1}{1-\alpha\eta}\int_0^1 (1-s) f(s,u)\, ds \\
&= -a_1 \int_0^{a_1} f(s,u)\, ds - \frac{\alpha a_1 \eta}{1-\alpha\eta}\int_0^{\eta} f(s,u)\, ds + \frac{a_1}{1-\alpha\eta}\int_0^1 f(s,u)\, ds \\
&\quad + \int_0^{a_1} s f(s,u)\, ds + \frac{\alpha a_1}{1-\alpha\eta}\int_0^{\eta} s f(s,u)\, ds - \frac{a_1}{1-\alpha\eta}\int_0^1 s f(s,u)\, ds \\
&= \frac{-a_1}{1-\alpha\eta}\int_0^{a_1} f(s,u)\, ds - \frac{\alpha a_1 \eta}{1-\alpha\eta}\int_{a_1}^{\eta} f(s,u)\, ds + \frac{a_1}{1-\alpha\eta}\int_0^1 f(s,u)\, ds \\
&\quad + \frac{1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} s f(s,u)\, ds + \frac{\alpha a_1}{1-\alpha\eta}\int_{a_1}^{\eta} s f(s,u)\, ds \\
&\quad - \frac{a_1}{1-\alpha\eta}\int_0^1 s f(s,u)\, ds \\
&= \frac{-\alpha a_1 \eta}{1-\alpha\eta}\int_{a_1}^{\eta} f(s,u)\, ds + \frac{a_1}{1-\alpha\eta}\int_{a_1}^{1} f(s,u)\, ds \\
&\quad + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} s f(s,u)\, ds \\
&\quad + \frac{\alpha a_1}{1-\alpha\eta}\int_{a_1}^{\eta} s f(s,u)\, ds - \frac{a_1}{1-\alpha\eta}\int_{a_1}^{1} s f(s,u)\, ds \\
&= \frac{a_1 - \alpha a_1 \eta}{1-\alpha\eta}\int_{a_1}^{\eta} f(s,u)\, ds + \frac{a_1}{1-\alpha\eta}\int_{\eta}^{1} f(s,u)\, ds \\
&\quad + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} s f(s,u)\, ds \\
&\quad + \frac{\alpha a_1 - a_1}{1-\alpha\eta}\int_{a_1}^{\eta} s f(s,u)\, ds - \frac{a_1}{1-\alpha\eta}\int_{\eta}^{1} s f(s,u)\, ds \\
&> a_1 \int_{a_1}^{\eta} (1-s) f(s,u)\, ds + \frac{a_1}{1-\alpha\eta}\int_{\eta}^{1} (1-s) f(s,u)\, ds \\
&\quad + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} s f(s,u)\, ds \\
&\geq \frac{a_1}{1-\alpha\eta}\int_{\eta}^{1} (1-s) f(s,u)\, ds + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} s f(s,u)\, ds \\
&> \frac{a_1(1-\eta)^2 + (1-a_1+\alpha a_1)a_1^2}{2(1-\alpha\eta)}\frac{b}{\delta} \geq b, \text{ taking into account that } f(s,u) > b\,\delta^{-1}.
\end{aligned}$$

If $\eta < a_1$, we have

$$
\begin{aligned}
Tu(a_1) &= \frac{\alpha a_1 \eta - a_1}{1-\alpha\eta}\int_\eta^{a_1} f(s,u)\,ds + \frac{a_1}{1-\alpha\eta}\int_\eta^1 f(s,u)\,ds \\
&\quad + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} sf(s,u)\,ds \\
&\quad + \frac{a_1-\alpha a_1}{1-\alpha\eta}\int_\eta^{a_1} sf(s,u)\,ds - \frac{a_1}{1-\alpha\eta}\int_\eta^1 sf(s,u)\,ds \\
&= \frac{\alpha a_1 \eta}{1-\alpha\eta}\int_\eta^{a_1} f(s,u)\,ds + \frac{a_1}{1-\alpha\eta}\int_{a_1}^1 f(s,u)\,ds \\
&\quad + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} sf(s,u)\,ds \\
&\quad - \frac{\alpha a_1}{1-\alpha\eta}\int_\eta^{a_1} sf(s,u)\,ds - \frac{a_1}{1-\alpha\eta}\int_{a_1}^1 sf(s,u)\,ds \\
&> \frac{\alpha a_1 \eta}{1-\alpha\eta}\int_\eta^{a_1} (1-s)f(s,u)\,ds + \frac{a_1}{1-\alpha\eta}\int_{a_1}^1 (1-s)f(s,u)\,ds \\
&\quad + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} sf(s,u)\,ds \\
&> \frac{a_1}{1-\alpha\eta}\int_{a_1}^1 (1-s)f(s,u)\,ds + \frac{1-a_1+\alpha a_1}{1-\alpha\eta}\int_0^{a_1} sf(s,u)\,ds \\
&> \frac{a_1(1-a_1)^2+(1-a_1+\alpha a_1)a_1^2}{2(1-\alpha\eta)}\frac{b}{\delta} \geq b.
\end{aligned}
$$

In a similar way we prove that $Tu(a_2) > k_2 > b$ for some k_2. Therefore, we obtain $\alpha(Tu) > k > b$ for all $u \in S(\alpha, b, b\,d/r)$ and some k.

Assume that $u \in S(\alpha, b, c)$. Then

$$
\alpha(Tu) \geq r\,\|Tu\|_\infty = \frac{b\,d}{b\,d/r}\,\|Tu\|_\infty > \frac{b}{b\,d/r}\,\|Tu\|_\infty\,.
$$

By Theorem 2.4.9, we conclude that T has at least three fixed points in K. Therefore, the boundary value problem (5.3.1) has at least three different nonnegative solutions. $\square$

The applicability of the *inviable* discontinuity curves has been shown by several examples along the document. In the following example, we illustrate the utility of the *viable* discontinuity curves.

Example 5.3.6. *Consider the problem*

$$
\begin{cases} u''(t) + f(t,u(t)) = 0, \quad t \in I = [0,1], \\ u(0) = 0, \ u(1/2) = u(1), \end{cases} \tag{5.3.6}
$$

that is, problem (5.3.1) *with $\alpha = 1$ and $\eta = 1/2$, and suppose that the nonlinearity f is given by*

$$
f(t,u) = \frac{1}{2\sqrt{t}}\sin^2\left(\frac{\pi}{2}\lfloor 1/(t^2+u)\rfloor\right) + u + 1,
$$

where $\lfloor x \rfloor$ *denotes the integer part of* x.

Notice that f *is unbounded, has infinitely many discontinuities in the second argument and it has a singularity at* $t = 0$.

For a.a. $t \in I$*, the function* $f(t,\cdot)$ *is continuous on* $\mathbb{R}_+ \setminus \bigcup_{\{n: t\in I_n\}} \{\gamma_n(t)\}$ *where for every* $n \in \mathbb{N}$,

$$\gamma_n(t) = -t^2 + n^{-1} \quad \textit{for all } t \in I_n = [0, n^{-1/2}].$$

The problem is that some of these discontinuity curves are not admissible. To solve this difficulty we consider a modified problem (5.3.6) *by replacing* f *by* $\tilde{f}$*, where for every* $n \in \mathbb{N}$*, we define*

$$\tilde{f}(t, \gamma_n(t)) = 2 (= -\gamma_n''(t)) \quad \textit{a.e. in } I_n,$$

and $\tilde{f}(t,u) = f(t,u)$ *otherwise. Therefore, now* γ_n *are viable discontinuity curves for* $\tilde{f}$ *for each* $n \in \mathbb{N}$.

We will check that Theorem 5.3.5 can be applied to the modified problem and then this problem has at least three solutions.

First, since $\alpha = 1$ *and* $\eta = 1/2$*, we have* $m = 1/2$ *and then for condition* $(H1)$ *we can take* $c = 10$ *and* $M(t) = \dfrac{1}{2\sqrt{t}} + 11$*. Indeed,* $\|M\|_{L^1} = 12$ *and then* $c > m\,\|M\|_{L^1}$.

Moreover, for verifying conditions 1. *and* 2. *in Theorem 5.3.5 it suffices to take, for example, the constants* $a = 2.01$*,* $b = 2.02$*,* $d = 9/8$*,* $a_1 = 8/10$ *and* $a_2 = 9/10$ *and the function* $M_a(t) = \dfrac{1}{2\sqrt{t}} + a + 1$*. In this case, we have* $r = 1/4$ *and* $\delta = 672/1000$*, and then it is immediate to check that the mentioned conditions are satisfied.*

Hence the modified problem (5.3.6) *with* f *replaced by* $\tilde{f}$ *has at least three positive solutions in* $W^{2,1}(I)$.

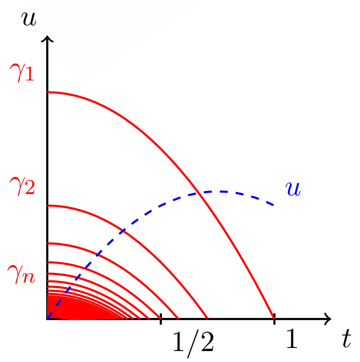

Figure 5.3.1: Possible solution u and discontinuity curves γ_n.

If u *is one of these solutions, then* u *is concave,* $u(0) = 0$ *and* $u(1/2) = u(1)$*, so* u *can only intersect each* γ_n *once, see Figure 5.3.1. Thus* u *is also a solution for our problem* (5.3.6) *which has at least three positive solutions too.*

5.4 Positive solutions to a one dimensional beam equation

We study the existence of positive solutions for the following fourth-order equation

$$\begin{cases} u^{(4)}(t) = g(t)f(u(t)), \quad t \in (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases} \tag{5.4.1}$$

where $g \geq 0$ a.e. on $I = [0,1]$ and $g \in L^1(I)$ and the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $f(u(\cdot))$ is measurable for every $u \in \mathcal{C}^2(I)$ and $f \in L^\infty_{loc}(\mathbb{R}_+)$.

Problem (5.4.1) was intensively studied in the literature (see, for example [31, 36, 43, 53, 142]) and it arises in many applications. For instance, fourth-order problems appear in nonlinear suspension bridge models (see [52, 142] and the references therein).

Following the spirit of this manuscript, our purpose is to remove the usual continuity assumptions regarding nonlinearity f. Therefore, in this section we achieve an existence result when f is not necessarily continuous with respect to the spatial variable but satisfies some superlinear or sublinear behavior at zero and infinity. Moreover, the fixed point results in Section 2.3 will be used to obtain the existence of a positive solution to the fourth order two-points boundary value problem (5.4.1) by assuming the existence of a strict upper solution for (5.4.1) and some asymptotic behavior near the origin or infinity. Therefore, Theorem 5.4.12 below is new even in the classical case of continuous right-hand sides in (5.4.1), see Corollary 5.4.13. Furthermore, a result concerning the existence of two positive solutions for problem (5.4.1) will be also obtained.

5.4.1 Existence results

In this section we shall work in the Banach space $(\mathcal{C}^2(I), \|\cdot\|)$, where

$$\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty, \|u''\|_\infty\}$$

and $\|\cdot\|_\infty$ is the usual supremum norm. We consider an operator T defined by

$$Tu(t) := \int_0^1 G(t,s)g(s)f(u(s))\,ds, \tag{5.4.2}$$

where G is the Green's function for problem (5.4.1). It is given by

$$G(t,s) = \begin{cases} \dfrac{1}{6}s(1-t)(2t-s^2-t^2), & s \leq t, \\ \dfrac{1}{6}t(1-s)(2s-t^2-s^2), & s > t, \end{cases}$$

which is nonnegative and satisfies (see [43, 142])

$$\begin{aligned} G(t,s) &\leq \Phi(s), && \text{for } t,s \in [0,1], \\ c\,\Phi(s) &\leq G(t,s), && \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right],\ s \in [0,1], \end{aligned}$$

where

$$\Phi(s) = \begin{cases} \dfrac{\sqrt{3}}{27} s(1-s^2)^{3/2}, & \text{for } 0 \le s \le \dfrac{1}{2}, \\ \dfrac{\sqrt{3}}{27}(1-s)s^{3/2}(2-s)^{3/2}, & \text{for } \dfrac{1}{2} \le s \le 1, \end{cases}$$

and $c = 45\sqrt{3}/128 \approx 0.608924$.

We will look for fixed points of T inside the cone

$$K = \left\{ u \in \mathcal{C}^2(I) : u \ge 0, \min_{t \in \left[\frac{1}{4}, \frac{3}{4}\right]} u(t) \ge \tilde{c}\, \|u\|_\infty \right\},$$

where $0 < \tilde{c} \le c$ will be fixed later.

Proposition 5.4.1. *The operator $T : K \to K$ is well-defined and maps bounded sets into relatively compact sets.*

Proof. The fact that $T\,K \subset K$ can be verified by using the properties of the Green's function G and the mapping Φ. In addition, from the hypotheses about f and g and the regularity of the Green's function it is routine to conclude that T maps bounded sets into relatively compact ones by means of the Áscoli–Arzela's theorem. □

Now we introduce the class of discontinuities that we will consider. The following definition is an adjustment of the admissible discontinuity curves of the previous chapters.

Definition 5.4.2. *An admissible discontinuity point is a nonnegative real number x satisfying one of the following conditions:*

(a) $f(x) = 0$ *(x is said a viable point),*

(b) *There exist $\varepsilon > 0$ and $\psi \in L^1(I)$, $\psi(t) > 0$ for a.a. $t \in [0,1]$ such that*

$$\psi(t) < g(t) f(y) \quad \textit{for a.a. } t \in [0,1] \textit{ and all } y \in [x-\varepsilon, x+\varepsilon] \quad (x \textit{ is inviable}). \tag{5.4.3}$$

Remark 5.4.3. *Notice that if $g \equiv 1$, then a nonnegative real number x is an admissible discontinuity point if and only if*

$$0 \in \mathcal{K}f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}}\, f(\overline{B}_\varepsilon(x)) \quad \textit{implies} \quad f(x) = 0;$$

which is the same condition that given in [84] for first-order discontinuous autonomous systems.

Moreover, if $g(t) > 0$ for a.a. $t \in [0,1]$, then the existence of $\varepsilon > 0$ such that

$$\inf_{y \in [x-\varepsilon, x+\varepsilon]} f(y) > 0$$

implies condition (5.4.3). *Similar assumptions were required in [18,19] in the study of second order problems.*

We state the following technical result whose proof is similar to that of Lemma 5.1.8.

Lemma 5.4.4. *If $M \in L^1(I)$, $M \geq 0$ almost everywhere, then the set*

$$Q = \left\{ u \in \mathcal{C}^3(I) : |u'''(t) - u'''(s)| \leq \int_s^t M(r)\,dr \quad \textit{whenever}\; 0 \leq s \leq t \leq 1 \right\},$$

is closed in $\mathcal{C}^2(I)$.

Moreover, if $u_n \in Q$ for all $n \in \mathbb{N}$ and $u_n \to u$ in the $\mathcal{C}^2$ norm, then there exists a subsequence $\{u_{n_k}\}$ which tends to u in the $\mathcal{C}^3$ norm.

Following the notation of [29], we define

$$\gamma_* = \inf_{t \in [1/4, 3/4]} \int_{1/4}^{3/4} G(t,s) g(s)\,ds, \qquad \gamma^* = \sup_{t\in[0,1]} \int_0^1 G(t,s) g(s)\,ds,$$

$$\gamma_1^* = \sup_{t\in[0,1]} \int_0^1 \left| \frac{\partial G}{\partial t}(t,s) \right| g(s)\,ds, \qquad \gamma_2^* = \sup_{t\in[0,1]} \int_0^1 \left| \frac{\partial^2 G}{\partial t^2}(t,s) \right| g(s)\,ds$$

and we suppose $\gamma_* > 0$.

Now we prove the first existence result of this section.

Theorem 5.4.5. *Assume that the functions f and g satisfy the following hypotheses:*

(H_1) *$g \geq 0$ a.e. on $I = [0,1]$ and $g \in L^1(I)$;*

(H_2) *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is such that*

- *$t \mapsto f(u(t))$ is measurable for every $u \in \mathcal{C}^2(I)$;*
- *f is locally bounded;*

(H_3) *There exist admissible discontinuity points $x_n \geq 0$ such that the function f is continuous in $\mathbb{R}_+ \setminus \bigcup_{n\in\mathbb{N}} \{x_n\}$.*

Moreover, assume that either

(i) $f_0 := \lim_{u\to 0^+} \dfrac{f(u)}{u} = +\infty$ *and* $f_\infty := \lim_{u\to\infty} \dfrac{f(u)}{u} = 0$ *(sublinear case); or*

(ii) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear case).*

Then BVP (5.4.1) *has at least one positive solution.*

Proof. The key of the proof is to show that operator T defined in (5.4.2) satisfies the conditions of Theorem 2.2.5. In the sequel we suppose that f satisfies (i) since the other case is analogous.

Claim 1: There exists $r_1 > 0$ such that $\|y\| < \|u\|$ for all $y \in \mathbb{T}\,u$ and all $u \in K$ with $\|u\| = r_1$.

Since $f_\infty = 0$, for each $L > 0$ there exists $M > 0$ such that

$$f(s) \leq M + Ls \quad \text{for } s \geq 0.$$

We can choose $L > 0$ small enough such that $5\max\{\gamma^*, \gamma_1^*, \gamma_2^*\}L < 2$ and $r_1 > 0$ large enough such that $2\max\{\gamma^*, \gamma_1^*, \gamma_2^*\}M < r_1$. Let $u \in K$ with $\|u\| = r_1$, then for every finite family $u_i \in \overline{B}_r(u) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1, 2, \dots, m)$, with $\sum \lambda_i = 1$ and $r = \|u\|_\infty /4$, we have

$$\begin{aligned} v(t) = \sum_{i=1}^m \lambda_i T u_i(t) &\le \sum_{i=1}^m \lambda_i \int_0^1 G(t,s) g(s) \left[M + L u_i(s)\right] ds \\ &\le \sum_{i=1}^m \lambda_i \gamma^* [M + L \|u_i\|_\infty] \le \gamma^* [M + 5L \|u\|_\infty /4] < \|u\| . \end{aligned}$$

In addition,

$$\begin{aligned} |v'(t)| = \left| \sum_{i=1}^m \lambda_i (T u_i)'(t) \right| &\le \sum_{i=1}^m \lambda_i \int_0^1 \left| \frac{\partial G}{\partial t}(t,s) \right| g(s) \left[M + L u_i(s)\right] ds \\ &\le \gamma_1^* [M + 5L \|u\|_\infty /4] < \|u\| , \end{aligned}$$

and

$$\begin{aligned} |v''(t)| = \left| \sum_{i=1}^m \lambda_i (T u_i)''(t) \right| &\le \sum_{i=1}^m \lambda_i \int_0^1 \left| \frac{\partial^2 G}{\partial t^2}(t,s) \right| g(s) \left[M + L u_i(s)\right] ds \\ &\le \gamma_2^* [M + 5L \|u\|_\infty /4] < \|u\| . \end{aligned}$$

Hence, if $y \in \mathbb{T} u$, then it is the limit of a sequence of functions v as above, so $\|y\| < \|u\|$ for all $y \in \mathbb{T} u$ and all $u \in K$ with $\|u\| = r_1$.

Notice that Claim 1 implies that $\lambda u \notin \mathbb{T}u$ for all $u \in K$ with $\|u\| = r_1$ and all $\lambda \ge 1$.

Claim 2: There exists $r_2 > 0$ such that $y \not\geq u$ for all $y \in \mathbb{T} u$ and all $u \in K$ with $\|u\| = r_2$.

As $f_0 = \infty$, then we can choose $L > 0$ large enough such that $\gamma_* L \tilde{c} > 2$ and $C > 0$ satisfying $f(s) \ge Ls$ provided that $0 \le s \le C$. Suppose that $u \in K$ with $\|u\| = C/2 =: r_2$, then for every finite family $u_i \in \overline{B}_r(u) \cap K$ and $\lambda_i \in [0,1]$ $(i = 1, 2, \dots, m)$, with $\sum \lambda_i = 1$ and $r = \|u\|_\infty /2$, we have $\|u_i\|_\infty \le 3r_2/2 < C$, so $0 \le u_i(t) \le C$ for all $t \in [1/4, 3/4]$ and

$$\begin{aligned} \sum_{i=1}^m \lambda_i T u_i(t) &\ge \sum_{i=1}^m \lambda_i \int_{1/4}^{3/4} G(t,s) g(s) f(u_i(s))\, ds \\ &\ge \gamma_* L \tilde{c} \sum_{i=1}^m \lambda_i \|u_i\|_\infty \ge \gamma_* L \tilde{c} \left(\|u\|_\infty - r\right) > \|u\|_\infty , \end{aligned}$$

which implies that $y \not\geq u$ for all $y \in \mathbb{T}u$ with $u \in K$ and $\|u\| = r_2$.

