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A Lipschitz condition along a transversal foliation implies local uniqueness for ODEs

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Abstract. We prove the following result: if a continuous vector field F is Lipschitz when restricted to the hypersurfaces determined by a suitable foliation and a transversal condition is satisfied at the initial condition, then F determines a locally unique integral curve. We also present some illustrative examples and sufficient conditions in order to apply our main result.

Keywords: uniqueness, Lipschitz condition, foliation, modulus of continuity, rotation formula.

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1 Introduction

Uniqueness for ODEs is an important and quite old subject, but still an active field of research [7–9], being Lipschitz uniqueness theorem the cornerstone on the topic. Besides the existence of many generalizations of that theorem, see [1,6,10], one recent and fruitful line of research has been the searching for alternative or weaker forms of the Lipschitz condition. For instance, let $U \subset \mathbb{R}^2$ be an open neighborhood of (t_0, x_0) and $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ be continuous and consider the scalar initial value problem

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1.1}$$

It was proved, independently by Mortici, [12], and Cid and Pouso [4,5], that local uniqueness holds provided that the following conditions are satisfied:

- $f(t, x)$ is Lipschitz with respect to t,
- $f(t_0, x_0) \neq 0$.

A more general result had been proved before by Stettner and Nowak [14], but in a paper restricted to German readers. They proved that if $U \subset \mathbb{R}^2$ is an open neighborhood of (t_0, x_0), $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous and $(u_1, u_2) \in \mathbb{R}^2$ such that

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- $|f(t,x) - f(t+ku_1, x+ku_2)| \le L|k|$ on D,
- $u_2 \neq f(t_0, x_0)u_1$,

then the scalar problem (1.1) has a unique local solution. By taking either $(u_1, u_2) = (0,1)$ or $(u_1, u_2) = (1,0)$ this result covers both the classical Lipschitz uniqueness theorem and the previous alternative version. Moreover this result has been remarkably generalized in [8] by Diblík, Nowak and Siegmund by allowing the vector (u_1, u_2) to depend on t.

Let us now consider the autonomous initial value problem for a system of differential equations

$$z'(t) = F(z(t)), \qquad z(t_0) = p_0, \tag{1.2}$$

where $n \in \mathbb{N}$, $F : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ and $p_0 \in U$.

Trough the paper we shall need the following definition: if $g : D \subset \mathbb{R}^{n+1} \to E$, where E is a normed space, we will say that g is *Lipschitz in D when fixing the first variable* if there exists $L > 0$ such that for all $(s, x_1, x_2, \dots, x_n), (s, y_1, y_2, \dots, y_n) \in D$ we have that

$$\|g(s, x_1, x_2, \dots, x_n) - g(s, y_1, y_2, \dots, y_n)\|_E \le L\|(x_1, x_2, \dots, x_n) - (y_1, y_2, \dots, y_n)\|,$$

and where $\|\cdot\|$ stands for any norm in $\mathbb{R}^n$. Moreover, for any function g with values in $\mathbb{R}^{n+1}$ we denote $g = (g_1, g_2, \dots, g_{n+1})$.

The following alternative version of Lipschitz uniqueness theorem for systems was proved by Cid in [3].

Theorem 1.1. *Let $U \subset \mathbb{R}^{n+1}$ an open neighborhood of p_0 and $F : U \to \mathbb{R}^{n+1}$ continuous. If moreover*

- *F is Lipschitz in U when fixing the first variable,*
- *$F_1(p_0) \neq 0$,*

then there exists $\alpha > 0$ such that problem (1.2) has a unique solution in $[t_0 - \alpha, t_0 + \alpha]$.

Remark 1.2. The classical Lipschitz theorem is included in the previous one. In order to see this, let $n \in \mathbb{N}$, $U \subset \mathbb{R}^{n+1}$ be an open set, $f : U \to \mathbb{R}^n$ and $(t_0, x_0) \in U$ and consider the non-autonomous problem

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0. \tag{1.3}$$

As it is well known, problem (1.3) is equivalent to the autonomous one (1.2), where

$$F(z_1, z_2, \dots, z_{n+1}) := (1, f(z_1, z_2, \dots, z_{n+1})),$$

and $p_0 := (t_0, x_0)$. Now, if $f(t,x)$ is Lipschitz with respect to x then $F(z_1, z_2, \dots, z_{n+1})$ is Lipschitz when fixing the first variable and moreover $F_1(p_0) = 1 \neq 0$, so Theorem 1.1 applies.

