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Research Article

Multiple Positive Solutions in the Sense of Distributions of Singular BVPs on Time Scales and an Application to Emden-Fowler Equations

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This paper is devoted to using perturbation and variational techniques to derive some sufficient conditions for the existence of multiple positive solutions in the sense of distributions to a singular second-order dynamic equation with homogeneous Dirichlet boundary conditions, which includes those problems related to the negative exponent Emden-Fowler equation.

1. Introduction

The Emden-Fowler equation,

$$u^{\Delta\Delta}(t) + q(t)\, u^{\alpha}\big(\sigma(t)\big) = 0, \quad t \in (0,1)_{\mathbb{T}}, \tag{1.1}$$

arises in the study of gas dynamics and fluids mechanics, and in the study of relativistic mechanics, nuclear physics, and chemically reacting system (see, e.g., [1] and the references therein) for the continuous model. The negative exponent Emden-Fowler equation ($\alpha < 0$) has been used in modeling non-Newtonian fluids such as coal slurries [2]. The physical interest lies in the existence of positive solutions. We are interested in a broad class of singular problem that includes those related with (1.1) and the more general equation

$$u^{\Delta\Delta}(t) + q(t)\, u^{\alpha}\big(\sigma(t)\big) = g\big(t, u^{\sigma}(t)\big), \quad t \in (0,1)_{\mathbb{T}}. \tag{1.2}$$

Recently, existence theory for positive solutions of second-order boundary value problems on time scales has received much attention (see, e.g., [3–6] for general case, [7] for the continuous case, and [8] for the discrete case).

In this paper, we consider the second-order dynamic equation with homogeneous Dirichlet boundary conditions:

$$(P)\begin{cases} -u^{\Delta\Delta}(t) = F(t, u^{\sigma}(t)), & \Delta\text{-a.e. } t \in (D^{\kappa})^{o}, \\ u(t) > 0, & t \in (a,b)_{\mathbb{T}}, \\ u(a) = 0 = u(b), \end{cases} \tag{1.3}$$

where we say that a property holds for Δ-a.e. $t \in A \subset \mathbb{T}$ or Δ-a.e. on $A \subset \mathbb{T}$, Δ-a.e., whenever there exists a set $E \subset A$ with null Lebesgue Δ-measure such that this property holds for every $t \in A \setminus E$, $\mathbb{T}$ is an arbitrary time scale, subindex $\mathbb{T}$ means intersection to $\mathbb{T}$, $a, b \in \mathbb{T}$ are such that $a < \rho(b)$, $D = [a,b]_{\mathbb{T}}$, $D^{\kappa} = [a, \rho(b)]_{\mathbb{T}}$, $D^{\kappa^2} = [a, \rho^2(b)]_{\mathbb{T}}$, $D^{o} = [a,b)_{\mathbb{T}}$, $(D^{\kappa})^{o} = [a, \rho(b))_{\mathbb{T}}$, and $F : D \times (0, +\infty) \to \overline{\mathbb{R}}$ is an L^1_{Δ}-Carathéodory function on compact subintervals of $(0, +\infty)$, that is, it satisfies the following conditions.

(C) (i) For every $x \in (0, +\infty)$, $F(\cdot, x)$ is Δ-measurable in D^{o}.

(ii) For Δ-a.e. $t \in D^{o}$, $F(t, \cdot) \in C((0, +\infty))$.

(C_c) For every $x_1, x_2 \in (0, +\infty)$ with $x_1 \leq x_2$, there exists $m_{(x_1,x_2)} \in L^1_{\Delta}(D^{o})$ such that

$$\left|F(t,x)\right| \leq m_{(x_1,x_2)}(t) \quad \text{for } \Delta\text{-a.e. } t \in D^{o},\ x \in [x_1, x_2]. \tag{1.4}$$

Moreover, in order to use variational techniques and critical point theory, we will assume that F satisfy the following condition.

(PM) For every $x \in (0, +\infty)$, function $P_F : D \times [0, +\infty) \to \overline{\mathbb{R}}$ defined for Δ-a.e. $t \in D$ and all $x \in [0, +\infty)$, as

$$P_F(t,x) := \int_0^x F(t,r)\, dr, \tag{1.5}$$

satisfies that $P_F(\cdot, x)$ is Δ-measurable in D^{o}.

We consider the spaces

$$\begin{aligned} C^1_{0,\mathrm{rd}}(D^{\kappa}) &:= C^1_{\mathrm{rd}}(D^{\kappa}) \cap C_0(D), \\ C^1_{\mathrm{c,rd}}(D^{\kappa}) &:= C^1_{\mathrm{rd}}(D^{\kappa}) \cap C_{\mathrm{c}}(D), \end{aligned} \tag{1.6}$$

where $C^1_{\mathrm{rd}}(D^{\kappa})$ is the set of all continuous functions on D such that they are Δ-differentiable on D^{κ} and their Δ-derivatives are rd-continuous on D^{κ}, $C_0(D)$ is the set of all continuous functions on D that vanish on the boundary of D, and $C_{\mathrm{c}}(D)$ is the set of all continuous functions on D with compact support on $(a,b)_{\mathbb{T}}$. We denote as $\|\cdot\|_{C(D)}$ the norm in $C(D)$, that is, the supremum norm.

On the other hand, we consider the first-order Sobolev spaces

$$\begin{aligned} H^1_{\Delta}(D) &:= \{v : D \longrightarrow \mathbb{R} : v \in \mathrm{AC}(D),\ v^{\Delta} \in L^2_{\Delta}(D^{o})\}, \\ H = H^1_{0,\Delta}(D) &:= \{v : D \longrightarrow \mathbb{R} : v \in H^1_{\Delta}(D),\ v(a) = 0 = v(b)\}, \end{aligned} \tag{1.7}$$

where AC(D) is the set of all absolutely continuous functions on D. We denote as

$$\int_{t_1}^{t_2} f(s)\,\Delta s = \int_{[t_1,t_2)_{\mathbb{T}}} f(s)\Delta s \quad \text{for } t_1, t_2 \in D,\ t_1 < t_2,\ f \in L^1_\Delta\left([t_1,t_2)_{\mathbb{T}}\right). \tag{1.8}$$

The set H is endowed with the structure of Hilbert space together with the inner product $(\cdot,\cdot)_H : H \times H \to \mathbb{R}$ given for every $(v,w) \in H \times H$ by

$$(v,w)_H := \left(v^\Delta, w^\Delta\right)_{L^2_\Delta} := \int_a^b v^\Delta(s) \cdot w^\Delta(s)\,\Delta s; \tag{1.9}$$

we denote as $\|\cdot\|_H$ its induced norm.