As shown in the proof of Corollary 2.2.3, it follows from Claim 2 that there exists $w \in K$ with $\|w\| \ne 0$ such that $x \notin \mathbb{T}x + \lambda w$ for every $\lambda \ge 0$ and all $x \in K$ with $\|x\| = r_2$.

Claim 3: The operator T satisfies the condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ for all $u \in \overline{K}_R$ with $R \ge r_1$.

The proof is similar to that in Theorem 5.1.9, so we omit it here.

Therefore the conditions of Theorem 2.2.5 are satisfied and we can ensure that BVP (5.4.1) has at least one positive solution. □

Remark 5.4.6. *Observe that the boundary conditions do not play an essential role when defining the admissible discontinuities of the function f. By this reason, the previous result can be generalized to other BCs whenever suitable sign conditions hold for the corresponding Green's functions, see [29, 142].*

We illustrate our theory with an example inspired by [43, Example 2].

Example 5.4.7. *Consider the BVP*

$$\begin{cases} u^{(4)} = \lfloor 7u^3 - 18u^2 + 12u \rfloor e^{-u} + \sqrt{u}, \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases}$$

where $\lfloor x \rfloor$ denotes the integer part of x.

The mapping $f(u) = \lfloor 7u^3 - 18u^2 + 12u \rfloor e^{-u} + \sqrt{u}$ is discontinuous at infinitely many points and these points are admissible inviable discontinuity points (it suffices to take $\psi \equiv 0.1$ and $\varepsilon = 0.05$ in Definition 5.4.2). In addition, it is not monotone and, clearly, $f_0 = +\infty$ and $f_\infty = 0$.

Therefore, Theorem 5.4.5 implies the existence of a positive solution for this problem.

Now we will use Theorem 2.3.2 in order to obtain the existence of positive solutions for problem (5.4.1) under monotonicity conditions. To do so, we will study the existence of upper fixed points for the operator T defined in (5.4.2) by means of maximum principles for the linear operator $L_M u := u^{(4)} + Mu$.

Definition 5.4.8. *Let $B \subset \mathcal{C}^4(I)$ and consider the operator $L_M : B \to \mathcal{C}(I)$ given by*

$$(L_M u)(t) := u^{(4)}(t) + Mu(t) \quad \textit{for all } t \in I.$$

We say that L_M is inverse positive in B if

$$u \in B, \ (L_M u)(t) \geq 0 \textit{ for all } t \in I \textit{ implies } u(t) \geq 0 \textit{ for all } t \in I,$$

and L_M is strongly inverse positive in B if it is inverse positive in B and

$$u \in B, \ L_M u \gneqq 0 \textit{ in } I \textit{ implies } u(t) > 0 \textit{ in } (0,1).$$

Proposition 5.4.9 ([31, Corollary 2.1])**.** *Let $M \geq 0$. Then the linear operator L_M is strongly inverse positive in the space*

$$\mathcal{W} = \{u \in \mathcal{C}^4(I) \, : \, u(0) \geq 0, \ u(1) \geq 0, \ u''(0) \leq 0, \ u''(1) \leq 0\}$$

if, and only if, $0 \leq M \leq c_1$, where $c_1 = 4k_1^4 \approx 125.137$ and k_1 is the smallest positive solution of the equation $\tan k = -\tanh k$.

Definition 5.4.10. *We say that $\beta \in W^{4,1}(I)$ is an upper solution for problem* (5.4.1) *if*

$$\begin{aligned}&\beta^{(4)}(t) \geq g(t)f(\beta(t)) \quad \textit{for a.a. } t \in I,\\ &\beta(0) \geq 0,\ \beta(1) \geq 0,\ \beta''(0) \leq 0,\ \beta''(1) \leq 0.\end{aligned}$$

Further, $\beta \in W^{4,1}(I)$ is a strict upper solution if it is an upper solution and, moreover, there exists an open subinterval $I_0 \subset I$ such that

$$\beta^{(4)}(t) > g(t)f(\beta(t)) \quad \textit{for a.a. } t \in I_0.$$

We are in a position to present some sufficient conditions for the existence of an upper fixed point for the operator T, that is, $\beta \in K$ such that $T\,\beta \preceq \beta$.

Lemma 5.4.11. *Suppose that one of the following two conditions holds:*

(i) *there exists $\beta > 0$ such that $\gamma^* f(\beta) < \beta$; or*

(ii) *there exists a strict upper solution β for problem* (5.4.1) *with $\min_{t\in I} \beta(t) > 0$.*

Then $T\beta \preceq \beta$ and there exists $R > 0$ such that $\overline{B}_R(\beta) \subset K$.

Proof. First, assume that condition (i) holds. By inequality $\gamma^* f(\beta) < \beta$, we obtain that

$$T\beta = \int_0^1 G(t,s)g(s)f(\beta)\,ds \leq \gamma^* f(\beta) < \beta.$$

Moreover, since $\|\beta - T\beta\|_\infty = \beta$ and there exists $0 < \tilde{c} < c$ such that $\beta - \gamma^* f(\beta) > \tilde{c}\,\beta$ we have for every $t \in [1/4, 3/4]$,

$$\beta(t) - T\beta(t) \geq \beta - \gamma^* f(\beta) > \tilde{c}\,\beta = \tilde{c}\,\|\beta - T\beta\|_\infty\,.$$

In addition, β is an interior point of K. Indeed, if $u \in \overline{B}_R(\beta)$ for $0 < R < \beta$ then $\|u - \beta\|_\infty \leq R$, that is, $\beta - R \leq u(t) \leq \beta + R$ for all $t \in [0,1]$, so $u(t) > 0$ for all $t \in [0,1]$ and whenever R is small enough we have

$$\min_{t\in[1/4,3/4]} u(t) \geq \beta - R \geq \tilde{c}(\beta + R) \geq \tilde{c}\,\|u\|_\infty\,.$$

Now, suppose that condition (ii) is satisfied. It ensures the existence of a nonnegative function $h \in L^1(I)$, $A, B \geq 0$ and $C, D \leq 0$ such that

$$\begin{aligned}&\beta^{(4)}(t) - g(t)f(\beta(t)) = h(t) \quad \text{for a.a. } t \in (0,1),\\ &\beta(0) = A,\ \beta(1) = B,\ \beta''(0) = C,\ \beta''(1) = D,\end{aligned}$$

or equivalently,

$$\beta(t) - T\beta(t) = \int_0^1 G(t,s)h(s)\,ds + \vartheta(t),$$

where ϑ is the unique solution of the problem

$$\begin{aligned}&y^{(4)}(t) = 0 \quad \text{for a.a. } t \in (0,1),\\ &y(0) = A,\ y(1) = B,\ y''(0) = C,\ y''(1) = D.\end{aligned}$$

Since $M = 0$, by Proposition 5.4.9, we deduce that $\beta(t) - T\beta(t) \geq 0$ in $[0,1]$ and, moreover, $\beta(t) - T\beta(t) > 0$ in $(0,1)$. Hence, there exists $0 < \tilde{c} < c$ small enough such that for $t \in [1/4, 3/4]$,

$$\beta(t) - T\beta(t) > \tilde{c} \, \|\beta - T\beta\|_\infty .$$

It can be shown in a similar as that done above that β is an interior point of K. □

Now we present a new existence result to (5.4.1).

Theorem 5.4.12. *Assume* $(H1)$*,* $(H2)$ *and*

$(\tilde{H}3)$ *There exist admissible discontinuity points $x_n \geq 0$ such that the function f is continuous in $\mathbb{R}_+ \setminus \bigcup_{n\in\mathbb{N}} \{x_n\}$ and there exists $r > 0$ such that f is right–continuous in* $[0, r]$.

Moreover, assume that

(i) *there exist $\beta \in K$, with $T\,\beta \preceq \beta$, and $R > 0$ such that $\overline{B}_R(\beta) \subset K$;*

(ii) *f is nondecreasing on* $[0, \|\beta\|_\infty]$*;*

(iii) $f_0 = +\infty$ *or* $f_\infty = +\infty$*.*

Then BVP (5.4.1) *has at least one positive solution.*

Proof. We will prove that the conditions of Theorem 2.3.2 are satisfied. Claims 1 and 3 are similar to the proof of [43, Theorem 3.1] and the last one is a technical result which follows the ideas of the proof of Theorem 5.1.9.

Claim 1: T is monotone nondecreasing in the set $\mathcal{P} = \{u \in K : u \preceq \beta\}$ and $T\mathcal{P}$ is relatively compact.

Since f is nondecreasing in $[0, \|\beta\|_\infty]$, it is clear that if we take $u, v \in K$ with $u(t) \leq v(t) \leq \beta(t)$ for all $t \in [0,1]$, we have $Tv(t) - Tu(t) \geq 0$ and for $t \in [1/4, 3/4]$ and $r \in [0,1]$,

$$\begin{aligned} Tv(t) - Tu(t) &= \int_0^1 G(t,s) g(s) \left[f(v(s)) - f(u(s))\right] ds \\ &\geq c \int_0^1 \Phi(s) g(s) \left[f(v(s)) - f(u(s))\right] ds \\ &\geq c \int_0^1 G(r,s) g(s) \left[f(v(s)) - f(u(s))\right] ds \geq \tilde{c} \left[Tv(r) - Tu(r)\right], \end{aligned}$$

so $\displaystyle\min_{t\in[1/4,3/4]} [Tv(t) - Tu(t)] \geq \tilde{c}\, \|Tv - Tu\|_\infty$. Therefore $Tv - Tu \in K$, i.e., $Tu \preceq Tv$. Thus T is nondecreasing.

Notice that $T\mathcal{P}$ is relatively compact because if $u \in \mathcal{P}$ then $0 \leq u(t) \leq \|\beta\|_\infty$ for all $t \in [0,1]$ so, since $f \in L^\infty_{loc}(\mathbb{R}_+)$, there exists $N > 0$ such that $f(u) \leq N$ for all $u \in \mathcal{P}$. Therefore the conclusion is easily obtained by Áscoli-Arzela's theorem.

Claim 2: The operator T satisfies that $\mathbb{T}u \subset \{Tu\} - K$ for all $u \in K$ with $\|u\| = r$.

Without loss of generality, assume that $r \leq \|\beta\|_\infty$. Let $u \in K$ with $\|u\| = r$ and $\varepsilon > 0$ given. By the right–continuity and the monotonicity of f in $[0, r]$, we have that for all $x \in [0, r]$ there exists $\delta > 0$ such that $-x \leq y - x < \delta$ implies that $f(y) \leq f(x) + \varepsilon/\gamma^*$. Therefore, for $\|v - u\| < \delta$, $v \in K$, we have

$$\begin{aligned} Tv(t) &= \int_0^1 G(t,s)g(s)f(v(s))\,ds \leq \int_0^1 G(t,s)g(s)\left[f(u(s)) + \frac{\varepsilon}{\gamma^*}\right] ds \\ &\leq \int_0^1 G(t,s)g(s)f(u(s))\,ds + \varepsilon = Tu(t) + \varepsilon, \end{aligned}$$

so $Tv \preceq Tu + \varepsilon$ and thus $Tv \in \overline{B}_\varepsilon(Tu) - K$. Now, using Remark 2.3.3, the conclusion is obtained.

Claim 3: There exists a bounded open set $V \subset K$ such that $i_K(T, V) = 0$ and either $\overline{V} \subset K_r$ or $\overline{K}_r \subset V$.

Suppose that $f_0 = \infty$ (the case $f_\infty = \infty$ is similar). In this case we can show that there exists a bounded open set $V \subset K$ such that $i_K(T, V) = 0$ and $\overline{V} \subset K_r$. Indeed, hypothesis (ii) guarantees that we can choose $L > 0$ large enough such that $\gamma_* L\tilde{c} > 2$ and choose $C > 0$ satisfying $f(s) \geq Ls$ provided that $0 \leq s \leq C$. Let $u \in K$ with $\|u\| = \min\{r/2, C/2\} =: \overline{r}$, then for every finite family $u_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, m)$, with $\sum \lambda_i = 1$ and $\varepsilon = \|u\|_\infty/2$, we have $\|u_i\|_\infty \leq 3\overline{r}/2 < C$, so $0 \leq u_i(t) \leq C$ for all $t \in [1/4, 3/4]$ and

$$\begin{aligned} \sum_{i=1}^m \lambda_i Tu_i(t) &\geq \sum_{i=1}^m \lambda_i \int_{1/4}^{3/4} G(t,s)g(s)f(u_i(s))\,ds \\ &\geq \gamma_* L\tilde{c} \sum_{i=1}^m \lambda_i \|u_i\|_\infty \geq \gamma_* L\tilde{c}\left(\|u\|_\infty - \varepsilon\right) > \|u\|_\infty, \end{aligned}$$

which implies that $y \npreceq u$ for all $y \in \mathbb{T}u$ with $u \in K$ and $\|u\| = \overline{r}$. By Corollary 2.2.3 we obtain that $i_K(T, K_{\overline{r}}) = 0$, so we can choose $V = K_{\overline{r}}$.

Claim 4: The operator T satisfies the condition $\{u\} \cap \mathbb{T}u \subset \{Tu\}$ for all $u \in \mathcal{P} \cup \overline{K}_R$.

The proof is similar to that in Theorem 5.1.9, so we omit it here.

Therefore, all the conditions of Theorem 2.3.2 are satisfied and we can ensure that BVP (5.4.1) has at least one positive solution. □

We emphasize that, even in the case of a continuous function f, Theorem 5.4.12 complements the existence results presented in [43]. As far as we know, the following corollary is new.

Corollary 5.4.13. *Assume that f is continuous and $g \geq 0$ a.e. on I and $g \in L^1(I)$. Moreover, assume that*

(i) *there exists a strict upper solution β for problem* (5.4.1) *with* $\min_{t\in I} \beta(t) > 0$*;*

(ii) *f is nondecreasing on $[0, \|\beta\|_\infty]$;*

(iii) $f_0 = +\infty$ *or* $f_\infty = +\infty$.

Then BVP (5.4.1) *has at least one positive solution.*

We illustrate our theory with an example which falls outside the scope of the fixed point theorems presented in [28–30, 43] because here the corresponding fixed point operator is not continuous.

Example 5.4.14. *Consider the BVP*

$$\begin{cases} u^{(4)} = \lfloor h_1(u) \rfloor + h_2(u), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases} \tag{5.4.4}$$

where $\lfloor x \rfloor$ *denotes the integer part of* x.

Assume that $h_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, *are continuous functions such that* $h_1(0) = 0$, $h_2(x) > 0$ *for all* $x \in (0, \infty)$, *both functions* h_1 *and* h_2 *are nondecreasing in* $[0, \beta]$ *for some* $\beta > 0$ *and*

$$\gamma^*(\lfloor h_1(\beta) \rfloor + h_2(\beta)) < \beta,$$

where $\gamma^* = 5/384$ *(see [43]). Moreover,*

$$\lim_{u \to \infty} \frac{h_1(u)}{u} = +\infty \quad \text{or} \quad \lim_{u \to \infty} \frac{h_2(u)}{u} = +\infty. \tag{5.4.5}$$

Then Theorem 5.4.12 guarantees the existence of a positive solution for problem (5.4.4).

Indeed, the mapping $f(u) = \lfloor h_1(u) \rfloor + h_2(u)$ *is discontinuous in* $h_1^{-1}(\mathbb{Z})$, *and so the positivity of the function* h_2 *implies that these points are admissible inviable discontinuity points (see Definition 5.4.2). In addition, the asymptotic condition* (5.4.5) *clearly guarantees that* $f_\infty = +\infty$.

For instance, we can choose $h_1(u) = 7u^3 - 18u^2 + 12u$ *and* $h_2(u) = \sqrt{u}$. *Then the previous conditions are satisfied by taking* $\beta = 0.69341$. *Notice that, in this case,* f *is not monotone in* $\mathbb{R}_+$, *but it is nondecreasing in* $[0, 0.69341]$.

5.4.2 A multiplicity result

Finally, we establish the existence of two positive solutions for problem (5.4.1). Our multiplicity result is based on the following Lemma and a suitable asymptotic behavior of the function f at zero and at infinity.

Lemma 5.4.15. *Assume that the functions* f *and* g *satisfy conditions* (H_1) *and* (H_2).

If there exist $r_1 > 0$ *and* $\varepsilon > 0$ *such that*

$$\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} \sup_{x \in [0, r_1 + \varepsilon]} f(x) < r_1, \tag{5.4.6}$$

then $\|y\| < \|u\|$ *for all* $y \in \mathbb{T}\, u$ *and all* $u \in K$ *with* $\|u\| = r_1$.

Proof. Let $u \in K$ with $\|u\| = r_1$. Then for every finite family $u_i \in \overline{B}_\varepsilon(u) \cap K$ and $\lambda_i \in [0, 1]$ $(i = 1, 2, \ldots, m)$, with $\sum \lambda_i = 1$, we have by virtue of condition (5.4.6) that

$$\begin{aligned} v(t) = \sum_{i=1}^m \lambda_i T u_i(t) &= \sum_{i=1}^m \lambda_i \int_0^1 G(t, s) g(s) f(u_i(s))\, ds \\ &\le \sum_{i=1}^m \lambda_i \gamma^* \sup_{x \in [0, r_1 + \varepsilon]} f(x) \le \gamma^* \sup_{x \in [0, r_1 + \varepsilon]} f(x) < r_1 = \|u\|. \end{aligned}$$

In addition,

$$|v'(t)| = \left|\sum_{i=1}^{m} \lambda_i (Tu_i)'(t)\right| \leq \sum_{i=1}^{m} \lambda_i \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)\right| g(s) f(u_i(s))\, ds$$
$$\leq \gamma_1^* \sup_{x\in[0,r_1+\varepsilon]} f(x) < r_1 = \|u\|\,,$$

and

$$|v''(t)| = \left|\sum_{i=1}^{m} \lambda_i (Tu_i)''(t)\right| = \sum_{i=1}^{m} \lambda_i \int_0^1 -\frac{\partial^2 G}{\partial t^2}(t,s) g(s) f(u_i(s))\, ds$$
$$\leq \gamma_2^* \sup_{x\in[0,r_1+\varepsilon]} f(x) < r_1 = \|u\|\,.$$

Hence, if $y \in \mathbb{T}\, u$, then it is the limit of a sequence of functions v as above, so $\|y\| < r_1$. □

Remark 5.4.16. *Notice that if f is a nondecreasing function, then condition* (5.4.6) *can be simply written as*

$$\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} f(r_1 + \varepsilon) < r_1.$$

Now we present our multiplicity result concerning the existence of a "small" and a "large" positive solutions for problem (5.4.1).

Theorem 5.4.17. *Assume that the functions f and g satisfy conditions (H_1)–(H_3). Moreover,*

(1) $f_0 = \infty$ *and* $f_\infty = \infty$;

(2) *there exist $r_1 > 0$ and $\varepsilon > 0$ such that*

$$\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} \sup_{x\in[0,r_1+\varepsilon]} f(x) < r_1.$$

Then problem (5.4.1) *has at least two positive solutions u_1 and u_2 such that $\|u_1\| < r_1$ and $\|u_2\| > r_1$.*

Proof. First, as in Claim 3 of Theorem 5.4.5, condition (H_3) guarantees that $u \in \mathbb{T}u$ implies $u = Tu$.

On the other hand, $f_0 = \infty$ implies that there exists $0 < r_2 < r_1$ such that $y \not\preceq u$ for all $y \in \mathbb{T}\, u$ and all $u \in K$ with $\|u\| = r_2$ (see Claim 2 in Theorem 5.4.5). Analogously, since $f_\infty = \infty$, there exists $R_2 > r_1$ such that $y \not\preceq u$ for all $y \in \mathbb{T}\, u$ and all $u \in K$ with $\|u\| = R_2$. Moreover, Lemma 5.4.15 implies that $\lambda u \notin \mathbb{T}u$ for all $u \in K$ with $\|u\| = r_1$ and all $\lambda \geq 1$.

Therefore, by applying Theorem 2.2.5 twice, we obtain that the operator T has at least two fixed points u_1 and u_2 such that $r_2 < \|u_1\| < r_1$ and $r_1 < \|u_2\| < R_2$. □

To finish we present a simple example which, as far as we are aware, is not covered by the previous literature.