Recently, Diblík, Nowak and Siegmund obtained in [13] a generalization of both [3] and [14]. Their result reads as follows.

Theorem 1.3. *Let $U \subset \mathbb{R}^{n+1}$ be an open neighborhood of p_0, $F : U \to \mathbb{R}^{n+1}$ be continuous and $\mathcal{V}$ a linear hyperplane in $\mathbb{R}^{n+1}$ such that*

- *F is Lipschitz continuous along $\mathcal{V}$, that is, there exists $L > 0$ such that if $x, y \in U$ and $x - y \in \mathcal{V}$, then*

$$\|F(x) - F(y)\| \le L\|x - y\|,$$

and the transversality condition

- $F(p_0) \notin \mathcal{V}$

holds. Then there exists $\alpha > 0$ such that problem (1.2) has a unique solution in $[t_0 - \alpha, t_0 + \alpha]$.

The previous theorem has the following geometric meaning: uniqueness for the autonomous system (1.2) follows provided that the continuous vector field F is Lipschitz when restricted to a family of parallel hyperplanes to $\mathcal{V}$ that covers U and that the vector field at the initial condition $F(p_0)$ is transversal to $\mathcal{V}$.

Our main goal in this paper is to extend Theorem 1.3 from the linear foliation generated by the hyperplane $\mathcal{V}$ to a general n-foliation. The paper is organized as follows: in Section 2 we present our main result which relies on an appropriate change of coordinates and Theorem 1.1. We will show by examples that our result is in fact a meaningful generalization of Theorem 1.3. In Section 3 we present some useful results about Lipschitz functions, including the definition of a modulus of Lipschitz continuity along a hyperplane that will be used in Section 4 for obtaining explicit sufficient conditions on F for the existence of a suitable n-foliation. Another key ingredient for that result shall be a general rotation formula proved too at Section 4.

Through the paper $\langle \cdot, \cdot \rangle$ shall denote the usual scalar product in the Euclidean space.

2 The main result: a general uniqueness theorem

Definition 2.1. Let $p_0 \in \mathbb{R}^{n+1}$. Assume there exist open subsets $V \subset \mathbb{R}^n$, $U \subset \mathbb{R}^{n+1}$, an open interval $J \subset \mathbb{R}$ with $0 \in J$ and a family of differentiable functions $\{g_s : V \to U\}_{s \in J}$ such that $g_0(0) = p_0 \in U$ and $\Phi : (s, y) \in J \times V \to g_s(y) \in U$ is a diffeomorphism. Then we say $\{g_s\}_{s \in J}$ is a *local n-foliation of U at p_0*.

Remark 2.2. An observation regarding notation. If $\Phi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a diffeomorphism, we denote by Φ' its derivative and by Φ^{-1} its inverse. Also, we write $(\Phi^{-1})'$ for the derivative of the inverse. Observe that Φ' takes values in $\mathcal{M}_{n+1}(\mathbb{R})$ so, although we cannot consider the functional inverse of Φ', we can consider the inverse matrix, whenever it exists, of every $\Phi'(x)$ for $x \in \mathbb{R}^{n+1}$. We denote this function by $(\Phi')^{-1}$. Clearly, the chain rule implies that

$$(\Phi')^{-1}(x) = (\Phi^{-1})'(\Phi(x)).$$

The following is our main result.