Moreover, we consider the sets

$$\begin{aligned} H_{0,\mathrm{loc}} &:= H^1_{\mathrm{loc},\Delta}(D) \cap C_0(D), \\ H_{\mathrm{c},\mathrm{loc}} &:= H^1_{\mathrm{loc},\Delta}(D) \cap C_{\mathrm{c}}(D), \end{aligned} \tag{1.10}$$

where $H^1_{\mathrm{loc},\Delta}(D)$ is the set of all functions such that their restriction to every closed subinterval J of $(a,b)_{\mathbb{T}}$ belong to the Sobolev space $H^1_\Delta(J)$.

We refer the reader to [9–11] for an introduction to several properties of Sobolev spaces and absolutely continuous functions on closed subintervals of an arbitrary time scale, and to [12] for a broad introduction to dynamic equations on time scales.

Definition 1.1. u is said to be a solution in the sense of distributions to (P) if $u \in H_{0,\mathrm{loc}}$, $u > 0$ on $(a,b)_{\mathbb{T}}$, and equality

$$\int_a^b \left[u^\Delta(s) \cdot \varphi^\Delta(s) - F\left(s, u^\sigma(s)\right) \cdot \varphi^\sigma(s)\right] \Delta s = 0 \tag{1.11}$$

holds for all $\varphi \in C^1_{\mathrm{c,rd}}(D^\kappa)$.

From the density properties of the first-order Sobolev spaces proved in [9, Seccion 3.2], we deduce that if u is solution in the sense of distributions, then, (1.11) holds for all $\varphi \in H_{\mathrm{c,loc}}$.

This paper is devoted to prove the existence of multiple positive solutions to (P) by using perturbation and variational methods.

This paper is organized as follows. In Section 2, we deduce sufficient conditions for the existence of solutions in the sense of distributions to (P). Under certain hypotheses, we approximate solutions in the sense of distributions to problem (P) by a sequence of weak solutions to weak problems. In Section 3, we derive some sufficient conditions for the existence of at least one or two positive solutions to (P).

These results generalize those given in [7] for $\mathbb{T} = [0,1]$, where problem (P) is defined on the whole interval $(0,1) \cap \mathbb{T}$ and the authors assume that $F \in C((0,1) \times (0,+\infty), \mathbb{R})$ instead of (C) and (PM). The sufficient conditions for the existence of multiple positive solutions obtained in this paper are applied to a great class of bounded time scales such as finite union of disjoint closed intervals, some convergent sequences and their limit points, or Cantor sets among others.

2. Approximation to (P) by weak problems

In this section, we will deduce sufficient conditions for the existence of solutions in the sense of distributions to (P), where $F = f + g$ and $f, g : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfy (C) and (PM), f satisfies ($\mathrm{C_c}$), and g satisfies the following condition.

($\mathrm{C_g}$) For every $p \in (0, +\infty)$, there exists $M_p \in L^1_\Delta(D^o)$ such that

$$\left|g(t,x)\right| \le M_p(t) \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in (0,p]. \tag{2.1}$$

Under these hypotheses, we will be able to approximate solutions in the sense of distributions to problem (P) by a sequence of weak solutions to weak problems.

First of all, we enunciate a useful property of absolutely continuous functions on Dwhose proof we omit because of its simplicity.

Lemma 2.1. *If $v \in AC(D)$, then $v^\pm := \max\{\pm v, 0\} \in AC(D)$,*

$$\left[(v^+)^\Delta - v^\Delta\right]\cdot (v^+)^\Delta \le 0, \qquad \left[(v^-)^\Delta + v^\Delta\right]\cdot (v^-)^\Delta \le 0, \tag{2.2}$$

Δ-a.e. on D^o.

We fix $\{\varepsilon_j\}_{j\ge 1}$ a sequence of positive numbers strictly decreasing to zero; for every $j \ge 1$, we define $f_j : D \times (0, +\infty) \to \overline{\mathbb{R}}$ as

$$f_j(t,x) = f\left(t, \max\left\{x, \varepsilon_j\right\}\right) \quad \text{for every } (t,x) \in D \times (0,+\infty). \tag{2.3}$$

Note that f_j satisfies (C) and (C_g); consider the following modified weak problem

$$(P_j)\ \begin{cases} -u^{\Delta\Delta}(t) = f_j\left(t, u^\sigma(t)\right) + g\left(t, u^\sigma(t)\right), & \Delta\text{-a.e. } t \in \left(D^\kappa\right)^o, \\ u(t) > 0, & t \in (a,b)_{\mathbb{T}}, \\ u(a) = 0 = u(b). \end{cases} \tag{2.4}$$

Definition 2.2. u is said to be a weak solution to (P_j) if $u \in H$, $u > 0$ on $(a,b)_{\mathbb{T}}$, and equality

$$\int_a^b \left[u^\Delta(s)\cdot\varphi^\Delta(s) - \left(f_j\left(s, u^\sigma(s)\right) + g\left(s, u^\sigma(s)\right)\right)\cdot\varphi^\sigma(s)\right]\Delta s = 0 \tag{2.5}$$

holds for all $\varphi \in C^1_{0,\mathrm{rd}}(D^\kappa)$.