Example 5.4.18. *Consider the problem*

$$\begin{cases} u^{(4)} = u^p + \lfloor u \rfloor^q, \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{cases} \tag{5.4.7}$$

where $0 < p < 1$ *and* $q > 1$*. Here* $g \equiv 1$ *and* $f(u) = u^p + \lfloor u \rfloor^q$*.*

Observe that f *is discontinuous at* $x_n = n$*,* $n \in \mathbb{N}$*, and for each* $n \in \mathbb{N}$*,*

$$0 < \inf \left\{ f(x) \, : \, x \in \left[\frac{1}{2}, \infty \right) \right\} \leq \inf \left\{ f(x) \, : \, x \in \left[n - \frac{1}{2}, n + \frac{1}{2} \right] \right\},$$

so the points x_n *are inviable, see Definition 5.4.2 and Remark 5.4.3.*

Since $0 < p < 1$ *and* $q > 1$*, we have that*

$$f_0 = \lim_{u \to 0^+} \frac{1}{u^{1-p}} + \frac{\lfloor u \rfloor^q}{u} = \infty, \qquad f_\infty = \lim_{u \to \infty} \frac{1}{u^{1-p}} + \frac{\lfloor u \rfloor^q}{u} \geq \lim_{u \to \infty} \frac{(u-1)^q}{u} = \infty.$$

Moreover, $\gamma^* = 5/384$*,* $\gamma_1^* \leq 1/6$ *and* $\gamma_2^* = 1/8$*, so* $\max\{\gamma^*, \gamma_1^*, \gamma_2^*\} \leq 1/6$*. By taking* $r_1 = 1/2$ *and* $\varepsilon = 1/2$*, condition* (2) *in Theorem 5.4.17 holds since* $\sup\{f(x) \, : \, x \in [0,1]\} = 2$ *and thus*

$$\frac{1}{6} \sup\{f(x) \, : \, x \in [0,1]\} < \frac{1}{2}.$$

Therefore, Theorem 5.4.17 ensures that problem (5.4.7) *has at least two positive solutions* u_1 *and* u_2 *such that* $\|u_1\| < 1/2$ *and* $\|u_2\| > 1/2$ *for any* $0 < p < 1$ *and* $q > 1$*.*

Chapter 6
Positive solutions for general problems

In this chapter, we establish existence and localization of positive solutions for general discontinuous problems for which a *Harnack type inequality* holds. In this way, a wide range of ordinary differential problems can be treated, as for example higher order boundary value problems or ϕ–Laplacian equations.

In the results of Chapter 5 we used intensively the properties of the corresponding Green's functions associated to each boundary value problem. These properties played an essential role in order to achieve the cone invariance and so apply Krasnosel'skiĭ type results. On the contrary, in this chapter we consider problems for which the Green's function may not exist and then the inequalities of Harnack type become a key ingredient in our reasonings.

More accurately, we study the existence of positive solutions for a general problem of the form

$$\begin{cases} Lu = f(t,u) & \text{a.e. on } I, \\ u \in \mathcal{B}, \end{cases} \tag{6.0.1}$$

where $I=[0,1]$, $\mathcal{B} \subset \mathcal{C}(I)$, $L : D(L) \subset \mathcal{C}(I) \to L^1(I)$ is a general operator not necessarily linear, and $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ is a function which may be discontinuous with respect to both variables.

The approach here is slightly different from that in Chapter 5: here we look for Krasovskij solutions of the general problem and then we impose conditions on f which ensure that all the Krasovskij solutions are, in fact, Carathéodory solutions. These ideas come from the papers [45, 59, 84] (see also the survey paper [14]). Following these previous works, where some *admissible* time–dependent discontinuity sets were presented, we study the mentioned general problem under suitable conditions for the operator L, namely, hypotheses (L_1)–(L_5) below and we allow the function f to be discontinuous over the graphs of a countable number of curves satisfying some 'transversality' condition.

The main results provide the existence and localization of positive solutions for problem (6.0.1), which allow also to derive multiplicity conclusions.

As a particular case of (6.0.1), we study the Dirichlet–Neumann problem involving the ϕ-Laplacian

$$\begin{cases} -(\phi(u'))' = f(t,u) & \text{a.e. on } [0,1], \\ u'(0) = u(1) = 0, \end{cases} \tag{6.0.2}$$

where $\phi : (-a,a) \to (-b,b)$ is an increasing homeomorphism such that $\phi(0)=0$, $0 < a, b \le \infty$, and the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ may have discontinuities. This approach allows to study in a unified scope the *classical* homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, the *singular* homeomorphism $\phi : (-a,a) \to \mathbb{R}$ and the *bounded* one $\phi : \mathbb{R} \to (-b,b)$, see [12].

In the last decades, ϕ-Laplacian equations have been extensively studied by different authors and by means of a variety of tools: fixed point theory [10,81], degree theory [12,127], lower and upper solutions [34, 114, 127] and variational methods [88]. Our approach is based on Bohnenblust–Karlin's fixed point theorem.

Remarkable particular cases of problem (6.0.2) are the p-Laplacian and the curvature operators in Euclidean and Minkowski spaces:

(1) $\phi(u) = |u|^{p-2}\, u$, with $p > 1$, where $a = b = \infty$;

(2) $\phi(u) = u/\sqrt{1+u^2}$, where $a = \infty$ and $b = 1$;

(3) $\phi(u) = u/\sqrt{1-u^2}$, where $a = 1$ and $b = \infty$.

To the best of our knowledge, the papers where a discontinuous ϕ-Laplacian equation was considered are either uniquely based on monotonicity hypotheses for the nonlinearity [33, 79] or the solutions are given in the sense of set-valued analysis, mainly, as Filippov or Krasovskij solutions, see [11, 23]. A remarkable exception is the paper [34], where a particular discontinuous problem can be transformed into a continuous one.

In particular, our general results (Theorems 6.2.8 and 6.2.11) yield the following new and simple existence theorem.

Theorem 6.0.1. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an almost everywhere continuous function such that*

(i) $f(u(\cdot))$ *is measurable whenever* $u \in \mathcal{C}(I)$*;*

(ii) *there exist* $c_1, c_2 \in \mathbb{R}_+$ *and* $p \geq 1$ *such that* $f(x) \leq c_1 x^p + c_2$ *for all* $x \in \mathbb{R}_+$*;*

(iii) $\liminf_{y\to x} f(y) = 0$ *implies* $f(x) = 0$ *for every* $x \in \mathbb{R}_+$.

Assume that there exists $0 < r < 2\sqrt{3}$ *such that*

$$r \leq \inf\{f(y) : r/16 \leq y \leq 1\}.$$

Then problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1-u'^2}}\right)' = f(u) \quad \text{a.e. on } [0,1], \\ u'(0) = u(1) = 0, \end{cases}$$

has at least one positive solution such that $\|u\|_\infty \geq 3r/16$.

As we said, the previous results were obtained by means of Bohnenblust–Karlin's fixed point theorem. Therefore, the readers may ask themselves if it is possible to apply a Krasnosel'skiĭ–type fixed point theorem with cone–compression and cone–expansion conditions in order to improve the results. The answer is affirmative: in the last section, we will employ a compression-expansion type fixed point theorem established in [48] for the composition of two multivalued operators (see also the coincidence point theorems in [20]) in order to obtain positive Krasovskij solutions for ϕ–Laplacian equations with Robin–Neumann boundary conditions.

The results presented here were published in the papers [121, 122].

6.1 Positive solutions for differential problems involving nonlinear operators

In this section we are concerned with the general form of problem (6.0.1).

Let P be the cone of all nonnegative functions of $L^1(I)$. It defines a partial order in $L^1(I)$ given by $v \le w$ if and only if $w - v \in P$. For any $0 < \rho \le \infty$, denote $P_\rho = \{u \in P : \|u\|_\infty < \rho\}$, where for a function $u \in L^1(I)$, by $\|u\|_\infty$ we mean the essential supremum of $|u|$.

We make the following assumptions concerning the operator L:

(L_1) There exists $0 < b \le \infty$ such that operator L is invertible in P_b, that is, for each $v \in P_b$ there is a unique $u =: L^{-1}v \in D(L) \cap \mathcal{B}$ such that $Lu = v$. Moreover, the operator $L^{-1} : P_b \to D(L) \cap \mathcal{B}$ can be written as $L^{-1} = A \circ B$, where

- (a) $B : L^1(I) \to \mathcal{C}(I)$ is a linear continuous operator, $B(P_b) \subset P_b^C := P_b \cap \mathcal{C}(I)$ and for each $\rho \in (0, b)$, $B(P_\rho)$ is relatively compact in $\mathcal{C}(I)$ and there exists $\tilde{\rho} \in (0, b)$ such that $B(P_\rho) \subset P_{\tilde{\rho}}^C$,
- (b) $A : P_b^C \to D(L) \cap \mathcal{B}$ is continuous and $A(P_\rho)$ is bounded in $\mathcal{C}(I)$ for every $0 < \rho < b$.

(L_2) A and B preserve the order in P_b^C and P_b, respectively, that is, if $0 \preceq v \preceq w$ then $0 \preceq Av \preceq Aw$ and $0 \preceq Bv \preceq Bw$.

(L_3) If $v \in P_b$, $v \ne 0$, then $L^{-1}v \ne 0$.

(L_4) There exist a constant $M > 0$ and a closed interval $I_0 \subset I$ such that for every $v \in P_b$ we have

$$\min_{t \in I_0} (L^{-1}v)(t) \ge M \left\| L^{-1}v \right\|_\infty .$$

Property (L_4) is called a *weak Harnack type inequality* for the operator L and it is usually required when looking for positive solutions, see [120].

Remark 6.1.1. *It is well–known that a wide class of boundary value problems can be reduced to integral equations involving operators of the form*

$$L^{-1}h(t) = \int_0^1 G(t,s)h(s)\,ds,$$

where G is the Green's function associated to the boundary value problem. In this case, if G is a nonnegative function and there exist $M > 0$, a closed interval $I_0 \subset I$ and a function $\Phi \in L^\infty(I)$ such that $\Phi > 0$ on $(0,1)$ and

$$\begin{array}{ll} G(t,s) \le \Phi(s) & \textit{for all } t, s \in I, \\ M\,\Phi(s) \le G(t,s) & \textit{for all } t \in I_0,\ s \in I, \end{array}$$

then conditions (L_3) and (L_4) hold, see e.g. [43, 142] and the problems studied in Chapter 5.

In other many cases, the operator L^{-1} can be split under the form $L^{-1} = AB$ with two operators A and B as required by condition (L_1). In particular, it happens if there exists the square root of L^{-1}, i.e., an operator A with $L^{-1} = A^2$. Such operators arise from some boundary value problems related to higher order differential equations. An example of this is given in [142].

When the operator L^{-1} exists, looking for positive solutions to problem (6.0.1) reduces to find a nonnegative function $u \in \mathcal{C}(I)$ such that $\mathcal{N}_f(u) \in P_b$ and $u = L^{-1}\mathcal{N}_f(u)$, where $\mathcal{N}_f$ is the Nemytskii operator $(\mathcal{N}_f u)(t) = f(t, u(t))$. Since f is not a continuous function, the fixed point operator $L^{-1}\mathcal{N}_f$ is not continuous in general and then most fixed point theorems cannot be applied to it. In order to overcome this difficulty it is usual in the literature, see e.g. [45, 68, 84] , to transform problem (6.0.1) into an inclusion

$$\begin{cases} Lu \in \mathcal{K}f(t,u) & \text{a.e. on } I, \\ u \in \mathcal{B}, \end{cases} \tag{6.1.1}$$

where $\mathcal{K}f : I \times \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$ is a regularization of the function f, often defined as

$$\mathcal{K}f(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}} f\left(t, \overline{B}_\varepsilon(x) \cap \mathbb{R}_+\right), \tag{6.1.2}$$

with $\overline{B}_\varepsilon(x) := [x-\varepsilon, x+\varepsilon]$. It is useful to recall that

$$\mathcal{K}f(t,x) = \left[\min\left\{f(t,x), \liminf_{y\to x} f(t,y)\right\}, \max\left\{f(t,x), \limsup_{y\to x} f(t,y)\right\}\right], \tag{6.1.3}$$

so $\mathcal{K}f(t,x) = \{f(t,x)\}$ provided that the function $f(t,\cdot)$ is continuous at x, see [8].

Thus, as in case of problem (6.0.1), in order to solve (6.1.1), it suffices to consider the inclusion operator $u \in L^{-1}\mathcal{N}_F(u)$, where

$$\mathcal{N}_F(u) = \{v \in L^1(I) : v(t) \in \mathcal{K}f(t, u(t)) \text{ for a.a. } t \in I\}. \tag{6.1.4}$$

Nevertheless, standard fixed point theorems for multivalued maps may not be applied to the operator $L^{-1}\mathcal{N}_F$. This is because L^{-1} is in general nonlinear, and so the values of $L^{-1}\mathcal{N}_F$ can be non–convex, even if $\mathcal{N}_F$ assumes convex values. To overcome this difficulty, following the ideas in [117], we consider the fixed point operator $\mathbb{T} : P^C \times P_b^C \to P^C \times \mathcal{P}(P^C)$ defined as

$$\mathbb{T}(u,v) = (Av,\ B\mathcal{N}_F(u)), \tag{6.1.5}$$

associated to the noncontinuous operator $T : P^C \times P_b^C \to P^C \times P^C$ given by

$$T(u,v) = (Av,\ B\mathcal{N}_f(u)), \tag{6.1.6}$$

and where A and B are given by condition (L_1).

Lemma 6.1.2. *Assume that the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following hypotheses:*

(h_1) *The composed function $f(\cdot, u(\cdot))$ is measurable for every $u \in \mathcal{C}(I)$;*

(h_2) *$f(t,u) < b$ on $I \times \mathbb{R}_+$, and if $b = \infty$, there exist $c_1, c_2 \in \mathbb{R}_+$ and $p \geq 1$ such that $f(t,u) \leq c_1 u^p + c_2$ for a.a. $t \in I$ and all $u \in \mathbb{R}_+$.*

Then the Nemytskii operator $\mathcal{N}_F : P^C \to \mathcal{P}(P)$ defined as in (6.1.4) *is an upper semicontinuous map from the topology of $\mathcal{C}(I)$ to that of $L^1(I)$.*

Proof. The mapping $\mathcal{K}f(t,\cdot)$ is upper semicontinuous for a.a. $t \in I$, see [8] or [45]. Then, according to [39, Theorem 1.1], the Nemytskii operator $\mathcal{N}_F$ is upper semicontinuous from the topology of $\mathcal{C}(I)$ to that of $L^1(I)$. □

Our aim is to guarantee the existence of fixed points for the operator $\mathbb{T}$ and then to show that they are fixed points for T or, equivalently, solutions to the problem (6.0.1). In order to do that, we assume that the operator L also satisfies the following condition:

(L_5) If $u, \gamma \in D(L)$ and $u(t) = \gamma(t)$ on $J \subset I$, then $Lu(t) = L\gamma(t)$ for a.a. $t \in J$.

Remark 6.1.3. *Observe that property (L_5) is satisfied by differential operators, since $u(t) = \gamma(t)$ on $J \subset I$ implies that $u'(t) = \gamma'(t)$ for a.a. $t \in J$, see Lemma 3.1.5.*

Let us introduce some notations. For $0 < r < R < b$ and $\varepsilon > 0$ small enough in order to ensure that $M\left\|L^{-1}(r\chi_{I_0})\right\|_\infty - \varepsilon > 0$, we let

$$f_{r,\varepsilon} := \inf\{f(t,y) \,:\, t \in I_0,\ M\left\|L^{-1}(r\chi_{I_0})\right\|_\infty - \varepsilon \leq y \leq \left\|L^{-1}R\right\|_\infty + \varepsilon\}, \tag{6.1.7}$$

$$f^R_\varepsilon := \sup\{f(t,y) \,:\, t \in I,\ 0 \leq y \leq \left\|L^{-1}R\right\|_\infty + \varepsilon\}, \tag{6.1.8}$$

where M and I_0 are given by condition (L_4) and χ denotes the characteristic function.

Theorem 6.1.4. *Suppose that the operator L fulfills conditions (L_1)–(L_5). Assume that the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the assumptions (h_1), (h_2) and*

(h_3) *There is a countable number of functions $\gamma_n \in D(L)$, $(n \in \mathbb{N})$, and a countable number of subintervals I_n of I such that*

$$\{L\gamma_n(t)\} \cap \mathcal{K}f(t,\gamma_n(t)) \subset \{f(t,\gamma_n(t))\} \quad \text{for a.a. } t \in I_n,\ \text{all } n \in \mathbb{N} \tag{6.1.9}$$

and

$$f(t,\cdot) \text{ is continuous on } \mathbb{R}_+ \setminus \bigcup_{\{n:\, t \in I_n\}} \{\gamma_n(t)\} \text{ for a.a. } t \in I.$$

Moreover, assume that there exist $0 < r < R < b$ and $\varepsilon > 0$ such that

$$f_{r,\varepsilon} \geq r \quad \text{and} \quad f^R_\varepsilon \leq R.$$

Then problem (6.0.1) *has at least one positive solution u such that*

$$L^{-1}(r\chi_{I_0}) \leq u \leq L^{-1}R.$$

Proof. Consider the sets

$$\begin{aligned} D_2^0 &= \{Bw : w \in P,\ w(t) \geq r \ \text{ for a.a. } \ t \in I_0,\ \|w\|_\infty \leq R\}, \\ D_2 &= \overline{D_2^0}, \\ D_1 &= \overline{\mathrm{co}}\{Av : v \in D_2^0\}, \end{aligned}$$

where the closures are taken in $\mathcal{C}(I)$. Since $R < b$, condition (L_1) (a) implies $D_2 \subset P_b^C$. Moreover, both D_1 and D_2 are convex sets. Now, assumption (L_1) guarantees that D_2 is compact in $\mathcal{C}(I)$ and, since A is continuous and maps bounded sets into bounded sets, D_1 is also compact.

Consider the operator $\mathbb{T} : D_1 \times D_2 \to P^C \times \mathcal{P}(P^C)$ defined as in (6.1.5). It is upper semicontinuous with respect to the product topology $\mathcal{C}(I) \times \mathcal{C}(I)$, since A is a single–valued operator, continuous from $\mathcal{C}(I)$ to $\mathcal{C}(I)$, $\mathcal{N}_F$ is upper semicontinuous from $\mathcal{C}(I)$ to $L^1(I)$ and B is continuous from $L^1(I)$ to $\mathcal{C}(I)$. Therefore, to apply Bohnenblust–Karlin's fixed point theorem, see Theorem A.14, it only remains to show that

$$\mathbb{T}(D_1 \times D_2) \subset D_1 \times D_2.$$

It is clear that $v \in D_2$ implies $Av \in D_1$. Let us show that $B\mathcal{N}_F(u) \in D_2$ for every $u \in D_1$. First, suppose that $u = Av$ for some $v \in D_2^0$. Let $v = Bw$ where $w \in P$ and $r\chi_{I_0} \preceq w \preceq R$. Since A and B preserve the ordering, we first have $B(r\chi_{I_0}) \preceq Bw \preceq BR$ and then $AB(r\chi_{I_0}) \preceq AB(w) \preceq AB(R)$. Observing that $ABw = u$ and $AB = L^{-1}$, one has

$$L^{-1}(r\chi_{I_0}) \preceq u \preceq L^{-1}R.$$

Clearly,

$$u(t) \leq (L^{-1}R)(t) \leq \left\|L^{-1}R\right\|_\infty . \tag{6.1.10}$$

Moreover, using condition (L_4), for $t \in I_0$ we obtain

$$u(t) \geq L^{-1}(r\chi_{I_0})(t) \geq M \left\|L^{-1}(r\chi_{I_0})\right\|_\infty . \tag{6.1.11}$$

Therefore, for $\tau \in [u(t) - \varepsilon, u(t) + \varepsilon] \cap \mathbb{R}_+$ we have that

$$\tau \in [M \left\|L^{-1}(r\chi_{I_0})\right\|_\infty - \varepsilon, \left\|L^{-1}R\right\|_\infty + \varepsilon] \cap \mathbb{R}_+,$$

and it follows that

$$f(t,\tau) \leq f_\varepsilon^R \leq R \quad \text{ for a.a. } t \in I,$$

and

$$r \leq f_{r,\varepsilon} \leq f(t,\tau) \quad \text{ for a.a. } t \in I_0.$$

Hence, for any $y \in \mathcal{N}_F(u)$ we have $\|y\|_\infty \leq R$ and $y(t) \geq r$ for $t \in I_0$ and so, $By \in D_2$. Thus, $B\mathcal{N}_F(u) \subset D_2$. Now, suppose that $u \in \mathrm{co}\{Av : v \in D_2^0\}$, that is, there exist $v_i \in D_2^0$ and $\lambda_i \in [0,1]$ $(i = 1, \ldots, m)$ such that $\sum_{i=1}^m \lambda_i = 1$ and $u = \sum_{i=1}^m \lambda_i A v_i$. The functions $u_i := Av_i$ are as in the previous case, so they satisfy (6.1.10) and (6.1.11). Then

$$u(t) = \sum \lambda_i u_i(t) \leq \left\|L^{-1}R\right\|_\infty \text{ for } t \in I, \tag{6.1.12}$$

and
$$u(t) = \sum \lambda_i u_i(t) \geq M \left\| L^{-1}(r\chi_{I_0}) \right\|_\infty \text{ for } t \in I_0, \tag{6.1.13}$$
so the conclusion that $B\mathcal{N}_F(u) \subset D_2$ follows in a similar way. Finally, if
$$u \in D_1 = \overline{\text{co}}\{Av : v \in D_2^0\},$$
then it is the limit of a sequence of functions satisfying (6.1.12) and (6.1.13) and, as a limit, it also satisfies the inequalities (6.1.12) and (6.1.13). Hence, for every $u \in D_1$ one has $B\mathcal{N}_F(u) \subset D_2$.