Theorem 2.3. *Let $U \subset \mathbb{R}^{n+1}$, $V \subset \mathbb{R}^n$ be open sets, $p_0 \in U$, $F : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ a continuous function and $\{g_s : V \to U\}_{s \in J}$ a local n-foliation of U at p_0 which defines the diffeomorphism $\Phi : J \times V \to U$. If the following assumptions hold,*

(C1) Transversality condition:

$$\left\langle \left(\frac{\partial \Phi_1^{-1}}{\partial z_1}(p_0), \dots, \frac{\partial \Phi_1^{-1}}{\partial z_{n+1}}(p_0) \right), F(p_0) \right\rangle \neq 0, \tag{2.1}$$

(C2) Lipschitz condition along the foliation: $F \circ \Phi$ and $(\Phi')^{-1}$ are Lipschitz in a neighborhood of zero when fixing the first variable,

then there exists $\alpha > 0$ such that problem (1.2) has a unique solution in $[t_0 - \alpha, t_0 + \alpha]$.

Proof. Consider the change of coordinates

$$z = (z_1, \dots, z_{n+1}) = \Phi(s, y_1, \dots, y_n) := g_s(y_1, \dots, y_n). \tag{2.2}$$

Since $\{g_s\}_{s\in J}$ is a foliation, Φ is a diffeomorphism. Then, considering $y = (s, y_1, \dots, y_n)$, differentiating (2.2) with respect to t and taking into account equation (1.2),

$$\frac{\mathrm{d}\, z}{\mathrm{d}\, t} = \Phi'(y)\frac{\mathrm{d}\, y}{\mathrm{d}\, t} = F(z) = (F \circ \Phi)(y). \tag{2.3}$$

Since Φ is a diffeomorphism, $\Phi'(y)$ is an invertible matrix for every y, so

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, t} = \Phi'(y)^{-1}(F \circ \Phi)(y).$$

By definition of g_s, $\Phi(0) = p_0$, so we can consider the problem

$$\frac{\mathrm{d}\, y}{\mathrm{d}\, t}(t) = h(y),\ y(t_0) = 0, \tag{2.4}$$

where

$$h(y) = \Phi'(y)^{-1}F(\Phi(y)).$$

Now, by (C2) we have that h is the product of locally Lipschitz functions when fixing the first variable. Furthermore, if $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$ and taking into account (C1),

$$h_1(0) = e_1^T\Phi'(0)^{-1}F(p_0) = e_1^T(\Phi^{-1})'(p_0)F(p_0) = \left\langle \left(\frac{\partial \Phi_1^{-1}}{\partial z_1}(p_0), \dots, \frac{\partial \Phi_1^{-1}}{\partial z_{n+1}}(p_0)\right), F(p_0)\right\rangle \neq 0.$$

Hence, we can apply Theorem 1.1 to problem (2.4) and conclude that problem (1.2) has, locally, a unique solution. □

Remark 2.4.

1) Condition (2.1) can be easily interpreted geometrically: the vector

$$\left(\frac{\partial \Phi_1^{-1}}{\partial z_1}(p_0), \dots, \frac{\partial \Phi_1^{-1}}{\partial z_{n+1}}(p_0)\right),$$

is normal to the hypersurface given by $g_0(V)$ at p_0. So, condition (2.1) means that the vector $F(p_0)$ is not tangent to that hypersurface, and therefore it is called the *'transversality condition'*.

2) Notice that, from [3, Example 3.1], we know that if the transversality condition (2.1) does not hold then the Lipschitz condition along the foliation, that is (C2), is not enough to ensure uniqueness. On the other hand, by [3, Example 3.4], we also know that (C1) and a Lipschitz condition along a local $(n-1)$-foliation do not imply uniqueness. So, in some sense, conditions (C1) and (C2) are sharp.

Theorem 2.3 generalizes the main result in [13], where only foliations consisting of hyperplanes are considered. In the next example we show the limitations of linear (or affine) coordinate changes which are used in [13].