$\underline{u}$ is said to be a weak lower solution to (P_j) if $\underline{u} \in H$ $\underline{u} > 0$ on $(a,b)_{\mathbb{T}}$, and inequality

$$\int_a^b \left[\underline{u}^\Delta(s)\cdot\varphi^\Delta(s) - \left(f_j\left(s, \underline{u}^\sigma(s)\right) + g\left(s, \underline{u}^\sigma(s)\right)\right)\cdot\varphi^\sigma(s)\right]\Delta s \le 0 \tag{2.6}$$

holds for all $\varphi \in C^1_{0,\mathrm{rd}}(D^\kappa)$ such that $\varphi \ge 0$ on D.

The concept of weak upper solution to (P_j) is defined by reversing the previous inequality.

We remark that the density properties of the first-order Sobolev spaces proved in [9, Seccion 3.2] allows to assert that relations in Definition 2.2 are valid for all $\varphi \in H$ and for all $\varphi \in H$ such that $\varphi \geq 0$ on D, respectively.

By standard arguments, we can prove the following result.

Proposition 2.3. *Assume that $f, g : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfy (C) and (PM), f satisfies (C_c), and g satisfies (C_g).*

Then, if for some $j \geq 1$ there exist $\underline{u}_j$ and $\overline{u}_j$ as a lower and an upper weak solution, respectively, to (P_j) such that $\underline{u}_j \leq \overline{u}_j$ on D, then (P_j) has a weak solution $u_j \in [\underline{u}_j, \overline{u}_j] := \{v \in H : \underline{u}_j \leq v \leq \overline{u}_j \text{ on } D\}$.

Next, we will deduce the existence of one solution in the sense of distributions to (P) from the existence of a sequence of weak solutions to (P_j). In order to do this, we fix $\{a_k\}_{k\geq1}, \{b_k\}_{k\geq1} \subset D$ two sequences such that $\{a_k\}_{k\geq1} \subset (a, (a+b)/2)_{\mathbb{T}}$ is strictly decreasing to a if $a = \sigma(a)$, $a_k = a$ for all $k \geq 1$ if $a < \sigma(a)$ and $\{b_k\}_{k\geq1} \subset ((a+b)/2, b)_{\mathbb{T}}$ is strictly increasing to b if $\rho(b) = b$, $b_k = b$ for all $k \geq 1$ if $\rho(b) < b$. We denote that $D_k := [a_k, b_k]_{\mathbb{T}}$, $k \geq 1$. Moreover, we fix $\{\delta_k\}_{k\geq1}$ a sequence of positive numbers strictly decreasing to zero such that

$$\left[\sigma(a_k), \rho(b_k)\right)_{\mathbb{T}} \subset \left[a + \delta_k, b - \delta_k\right)_{\mathbb{T}}, \quad \delta_k \leq \frac{b-a}{2} \text{ for } k \geq 1. \tag{2.7}$$

Proposition 2.4. *Suppose that $F = f + g$ and $f, g : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfy (C) and (PM), f satisfies (C_c), and g satisfies (C_g).*

Then, if for every $j \geq 1$, $u_j \in H$ is a weak solution to (P_j) and

$$\nu_\delta := \inf_{j\geq1} \min_{[a+\delta, b-\delta]_{\mathbb{T}}}, \quad u_j > 0 \quad \forall \delta \in \left(0, \frac{b-a}{2}\right], \tag{2.8}$$

$$M := \sup_{j\geq1} \max_{D} u_j < \infty, \tag{2.9}$$

then a subsequence of $\{u_j\}_{j\geq1}$ converges pointwise in D to a solution in the sense of distributions u_1 to (P).

Proof. Let $k \geq 1$ be arbitrary; we deduce, from (2.2), (2.7), (2.8), and (2.9), that there exists a constant $K_k \geq 0$ such that for all $j \geq 1$,

$$\begin{aligned} \int_{a_k}^{b_k} \left(u_j^\Delta(s)\right)^2 \Delta s &= \left(u_j^\Delta(a_k)\right)^2 \cdot \mu(a_k) + \left(u_j^\Delta(\rho(b_k))\right)^2 \cdot \mu(\rho(b_k)) \\ &\quad + \int_{\sigma(a_k)}^{\rho(b_k)} u_j^\Delta(s) \cdot \left(\left(u_j - \nu_{\delta_k}\right)^+\right)^\Delta(s)\, \Delta s \\ &\leq K_k + \left(u_j, \left(u_j - \nu_{\delta_k}\right)^+\right)_H. \end{aligned} \tag{2.10}$$

Therefore, for all $j \geq 1$ so large that $\varepsilon_j < \nu_{\delta_1}$, as u_j is a weak solution to (P_j), by taking $\widetilde{\varphi}_1 := (u_j - \nu_{\delta_1})^+ \in H$ as the test function in (2.5), from (2.9), (C_c) and (C_g), we can assert that there exists $l \in L^1_\Delta(D^o)$ such that

$$\begin{aligned}\int_{a_1}^{b_1} \left(u_j^\Delta(s)\right)^2 \Delta s &\leq K_1 + \int_a^b F\left(s, u_j^\sigma(s)\right) \cdot \widetilde{\varphi}_1^\sigma(s) \Delta s \\ &\leq K_1 + M \int_a^b l(s) \Delta s,\end{aligned} \tag{2.11}$$

that is, $\{u_j\}_{j\geq 1}$ is bounded in $H^1_\Delta(D_1)$ and hence, there exists a subsequence $\{u_{1_j}\}_{j\geq 1}$ which converges weakly in $H^1_\Delta(D_1)$ and strongly in $C(D_1)$ to some $u^1 \in H^1_\Delta(D_1)$.

For every $k \geq 1$, by considering for each $j \geq 1$ the weak solution to $(P_{k_j}) u_{k_j}$ and by repeating the previous construction, we obtain a sequence $\{u_{(k+1)_j}\}_{j\geq 1}$ which converges weakly in $H^1_\Delta(D_{k+1})$ and strongly in $C(D_{k+1})$ to some $u^{k+1} \in H^1_\Delta(D_{k+1})$ with $\{u_{(k+1)_j}\}_{j\geq 1} \subset \{u_{k_j}\}_{j\geq 1}$. By definition, we know that for all $k \geq 1$, $u^{k+1}|_{D_k} = u^k$.