Therefore, the operator $\mathbb{T}$ has a fixed point in $D_1 \times D_2$ as a consequence of Bohnenblust–Karlin's fixed point theorem. Let us prove now that
$$\text{Fix}(\mathbb{T}) \subset \text{Fix}(T)$$
where T is the operator given in (6.1.6).

For $(u, v) \in \text{Fix}(\mathbb{T})$, we define
$$J_n := \{t \in I_n : u(t) = \gamma_n(t)\}, \ n \in \mathbb{N}.$$
By property (L_5), $Lu(t) = L\gamma_n(t)$ for a.a. $t \in J_n$. Hence,
$$L\gamma_n(t) \in \mathcal{K}f(t, u(t)) = \mathcal{K}f(t, \gamma_n(t)) \text{ for a.a. } t \in J_n,$$
which implies that $L\gamma_n(t) = f(t, u(t))$ for a.a. $t \in J_n$, by condition (6.1.9). Therefore, $Lu(t) = f(t, u(t))$ for a.a. $t \in J = \bigcup_{n\in\mathbb{N}} J_n$. Since $\mathcal{K}f(t, u(t)) = \{f(t, u(t))\}$ for $t \in I \setminus J$, then $Lu(t) = f(t, u(t))$ for a.a. $t \in I$ and, thus, $(u, v) \in \text{Fix}(T)$.

In conclusion, the operator T has a fixed point $(u, v) \in D_1 \times D_2$, that is, $Av = u$ and $B\mathcal{N}_f(u) = v$. Consequently, $u = AB\mathcal{N}_f(u)$, i.e., $u = L^{-1}\mathcal{N}_f(u)$, and so u is a positive solution for problem (6.0.1). □

Remark 6.1.5. *For the autonomous case $f(t, u) = f(u)$, if f is a nondecreasing function on $\mathbb{R}_+$, then we have that*
$$f_{r,\varepsilon} = f(M \left\| L^{-1}(r\chi_{I_0}) \right\|_\infty - \varepsilon) \quad \text{and} \quad f_\varepsilon^R = f(\left\| L^{-1}R \right\|_\infty + \varepsilon).$$

Remark 6.1.6. *We emphasize that condition (h_3) does not mean that the set of all discontinuity points of $f(t, \cdot)$ is equal to $\bigcup_{\{n:\, t\in I_n\}} \{\gamma_n(t)\}$, but only that it is contained in $\bigcup_{\{n:\, t\in I_n\}} \{\gamma_n(t)\}$. Therefore, the set of discontinuity points of $f(t, \cdot)$ needs not be the same for all values of t (the value of the function γ_n changes with t), but for each $t \in I$, it is at most countable.*

We allow the function f to be discontinuous over the graphs of a countable number of curves satisfying the "transversality" condition (6.1.9). *It is clear that such a condition is necessary, otherwise the existence result is not guaranteed, see Example 6.2.6.*

Remark 6.1.7. *Condition* (6.1.9) *was inspired by the following one:*
$$\mathcal{K}f(t, x) \cap DK_n(t, x)(1) \subset \{f(t, x)\} \quad \textit{for } x \in K_n(t), \tag{6.1.14}$$

where $K_n : I \to \mathcal{P}(\mathbb{R}^n)$ and $DK_n(t,x)$ denotes the contingent derivative of K_n at the point (t,x), see [8, 45]. Notice that if $K_n(t) = \{\gamma_n(t)\}$ for some $\gamma_n : I \to \mathbb{R}^n$ and γ_n is differentiable, then $DK_n(t,\gamma_n(t))(1) = \{\gamma_n'(t)\}$ and so condition (6.1.14) *can be written as*

$$\mathcal{K}f(t,\gamma_n(t)) \cap \{\gamma_n'(t)\} \subset \{f(t,\gamma_n(t))\}.$$

Condition (6.1.14) *was presented in [45] in the study of first–order initial discontinuous systems and it was also applied in [59] to first–order scalar functional boundary value problems. Our condition* (6.1.9) *is also related to the admissible discontinuity curves defined in the previous chapters, as shown by Remark 6.2.5.*

6.2 Positive solutions for the Dirichlet–Neumann problem involving the ϕ–Laplacian

We illustrate the applicability of the general theory from Section 6.1 to the existence of positive solutions for the differential problem

$$\begin{cases} [Lu](t) := -(\phi(u'))'(t) = f(t,u(t)) \quad \text{a.e. on } I := [0,1], \\ u'(0) = u(1) = 0, \end{cases} \tag{6.2.1}$$

where $\phi : (-a,a) \to (-b,b)$ is an increasing homeomorphism such that $\phi(0) = 0$, $0 < a, b \le \infty$, and the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ may have discontinuities.

By a *positive solution* of problem (6.2.1) we mean a function $u \in C^1(I)$, $u \ge 0$, $u \not\equiv 0$, with $u'(0) = u(1) = 0$, such that $u'(t) \in (-a,a)$ for all $t \in I$, $\phi \circ u' \in W^{1,1}(I)$ and $-(\phi(u'))'(t) = f(t,u(t))$ for a.a. $t \in I$.

In this case, $D(L)$ is the set of all functions $u \in C^1(I)$ such that $u'(t) \in (-a,a)$ for all $t \in I$ and $\phi \circ u' \in W^{1,1}(I)$. On the other hand, $\mathcal{B}$ is the set of functions $u \in C^1(I)$ with $u'(0) = u(1) = 0$. It is easy to check that operator L satisfies assumptions (L_1)–(L_3) and (L_5). In particular, L is invertible in P_b and

$$L^{-1}v(t) = -\int_t^1 \phi^{-1}\left(-\int_0^s v(\tau)\,d\tau\right) ds, \quad t \in I, \quad v \in P_b. \tag{6.2.2}$$

Moreover, $L^{-1} = A \circ B$ with

$$Bw(t) = \int_0^t w(\tau)\,d\tau$$

and

$$Av(t) = -\int_t^1 \phi^{-1}(-v(s))\,ds.$$

Finally, condition (L_4) holds according to the following result from [80, 83].

Proposition 6.2.1. *For each $c \in (0,1)$ and any $u \in D(L) \cap \mathcal{B}$ with $u \ge 0$ in I and $Lu \ge 0$ a.e. in I, we have that*

$$u(t) \ge M\,\|u\|_\infty \quad \textit{for all } t \in [0,c],$$

where $M = 1 - c$.

As a direct consequence of Theorem 6.1.4, we obtain the following result concerning problem (6.2.1). From now on, the value $c \in (0,1)$ is fixed.

Theorem 6.2.2. *Assume that the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following hypotheses:*

(H_1) *The composed function $f(\cdot, u(\cdot))$ is measurable for every $u \in \mathcal{C}(I)$;*

(H_2) *$f(t,u) < b$ on $I \times \mathbb{R}_+$, and if $b = \infty$, there exist $c_1, c_2 \in \mathbb{R}_+$ and $p \geq 1$ such that $f(t,u) \leq c_1 u^p + c_2$ for a.a. $t \in I$ and all $u \in \mathbb{R}_+$.*

(H_3) *There is a countable number of functions $\gamma_n \in \mathcal{C}^1(I)$, $(n \in \mathbb{N})$, with $\phi \circ \gamma_n' \in W^{1,1}(I)$, and a countable number of closed subintervals I_n of I such that*

$$\{-(\phi(\gamma_n'))'(t)\} \cap \mathcal{K}f(t,\gamma_n(t)) \subset \{f(t,\gamma_n(t))\} \quad \textit{for a.a. } t \in I_n, \textit{ all } n \in \mathbb{N} \tag{6.2.3}$$

and

$$f(t,\cdot) \textit{ is continuous on } \mathbb{R}_+ \setminus \bigcup_{\{n:\, t \in I_n\}} \{\gamma_n(t)\} \quad \textit{for a.a. } t \in I.$$

Moreover, assume that there exist $0 < r < R < b$ and $\varepsilon > 0$ such that

$$f_{r,\varepsilon} \geq r \quad \textit{and} \quad f_\varepsilon^R \leq R. \tag{6.2.4}$$

Then problem (6.2.1) *has at least one positive solution such that*

$$L^{-1}(r\chi_{[0,c]}) \leq u \leq L^{-1}R.$$

Remark 6.2.3. *In the conditions of this theorem, the interval $[0,c]$ plays the role of I_0 in* (6.1.7)–(6.1.8).

Remark 6.2.4. *Analogous results to Theorem 6.2.2 can be derived from Theorem 6.1.4 for some other boundary value problems with ϕ–Laplacian for which Harnack–type inequalities are known and the split of the solution operator L^{-1} as in (L_1) holds, see [80–82].*

Remark 6.2.5. *Assumption* (6.2.3) *is satisfied if either of the following conditions holds:*

(1) $-(\phi(\gamma_n'))'(t) = f(t,\gamma_n(t))$ *for a.a. $t \in I_n$;*

(2) $\{-(\phi(\gamma_n'))'(t)\} \not\in \mathcal{K}f(t,\gamma_n(t))$ *for a.a. $t \in I_n$.*

Alternative (1) *means that the curve γ_n solves the differential equation in the interval I_n, so it is usually said to be viable. On the other hand, for a curve γ_n as in case* (2), *$-(\phi(\gamma_n'))'(t)$ cannot coincide neither with $f(t,\gamma_n(t))$ nor with any limit of f when the variables tend to $(t,\gamma_n(t))$. In this case, we say that γ_n is an inviable curve. Let us point out the following sufficient condition for a curve $\gamma : I \to \mathbb{R}$ to be inviable on some closed subinterval $\tilde{I} \subset I$: there exist $\delta, \epsilon > 0$ such that*

$$-\left(\phi(\gamma')\right)'(t) + \delta \leq f(t,y) \quad \textit{for a.a. } t \in \tilde{I} \textit{ and all } y \in [\gamma(t)-\epsilon, \gamma(t)+\epsilon]\,, \tag{6.2.5}$$

or

$$-\left(\phi(\gamma')\right)'(t) - \delta \geq f(t,y) \quad \textit{for a.a. } t \in \tilde{I} \textit{ and all } y \in [\gamma(t)-\epsilon, \gamma(t)+\epsilon]\,. \tag{6.2.6}$$

The existence of solutions is not guaranteed if the function f is discontinuous over a single curve γ and condition (6.2.3) fails, as shown by the following example.

Example 6.2.6. *Consider the problem* $-(|u'|^{p-2}\,u')' = f(t,u)$, $u'(0) = u(1) = 0$, *where* $p > 1$ *and*

$$f(t,u) = \begin{cases} 1, & \text{if } u \geq \left(\frac{p-1}{p}\right) 2^{\frac{1}{p-1}} \left(1 - t^{\frac{p}{p-1}}\right), \\ 3, & \text{if } u < \left(\frac{p-1}{p}\right) 2^{\frac{1}{p-1}} \left(1 - t^{\frac{p}{p-1}}\right). \end{cases}$$

Here, the discontinuity curve is

$$\gamma(t) = \left(\frac{p-1}{p}\right) 2^{\frac{1}{p-1}} \left(1 - t^{\frac{p}{p-1}}\right) \text{ for all } t \in I,$$

and it satisfies that $-(|\gamma'|^{p-2}\,\gamma')'(t) = 2$, *but* $\mathcal{K}f(t,\gamma(t)) = [1,3]$ *for all* $t \in I$, *so*

$$\{-(|\gamma'|^{p-2}\,\gamma')'(t)\} \cap \mathcal{K}f(t,\gamma(t)) = \{2\} \not\subset \{f(t,\gamma(t))\}.$$

Notice that γ is not a solution to the problem.

In fact it has no solutions on the interval $I = [0,1]$. Indeed, suppose that u is a solution to the problem such that $u(0) > (p-1)\,2^{1/(p-1)}/p$. Then there exists $t_0 > 0$ such that $-(|u'|^{p-2}\,u')'(t) = 1$ for a.a. $t \in [0,t_0]$. Hence, $u'(t) = -t^{1/(p-1)}$ for all $t \in [0,t_0]$, which implies that $u'(t) \geq -(2t)^{1/(p-1)} = \gamma'(t)$ and thus $u \geq \gamma$ on I. By integration we obtain that $u(t) = -(p-1)t^{p/(p-1)}/p + k$ for some $k > 1$ and so $u(1) > 0$. Therefore, u is not a solution to the problem. In a similar way we can prove that there is no solutions such that $u(0) \leq (p-1)\,2^{1/(p-1)}/p$.

If no time–dependent discontinuities are considered (which is the case for autonomous equations) or, equivalently, we assume the discontinuity functions γ_n in Theorem 6.2.2 are constants, then we get the following result.

Corollary 6.2.7. *Assume that the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies hypotheses (H_1), (H_2) and*

(H_3^*) *There exists a sequence of numbers $x_n \in \mathbb{R}_+$, $(n \in \mathbb{N})$, such that*

$$\{0\} \cap \mathcal{K}f(t,x_n) \subset \{f(t,x_n)\} \tag{6.2.7}$$

and

$$f(t,\cdot) \text{ is continuous on } \mathbb{R}_+ \setminus \bigcup_{n\in\mathbb{N}} \{x_n\}$$

for a.a. $t \in I$.

Moreover, assume that there exist $0 < r < R < b$ and $\varepsilon > 0$ satisfying condition (6.2.4). *Then problem* (6.2.1) *has at least one positive solution such that*

$$L^{-1}(r\chi_{[0,c]}) \leq u \leq L^{-1}R.$$

In this context, Corollary 6.2.7 can be improved in order to allow an uncountable number of discontinuities with respect to the spatial variable.

Theorem 6.2.8. *Assume that the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies hypotheses (H_1), (H_2) and*

(H_3^{**}) *There is a set $A \subset \mathbb{R}_+$ of null Lebesgue measure such that*

$$\{0\} \cap \mathcal{K}f(t,x) \subset \{f(t,x)\} \quad \textit{for all } x \in A \tag{6.2.8}$$

and

$$f(t,\cdot) \textit{ is continuous on } \mathbb{R}_+ \setminus A$$

for a.a. $t \in I$.

Moreover, assume that there exist $0 < r < R < b$ and $\varepsilon > 0$ satisfying (6.2.4). Then problem (6.2.1) has at least one positive solution such that

$$L^{-1}(r\chi_{[0,c]}) \le u \le L^{-1}R.$$

Proof. It is clear, as a consequence of Theorems 6.1.4 and 6.2.2, that the multivalued operator $\mathbb{T}$ defined in (6.1.5), where L^{-1} is the integral operator given in (6.2.2), has a fixed point. Hence, it only remains to verify that $\mathrm{Fix}(\mathbb{T}) \subset \mathrm{Fix}(T)$.

For $(u,v) \in \mathrm{Fix}(\mathbb{T})$, we define $J := \{t \in I : u(t) \in A\}$. By Lemma 3.1.5, $u'(t) = 0$ for a.a. $t \in J$ and thus $-(\phi(u'))'(t) = 0$ for a.a. $t \in J$. Hence, $0 \in \mathcal{K}f(t,u(t))$ for a.a. $t \in J$ what implies that $0 = f(t,u(t))$ for a.a. $t \in J$, by condition (6.2.8). Therefore, $-(\phi(u'))'(t) = f(t,u(t))$ for a.a. $t \in J$. Since $\mathcal{K}f(t,u(t)) = \{f(t,u(t))\}$ for $t \in I \setminus J$, then $-(\phi(u'))'(t) = f(t,u(t))$ for a.a. $t \in I$ and, thus, $(u,v) \in \mathrm{Fix}(T)$. □

Notice that some examples of functions with an uncountable null measure set of discontinuity points are known, see e.g. Proposition 3.2.15 and [21, Example 3.2].

Remark 6.2.9. *We note that condition (6.2.8) is simply equivalent, in virtue of (6.1.3), to the following one:*

$$\liminf_{y \to x} f(t,y) = 0 \quad \textit{implies} \quad f(t,x) = 0.$$

Hence, it is trivially satisfied if $\inf_{t \in I, x \in \mathbb{R}_+}\{f(t,x)\} > 0$.

Remark 6.2.10. *Condition (H_3^{**}) is similar to that employed recently by Bonanno and collaborators in the study of the existence and multiplicity of solutions for Sturm–Liouville equations, see [18, 19] and the references therein. Their approach is based on critical point theory for non–differentiable functions. Nevertheless, no time–dependent discontinuity sets are considered in their results.*

The definitions of $f_{r,\varepsilon}$ and f_ε^R make condition (6.2.4) difficult to be applied in concrete problems since they are given in terms of the operator L^{-1}. However, for the cases $b = \infty$ or $a = \infty$ we are able to give some sufficient conditions which are easier to check.

Theorem 6.2.11. *Assume that* $\phi : (-a, a) \to \mathbb{R}$ *is an increasing homeomorphism such that* $\phi(-\tau) = -\phi(\tau)$ *for all* $\tau \in (-a, a)$*,* ϕ *is a convex function on* $(0, a)$*,* $0 < a \le \infty$ *and there exists* $\tau_0 \in (0, a)$ *such that* $\phi(\tau) \le \tau$ *for all* $\tau \in [0, \tau_0]$.

Then condition (6.2.4) *holds if there exist* $0 < r < R$ *and* $\varepsilon > 0$ *such that* $r \le \tau_0/c$ *and*

$$\inf\{f(t, y) \,:\, t \in [0, c],\ r\tilde{c} \le y \le \phi^{-1}(R) + \varepsilon\} \ge r, \tag{6.2.9}$$

$$\sup\{f(t, y) \,:\, t \in [0, 1],\ 0 \le y \le \phi^{-1}(R) + \varepsilon\} \le R, \tag{6.2.10}$$

where $\tilde{c} > 0$ *is a fixed number such that* $\tilde{c} < c(2-c)(1-c)/2$.

In particular, if $f(t, u) = f(u)$ *and* f *is a nondecreasing map, then the following asymptotic conditions guarantee that* (6.2.4) *is satisfied:*

$$\liminf_{y \to a} \frac{f(y)}{\phi(y)} < 1 \quad \textit{and} \quad \limsup_{y \to 0} \frac{f(y)}{y} > \frac{1}{\tilde{c}}.$$

Proof. Let us show that if there exist $0 < r < R$ and $\varepsilon > 0$ small enough such that $r \le \tau_0/c$ and (6.2.9) and (6.2.10) hold, then

$$f_{r,\varepsilon} := \inf\{f(t, y) \,:\, t \in [0, c],\ (1-c)\left\|L^{-1}(r\chi_{[0,c]})\right\|_\infty - \varepsilon \le y \le \left\|L^{-1}R\right\|_\infty + \varepsilon\} \ge r,$$
$$f_\varepsilon^R := \sup\{f(t, y) \,:\, t \in [0, 1],\ 0 \le y \le \left\|L^{-1}R\right\|_\infty + \varepsilon\} \le R,$$

where L^{-1} is the operator defined in (6.2.2).

Let us compute the values of $\left\|L^{-1}(r\chi_{[0,c]})\right\|_\infty$ and $\left\|L^{-1}R\right\|_\infty$. Since ϕ is assumed to be an odd function, we have

$$(L^{-1}R)(t) = \int_t^1 \phi^{-1}\left(\int_0^s R\, d\tau\right) ds = \int_t^1 \phi^{-1}(Rs)\, ds,$$

so

$$\left\|L^{-1}R\right\|_\infty = (L^{-1}R)(0) = \int_0^1 \phi^{-1}(Rs)\, ds \le \int_0^1 \phi^{-1}(R)\, ds = \phi^{-1}(R).$$

On the other hand, we have

$$L^{-1}(r\chi_{[0,c]})(t) = \int_t^1 \phi^{-1}\left(r \int_0^s \chi_{[0,c]}\, d\tau\right) ds.$$

Hence, since ϕ^{-1} is an increasing function,

$$\left\|L^{-1}(r\chi_{[0,c]})\right\|_\infty = \int_0^1 \phi^{-1}\left(r\int_0^s \chi_{[0,c]}\, d\tau\right) ds = \int_0^c \phi^{-1}(rs)\, ds + \int_c^1 \phi^{-1}(rc)\, ds,$$

and thus the facts that $\phi^{-1}(\tau) \ge \tau$ for all $\tau \in [0, \tau_0]$ and $rc \le \tau_0$ imply that

$$\left\|L^{-1}(r\chi_{[0,c]})\right\|_\infty \ge \int_0^c rs\, ds + \int_c^1 rc\, ds = r\frac{c^2}{2} + (1-c)rc = r\frac{c(2-c)}{2}.$$

Notice that for a fixed $\tilde{c} < c(2-c)(1-c)/2$ it is possible to choose $\varepsilon > 0$ small enough in order to guarantee that

$$\tilde{c}r \leq rc(2-c)(1-c)/2 - \varepsilon \leq (1-c)\left\|L^{-1}(r\chi_{[0,c]})\right\|_\infty - \varepsilon$$

and then

$$f_{r,\varepsilon} \geq \inf\{f(t,y)\,:\, t\in[0,c],\ r\tilde{c} \leq y \leq \phi^{-1}(R)+\varepsilon\},$$
$$f_\varepsilon^R \leq \sup\{f(t,y)\,:\, t\in[0,1],\ 0 \leq y \leq \phi^{-1}(R)+\varepsilon\},$$

so the conclusion is obtained. □

Remark 6.2.12. *Note that it is not a restriction to assume, in Theorem 6.2.11, that there exists $\tau_0 \in (0,a)$ such that $\phi(\tau) \leq \tau$ for all $\tau \in [0,\tau_0]$. Indeed, because of being ϕ odd and convex in $(0,a)$, for any $\tau_0 \in (0,a)$ there exists a constant $m > 0$ such that $\phi(\tau) \leq m\tau$ for all $\tau \in [0,\tau_0]$. Then problem* (6.2.1) *is equivalent to the following one:*

$$\begin{cases} -(\tilde{\phi}(u'))'(t) = \tilde{f}(t,u(t)) & \text{a.e. on } I, \\ u'(0) = u(1) = 0, \end{cases}$$

where $\tilde{\phi}(\tau) = m^{-1}\phi(\tau)$ for all $\tau \in (-a,a)$ and $\tilde{f}(t,u) = m^{-1}f(t,u)$ for all $t \in I$ and $u \in \mathbb{R}_+$. It is clear that $\tilde{\phi}(\tau) \leq \tau$ for all $\tau \in [0,\tau_0]$. Of course, in this case, the estimates from above on f are different, and then in condition (6.2.9) *it is necessary to take $\tilde{c} < c(2-c)(1-c)/2m$.*

Now we are able to give the proof of Theorem 6.0.1, which we recall again now.

Theorem 6.2.13. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an almost everywhere continuous function such that*

(i) *$f(u(\cdot))$ is measurable whenever $u \in \mathcal{C}(I)$;*

(ii) *there exist $c_1, c_2 \in \mathbb{R}_+$ and $p \geq 1$ such that $f(x) \leq c_1x^p + c_2$ for all $x \in \mathbb{R}_+$;*

(iii) *$\liminf_{y\to x} f(y) = 0$ implies $f(x) = 0$ for every $x \in \mathbb{R}_+$.*

Assume that there exists $0 < r < 2\sqrt{3}$ such that

$$r \leq \inf\{f(y) : r/16 \leq y \leq 1\}.$$

Then problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1-u'^2}}\right)' = f(u) & \textit{a.e. on } [0,1], \\ u'(0) = u(1) = 0, \end{cases}$$

has at least one positive solution such that $\|u\|_\infty \geq 3r/16$.

Proof. It is clear that $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies conditions (H_1) and (H_2) in Theorem 6.2.8. By Remark 6.2.9, it also fulfills condition (H_3^{**}). Hence it only remains to see that there exist $0 < r < R$ and $\varepsilon > 0$ satisfying (6.2.4). In order to do that, we take into account

Theorem 6.2.11 and Remark 6.2.12. Since $\phi^{-1}(\tau) = \tau/\sqrt{1+\tau^2}$ is bounded and f is a locally bounded function, condition (6.2.10) is satisfied for $R > 0$ big enough. On the other hand, the homeomorphism $\phi(\tau) = \tau/\sqrt{1-\tau^2}$ satisfies $\phi(\tau) \leq 2\tau$ for all $\tau \in [0, \sqrt{3}/2]$. Hence, if we choose $c = 1/2$ and then $\tilde{c} = 1/16$, the conclusion is obtained since

$$\left\| L^{-1}(r\chi_{[0,c]}) \right\|_\infty \geq r\frac{c(2-c)}{2m} = \frac{3}{16}r.$$

□

Now we present an analogous result to Theorem 6.2.11 concerning the case $a = \infty$.

Theorem 6.2.14. *Assume that $\phi : \mathbb{R} \to (-b, b)$ $(0 < b \leq \infty)$ is an increasing homeomorphism such that $\phi(-\tau) = -\phi(\tau)$ for all $\tau \in \mathbb{R}$, ϕ is a concave function on $\mathbb{R}_+$ and $\phi(\tau) \leq \tau$ for all $\tau \in \mathbb{R}_+$.*

Then condition (6.2.4) *holds if there exist $0 < r < R < b$ and $\varepsilon > 0$ satisfying* (6.2.9) *and* (6.2.10).

Proof. Reasoning as in Theorem 6.2.11, it is easy to see that

$$\left\| L^{-1}R \right\|_\infty \leq \phi^{-1}(R) \quad \text{and} \quad \left\| L^{-1}(r\chi_{[0,c]}) \right\|_\infty \geq \frac{rc(2-c)}{2},$$

since $\tau \leq \phi^{-1}(\tau)$ for all $\tau \in \mathbb{R}_+$. □

Remark 6.2.15. *Observe that it is enough to assume in Theorem 6.2.14 that there exists $m > 0$ such that $\phi(\tau) \leq m\tau$ for all $\tau \in \mathbb{R}_+$ and then follow the explanation in Remark 6.2.12.*

Remark 6.2.16. *Theorems 6.2.11 and 6.2.14 also provide a localization result for the solution established by Theorem 6.2.2, with bounds which are not given in terms of L^{-1}. Indeed, the solution u satisfies*

$$\frac{rc(2-c)}{2} \leq \left\| L^{-1}(r\chi_{[0,c]}) \right\|_\infty \leq \|u\|_\infty \leq \left\| L^{-1}R \right\|_\infty \leq \phi^{-1}(R).$$

The previous results allow us to obtain multiple solutions whenever there exist several couples of numbers r_n, R_n satisfying

$$r_n < R_n, \quad r_n \leq \frac{\tau_0}{c} \quad (\tau_0 = \infty \text{ if } b < \infty) \text{ and } \quad \phi^{-1}(R_{n+1}) < r_n \frac{c(2-c)}{2}.$$

In particular, we state the following result concerning the existence of infinitely many solutions for problem (6.2.1).

Theorem 6.2.17. *Assume that the hypotheses of Theorems 6.2.11 or 6.2.14 are satisfied and that $f = f(u)$ is a nondecreasing function satisfying (H_1), (H_2) and (H_3^{**}). If, in addition,*

$$\liminf_{y\to 0} \frac{f(y)}{\phi(y)} < 1 \quad \textit{and} \quad \limsup_{y\to 0} \frac{f(y)}{y} > \frac{1}{\tilde{c}},$$

then problem (6.2.1) *has a sequence of positive solutions $(u_n)_{n\geq 1}$ which moreover satisfy $\|u_n\|_\infty \to 0$ as $n \to \infty$.*

Proof. The asymptotic behavior of f near 0 guarantees the existence of two decreasing sequences of numbers $(r_n)_{n\geq 1}$ and $(R_n)_{n\geq 1}$ such that

$$0 < r_n < R_n \text{ and } \phi^{-1}(R_{n+1}) < r_n \frac{c(2-c)}{2} \text{ for every } n \geq 1, \text{ and } R_n \to 0 \text{ as } n \to \infty,$$

and they satisfy conditions (6.2.9) and (6.2.10) replacing r and R by r_n and R_n, respectively. Therefore, for each $n \geq 1$, there exists a positive solution u_n such that

$$r_n \frac{c(2-c)}{2} \leq \|u_n\|_\infty \leq \phi^{-1}(R_n).$$

Since $\phi^{-1}(R_n) \to 0$ as $n \to \infty$, we have $\|u_n\|_\infty \to 0$ as $n \to \infty$. □

To finish this section, we illustrate our existence results with two examples.

Example 6.2.18. *Consider the problem*

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = \sqrt{u}e^{-u} + \dfrac{1}{2}H(u-kt^2) \quad a.e \text{ on } I, \\ u'(0) = u(1) = 0, \end{cases} \tag{6.2.11}$$

where $k > 0$ and $H : \mathbb{R} \to \mathbb{R}$ is the Heaviside step function given by

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0. \end{cases}$$

Notice that the homeomorphism $\phi : \mathbb{R} \to (-1, 1)$ given by $\phi(\tau) = \tau/\sqrt{1+\tau^2}$ satisfies the hypotheses in Theorem 6.2.14 and that $f(t,u) < 1$. Moreover, for a.a. $t \in I$, the function $u \mapsto f(t,u) = \sqrt{u}e^{-u} + H(u-kt^2)/2$ is continuous on $\mathbb{R}_+ \setminus \{kt^2\}$. The curve $\gamma(t) = kt^2$, $t \in I$, satisfies that

$$-(\phi(\gamma'))'(t) \leq -2k/(1+4k^2)^{3/2},$$

so condition (6.2.5) *in Remark 6.2.5 holds since $f(t,u) \geq 0$ for a.a. $t \in I$ and all $u \in \mathbb{R}_+$. On the other hand, if we take $c = 1/2$ and then $\tilde{c} = 1/8$, it is easy to check that the function f satisfies conditions* (6.2.9) *and* (6.2.10) *for $r = 1/32$, $R = 19/20$ and $\varepsilon > 0$ small enough. Therefore, Theorem 6.2.2 and Remark 6.2.16 ensure the existence of a positive solution for problem* (6.2.11) *such that*

$$\frac{3}{256} \leq \|u\|_\infty \leq \frac{19\sqrt{39}}{39}.$$

Example 6.2.19. *Consider the differential problem*

$$\begin{cases} -(u'\,|u'|^2)' = (2 - \cos\lfloor 1/(u-t+1)\rfloor)\,(u+1)^2 \quad a.e. \text{ on } I, \\ u'(0) = u(1) = 0, \end{cases} \tag{6.2.12}$$

where $\lfloor x \rfloor$ denotes the integer part of x. First, observe that the homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ defined as $\phi(\tau) = \tau\,|\tau|^2$ satisfies $\phi(\tau) \leq \tau$ for all $\tau \in [0,1]$ and $\phi^{-1}(\tau) = |\tau|^{1/3}\operatorname{sgn}(\tau)$. For this example, the discontinuity curves γ_n are given by

$$\gamma_n(t) = t - 1 + \frac{1}{n}$$

with domain

$$I_n = \left[1 - \frac{1}{n},\, 1\right], \qquad n \in \mathbb{N}.$$

Clearly, the function $u \mapsto f(t,u) = (2 - \cos\lfloor 1/(u - t + 1)\rfloor)\,(u+1)^2$ *is continuous on* $\mathbb{R}_+ \setminus \bigcup_{\{n:\, t\in I_n\}}\{\gamma_n(t)\}$ *for a.a.* $t \in I$. *Note that* $-(\phi(\gamma_n'))'(t) = 0$ *for a.a.* $t \in I_n$ *and* $n \in \mathbb{N}$. *In addition,* $f(t,u) \geq 1$ *for a.a.* $t \in I$ *and all* $u \in \mathbb{R}_+$, *and thus condition* (6.2.5) *in Remark 6.2.5 is satisfied. To finish, it is sufficient to choose* $c = 1/2$, $\tilde{c} = 1/8$, $R = 64$, $r = 1$ *and* $\varepsilon > 0$ *small enough to guarantee that conditions* (6.2.9) *and* (6.2.10) *in Theorem 6.2.11 hold. Therefore, Theorem 6.2.2 gives a positive solution for problem* (6.2.12) *and by Remark 6.2.16 one has*

$$\frac{3}{8} \leq \|u\|_\infty \leq 4.$$

We underline that this positive solution must cross infinitely many discontinuity curves γ_n, *since it satisfies the final condition* $u(1) = 0$ *and the graphs of* γ_n *restricted to* I_n *tend to the point* $(1,0)$ *as* n *goes to infinity, as shown in Figure 6.2.1.*

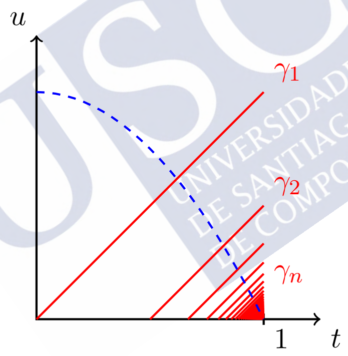

Figure 6.2.1: Discontinuity curves γ_n.

6.3 Positive solutions for the Neumann–Robin problem involving the ϕ–Laplacian

In this section we establish new existence, localization and multiplicity results of positive solutions for the problem

$$\begin{cases} -(\phi(u'))'(t) = f(t,u(t)) & \text{a.e. on } I := [0,1], \\ u(0) - \alpha u'(0) = u'(1) = 0, \end{cases} \tag{6.3.1}$$

where $\alpha \geq 0$ and, as in the previous section, $\phi : (-a,a) \to (-b,b)$ is an increasing homeomorphism such that $\phi(0) = 0$, $0 < a,\ b \leq \infty$.

Unlike the previous section where the results are based on the application of Bohnenblust–Karlin's fixed point theorem, here they are obtained via a suitable version of Krasnosel'skiĭ's compression–expansion fixed point theorem in cones. This allows us to consider a more general class of nonlinearities.

We say that u is a *positive solution* to problem (6.3.1) if it is a function $u \in \mathcal{C}^1(I)$, $u \geq 0$, $u \not\equiv 0$, with $u(0) - \alpha u'(0) = u'(1) = 0$, such that

$$\phi \circ u' \in W^{1,1}(I) \quad \text{and} \quad -(\phi(u'))'(t) = f(t, u(t)) \;\text{ for a.a. }\; t \in I.$$

Positive solutions for this problem coincide with fixed points of the integral operator $T : P \to P$ given by

$$Tu(t) = \alpha\phi^{-1}\left(\int_0^1 f(s, u(s))\, ds\right) + \int_0^t \phi^{-1}\left(\int_r^1 f(s, u(s))\, ds\right) dr,$$

where P is the cone of nonnegative functions in the Banach space of the continuous functions with the maximum norm $(\mathcal{C}(I), \|\cdot\|_\infty)$.

Since we want to consider discontinuous nonlinearities, we will first study the regularized problem in the Filippov sense [68], namely, the boundary value problem for a differential inclusion:

$$\begin{cases} -(\phi(u'))'(t) \in \mathcal{K}f(t, u(t)) & \text{a.e. on } I, \\ u(0) - \alpha u'(0) = u'(1) = 0, \end{cases} \tag{6.3.2}$$

where the multivalued map $\mathcal{K}f : I \times \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$ is defined as in (6.1.2).

As mentioned above, since f is not necessarily continuous, the operator T may be discontinuous and the usual compression–expansion type results are not applicable. This is the motivation to consider the inclusion (6.3.2) and to look for solutions of this problem by means of the multivalued operator $\mathbb{T} : P \to \mathcal{P}(P)$ defined as

$$\mathbb{T}u(t) = \alpha\phi^{-1}\left(\int_0^1 \mathcal{K}f(s, u(s))\, ds\right) + \int_0^t \phi^{-1}\left(\int_r^1 \mathcal{K}f(s, u(s))\, ds\right) dr.$$

Unfortunately, the standard generalization of Krasnosel'skiĭ's fixed point theorem to upper semicontinuous multivalued maps with convex values, due to Fitzpatrick and Petryshyn [69], is not applicable to $\mathbb{T}$. The reason is that the values of the integral operator are not convex in general, due to the nonlinearity of ϕ. To overcome this difficulty, we will apply a compression-expansion fixed point theorem established in [48] for the composition of two multivalued operators (see Appendix B).

Notice that the operator $\mathbb{T}$ can be decomposed as

$$\mathbb{T} = \Psi\Phi,$$

where for every $v \in P$,

$$\Psi v(t) = \alpha\phi^{-1}(v(0)) + \int_0^t \phi^{-1}(v(s))\, ds, \quad t \in I, \tag{6.3.3}$$

and

$$\Phi v(t) = \Lambda\,\mathcal{N}_F v(t), \tag{6.3.4}$$

with

$$\Lambda w(t) = \int_t^1 w(s)\,ds$$

and $\mathcal{N}_F$ the Nemytskii operator defined as in (6.1.4).

In order to apply Theorem B.2 we need the following Harnack–type inequality established in [81, 83] for the case $a = \infty$. Notice that, with the same proof, the result remains true for $a < \infty$.

Lemma 6.3.1. *Let $u \in \mathcal{C}^1(I)$, $u \geq 0$, be such that $u(0) - \alpha u'(0) = u'(1) = 0$ and $\phi \circ u'$ is nonincreasing in I. Then for each $c \in (0,1)$ we have*

$$u(t) \geq M \, \|u\|_\infty \quad \textit{for all } t \in [c,1],$$

where $M = (\alpha + c) / (\alpha + 1)$.

From now on, the value $c \in (0,1)$ is fixed. The essential properties of the operators Φ and Ψ introduced in (6.3.3)–(6.3.4) are given by the following theorem involving two subcones of P, namely

$$K_1 = \{u \in P \ : \ u(t) \geq M \, \|u\|_\infty \ \text{ for all } t \in [c,1]\},$$
$$K_2 = \{u \in P \ : \ u \text{ is nonincreasing, } \ u(1) = 0\}.$$

Theorem 6.3.2. *Assume that $f : I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfies the following conditions:*

(H_1) *The function $f(\cdot, u(\cdot))$ is measurable for every $u \in P$;*

(H_2) *$f(t,u) < b$ on $I \times \mathbb{R}_+$, and if $b = \infty$, there exist $c_1, c_2 \in \mathbb{R}_+$ and $p \geq 1$ such that $f(t,u) \leq c_1 u^p + c_2$ for a.a. $t \in I$ and all $u \in \mathbb{R}_+$.*

Then the operators

$$\Phi : K_1 \to \mathcal{P}(K_2) \quad \textit{and} \quad \Psi : K_2 \to K_1$$

are well–defined. Moreover, Φ is usc with nonempty, closed and convex values and maps bounded sets into relatively compact sets; and Ψ is a single–valued continuous operator which maps bounded sets into relatively compact sets.

Proof. Since $\Phi = \Lambda \mathcal{N}_F$, it follows from the definition of the operator Λ that $\Phi(K_1) \subset K_2$. To show that $\Psi(K_2) \subset K_1$, take any $v \in K_2$ and let $u := \Psi(v)$. Clearly, $u \in P$. Also $\phi \circ u' = v$ and so $\phi \circ u'$ is nonincreasing in I. Moreover, $u(0) - \alpha u'(0) = u'(1) = 0$. Consequently, by Lemma 6.3.1, $u(t) \geq M \|u\|_\infty$ for all $t \in [c,1]$. Hence $u \in K_1$, as desired.