Let $u_1 : D \to \mathbb{R}$ be given by $u_1 := u^k$ on D_k for all $k \geq 1$ and $u_1(a) := 0 =: u_1(b)$ so that $u_1 > 0$ on $(a,b)_\mathbb{T}$, $u_1 \in H^1_{\mathrm{loc},\Delta}(D) \cap C((a,b)_\mathbb{T})$, u_1 is continuous in every isolated point of the boundary of D, and $\{u_{k_k}\}_{k\geq 1}$ converges pointwise in D to u_1.

We will show that $u_1 \in C_0(D)$; we only have to prove that u_1 is continuous in every dense point of the boundary of D. Let $0 < \varepsilon < M$ be arbitrary, it follows from (C_c) and (C_g) that there exist $m_\varepsilon \in L^1_\Delta(D^o)$ such that $m_\varepsilon \geq 0$ on D^o and $F(t,x) \leq m_\varepsilon(t)$ for Δ-a.e. $t \in D^o$ and all $x \in [\varepsilon, M]$; let $\varphi_\varepsilon \in H$ be the weak solution to

$$-\varphi_\varepsilon^{\Delta\Delta}(t) = m_\varepsilon(t), \quad \Delta\text{-a.e. } t \in \left(D^\kappa\right)^o, \qquad \varphi_\varepsilon(a) = 0 = \varphi_\varepsilon(b); \tag{2.12}$$

we know (see [4]) that $\varphi_\varepsilon > 0$ on $(a,b)_\mathbb{T}$.

For all $k \geq 1$ so large that $\varepsilon_{k_k} < \varepsilon$, since u_{k_k} and φ_ε are weak solutions to some problems, by taking $\widetilde{\varphi}_2 = (u_{k_k} - \varepsilon - \varphi_\varepsilon)^+ \in H$ as the test function in their respective problems, we obtain

$$\begin{aligned}\left(u_{k_k}, \widetilde{\varphi}_2\right)_H &= \int_a^b F\left(s, u_{k_k}^\sigma(s)\right) \cdot \widetilde{\varphi}_2^\sigma(s)\, \Delta s \\ &\leq \int_a^b m_\varepsilon(s) \cdot \widetilde{\varphi}_2^\sigma(s)\, \Delta s = \left(\varphi_\varepsilon, \widetilde{\varphi}_2\right)_H;\end{aligned} \tag{2.13}$$

thus, (2.2) yields to

$$\left\|\widetilde{\varphi}_2\right\|_H^2 \leq \left(u_{k_k} - \varphi_\varepsilon, \widetilde{\varphi}_2\right)_H \leq 0, \tag{2.14}$$

which implies that $0 \leq u_{k_k} \leq \varepsilon + \varphi_\varepsilon$ on D and so $0 \leq u_1 \leq \varepsilon + \varphi_\varepsilon$ on D. Thereby, the continuity of φ_ε in every dense point of the boundary of D and the arbitrariness of ε guarantee that $u_1 \in C_0(D)$.

Finally, we will see that (1.11) holds for every test function $\varphi \in C^1_{c,\mathrm{rd}}(D^\kappa)$; fix one of them.

For all $k \geq 1$ so large that $\operatorname{supp} \varphi \subset (a_k, b_k)_\mathbb{T}$ and all $j \geq 1$ so large that $\varepsilon_{k_j} < \nu_{\delta_k}$, as u_{k_j} is a weak solution to (P_{k_j}), by taking $\varphi \in C^1_{c,\mathrm{rd}}(D^\kappa) \subset C^1_{0,\mathrm{rd}}(D^\kappa)$ as the test function in (2.5) and bearing in mind (2.7), we have

$$\int_{a_k}^{b_k} u_{k_j}^\Delta(s) \cdot \varphi^\Delta(s) \Delta s = \left(u_{k_j}, \varphi\right)_H = \int_{a_k}^{b_k} F\left(s, u_{k_j}^\sigma(s)\right) \cdot \varphi^\sigma(s) \Delta s, \tag{2.15}$$

whence it follows, by taking limits, that

$$\int_{a_k}^{b_k} \left((u^k)^{\Delta}(s) \cdot \varphi^{\Delta}(s) - F\left(s, (u^k)^{\sigma}(s)\right) \cdot \varphi^{\sigma}(s) \right) \Delta s = 0, \tag{2.16}$$

which is equivalent because $u_1|_{D_k} = u^k$ and $\varphi = 0 = \varphi^{\sigma}$ on $D^o \setminus D_k^o$ to

$$\int_a^b \left(u_1^{\Delta}(s) \cdot \varphi^{\Delta}(s) - F\left(s, u_1^{\sigma}(s)\right) \cdot \varphi^{\sigma}(s) \right) \Delta s = 0, \tag{2.17}$$

and the proof is therefore complete. □

Propositions 2.3 and 2.4 lead to the following sufficient condition for the existence of at least one solution in the sense of distributions to problem (P).

Corollary 2.5. *Let $F = f + g$ be such that $f, g : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfy (C) and (PM), f satisfies (C_c), and g satisfies (C_g).*

Then, if for each $j \geq 1$ there exist $\underline{u}_j$ and $\overline{u}_j$ a lower and an upper weak solution, respectively, to (P_j) such that $\underline{u}_j \leq \overline{u}_j$ on D and

$$\inf_{j\geq 1} \min_{[a+\delta, b-\delta]_{\mathbb{T}}} \underline{u}_j > 0 \quad \forall \delta \in \left(0, \frac{b-a}{2}\right], \qquad \sup_{j\geq 1} \max_{D} \overline{u}_j < \infty, \tag{2.18}$$

then (P) has a solution in the sense of distributions u_1.

Finally, fixed $u_1 \in H_{0,\mathrm{loc}}$is a solution in the sense of distributions to (P) with $F = f + g$, we will derive the existence of a second solution in the sense of distributions to (P)greater than or equal to u_1 on D. For every $k \geq 1$, consider the weak problem

$$(\widetilde{P}_k) \begin{cases} -v^{\Delta\Delta}(t) = F\left(t, (u_1 + v^+)^{\sigma}(t)\right) - F\left(t, u_1^{\sigma}(t)\right), & \Delta\text{-a.e. } t \in \left(D_k^{\kappa}\right)^o, \\ v(a_k) = 0 = v(b_k). \end{cases} \tag{2.19}$$

For every $k \geq 1$, consider $H_k := H^1_{0,\Delta}(D_k)$ as a subspace of H by defining it for every $v \in H_k$ as $v = 0$ on $D \setminus D_k$ and define the functional $\Phi_k : H_k \subset H \to \mathbb{R}$ for every $v \in H_k$ as

$$\Phi_k(v) := \frac{1}{2}\|v\|_H^2 - \int_{a_k}^{b_k} G\left(s, (v^+)^{\sigma}(s)\right) \Delta s, \tag{2.20}$$

where function $G : D \times [0, +\infty) \to \overline{\mathbb{R}}$ is defined for Δ-a.e. $t \in D$ and all $x \in [0, +\infty)$ as

$$G(t, x) := \int_0^x \left(F\left(t, u_1^{\sigma}(t) + r\right) - F\left(t, u_1^{\sigma}(t)\right) \right) dr. \tag{2.21}$$

As a consequence of Lemma 2.1, we deduce that every weak solution to $(\widetilde{P}_k)$ is nonnegative on D_k and by reasoning as in [4, Section 3], one can prove that Φ_k is weakly lower semicontinuous, Φ_k is continuously differentiable in H_k, for every $v, w \in H_k$,

$$\Phi_k'(v)(w) = (v, w)_H - \int_{a_k}^{b_k} \left(F\left(s, (u_1 + v^+)^{\sigma}(s)\right) - F\left(s, u_1^{\sigma}(s)\right) \right) \cdot w^{\sigma}(s) \Delta s, \tag{2.22}$$

and weak solutions to $(\widetilde{P}_k)$ match up to the critical points of Φ_k.

Next, we will assume the following condition.

(NI) For Δ-a.e. $t \in D^o$, $f(t, \cdot)$ is nonincreasing on $(0, +\infty)$.

Proposition 2.6. *Suppose that $F = f + g$ is such that $f, g : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfy* (C) *and* (PM)*, f satisfies* (C_c) *and* (NI)*, and g satisfies* (C_g)*.*

If $\{v_k\}_{k\geq 1} \subset H$, $v_k \in H_k$ is a bounded sequence in H such that

$$\inf_{k\geq 1} \Phi_k(v_k) > 0, \qquad \lim_{k\to+\infty} \left\|\Phi'_k(v_k)\right\|_{H_k^*} = 0, \tag{2.23}$$

then $\{v_k\}_{k\geq 1}$ has a subsequence convergent pointwise in D to a nontrivial function $v \in H$ such that $v \geq 0$ in D and $u_2 := u_1 + v$ is a solution in the sense of distributions to (P)*.*

Proof. Since $\{v_k\}_{k\geq 1}$ is bounded in H, it has a subsequence which converges weakly in H and strongly in $C_0(D)$ to some $v \in H$.

For every $k \geq 1$, by (2.2), we obtain

$$\left\|v_k^-\right\|_H \leq \left\|\Phi'_k(v_k)\right\|_{H_k^*}, \tag{2.24}$$

which implies, from (2.23), that $v \geq 0$ on D and so $u_2 := u_1 + v > 0$ on $(a, b)_{\mathbb{T}}$.

In order to show that $u_2 := u_1 + v \in H_{0,\mathrm{loc}}$ is a solution in the sense of distributions to (P), fix $\varphi \in C^1_{\mathrm{c,rd}}(D^k)$ arbitrary and choose $k \geq 1$ so large that supp $\varphi \subset (a_k, b_k)_{\mathbb{T}}$, bearing in mind that u_1 is a solution in the sense of distributions to (P), and the pass to the limit in (2.22) with $v = v_k$ and $w = \varphi$ yields to

$$\begin{aligned} 0 &= \int_a^b \left[v^\Delta(s)\cdot\varphi^\Delta(s) - \left(F\left(s, (u_1+v)^\sigma(s)\right) - F\left(s, u_1^\sigma(s)\right)\right)\cdot\varphi^\sigma(s)\right]\Delta s \\ &= \int_a^b \left[u_2^\Delta(s)\cdot\varphi^\Delta(s) - F\left(s, u_2^\sigma(s)\right)\cdot\varphi^\sigma(s)\right]\Delta s; \end{aligned} \tag{2.25}$$

thus, u_2 is a solution in the sense of distributions to (P).

Finally, we will see that v is not the trivial function; suppose that $v = 0$ on D. Condition (NI)ensures that function G defined in (2.21) satisfies for every $k \geq 1$ and Δ-a.e. $s \in D^o$,

$$\begin{aligned} G\left(s, (v_k^+)^\sigma(s)\right) \geq{} & \left(f\left(s, (u_1+v_k^+)^\sigma(s)\right) - f\left(s, u_1^\sigma(s)\right)\right)\cdot(v_k^+)^\sigma(s) \\ & + \int_0^{(v_k^+)^\sigma(s)} \left(g\left(s, u_1^\sigma(s)+r\right) - g\left(s, u_1^\sigma(s)\right)\right)dr, \end{aligned} \tag{2.26}$$

so that, by (2.20) and (2.22), we have, for every $k \geq 1$,

$$\begin{aligned} \Phi_k(v_k) \leq{} & \frac{1}{2}\left\|v_k\right\|_H^2 - \left(v_k, v_k^+\right)_H + \Phi'_k(v_k)(v_k^+) \\ & - \int_a^b \left(g\left(s, (u_1+v_k^+)^\sigma(s)\right) - g\left(s, u_1^\sigma(s)\right)\right)\cdot(v_k^+)^\sigma(s)\Delta s \\ & + \int_a^b \left[\int_0^{(v_k^+)^\sigma(s)} \left(g\left(s, u_1^\sigma(s)+r\right) - g\left(s, u_1^\sigma(s)\right)\right)dr\right]\Delta s; \end{aligned} \tag{2.27}$$