In addition, Λ, as a linear operator from $L^1(I)$ to $\mathcal{C}(I)$ is compact, while in view of Lemma 6.1.2 , $\mathcal{N}_F$ is usc from the topology of $\mathcal{C}(I)$ to that of $L^1(I)$. Thus Φ is usc and maps bounded sets into relatively compact sets.

Clearly, Φ has convex values. To show that its values are also closed in $\mathcal{C}(I)$, take any element $u \in K_1$ and any sequence $v_n \in \Phi u$ with $v_n \to v$ in $\mathcal{C}(I)$. Then $v_n = \Lambda w_n$ for some $w_n \in \mathcal{N}_F(u)$. From the definition of $\mathcal{K}f$, we have that $\mathcal{N}_F(u)(t)$ is uniformly bounded on I. As a result, the sequence w_n is bounded in $L^p(I)$ for any (fixed) $p \in (1, \infty)$, and so we can assume without loss of generality that w_n is weakly convergent in $L^p(I)$ to some w. It is easy to see that $w \in \mathcal{N}_F(u)$. Then there is a sequence $\overline{w}_n$ of convex combinations of w_n which strongly converges in $L^p(I)$, and consequently in $L^1(I)$, to w. From the fact that $\overline{v}_n = \Lambda\overline{w}_n$, we deduce that the corresponding sequence $\overline{v}_n$ of convex combinations of v_n converges in $\mathcal{C}(I)$ to Λw. But since $v_n \to v$, the limit of $\overline{v}_n$ is v. Then $v = \Lambda w$, where $w \in \mathcal{N}_F(u)$, which proves that $v \in \Phi u$, as desired.

Finally, the continuity and the compactness of the operator Ψ are standard consequences of Lebesgue's dominated convergence theorem and Áscoli–Arzela's theorem. □

Now we are ready to state and prove the main result about the existence and localization of positive solutions to the problem (6.3.1).

Theorem 6.3.3. *Assume that the function f satisfies conditions (H_1), (H_2) and*

(H_3) *There exists a countable number of functions $\gamma_n \in \mathcal{C}^1(I)$, $(n \in \mathbb{N})$, with $\phi \circ \gamma_n' \in W^{1,1}(I)$, and a countable number of closed subintervals I_n of I such that*

$$\{-(\phi(\gamma_n'))'(t)\} \cap \mathcal{K}f(t,\gamma_n(t)) \subset \{f(t,\gamma_n(t))\} \quad \text{for a.a. } t \in I_n, \text{ all } n \in \mathbb{N} \tag{6.3.5}$$

and

$$f(t,\cdot) \text{ is continuous on } \mathbb{R}_+ \setminus \bigcup_{\{n:\, t \in I_n\}} \{\gamma_n(t)\} \quad \text{for a.a. } t \in I. \tag{6.3.6}$$

In addition assume that there exist $0 < r_1,\, r_2,\, r_1 \neq r_2$ and $\varepsilon > 0$ such that

$$\alpha\phi^{-1}\left(\int_0^1 \Gamma_{r_1}^{\varepsilon}(s)\,ds\right) + \int_0^1 \phi^{-1}\left(\int_r^1 \Gamma_{r_1}^{\varepsilon}(s)\,ds\right) dr \le r_1, \tag{6.3.7}$$

$$\alpha\phi^{-1}\left(\int_c^1 \Gamma_{r_2,\varepsilon}(s)\,ds\right) + \int_c^1 \phi^{-1}\left(\int_r^1 \Gamma_{r_2,\varepsilon}(s)\,ds\right) dr \ge r_2, \tag{6.3.8}$$

where

$$\Gamma_{r_1}^{\varepsilon}(s) = \max_{x \in [0,\, r_1+\varepsilon]} f(s,x) \quad \text{and} \quad \Gamma_{r_2,\varepsilon}(s) = \min_{x \in [(r_2-\varepsilon)M,\, r_2+\varepsilon]} f(s,x).$$

Then problem (6.3.1) *has at least one positive solution u such that*

$$\min\{r_1, r_2\} \le \|u\|_\infty \le \max\{r_1, r_2\}. \tag{6.3.9}$$

Proof. We apply Theorem B.2. By virtue of Theorem 6.3.2, it only remains to prove that the operator $\mathbb{T} = \Psi\Phi$ satisfies the compression–expansion conditions given in (B.3).

We first show that

$$\|v\|_\infty \le r_1 \quad \text{for all } v \in \mathbb{T}u \quad \text{and all } u \in K_1 \text{ with } \|u\|_\infty = r_1,$$

which implies that

$$u \notin \lambda\mathbb{T}u \quad \text{for all } \lambda \in (0,1) \text{ and all } u \in K_1 \text{ with } \|u\|_\infty = r_1.$$

Assume on the contrary that there exists $v \in \mathbb{T}u$ and $u \in K_1$ with $\|u\|_\infty = r_1$ such that $r_1 < \|v\|_\infty$. Notice that for $\varepsilon > 0$, if $w \in \mathcal{N}_F(u)$ and $\|u\|_\infty = r_1$, then we have that $w(s) \leq \max_{x\in[0,r_1+\varepsilon]} f(s,x) =: \Gamma^\varepsilon_{r_1}(s)$ for all $s \in I$. Hence, by the fact that $v \in \mathbb{T}u$ and (6.3.7), we obtain that

$$\|v\|_\infty \leq \alpha\phi^{-1}\left(\int_0^1 \Gamma^\varepsilon_{r_1}(s)\,ds\right) + \int_0^1 \phi^{-1}\left(\int_r^1 \Gamma^\varepsilon_{r_1}(s)\,ds\right)dr \leq r_1,$$

which yields the contradiction $r_1 < r_1$.

Next we have to show that

$$r_2 \leq \|v\|_\infty \quad \text{for all } \ v \in \mathbb{T}u \ \text{ and all } \ u \in K_1 \ \text{ with } \ \|u\|_\infty = r_2,$$

which implies that

$$u \notin \mathbb{T}u + \mu \quad \text{for all } \ \mu > 0 \ \text{ and all } \ u \in K_1 \ \text{ with } \ \|u\|_\infty = r_2.$$

Assume on the contrary that there exists $v \in \mathbb{T}u$ and $u \in K_1$ with $\|u\|_\infty = r_2$ such that $\|v\|_\infty < r_2$. Observe that for $\varepsilon > 0$, if $w \in \mathcal{N}_F(u)$ and$\|u\|_\infty = r_2$, then $w(s) \geq \Gamma_{r_2,\varepsilon}(s)$ for all $s \in [c,1]$. Now, by (6.3.8),

$$r_2 \leq \alpha\phi^{-1}\left(\int_c^1 \Gamma_{r_2,\varepsilon}(s)\,ds\right) + \int_c^1 \phi^{-1}\left(\int_r^1 \Gamma_{r_2,\varepsilon}(s)\,ds\right)dr \leq \|v\|_\infty\,,$$

a contradiction.

Therefore, Theorem B.2 applies and yields the existence of a fixed point $u \in P$ for the operator $\mathbb{T}$ satisfying (6.3.9). Then

$$-\,(\phi(u'))'(t) \in \mathcal{K}f(t,u(t)) \quad \text{for a.a. } \ t \in I. \tag{6.3.10}$$

Now we prove that u (in fact, any fixed point of $\mathbb{T}$) solves the former discontinuous problem (6.3.1). To this aim, define

$$J_n := \{t \in I_n \,:\, u(t) = \gamma_n(t)\}\,, \qquad n \in \mathbb{N}.$$

Clearly

$$-(\phi(u'))'(t) = -(\phi(\gamma_n'))'(t) \quad \text{for a.a. } \ t \in J_n.$$

Hence, by (6.3.10),

$$-(\phi(\gamma_n'))'(t) \in \mathcal{K}f(t,u(t)) = \mathcal{K}f(t,\gamma_n(t)) \quad \text{for a.a. } \ t \in J_n.$$

This, based on condition (6.3.5), implies

$$-(\phi(\gamma_n'))'(t) = f(t,\gamma_n(t)) \qquad \text{for a.a. } t \in J_n,$$

equivalently

$$-(\phi(u'))'(t) = f(t,u(t)) \qquad \text{for a.a. } t \in J_n.$$

Thus u satisfies the discontinuous differential equation a.e. on $J = \bigcup_{n\in\mathbb{N}} J_n$. Finally, from (6.3.6) one has

$$\mathcal{K}f(t,u(t)) = \{f(t,u(t))\} \quad \text{for } t \in I \setminus J.$$

This together with (6.3.10) shows that u also satisfies the discontinuous differential equations a.e. on $I \setminus J$. Therefore u solves (6.3.1) on I. $\square$

Remark 6.3.4. *If the function f is nondecreasing with respect to the second variable, then conditions* (6.3.7) *and* (6.3.8) *can be written as*

$$\alpha\phi^{-1}\left(\int_0^1 f(s,r_1+\varepsilon)\,ds\right)+\int_0^1\phi^{-1}\left(\int_r^1 f(s,r_1+\varepsilon)\,ds\right)dr\le r_1,$$

$$\alpha\phi^{-1}\left(\int_c^1 f(s,M(r_2-\varepsilon))\,ds\right)+\int_c^1\phi^{-1}\left(\int_r^1 f(s,M(r_2-\varepsilon))\,ds\right)dr\ge r_2,$$

and they are analogous to those considered in [81] for the case of a continuous nonlinearity.

Remark 6.3.5 (Asymptotic conditions)**.** *If the function f is nondecreasing with respect to the second variable, the existence of two numbers r_1 and r_2 satisfying* (6.3.7) *and* (6.3.8) *is guaranteed by any one of the following two conditions:*

(a)

$$\liminf_{x\to 0}\frac{\alpha\phi^{-1}\left(\int_0^1 f(s,x)\,ds\right)+\int_0^1\phi^{-1}\left(\int_r^1 f(s,x)\,ds\right)dr}{x}<1,$$

$$\limsup_{x\to\infty}\frac{\alpha\phi^{-1}\left(\int_c^1 f(s,Mx)\,ds\right)+\int_c^1\phi^{-1}\left(\int_r^1 f(s,Mx)\,ds\right)dr}{x}>1;$$

(b)

$$\liminf_{x\to \infty}\frac{\alpha\phi^{-1}\left(\int_0^1 f(s,x)\,ds\right)+\int_0^1\phi^{-1}\left(\int_r^1 f(s,x)\,ds\right)dr}{x}<1,$$

$$\limsup_{x\to 0}\frac{\alpha\phi^{-1}\left(\int_c^1 f(s,Mx)\,ds\right)+\int_c^1\phi^{-1}\left(\int_r^1 f(s,Mx)\,ds\right)dr}{x}>1.$$

Observe that the first case is only possible if $a=+\infty$, otherwise ϕ^{-1} is bounded.

We illustrate the applicability of our existence result with an example.

Example 6.3.6. *Consider the differential problem involving the curvature operator in Euclidean space*

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)'=f(t,u):=\sqrt[3]{u}e^{-u}+\dfrac{1}{2}\cos^2\left(\lfloor 1/(u+t)\rfloor\right) & \text{a.e. on } I,\\ u(0)=u'(1)=0, \end{cases}\tag{6.3.11}$$

where $\lfloor x\rfloor$ denotes the integer part of x. Here, $\phi:\mathbb{R}\to(-1,1)$ is given by

$$\phi(\tau)=\tau/\sqrt{1+\tau^2}\quad \textit{and}\quad \phi^{-1}(\tau)=\tau/\sqrt{1-\tau^2}.$$

First, standard arguments show that condition (H_1) holds, and (H_2) is trivially satisfied because $f(t,u)<1$ for a.a. $t\in I$ and all $u\in\mathbb{R}_+$.

On the other hand, condition (H_3) *holds with*

$$\gamma_n = -t + \frac{1}{n} \quad \textit{and} \quad I_n = \left[0, \frac{1}{n}\right], \quad n \in \mathbb{N}.$$

Indeed, the function $u \mapsto f(t,u)$ *is continuous on* $\mathbb{R}_+ \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$ *for a.a.* $t \in I$. *Note that*

$$-(\phi(\gamma_n'))'(t) = 0 \quad \textit{for a.a. } t \in I_n \textit{ and } n \in \mathbb{N},$$

so condition (6.2.5) *is obviously satisfied for* $\delta, \varepsilon > 0$ *small enough, since* $\cos^2 n > 0$ *for all* $n \in \mathbb{N}$.

Finally, if we take $c = 1/2$, *then* $M = 1/2$ *and it is easy to verify that conditions* (6.3.7) *and* (6.3.8) *(with* $\alpha = 0$*) hold for* $r_1 = 1$ *and* $r_2 = 1/50$, *respectively.*

Therefore, Theorem 6.3.3 ensures the existence of a positive solution to problem (6.3.11) *such that* $1/50 \le \|u\|_\infty \le 1$.

The existence and localization result, Theorem 6.3.3, yields multiplicity results for problem (6.3.1) when several couples of numbers r_1 and r_2 satisfying conditions (6.3.7) and (6.3.8) exist such that the corresponding intervals (r, R) are disjoint.

Taking this into account and using asymptotic conditions, it is possible to derive a multiplicity result concerning the existence of infinitely many positive solutions for problem (6.3.1).

Theorem 6.3.7. *Assume that the function* f *satisfies conditions* (H_1)–(H_3) *and it is nondecreasing with respect to its second variable. If the following asymptotic condition*

(c)

$$\liminf_{x\to\infty} \frac{\alpha\phi^{-1}\left(\int_0^1 f(s,x)\,ds\right) + \int_0^1 \phi^{-1}\left(\int_r^1 f(s,x)\,ds\right) dr}{x} < 1,$$

$$\limsup_{x\to\infty} \frac{\alpha\phi^{-1}\left(\int_c^1 f(s,Mx)\,ds\right) + \int_c^1 \phi^{-1}\left(\int_r^1 f(s,Mx)\,ds\right) dr}{x} > 1$$

holds, then problem (6.3.1) *has a sequence of positive solutions* $(u_n)_{n\in\mathbb{N}}$ *satisfying* $\|u_n\|_\infty \to \infty$ *as* $n \to \infty$.

If the condition

(d)

$$\liminf_{x\to 0} \frac{\alpha\phi^{-1}\left(\int_0^1 f(s,x)\,ds\right) + \int_0^1 \phi^{-1}\left(\int_r^1 f(s,x)\,ds\right) dr}{x} < 1,$$

$$\limsup_{x\to 0} \frac{\alpha\phi^{-1}\left(\int_c^1 f(s,Mx)\,ds\right) + \int_c^1 \phi^{-1}\left(\int_r^1 f(s,Mx)\,ds\right) dr}{x} > 1$$

holds, then problem (6.3.1) *has a sequence of positive solutions* $(u_n)_{n\in\mathbb{N}}$ *satisfying* $\|u_n\|_\infty \to 0$ *as* $n \to \infty$.

Proof. Notice that, in view of Remarks 6.3.4 and 6.3.5, condition (c) ensures that there exist two sequences $(r_{1,n})_{n\in\mathbb{N}}$ and $(r_{2,n})_{n\in\mathbb{N}}$ such that

$$0 < r_{1,n} < r_{2,n} < r_{1,n+1} \quad \text{for every } \ n \in \mathbb{N}, \quad r_{1,n} \to \infty \text{ as } n \to \infty, \tag{6.3.12}$$

and for each $n \in \mathbb{N}$ the numbers $r_{1,n}$ and $r_{2,n}$ satisfy (6.3.7) and (6.3.8). Therefore, for each $n \in \mathbb{N}$ Theorem 6.3.3 ensures the existence of a solution u_n such that

$$r_{1,n} \leq \|u_n\|_\infty \leq r_{2,n}. \tag{6.3.13}$$

Now (6.3.12) and (6.3.13) show that the solutions u_n are distinct being located in disjoint annular sets and, moreover, $\|u_n\|_\infty \to \infty$ as $n \to \infty$.

In a similar way, condition (d) and Theorem 6.3.3 ensure the existence of a sequence of positive solutions $(u_n)_{n\in\mathbb{N}}$ in the conditions of the statement. □

We illustrate Theorem 6.3.7 with two examples where the nonlinearities are discontinuous perturbations of those in [81, Addendum, Examples 4.2 and 4.3].

Example 6.3.8. *Consider the problem*

$$\begin{cases} -u'' = f(u), \\ u(0) = u'(1) = 0, \end{cases} \tag{6.3.14}$$

with

$$f(u) := \alpha u + \beta \lfloor u \rfloor + \rho u \sin(\delta \ln(u+1)),$$

and where α, β, ρ and δ are positive constants, and $\lfloor x \rfloor$ denotes the integer part of x.

Assume that

$$\alpha \geq \rho(\delta + 1). \tag{6.3.15}$$

Then, it is easy to verify that f takes nonnegative values and is nondecreasing on $\mathbb{R}_+$. On the other hand, f is continuous in

$$\mathbb{R}_+ \setminus \bigcup_{\{n: t \in I_n\}} \{\gamma_n(t)\},$$

with

$$\gamma_n(t) \equiv n \quad \text{and} \quad I_n = [0,1] \quad \text{for } n \in \mathbb{N}.$$

Observe that $\inf_{u\in[1/2,\,\infty)} f(u) > 0$ *and thus condition* (6.2.5) *in Remark 6.2.5 is clearly satisfied for the functions γ_n, $n \in N$, and $\phi(u) = u$ ($a = b = +\infty$).*

Now we compute the limits in condition (c) from Theorem 6.3.7, and we find

$$\begin{aligned} \liminf_{x\to\infty} \frac{\int_0^1 \int_r^1 f(x)\, ds dr}{x} &\leq \frac{1}{2}(\alpha + \beta - \rho), \\ \limsup_{x\to\infty} \frac{\int_c^1 \int_r^1 f(cx)\, ds dr}{x} &\geq \frac{c(1-c)^2}{2} \left(\alpha + \frac{\beta}{2} + \rho \right). \end{aligned}$$

Hence, if we choose $c = 1/2$*, then the following inequalities*

$$\alpha + \frac{\beta}{2} + \rho > 16 \quad \text{and} \quad \alpha + \beta - \rho < 2 \tag{6.3.16}$$

guarantee that condition (c) *holds. Therefore, under conditions* (6.3.15) *and (6.3.16), Theorem 6.3.7 ensures that problem* (6.3.14) *has a sequence of positive solutions* $(u_n)_{n\in\mathbb{N}}$ *with* $\|u_n\|_\infty \to \infty$ *as* $n \to \infty$*.*

For example, conditions (6.3.15) *and* (6.3.16) *hold for the following values of parameters:*

$$\alpha = 8.5, \quad \beta = 0.5, \quad \rho = 8, \quad \delta = 0.05.$$

In the next example, ϕ is not the identity function, while the right–hand side is a slight modification of that from Example 6.3.8.