moreover, as we know that $v_k^+ \leq p$ on D for some $p > 0$, it follows from ($\mathrm{C_g}$) that there exists $m \in L^1_\Delta(D^o)$ such that

$$\begin{aligned} \Phi_k(v_k) &\leq \frac{1}{2}\left(\left\|v_k^-\right\|_H^2 - \left\|v_k^+\right\|_H^2\right) + \Phi'_k(v_k)(v_k^+) + 2\int_a^b m(s)\cdot(v_k^+)^\sigma(s)\Delta s \\ &\leq \frac{1}{2}\left\|v_k^-\right\|_H^2 + \left\|\Phi'_k(v_k)\right\|_{H_k^*}\cdot\left\|v_k^+\right\|_H + 2\int_a^b m(s)\cdot(v_k^+)^\sigma(s)\Delta s, \end{aligned} \tag{2.28}$$

and hence, since $\{v_k^+\}_{k\geq 1}$ is bounded in H and converges pointwise in D to the trivial function v, we deduce, from the second relation in (2.23) and (2.24), that $\lim_{k\to\infty}\Phi_k(v_k) \leq 0$ which contradicts the first relation in (2.23). Therefore, v is a nontrivial function. □

3. Results on the existence and uniqueness of solutions

In this section, we will derive the existence of solutions in the sense of distributions to (P) where $F = f + g_0 + \eta g_1$, $\eta \geq 0$ is a small parameter, and $f, g_0, g_1 : D \times (0,+\infty) \to \overline{\mathbb{R}}$ satisfy (C), (PM) as well as the following conditions.

(H_1) There exists a constant $x_0 \in (0,+\infty)$ and a nontrivial function $f_0 \in L^1_\Delta(D^o)$ such that $f_0 \geq 0$ Δ-a.e. on D^o and

$$f(t,x) \geq f_0(t), \quad g_0(t,x), g_1(t,x) \geq 0 \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in \big(0, x_0\big]. \tag{3.1}$$

(H_2) For every $p \in (0,+\infty)$, there exist $m_p \in L^1_\Delta(D^o)$ and $K_p \geq 0$ such that

$$\begin{gathered} \big|f(t,x)\big| \leq m_p(t) \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in [p,+\infty), \\ \big|g_1(t,x)\big| \leq K_p \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in (0,p]. \end{gathered} \tag{3.2}$$

(H_3) There are $m_0 \in L^2_\Delta(D^o)$ such that

$$\big|g_0(t,x)\big| \leq \lambda x + m_0(t) \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in (0,+\infty), \tag{3.3}$$

for some $\lambda < \lambda_1$, where λ_1 is the smallest positive eigenvalue of problem

$$\begin{gathered} -u^{\Delta\Delta}(t) = \lambda u^\sigma(t), \quad t \in D^{\kappa^2}, \\ u(a) = 0 = u(b). \end{gathered} \tag{3.4}$$

3.1. Existence of one solution. Uniqueness

Theorem 3.1. *Suppose that $f, g_0, g_1 : D \times (0,+\infty) \to \overline{\mathbb{R}}$ satisfy (C), (PM), and (H_1)–(H_3). Then, there exists a $\eta_0 > 0$ such that for every $\eta \in [0,\eta_0)$, problem (P) with $F = f + g_0 + \eta g_1$ has a solution in the sense of distributions u_1.*

Proof. Let $\eta \geq 0$ be arbitrary; conditions (H_1)–(H_3) guarantee that $g := g_0 + \eta g_1$ satisfies (C_g). We will show that there exists a $\eta_0 > 0$ such that for every $\eta \in [0,\eta_0)$, hypotheses in Corollary 2.5 are satisfied.

Let x_0 and f_0 be given in (H_1), we know, from [4, Proposition 2.7], that we can choose $\varepsilon \in (0,1]$ so small that the weak solution $\underline{u} \in H$ to

$$-u^{\Delta\Delta}(t) = \varepsilon f_0(t), \quad \Delta\text{-a.e. } t \in \big(D^\kappa\big)^o, \qquad u(a) = 0 = u(b), \tag{3.5}$$

satisfies that $\underline{u} > 0$ on $(a,b)_{\mathbb{T}}$ and $\underline{u} \leq x_0$ on D.

Let $j \geq 1$ be so large that $\varepsilon_j < x_0$, we obtain, by (H_1), that

$$-\underline{u}^{\Delta\Delta}(t) \leq f_0(t) \leq f_j\big(t, \underline{u}^\sigma(t)\big) + g\big(t, \underline{u}^\sigma(t)\big), \quad \Delta\text{-a.e. } t \in D^o, \tag{3.6}$$

whence it follows that $\underline{u}$ is a weak lower solution to (P_j).

As a consequence of (C), (PM), and (H_1)–(H_3), by reasoning as in [4, Theorem 4.2], we deduce that problem

$$-u^{\Delta\Delta}(t) = f_j(t, u^\sigma(t)) + g_0(t, u^\sigma(t)) + 1, \quad \Delta\text{-a.e. } t \in (D^\kappa)^o,$$
$$u(t) > 0, \quad t \in (a,b)_{\mathbb{T}}, \tag{3.7}$$
$$u(a) = 0 = u(b)$$

has some weak solution $\overline{u}_j \in H$ which, from Lemma 2.1 and (H_1), satisfies that $\underline{u} \le \overline{u}_j$ on D. We will see that $\{\overline{u}_j\}_{j\ge1}$ is bounded in $C_0(D)$, by taking $\varphi_j := (\overline{u}_j - x_0)^+ \in H$ as the test function, we know from (2.2), (H_2), and (H_3) that there exist $m_{x_0} \in L^2_\Delta(D^o)$ such that

$$\begin{aligned}\|\varphi_j\|_H^2 &\le (\overline{u}_j - x_0, \varphi_j)_H \\ &= \int_a^b \left(f_j(s, \overline{u}_j^\sigma(s)) + g_0(s, \overline{u}_j^\sigma(s)) + 1\right) \cdot \varphi_j^\sigma(s)\Delta s \\ &\le \int_a^b \left(\lambda \overline{u}_j^\sigma(s) + m_{x_0}(s) + m_0(s) + 1\right) \cdot \varphi_j^\sigma(s)\Delta s;\end{aligned} \tag{3.8}$$

so that, it follows from the fact that the immersion from H into $C_0(D)$ is compact, see [9, Proposition 3.7], Wirtinger's inequality [10, Corollary 3.2] and relation $\lambda < \lambda_1$ that $\{\varphi_j\}_{j\ge1}$ is bounded in H and, hence, $\{\overline{u}_j\}_{j\ge1}$ is bounded in $C_0(D)$. Thereby, condition (H_2) allows to assert that there exists $\eta_0 \ge 0$, such that for all $\eta \in [0, \eta_0)$

$$-\overline{u}_j^{\Delta\Delta}(t) \ge f_j(t, \overline{u}_j^\sigma(t)) + g_0(t, \overline{u}_j^\sigma(t)) + \eta g_1(t, \overline{u}_j^\sigma(t)), \quad \Delta\text{-a.e. } t \in D^o, \tag{3.9}$$

holds, which implies that $\overline{u}_j$ is a weak upper solution to (P_j).

Therefore, for every $j \ge 1$ so large, we have a lower and an upper solution to (P_j), respectively, such that (2.2) is satisfied and so, Corollary 2.5 guarantees that problem (P) has at least one solution in the sense of distributions u_1. □

Theorem 3.2. *If $f : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfies (C), (C_c), and (NI), then, (P) with $F = f$ has at most one solution in the sense of distributions.*

Proof. Suppose that (P) has two solutions in the sense of distributions $u_1, u_2 \in H_{0,\text{loc}}$. Let $\varepsilon > 0$ be arbitrary, take $\varphi = (u_1 - u_2 - \varepsilon)^+ \in H_{c,\text{loc}}$ as the test function in (1.11), by (2.2) and (NI), we have

$$\|\varphi\|_H^2 \le (u_1 - u_2 - \varepsilon, \varphi)_H = \int_a^b \left(f(s, u_1^\sigma(s)) - f(s, u_2^\sigma(s))\right) \cdot \varphi^\sigma(s)\Delta s \le 0, \tag{3.10}$$

thus, $u_1 \le u_2 + \varepsilon$ on D. The arbitrariness of ε leads to $u_1 \le u_2$ on D and by interchanging u_1 and u_2, we conclude that $u_1 = u_2$ on D. □

Corollary 3.3. *If $f : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfies (C), (PM), (NI), and (H_1)-(H_2) with $g_0 = 0 = g_1$, then (P) with $F = f$ has a unique solution in the sense of distributions.*

3.2. Existence of two ordered solutions

Next, by using Theorem 3.1 which ensures the existence of a solution in the sense of distributions to (P), we will deduce, by applying Proposition 2.6, the existence of a second one greater than or equal to the first one on the whole interval D; in order to do this, we will assume that $f, g_0, g_1 : D \times (0, +\infty) \to \overline{\mathbb{R}}$ satisfy (C), (PM), (H_1)–(H_3), as well as the following conditions.

(H_4) For Δ-a.e. $t \in D^o$, $f(t,\cdot)$ is nonincreasing and convex on $(0, x_0]$ with x_0 given in (H_1).

(H_5) There are constants $\theta > 2$, $C_1, C_2 \geq 0$ and $x_1 > 0$ such that

$$\begin{gathered} \left|g_1(t,x)\right| \leq C_1 x^{\theta-1} + C_2 \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in (0,+\infty), \\ 0 < \int_0^x g_1(t,r)dr \leq \frac{1}{\theta} x g_1(t,x) \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in \left[x_1, +\infty\right). \end{gathered} \tag{3.11}$$

We will use the following variant of the mountain pass, see [13].

Lemma 3.4. *If Φ is a continuously differentiable functional defined on a Banach space H and there exist $v_0, v_1 \in H$ such that*

$$c := \inf_{\gamma \in \Gamma} \max_{v \in \gamma([0,1])} \Phi(v) > \Phi(v_0), \Phi(v_1), \tag{3.12}$$

where Γ is the class of paths in H joining v_0 and v_1, then there is a sequence $\{v_k\}_{k\geq 1} \subset H$ such that

$$\lim_{k\to+\infty} \Phi(v_k) = c, \qquad \lim_{k\to+\infty} (1 + \|v_k\|_H)\|\Phi'(v_k)\|_{H^*} = 0. \tag{3.13}$$

Theorem 3.5. *Let $f, g_0, g_1 : D \times (0,+\infty) \to \overline{\mathbb{R}}$ be such that (C), (PM), and (H_1)–(H_5) hold. Then, there exists an $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0)$, problem (P) with $F = f + g_0 + \eta g_1$ has two solutions in the sense of distributions u_1, u_2 such that $u_1 \leq u_2$ on D and $u_2 - u_1 \in H$.*

Proof. Conditions (H_1)–(H_4) allow to suppose that for Δ-a.e. $t \in D^o$, $f(t,\cdot)$ is nonnegative, nonincreasing, and convex on $(0,+\infty)$ because these conditions can be obtained by simply replacing on $D \times (x_0, +\infty) f$ and g_0 with $f(t, x_0)$ and $g_0(t,x) + f(t,x) - f(t,x_0)$, respectively.

Let u_1 be a solution in the sense of distributions to (P), its existence is guaranteed by Theorem 3.1, and let $\eta > 0$ be arbitrary; it is clear that $F = f + g$ with $g := g_0 + \eta g_1$satisfies hypothesis in Proposition 2.6; we will derive the existence of an $\eta_0 > 0$ such that for every $\eta \in (0, \eta_0)$, we are able to construct a sequence $\{v_k\}_{k\geq 1} \subset H$ in the conditions of Proposition 2.6.