Example 6.3.9. *Consider the problem with* p*–Laplacian*

$$\begin{cases} -(|u'|^{p-2}\, u')' = f(u), \\ u(0) = u'(1) = 0, \end{cases} \tag{6.3.17}$$

where $p > 1$*,*

$$f(u) := g(u)^{p-1},$$

$$g(u) = \alpha u + \beta \lfloor u \rfloor + \rho u \sin\left(\delta \ln \frac{1}{u}\right) \quad \textit{for } u > 0, \quad g(0) = 0,$$

and α*,* β*,* ρ*,* δ *are positive constants.*

Here, $\phi : \mathbb{R} \to \mathbb{R}$ *is the homeomorphism given by*

$$\phi(x) = |x|^{p-2}\, x \quad \textit{and} \quad \phi^{-1}(x) = |x|^{\frac{1}{p-1}} \operatorname{sgn}(x).$$

As in the previous example, under assumption (6.3.15)*,* f *is nonnegative and nondecreasing on* $\mathbb{R}_+$*. Moreover, it is discontinuous exactly at the natural numbers, which are admissible discontinuity points.*

If we compute the limits in condition (c) *of Theorem 6.3.7, we obtain*

$$\begin{aligned} \liminf_{x\to\infty} \frac{\int_0^1 \phi^{-1}\left(\int_r^1 f(x)\, ds\right) dr}{x} &= \frac{p-1}{p} \liminf_{x\to\infty} \frac{g(x)}{x} \\ &\le \frac{p-1}{p}(\alpha + \beta - \rho), \\ \limsup_{x\to\infty} \frac{\int_c^1 \phi^{-1}\left(\int_r^1 f(cx)\, ds\right) dr}{x} &= \frac{p-1}{p}(1-c)^{\frac{p}{p-1}} \limsup_{x\to\infty} \frac{g(cx)}{x} \\ &\ge \frac{p-1}{p}(1-c)^{\frac{p}{p-1}} c\left(\alpha + \frac{\beta}{2} + \rho\right). \end{aligned}$$

Therefore, if we choose $c = 1/2$*, then the following inequalities*

$$\alpha + \frac{\beta}{2} + \rho > \frac{4p}{p-1} 2^{\frac{1}{p-1}} \quad \textit{and} \quad \alpha + \beta - \rho < \frac{p}{p-1} \tag{6.3.18}$$

guarantee that condition (c) *holds. Hence, under conditions* (6.3.15) *and* (6.3.18)*, Theorem 6.3.7 ensures that problem* (6.3.17) *has a sequence of positive solutions* $(u_n)_{n\in\mathbb{N}}$ *with* $\|u_n\|_\infty \to \infty$ *as* $n \to \infty$*.*

Moreover, we have

$$\liminf_{x\to 0} \frac{\int_0^1 \phi^{-1}\left(\int_r^1 f(x)\,ds\right)dr}{x} = \frac{p-1}{p}(\alpha-\rho),$$
$$\limsup_{x\to 0} \frac{\int_c^1 \phi^{-1}\left(\int_r^1 f(cx)\,ds\right)dr}{x} = \frac{p-1}{p}(1-c)^{\frac{p}{p-1}}c\,(\alpha+\rho)\,.$$

If we choose $c = 1/2$*, then the following inequalities*

$$\alpha+\rho > \frac{4p}{p-1}2^{\frac{1}{p-1}} \quad \text{and} \quad \alpha-\rho < \frac{p}{p-1} \tag{6.3.19}$$

guarantee that condition (d) *holds. Hence, under conditions* (6.3.15) *and* (6.3.19)*, Theorem 6.3.7 implies that problem* (6.3.17) *has a sequence of positive solutions* $(v_n)_{n\in\mathbb{N}}$ *with* $\|v_n\|_\infty \to 0$ *as* $n \to \infty$*.*

Let us remark that both conditions (6.3.18) *and* (6.3.19) *hold if*

$$\alpha+\rho > \frac{4p}{p-1}2^{\frac{1}{p-1}} \quad \text{and} \quad \alpha+\beta-\rho < \frac{p}{p-1}, \tag{6.3.20}$$

when problem (6.3.17) *has two sequences of positive solutions* $(u_n)_{n\in\mathbb{N}}$*,* $(v_n)_{n\in\mathbb{N}}$ *such that* $\|u_n\|_\infty \to \infty$ *and* $\|v_n\|_\infty \to 0$ *as* $n \to \infty$*.*

Notice that for $p \geq 2$ *(since* $2^{1/(p-1)} \leq 2$ *and* $1 < p/(p-1) \leq 2$*), in order to fulfill conditions* (6.3.20)*, it suffices that*

$$\alpha+\rho > 16 \quad \text{and} \quad \alpha+\beta-\rho \leq 1.$$

For example, these inequalities and (6.3.15) *are satisfied for the following values of parameters:*

$$\alpha = 8.5, \quad \beta = 0.25, \quad \rho = 8, \quad \delta = 0.05.$$

Appendix

A.
Multivalued mappings

This Appendix is devoted to introduce some definitions and previous results about multivalued mappings which are used along the document, see [3, 8, 49, 85]. In particular, we recall the topological degree for multivalued mappings defined in [40, 69, 109, 139] (see also [49, 113]), as well as Bohnenblust–Karlin's fixed point theorem and Fitzpatrick–Petryshyn's compression–expansion result, see [69, 144].

Let X and Y be normed spaces and $T : X \longrightarrow 2^Y$ be a multivalued mapping, that is, Tx is a subset of Y for every $x \in X$. Let us assume that $Tx \neq \emptyset$ for every $x \in X$.

Definition A.1. *T is said to be upper semicontinuous at x_0 if for any open neighborhood V of Tx_0 there exists an open neighborhood U of x_0 such that $TU \subset V$. In addition, T is said to be upper semicontinuous (usc, for short) if it is upper semicontinuous at every $x \in X$.*

The following equivalent statement is useful to check the upper semicontinuity of a multivalued operator which assumes compact values by means of sequences. Its proof can be looked up in [3, Theorem 17.20].

Proposition A.2. *Let X and Y be normed spaces and $T : D \subset X \longrightarrow 2^Y$ be a multivalued mapping. The following statements are equivalent:*

(i) *T is upper semicontinuous at $x \in D$ and Tx is compact.*

(ii) *If $x_n \to x$ in D, $y_n \in Tx_n$ for every $n \in \mathbb{N}$ with $y_n \to y$, then $y \in Tx$.*

The graph of $T : D \longrightarrow 2^Y$ is the subset of pairs $(x, y) \in D \times Y$ where $y \in T(x)$:

$$\operatorname{Graph}(T) = \{(x, y) \in D \times Y \, : \, y \in T(x)\}.$$

Proposition A.3. *The graph of an usc multivalued map $T : D \longrightarrow 2^Y$ with compact values is closed.*

A simple consequence of the definition is contained in the following result, see e.g. [3, Lemma 17.8] or [49, Proposition 24.1].

Proposition A.4. *Let K be a compact subset of the normed space X and $T : K \longrightarrow 2^Y$ be an usc multivalued mapping such that Tx is compact for every $x \in K$. Then TK is compact.*

Moreover, the composition of usc mappings is usc, see [3, Theorem 17.23] or [8, Proposition 1, p. 41].

Proposition A.5. *Given two multivalued maps $T : X \longrightarrow 2^Y$ and $S : Y \longrightarrow 2^Z$, we define the composition $S \circ T : X \longrightarrow 2^Z$ as*

$$(S \circ T)(x) = \bigcup_{y \in T(x)} S(y).$$

If T and S are upper semicontinuous, then $S \circ T$ is upper semicontinuous.

The topological degree for multivalued mappings is defined for operators of the form $Id - T$, where Id denotes the identity function and T is an usc multivalued mapping with closed convex values. This class of mappings satisfies the following useful property, see [74, Theorem 28.2].

Theorem A.6. *Let U be a subset of a normed space X and $T : U \longrightarrow 2^X$ be an usc multivalued mapping such that $T\,U$ is relatively compact. Then $Id - T$ is a closed mapping, that is, for every closed $A \subset U$ the set $(Id - T)(A)$ is closed.*

The main idea in the definition of the topological degree for multivalued mappings consists in approximating them by single–valued compact mappings for which Leray–Schauder's degree is available.

Lemma A.7. *Let X and Y be Banach spaces and $T : X \longrightarrow 2^Y$ an usc multivalued mapping with closed and convex values. Then for any $\varepsilon > 0$ there exists a continuous mapping $f_\varepsilon : X \longrightarrow \operatorname{co}(TX)$ such that for every $x \in X$ there exist $y \in X$ and $z \in Ty$ satisfying*

$$\|x - y\| < \varepsilon, \qquad \|f_\varepsilon x - z\| < \varepsilon.$$

Lemma A.8. *Let X be a real Banach space, $\Omega \subset X$ a bounded open subset and $T : \overline{\Omega} \longrightarrow 2^X$ an usc mapping with closed and convex values. Assume that $T\,\overline{\Omega}$ is relatively compact and $0 \notin (Id - T)(\partial\,\Omega)$. Then there exists $\varepsilon_0 > 0$ such that $0 \notin (Id - f_\varepsilon)(\partial\,\Omega)$ for $\varepsilon \in (0, \varepsilon_0)$, where f_ε is defined as in Lemma A.7.*

Based on the previous results, a degree theory is defined for usc multivalued mappings [40, 139].

Definition A.9. *Let X be a real Banach space, $\Omega \subset X$ a bounded open subset and $T : \overline{\Omega} \longrightarrow 2^X$ an usc mapping with closed and convex values. Assume that $T\,\overline{\Omega}$ is relatively compact and $0 \notin (Id - T)(\partial\,\Omega)$. Then*

$$\deg\left(Id - T, \Omega, 0\right) = \lim_{\varepsilon \to 0} \deg\left(Id - f_\varepsilon, \Omega, 0\right),$$

where f_ε is defined as in Lemma A.7.

Observe that $\deg\left(Id - f_\varepsilon, \Omega, 0\right)$ is well–defined as the Leray–Schauder's degree [49,100] for any $\varepsilon \in (0, \varepsilon_0)$. As usual, we will use the notation

$$\deg\left(Id - T, \Omega\right) \equiv \deg\left(Id - T, \Omega, 0\right).$$

Remark A.10. *We have assumed that Ω is bounded, but it is possible to consider unbounded domains by imposing some compactness condition on T, see [49, 109].*

Now we present the main properties of this topological degree. The interested reader can find their detailed proofs in [85, p. 459] or [139].

Theorem A.11. *Let T be under the hypotheses of Definition A.9. Then the following properties are satisfied:*

1. *(Homotopy invariance) Let $H : \overline{\Omega} \times [0,1] \longrightarrow 2^X$ be an usc mapping with closed convex values such that $H(\overline{\Omega} \times [0,1])$ is relatively compact and $x \notin H(x,t)$ for all $(x,t) \in \partial\,\Omega \times [0,1]$. Let us denote $H_t(x) = H(x,t)$ with $x \in \overline{\Omega}$, $t \in [0,1]$. Then the degree $\deg(Id - H_t, \Omega)$ does not depend on t.*

2. *(Additivity) Let Ω_1 and Ω_2 be two disjoint open sets such that $\Omega_1 \cup \Omega_2 \subset \Omega$.*
   If $0 \notin (Id - T)\left(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)\right)$, then
$$\deg\left(Id - T, \Omega\right) = \deg\left(Id - T, \Omega_1\right) + \deg\left(Id - T, \Omega_2\right).$$

3. *(Existence) If $\deg\left(Id - T, \Omega\right) \neq 0$, then T has a fixed point, i.e., there exists $x \in \Omega$ such that $x \in Tx$.*

4. *(Normalization) $\deg\left(Id, \Omega\right) = 1$ if, and only if, $0 \in \Omega$.*

Remark A.12. *The well–known excision property is a particular case of the additivity property as stated in Theorem A.11.*

The excision property can be stated as follows for an operator T in the conditions of Definition A.9: if $A \subset \overline{\Omega}$ is closed and $0 \notin (Id - T)(A) \cup (Id - T)(\partial\,\Omega)$, then
$$\deg\left(Id - T, \Omega\right) = \deg\left(Id - T, \Omega \setminus A\right).$$
To prove it, take $\Omega_1 = \Omega \setminus A$, $\Omega_2 = \emptyset$ and then use the additivity property.

The following proposition states the fact that, from degree's point of view, "only what happens in the boundary matters".

Proposition A.13. *Let $T, S : \overline{\Omega} \longrightarrow 2^X$ be under the hypotheses of Definition A.9. If $S(x) = T(x)$ for all $x \in \partial\,\Omega$ and $0 \notin (Id - T)(\partial\,\Omega)$, then*
$$\deg\left(Id - T, \Omega\right) = \deg\left(Id - S, \Omega\right).$$

Finally we present two well–known fixed point theorems. The first one is the extension of Schauder's fixed point theorem to the class of usc multivalued maps. Its finite dimensional version is usually called Kakutani's fixed point theorem [49].

Theorem A.14 (Bohnenblust–Karlin)**.** *Let K be a nonempty, compact and convex subset of a Banach space X. Let $T : K \longrightarrow 2^K$ be an usc multivalued mapping with nonempty, closed and convex values. Then T has a fixed point.*

The second result is the version of Fitzpatrick–Petryshyn's compression–expansion fixed point theorem in Banach spaces [69, Theorem 3.2].

Theorem A.15. *Let X be a Banach space with a cone $K \subset X$. Let $r_1, r_2 \in (0, \infty)$, $r_1 \neq r_2$, $r = \max\{r_1, r_2\}$ and $T : \overline{B}_r(0) \cap K \longrightarrow 2^K$ be an usc mapping with nonempty, closed and convex values such that $T(\overline{B}_r(0) \cap K)$ is relatively compact. Moreover, assume that T satisfies:*

(a) *there is some $h \in K \setminus \{0\}$ such that $x \notin Tx + \mu h$ for any $\mu > 0$ and $x \in K$ with $\|x\| = r_1$;*

(b) *$\lambda x \notin Tx$ for any $\lambda > 1$ and $x \in K$ with $\|x\| = r_2$.*

Then T has a fixed point x_0 with

$$\min\{r_1, r_2\} \leq \|x_0\| \leq \max\{r_1, r_2\}.$$

B. Fixed point theorems for decomposable mappings

In [48, 117], an existence theory for the inclusion operator

$$x \in \Psi\Phi x, \tag{B.1}$$

where Ψ and Φ are two single or multivalued operators, was developed.

As explained in [117], several difficulties appear when Φ is multivalued and we try to apply the usual fixed point theorems for multivalued mappings (namely, Bohnenblust–Karlin's theorem and Fitzpatrick–Petryshyn's compression–expansion result) to $\mathcal{F} = \Psi\Phi$. For instance, even if the values of Φ are convex and Ψ is nonlinear single–valued, the values of $\mathcal{F} = \Psi\Phi$ may not be convex. The main idea in [117] to overcome this difficulty consists in considering the map

$$\Pi(x, y) = \Psi y \times \Phi x$$

which assumes convex values in the product space.

Notice that if (x, y) is a fixed point of Π, then x is a fixed point of $\mathcal{F} = \Psi\Phi$.

Let X be a normed linear space and let us introduce the following notation:

$$P_{fc}(X) = \{A \subset X \ : \ A \text{ is nonempty, closed and convex}\},$$
$$P_{k^w c}(X) = \{A \subset X \ : \ A \text{ is nonempty, weakly compact and convex}\}.$$

Also recall that a multivalued operator Φ from a subset D of a normed linear space to another normed linear space is said to be

- *upper semicontinuous* (usc, for short) on D if for every closed subset C of D, the set

  $$\Phi^{-}(C) = \{x \in D : \ C \cap \Phi x \neq \emptyset\}$$

  is closed in D.

- *sequentially weakly upper semicontinuous* (w-usc, for short) on D if for every weakly closed subset C of D, the set $\Phi^{-}(C)$ is sequentially closed for the weak topology on D.

Observe that this notion of upper semicontinuity is equivalent to that in Definition A.1 as shown by [3, Lemma 17.4] or [49, Proposition 24.1].

Now we present an extension of the Bohnenblust–Karlin's fixed point theorem to decomposable maps, see [117, Theorem 2.1].

Theorem B.1. *Let X and Y be normed linear spaces and A and B nonempty weakly compact convex subsets of X and Y, respectively. Let $\Phi : A \longrightarrow P_{fc}(B)$ and $\Psi : B \longrightarrow P_{fc}(A)$ be two sequentially w-usc multivalued maps. Then there exists at least one $x \in A$ such that $x \in \Psi\Phi x$ and, equivalently, there exists at least one $y \in B$ with $y \in \Phi\Psi y$.*

Next we state the compression–expansion fixed point theorem for the inclusion (B.1).

Theorem B.2 ([48, Theorem 2.3])**.** *Let $(X, \|\cdot\|)$ and Y be normed linear spaces and K a wedge of X. Let $\Phi : K \longrightarrow P_{k^w c}(Y)$, $\Psi : C \longrightarrow P_{fc}(K)$ be two bounded multivalued maps, where $C = \overline{\text{co}}(\{0\} \cup \Phi(K))$. Assume that*

$$\begin{array}{l} \textit{if } A \subset K,\ A = \overline{\text{co}}(\{0\} \cup \Psi(\overline{\text{co}}(\{0\} \cup \Phi(A)))),\ \textit{ then } A \textit{ is weakly compact} \\ \textit{and } \Phi,\, \Psi \textit{ are w-usc on } A \textit{ and } \overline{\text{co}}(\{0\} \cup \Phi(A)),\ \textit{ respectively.} \end{array} \tag{B.2}$$

In addition, assume that there exist $r_1, r_2 > 0$, $r_1 \neq r_2$, and $h \in K \setminus \{0\}$ such that

$$\begin{array}{ll} x \notin \lambda\Psi\Phi x & \textit{for } \lambda \in (0,1) \textit{ and } x \in K \textit{ with } \|x\| = r_1; \textit{ and} \\ x \notin \Psi\Phi x + \mu h & \textit{for } \mu > 0 \textit{ and } x \in K \textit{ with } \|x\| = r_2. \end{array} \tag{B.3}$$

Then there exists at least one $x \in K$ with $x \in \Psi\Phi x$ such that

$$\min\{r_1, r_2\} \leq \|x\| \leq \max\{r_1, r_2\}.$$

Remark B.3. Since any usc map on a compact set is sequentially w-usc, Theorem B.2 remains true if instead of (B.2) we assume the condition

$$\begin{array}{l} \text{if } A \subset K,\ A = \overline{\text{co}}(\{0\} \cup \Psi(\overline{\text{co}}(\{0\} \cup \Phi(A)))),\ \text{ then } A \text{ is compact} \\ \text{and } \Phi,\, \Psi \text{ are usc on } A \text{ and } \overline{\text{co}}(\{0\} \cup \Phi(A)),\ \text{ respectively.} \end{array}$$

Resumo

Esta Tese, titulada *Métodos topolóxicos para operadores descontinuos e aplicacións*, contén a meirande parte do traballo de investigación levado a cabo polo autor nos últimos anos.

Debe resaltarse dende o comezo que a palabra clave no título é *descontinuo* xa que é esa falta de continuidade dos operadores estudados ao longo do texto o que motiva a nosa investigación.

Como unha motivación para empezar co estudo dos métodos topolóxicos e as súas aplicacións ás ecuacións diferenciais descontinuas, presentamos un resumo dos capítulos incluídos neste manuscrito. Este podería dividirse en dúas partes: unha primeira, que abarca os dous primeiros capítulos, céntrase no desenvolvemento dunha teoría de punto fixo para certa clase de operadores descontinuos baseada na de aplicacións multivaluadas. A segunda parte, que vai do Capítulo 3 ao Capítulo 6, dedícase ao estudo de problemas diferenciais con parte non lineal descontinua.

O propósito da primeira parte é construir unha maquinaria aplicable a operadores descontinuos e que deste xeito sexa útil para establecer novos resultados de existencia para problemas diferenciais. A simplicidade da idea detrás dos resultados de punto fixo ben podería esconder a súa utilidade e esa é a razón pola cal ambas partes deben verse como un todo en vez de como apartados independentes.

Agora presentamos brevemente as ideas principais de cada capítulo. Os nosos resultados aparece recollidos en varios artigos, ver [61–66, 102–105, 121, 122, 128, 129].

Capítulo 1: Teoría do grao para unha clase de operadores descontinuos

O estudo de problemas diferenciais lévase a cabo con frecuencia por medio da teoría do grao topolóxico e isto implica tratar cos operadores de punto fixo asociados, que están definidos entre espazos normados. No caso de problemas diferenciais con parte non lineal descontinua, que son o tipo de problemas nos que estamos interesados, os operadores non son continuos e polo tanto a teoría clásica do grao de Leray–Schauder non resulta útil. Esa é a nosa principal motivación para considerar unha teoría para operadores descontinuos.

Dado un operador $T : D \subset X \longrightarrow X$, non necesariamente continuo, consideremos a súa envoltura pechada e convexa $\mathbb{T} : D \subset X \longrightarrow 2^X$ definida como

$$\mathbb{T}x = \bigcap_{\varepsilon>0} \overline{\mathrm{co}}\, T\left(\overline{B}_\varepsilon(x) \cap D\right) \quad \text{para cada } x \in D,$$

onde D é un subconxunto non baleiro do espazo normado X.

Se o conxunto D é a clausura dun subconxunto non baleiro, aberto e limitado Ω dun espazo de Banach X, entón o grao topolóxico de $\mathbb{T}$ está ben definido como o grao para aplicacións multivaluadas superiormente semicontinuas [40, 139] baixo hipóteses razoables sobre

T. Isto proporciona un modo de definir o concepto de grao para un operador descontinuo T por medio do grao da súa envoltura pechada e convexa, $\mathbb{T}$.