For every $k \geq 1$ and $v \in H_k$, as a straight-forward consequence of (NI), (H_3), (H_5), and the compact immersion from H into $C_0(D)$, we deduce that there exist two constants $C_3, C_4 \geq 0$ such that function G, defined in (2.21), satisfies for Δ-a.e. $s \in D^o$,

$$G\left(s, (v^+)^\sigma(s)\right) \leq \frac{\lambda}{2}(v^\sigma)^2(s) + C_3\left(m_0(s)+1\right)\|v\|_H + \eta C_4\left(1 + \|v\|_H\right)^{\theta-1}\|v\|_H, \tag{3.14}$$

which implies, by (2.20) and Wirtinger's inequality [10, Corollary 3.2], that there exists a constant $C_5 \geq 0$ such that

$$\begin{aligned} \Phi_k(v) &= \frac{1}{2}\|v\|_H^2 - \int_{a_k}^{b_k} G\left(s, (v^+)^\sigma(s)\right)\Delta s \\ &\geq \frac{1}{2}\left(1 - \frac{\lambda}{\lambda_1}\right)\|v\|_H^2 - C_5\left(1 + \eta\left(1 + \|v\|_H\right)^{\theta-1}\right)\|v\|_H. \end{aligned} \tag{3.15}$$

Thereby, as $\lambda < \lambda_1$, there exist constants $R, \eta_0, c_0 > 0$ such that

$$\inf_{\substack{v\in H_k\\ \|v\|_H = R}} \Phi_k(v) \geq c_0 > 0 \quad \forall k \geq 1,\ \eta \in (0, \eta_0). \tag{3.16}$$

Let $\eta \in (0, \eta_0)$ be arbitrary. From the second relation in (H_5), we obtain that

$$g_1(t, x) \geq C_6 x^{\theta-1} \quad \text{for } \Delta\text{-a.e. } t \in D^o,\ x \in [x_1, +\infty), \tag{3.17}$$

for some constant $C_6 > 0$; thus, it is not difficult to prove that there is a $v_1 \in H_1$ such that $v_1 > 0$ on $(a, b)_{\mathbb{T}}$, $\|v_1\|_H > R$ and $\Phi_1(v_1) < 0$ and hence, since $\Phi_1(0) = 0$, by denoting as Γ_1 the class of paths in H_1 joining 0 and v_1, it follows from (3.16) that

$$c_1 := \inf_{\gamma\in\Gamma_1} \max_{v\in\gamma([0,1])} \Phi_1(v) \geq c_0 > \Phi_1(0), \Phi_1(v_1), \tag{3.18}$$

hence, Lemma 3.4 establishes the existence of a sequence $\{v_k\}_{k\geq 1} \subset H_1$ such that

$$\lim_{k\to+\infty} \Phi_1(v_k) = c_1, \qquad \lim_{k\to+\infty} \left(1 + \|v_k\|_H\right) \left\|\Phi_1'(v_k)\right\|_{H_1^*} = 0. \tag{3.19}$$

Consequently, bearing in mind that $H_1 \subset H_k$ and $\Phi_k|_{H_1} = \Phi_1$ for all $k \geq 1$ and by removing a finite number of terms if it is necessary, we obtain a sequence $\{v_k\}_{k\geq 1} \subset H$ such that $v_k \in H_k$ for every $k \geq 1$ and

$$0 < \frac{c_0}{2} \leq \Phi_k(v_k) \leq \ k \geq 1, \qquad \lim_{k\to+\infty} \left(1 + \|v_k\|_H\right) \left\|\Phi_k'(v_k)\right\|_{H_k^*} = 0, \tag{3.20}$$

we will show that this sequence is bounded in H.

From (2.2), we deduce that

$$0 \leq \lim_{k\to+\infty} \left\|v_k^-\right\|_H \leq \lim_{k\to+\infty} \left\|\Phi_k'(v_k)\right\|_{H_k^*} = 0, \tag{3.21}$$

For every $k \geq 1$, from (2.2), (2.20), and (2.22), we have that

$$\Phi_k(v_k) - \frac{1}{2}\Phi_k'(v_k)(v_k^+) \geq \frac{1}{2}\left\|v_k^-\right\|_H^2 + \int_a^b H_F\left(s, (v_k^+)^\sigma(s)\right)\Delta s, \tag{3.22}$$

where, for Δ-a.e. $s \in D^o$,

$$\begin{aligned} &H_F\left(s, (v_k^+)^\sigma(s)\right) \\ &\quad = \frac{1}{2}\left(F\left(s, (u_1 + v_k^+)^\sigma(s)\right) + F\left(s, u_1^\sigma(s)\right)\right) \cdot (v_k^+)^\sigma(s) - \int_{u_1^\sigma(s)}^{(u_1+v_k^+)^\sigma(s)} F(s, r)\, dr; \end{aligned} \tag{3.23}$$

as a straight-forward consequence of the convexity of f and conditions (H_2), (H_3), (H_5), and (3.17), we deduce that there exist constants $C_7 > 0$ and $C_8, C_9 \geq 0$ such that

$$\int_a^b H_F\left(s, (v_k^+)^\sigma(s)\right) \Delta s \geq C_7 \left\|(v_k^+)^\sigma\right\|_{L_\Delta^\theta}^\theta - C_8\left(\left\|(v_k^+)^\sigma\right\|_{L_\Delta^2}^2 + 1\right) - C_9. \tag{3.24}$$

Therefore, relations (3.20), (3.21), (3.22), and (3.24) allow to assert that sequence $\{(v_k^+)^\sigma\}_{k\geq 1}$ is bounded in $L_\Delta^\theta(D^o)$ and so, as for every $k \geq 1$,

$$\frac{1}{2}\|v_k\|_H^2 \leq \Phi_k(v_k) + \int_a^b \left[\int_0^{(v_k^+)^\sigma(s)} \left(g\left(s, u_1^\sigma(s) + r\right) - g\left(s, u_1^\sigma(s)\right)\right) dr\right] \Delta s. \tag{3.25}$$

We conclude by (3.20), (H_3), and (H_5) that $\{v_k\}_{k\geq 1}$ is bounded in H and Proposition 2.6 leads to the result. □

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