Definición. *Sexa Ω un subconxunto aberto e limitado dun espazo de Banach X e sexa $T : \overline{\Omega} \longrightarrow X$ tal que $T\overline{\Omega}$ é relativamente compacto, $Tx \neq x$ para todo $x \in \partial\,\Omega$, e*

$$x \in \mathbb{T}x \ \textit{ implica } \ x = Tx \ \textit{ para todo } \ x \in \overline{\Omega}.$$

Definimos o grao de $Id - T$ en Ω con respecto a $0 \in X$ como segue:

$$\deg\left(Id - T, \Omega, 0\right) = \deg\left(Id - \mathbb{T}, \Omega, 0\right).$$

No caso de que T sexa continuo, entón $\mathbb{T}x = \{Tx\}$ para todo $x \in \overline{\Omega}$ e deste xeito o grao definido arriba coincide co de Leray–Schauder. Ademais, obsérvese que a hipótese de continuidade sobre o operador T se trocou pola seguinte condición máis xeral

$$x \in \mathbb{T}x \quad \text{implica} \quad x = Tx \quad \text{para todo} \quad x \in \overline{\Omega},$$

que significa que tódolos puntos fixos de $\mathbb{T}$ son puntos fixos de T. Esta hipótese permite probar que o novo grao herda as propiedades básicas do grao para aplicacións multivaluadas, é dicir, invarianza baixo homotopía, aditividade, normalización e existencia.

Como ocorre no caso clásico, o grao tan só pode aplicarse a operadores definidos na clausura de conxuntos abertos, o que supón unha importante restrición á hora de aplicalo a problemas diferenciais, xa que con frecuencia é conveniente traballar en conxuntos con interior baleiro, como por exemplo, algún tipo de cono. Para superar esta dificultade introduciuse o concepto de *índice de punto fixo* baseado na teoría do grao definida previamente.

Capítulo 2: Teoremas de punto fixo para certos operadores descontinuos

Este capítulo está dedicado a xeneralizar algúns teoremas de punto fixo a unha clase de operadores descontinuos usando o mesmo truco que na definición do grao: a hipótese de continuidade substitúese pola condición que, concretamente, significa que os puntos fixos do operador descontinuo coinciden cos da súa envoltura pechada e convexa. A maior parte destes teoremas de punto fixo obtéñense como consecuencia do cálculo do grao en conxuntos adecuados e é aquí onde as propiedades deste son de grande utilidade.

Unha primeira mirada vai dirixida ao coñecido teorema de punto fixo de Schauder, cuxa extensión enunciamos a continuación.

Teorema. *Sexa K un subconxunto non baleiro, convexo e compacto do espazo de Banach X. Sexa $T : K \longrightarrow K$ unha aplicación que satisface a condición: $x \in \mathbb{T}x$ implica $x = Tx$ para todo $x \in K$. Entón T ten un punto fixo en K.*

A simplicidade do teorema anterior convérteo nunha ferramenta especialmente útil á hora de buscar solucións para problemas non lineais. Porén, con frecuencia é conveniente obter información adicional sobre ditas solucións, como por exemplo, o feito de que sexan positivas ou a súa localización, e é entón cando os teoremas de punto fixo en conos adquiren gran importancia. Os teoremas de punto fixo en conos tipo compresión–expansión de Krasnosel'skiĭ úsanse intensivamente con este fin e aquí adaptámolos ao marco dos operadores descontinuos.

Teorema. *Sexa K un cono, $0 < r_1, r_2 \leq R$, $r_1 \neq r_2$ e sexa $T : \overline{B}_R(0) \cap K \longrightarrow K$ unha aplicación tal que $T(\overline{B}_R(0) \cap K)$ é relativamente compacto e satisface a condición: $x \in \mathbb{T}x$ implica $x = Tx$ para todo $x \in \overline{B}_R(0) \cap K$.*

Supoñamos que

(a) *$\lambda x \notin \mathbb{T}x$ para todo $x \in K$ con $\|x\| = r_1$ e todo $\lambda \geq 1$,*

(b) *existe $w \in K$ con $\|w\| \neq 0$ tal que $x \notin \mathbb{T}x + \mu w$ para todo $\mu \geq 0$ e todo $x \in K$ con $\|x\| = r_2$.*

Entón T ten un punto fixo $x \in K$ tal que

$$\min\{r_1, r_2\} < \|x\| < \max\{r_1, r_2\}.$$

Dun xeito similar no Capítulo 2 xeneralízanse outros teoremas de punto fixo. Algúns deles permiten deducir a existencia de múltiples puntos fixos, como no caso do teorema de Leggett–Williams, mentres que outros proporcionan información máis precisa sobre a súa localización, como ocorre coa versión vectorial do teorema de Krasnosel'skiĭ. Todos eles foron aplicados nos capítulos seguintes para obter novos resultados de existencia para problemas diferenciais.

A idea principal detrás destes novos resultados vai máis aló dos teoremas concretos que se xeneralizan no Capítulo 2 e proporciona un método para adaptar gran parte dos teoremas de punto fixo para operadores compactos a esta clase de operadores descontinuos.

Capítulo 3: Problemas de primeira orde

Este capítulo versa sobre a existencia de solucións para problemas de primeira orde, tanto escalares como sistemas, suxeitos a condicións iniciais ou funcionais.

Primeiro considérase o problema de valor inicial con condición funcional

$$x'(t) = f(t, x(t)) \text{ para c.t.p. } t \in I = [0, L], \quad x(0) = F(x),$$

onde supoñemos que $F : \mathcal{C}(I) \to \mathbb{R}$ é continua e a parte non lineal f cumpre as seguintes condicións:

(H1) Existen $r < R$, $N \geq 0$ e $M \in L^1(I)$ tales que $N + \|M\|_{L^1} < r$, $|F(x)| \leq N$ se $\|x\|_\infty \leq R$, e para case todo punto (c.t.p.) $t \in I$ e todo $x \in [-R, R]$ tense que $|f(t, x)| \leq M(t)$.

(H2) Calquera composición $t \in I \mapsto f(t, x(t))$ é medible se $x \in \mathcal{C}(I)$ e $\|x\|_\infty \leq R$.

(H3) Existen *curvas de descontinuidade admisible* $\gamma_n : I_n \longrightarrow \mathbb{R}$ $(n \in \mathbb{N})$ tales que para c.t.p. $t \in I$ a función $x \mapsto f(t, x)$ é continua en $[-R, R] \setminus \bigcup_{\{n : t \in I_n\}} \{\gamma_n(t)\}$.

Baixo estas hipóteses, o problema anterior ten polo menos unha solución absolutamente continua. Obviamente as curvas de descontinuidade deben cumprir algún tipo de condición de *transversalidade* para ser curvas *admisibles*. Este concepto de *curvas de descontinuidade admisible* será clave ao longo dos seguintes capítulos. Polo tanto, condicións do tipo (H3) empregaranse na meirande parte dos resultados de existencia deste manuscrito, o que os distingue dos resultados clásicos para problemas con parte non lineal de tipo Carathéodory.

A existencia de solución será unha consecuencia da aplicación da teoría do grao introducida no Capítulo 1 ao operador integral

$$Tx(t) = F(x) + \int_0^t f(s, x(s))\, ds \quad (t \in I,\ x \in \mathcal{C}(I)).$$

Logo, investígase a existencia de solucións absolutamente continuas para o sistema

$$x'(t) = f(t, x(t)) \text{ para c.t.p. } t \in I = [0, L], \quad x(0) = x_0 \in \mathbb{R}^n.$$

De novo a parte non lineal $f : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ non necesita ser continua.

A hipótese principal no resultado de existencia consiste en supoñer que a función f pode expresarse da seguinte forma:

$$f(t, x) = F(t, g_1(\tau_1(t, x), x), g_2(\tau_2(t, x), x), \dots),$$

onde para cada $i \in N$,

(i) a) Cada función $\tau_i : I \times \mathbb{R}^n \longrightarrow \mathbb{R}$ é diferenciable;

b) Cada función $g_i : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ é continua en $(\mathbb{R} \setminus A_i) \times \mathbb{R}^n$, onde A_i é un conxunto de medida nula;

c) Para c.t.p. $t \in I$ e todo $x \in \mathbb{R}^n$, a condición $\tau_i(t, x) \in \mathbb{R} \setminus A_i$ para todo $i \in N$ implica que $f(t, \cdot)$ é continua en x.

(ii) Para cada $(t, x) \in \tau_i^{-1}(A_i)$ temos

$$\nabla \tau_i(t, x) \cdot (1, z) \neq 0 \quad \text{para todo } z \in \mathcal{K}f(t, x),$$

onde $\mathcal{K}f(t, x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} f\left(t, \overline{B}_\varepsilon(x)\right)$ para todo $(t, x) \in I \times \mathbb{R}^n$.

Baixo hipóteses adicionais sobre o carácter medible e limitado de f, a existencia de solucións obtense do seguinte modo: primeiro, próbase que a inclusión

$$x'(t) \in \mathcal{K}f(t, x(t)) \text{ para c.t.p. } t \in I, \quad x(0) = x_0,$$

ten polo menos unha solución, e entón a hipótese enunciada arriba permite probar que é, de feito, unha solución do sistema diferencial descontinuo previo.

Ademais, no caso escalar, tamén se deduce a existencia de solucións extremais. Este feito xunto cun método iterativo para operadores descontinuos e o de subsolucións e sobresolucións son as ferramentas empregadas na Sección 3.3 para probar a existencia de solucións extremais absolutamente continuas para o problema funcional

$$x'(t) = f(t, x(t), x) \text{ para c.t.p. } t \in I, \quad B(x(0), x) = 0,$$

onde tanto f coma B poden ser descontinuas con respecto a todas as súas variables, pero é necesario impoñer algunha hipótese de monotonía con respecto ás variables funcionais.

Algúns problemas de segunda orde poden reducirse á clase de problemas de primeira orde con dependencia funcional considerada arriba e, en consecuencia, dedúcense novos resultados de existencia para eles.

Capítulo 4: Problemas de segunda orde e sub e sobresolucións

Este capítulo está dedicado á existencia e multiplicidade de solucións para a seguinte clase de problemas de fronteira de segunda orde

$$\left\{\begin{array}{rcl} x''(t) & = & f(t,x(t),x'(t)) \quad \text{para c.t.p. } t\in I=[a,b], \\ 0 & = & L_1(x(a),x(b),x'(a),x'(b),x), \\ 0 & = & L_2(x(a),x(b)). \end{array}\right.$$

O método de subsolucións e sobresolucións combínase aquí coa teoría do grao do Capítulo 1 e os teoremas de punto fixo tipo Schauder probados no Capítulo 2 para lograr os resultados de existencia.

Primeiro, supoñemos que existen unha subsolución e unha sobresolución ben ordenadas, isto é, a subsolución é máis pequena que a sobresolución, e establécese a existencia dunha solución entre ambas no espazo $W^{2,1}$. Para isto supoñemos que a parte non lineal cumpre unha condición de Nagumo, o que permite obter unha cota *a priori* para a derivada de calquera solución. Para probar o resultado de existencia, buscamos puntos fixos en $\mathcal{C}^1(I)$ do operador integral asociado a un problema modificado axeitado.

A principal diferenza con respecto aos outros traballos onde este tipo de problemas foron estudados mediante técnicas similares é que aquí a parte non lineal pode ser descontinua sobre os grafos de como máximo unha cantidade numerable de *curvas de descontinuidade admisible*. Ademais, tamén se establece a existencia de solucións extremais entre unha subsolución e unha sobresolución.

Por outra parte, se existen unha subsolución e unha sobresolución, que non están ben ordenadas, entón é posible construir un par de subsolución e sobresolución constantes e ben ordenadas e, polo tanto, a existencia de solucións obtense como consecuencia dos resultados anteriores. Para iso, consideramos un problema máis restrictivo con $L_2(x,y)=x-y$ e hipóteses máis fortes sobre a parte non lineal.

Como consecuencia dos resultados previos e supoñendo que existen máis dun par de subsolucións e sobresolucións con certas relacións de orde entre elas, obtéñense varios resultados de multiplicidade.

Finalmente, na Sección 4.4, presentamos un resultado de existencia novo para o seguinte problema de segunda orde na semirrecta

$$\left\{\begin{array}{l} x''(t)=f(t,x(t),x'(t)) \quad \text{para c.t.p. } t\in\mathbb{R}_+, \\ L(x(0),x'(0),x)=0, \quad \lim\limits_{t\to+\infty} x'(t)=B. \end{array}\right.$$

A filosofía empregada é similar á descrita arriba no caso de dominios limitados, pero agora buscamos puntos fixos nun espazo de Banach diferente

$$X=\left\{x\in\mathcal{C}^1(\mathbb{R}_+)\,:\, \lim_{t\to\infty}\frac{x(t)}{1+t}\in\mathbb{R} \text{ e } \lim_{t\to\infty}x'(t)\in\mathbb{R}\right\}$$

cunha norma tipo Bielecki,

$$\|x\| := \max\{\|x\|_0, \|x\|_1\},$$

onde

$$\|x\|_0 = \sup_{0\le t<\infty} \frac{|x(t)|}{1+t} \quad \text{e} \quad \|x\|_1 = \sup_{0\le t<\infty} |x'(t)|.$$

Cabe destacar que os nosos resultados melloran os existentes na literatura incluso no caso de problemas con parte non lineal de Carathéodory xa que relaxamos as nocións usuais de subsolución e sobresolución e, ademais, probamos a existencia de solucións extremais entre elas.

Capítulo 5: Solucións positivas para problemas de segunda orde e orde superior

A existencia de solucións *positivas* para ecuacións diferenciais e integrais adquire gran importancia debido a que na maioría das aplicacións as ecuacións diferenciais modelan procesos físicos ou biolóxicos onde as magnitudes non poden tomar valores negativos. Neste capítulo buscamos solucións positivas para diferentes problemas de segunda e cuarta orde con partes non lineais que presentan descontinuidades.

No noso contexto, as solucións dos problemas diferenciais correspóndense cos puntos fixos dun operador tipo Hammerstein

$$Tu(t) := \int_0^1 G(t,s)g(s)f(s,u(s))\,ds,$$

onde G é a función de Green asociada ao problema diferencial. Agora a idea é aplicarlle ao operador T os teoremas de punto fixo en conos obtidos no Capítulo 2. Deste xeito deducimos a existencia de solucións positivas para os problemas diferenciais. De novo a parte non lineal f pode ser descontinua sobre *curvas de descontinuidade admisible*.

A técnica dos teoremas de punto fixo en conos require a construcción dun cono de funcións adecuado (que pode variar dependendo de cal sexa o problema diferencial a considerar). Neste punto as propiedades da función de Green xogan un papel clave. Para os problemas estudados, existen unha función continua $\Phi : I \to \mathbb{R}_+$ e unha constante $c \in (0,1]$ tales que

$$\begin{array}{ll} G(t,s) \le \Phi(s) & \text{para calquera } t,s \in I, \\ c\,\Phi(s) \le G(t,s) & \text{para calquera } t \in [a,b],\ s \in I, \end{array}$$

onde $[a,b] \subset I$. Estas cotas para a función de Green permiten probar que o operador T leva o cono

$$K = \left\{ u \in \mathcal{C}(I) :\ u \ge 0,\ \min_{t\in[a,b]} u(t) \ge c\,\|u\|_\infty \right\},$$

en sí mesmo. Este cono K é adecuado para aplicar teoremas de punto fixo do tipo do de Krasnosel'skiĭ.

Por outro lado, cando buscamos resultados de multiplicidade por medio de teoremas de punto fixo do tipo do de Leggett–Williams usamos o cono das funcións non negativas $P = \{u \in \mathcal{C}(I) :\ u \ge 0\}$.

Capítulo 6: Solucións positivas para problemas xerais

O obxectivo deste capítulo é obter a existencia de solucións positivas para problemas diferenciais para os que a función de Green podería non existir, pero no seu lugar temos unha *desigualdade tipo Harnack*. Este feito supón a maior diferenza con respecto aos problemas estudados no capítulo previo. Como un exemplo do tipo de problemas susceptibles de ser considerados mencionamos as ecuacións ϕ–Laplacianas.

Máis concretamente estudamos a existencia de solucións positivas para un problema xeral da forma

$$\begin{cases} Lu(t) = f(t, u(t)) & \text{para c.t.p. } t \in I = [0, 1], \\ u \in \mathcal{B}, \end{cases}$$

onde $\mathcal{B} \subset \mathcal{C}(I)$ e $L : D(L) \subset \mathcal{C}(I) \longrightarrow L^1(I)$ é un operador xeral non necesariamente lineal.

O método usado tamén difire do empregado no capítulo anterior. Aquí consideramos primeiro o problema auxiliar

$$\begin{cases} Lu(t) \in \mathcal{K}f(t, u(t)) & \text{para c.t.p. } t \in I, \\ u \in \mathcal{B}. \end{cases}$$

Unha vez que garantimos a existencia de solucións para a inclusión diferencial, unha condición de transversalidade axeitada sobre as descontinuidades de f permite probar que estas tamén son solucións para o problema de partida.

Como consecuencia dos resultados obtidos podemos deducir novos teoremas de existencia para as ecuacións ϕ–Laplacianas con condicións de fronteira tipo Dirichlet–Neumann

$$\begin{cases} -(\phi(u'))'(t) = f(t, u(t)) & \text{para c.t.p. } t \in I, \\ u'(0) = u(1) = 0, \end{cases}$$

onde $\phi : (-a, a) \to (-b, b)$ é un homeomorfismo crecente tal que $\phi(0) = 0$ e $0 < a, b \leq \infty$.

Ademais, as conclusións sobre a localización das solucións permiten obter resultados de multiplicidade. No caso de partes non lineais cun comportamento altamente oscilatorio en cero ou infinito pódese deducir a existencia de infinitas solucións positivas.

Conclusións e traballo futuro

Ao longo desta tese desenvolvemos un método que permite abordar problemas diferenciais con partes non lineais que presentan descontinuidades. Dito método está esencialmente baseado na teoría de punto fixo para aplicacións superiormente semicontinuas e en condicións de transversalidade sobre as partes non lineais dos problemas diferenciais. O concepto de curva de descontinuidade admisible adaptouse a un gran número de problemas de fronteira, co cal se obtiveron novos resultados de existencia para eles. Estes ilustráronse con diferentes exemplos.

A pesar de toda a teoría desenvolvida sobre operadores e ecuacións diferenciais descontinuos ao longo destas páxinas, un número enorme de problemas aínda permanecen abertos esperando a ser resoltos. Hai varias direccións de cara a traballo futuro que nos gustaría resaltar aquí e que, de algún modo, complementarían ou mellorarían os nosos resultados actuais.

Un problema estimulante é a definición dunha teoría do grao para operadores descontinuos sen usar a de aplicacións multivaluadas. Incluso no caso de dimensión finita a extensión do grao de Brouwer a unha clase de operadores descontinuos parece unha tarea ardua. Evidentemente a continuidade do operador debe substituirse por outra condición (máis débil) para obter unha teoría sólida.

A nosa teoría de punto fixo para operadores descontinuos depende en gran medida da definición da envoltura pechada e convexa. Esta é a aplicación superiormente semicontinua con valores pechados e convexos máis pequena que contén ao operador descontinuo coma unha selección. Porén, podería ser interesante analizar outras regularizacións que non conteñan ao operador estudado, pero que proporcionen a existencia de puntos fixos baixo condicións máis febles.

Un concepto clave ao longo do texto é o das curvas de descontinuidade admisible. A súa definición preséntase para ecuacións diferenciais, pero sería natural tamén preguntarse se é posible unha noción similar para ecuacións integrais. Ademais, nos nosos resultados de existencia supoñemos que pode existir como máximo unha cantidade numerable destas curvas, así que resulta razoable preguntarse se seguen sendo certos para certa cantidade infinita non numerable delas.

Outro problema interesante, que non estudamos na tese, é a unicidade de solución para ecuacións diferenciais con descontinuidades. No caso de sistemas de primeira orde, o artigo de Bressan e Shen [25], do cal xeneralizamos o seu resultado de existencia na Sección 3.2, podería ser un bo punto de partida.

No Capítulo 4, para lograr a existencia de solucións no caso de subsolucións e sobresolucións que non están ben ordenadas traballamos cunha clase de problemas de fronteira máis reducida ca no caso ben ordenado, así que queda aberto se o resultado é certo para o resto de condicións de fronteira. Ademais, unha extensión común dos resultados do Capítulo 4 consiste en considerar as ecuacións ϕ–Laplacianas. Porén, no caso das ecuacións ϕ–Laplacianas non fomos capaces de probar que os puntos fixos do operador integral asociado ao problema diferencial coinciden cos da súa envoltura pechada e convexa. Esa é a razón pola que no Capítulo 6 empregamos unha técnica distinta. Esa técnica usa a teoría de punto fixo para a composición de aplicacións multivaluadas, pero parece que non é suficiente para este propósito, polo que sería de axuda desenvolver unha teoría do grao para esta clase de aplicacións.

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