Example 2.5. Let $F(x,y) := 1 + (y - x^2)^{\frac{2}{3}}$. Is there a linear change of coordinates Φ such that $F \circ \Phi$ is Lipschitz in a neighborhood of zero when fixing the first variable? The answer is no. Any linear change of variables Φ will be given by two linearly independent vectors $v, w \in \mathbb{R}^2$ as $\Phi(z,t) = zw + tv$. If $F \circ \Phi$ is Lipschitz in a neighborhood of zero when fixing the first variable, that is, z, that implies that the directional derivative of F at any point of the neighborhood in the direction of v, whenever it exists, is a lower bound for any Lipschitz constant. To see that this cannot happen, take $S = \{(x,y) \in \mathbb{R}^2 \, : y = x^2\}$ and realize that F is differentiable in $\mathbb{R}^2 \backslash S$, with

$$\nabla F(x,y) = \frac{2}{3}(y - x^2)^{-\frac{1}{3}}(-2x, 1), \quad \text{for } (x,y) \in \mathbb{R}^2 \backslash S.$$

Let $v = (v_1, v_2) \in \mathbb{R}^2$. The directional derivative of F at (x,y) in the direction of v is

$$D_v F(x,y) = \langle \nabla F(x,y), v \rangle = \frac{2}{3}(y - x^2)^{-\frac{1}{3}}(v_2 - 2v_1 x), \quad \text{for } (x,y) \in \mathbb{R}^2 \backslash S.$$

Now consider a neighborhood N of 0. In particular, we can consider the points of the form $(x,y) = (\lambda, \lambda^2 + \mu) \in N \backslash S$ for $\mu \neq 0$ and $\lambda \in (-\epsilon, \epsilon)$, so

$$D_v F(x,y) = \frac{2}{3} \frac{v_2 - 2\lambda v_1}{\mu^{1/3}}.$$

This quantity is unbounded in $N \backslash S$ unless the numerator is 0 for every $\lambda \in (-\epsilon, \epsilon)$, but that means that $v = 0$, so v and w cannot be linearly independent. Hence, no linear change of coordinates Φ makes $F \circ \Phi$ Lipschitz in a neighborhood of zero when fixing the first variable.

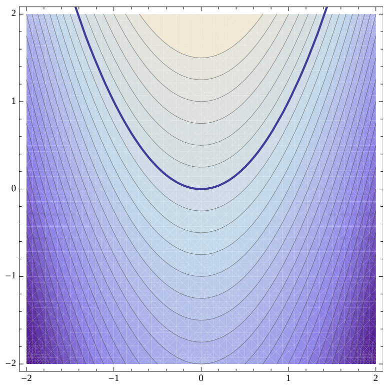

Figure 2.1: The parabolas $g_z(t)$ foliating the plane, where $g_0(t)$ is the thicker one.

Nevertheless, take $(x,y) = \Phi(z,t) = g_z(t) = (t, z + t^2)$. We have $\Phi^{-1}(x,y) = (y - x^2, x)$ and both are differentiable, so Φ is a diffeomorphism. Now, $(F \circ \Phi)(z,t) = 1 + z^{\frac{2}{3}}$, which is clearly Lipschitz when fixing the first variable.

Example 2.6. With what we learned from Example 2.5, it is easy to see that uniqueness for the scalar initial value problem

$$x'(t) = 1 + (x(t) - t^2)^{\frac{2}{3}}, \qquad x(0) = 0, \tag{2.5}$$

can not be dealt with [13, Theorem 2] neither with [8, Theorem 1]. However, by using the local 1-foliation associated to diffeomorphism Φ given in Example 2.5, it is easy to show that conditions (C1) and (C2) of Theorem 2.3 are satisfied. Therefore, we have the local uniqueness of solution.

3 Some results about Lipschitz functions

We will now establish some properties of Lipschitz functions that will be useful for checking condition (C2) in Theorem 2.3. Before that, consider the following lemma.

Lemma 3.1. *Let $A, B, C \in \mathcal{M}_n(\mathbb{R})$, A and C invertible. Then*

$$\|ABC\| \geq \frac{\|B\|}{\|A^{-1}\|\|C^{-1}\|},$$

where $\|\cdot\|$ is the usual matrix norm.

Proof. It is enough to observe that

$$\|B\| = \|A^{-1}ABCC^{-1}\| \leq \|A^{-1}\|\|ABC\|\|C^{-1}\|. \qquad \square$$

Lemma 3.2. *Let U be an open subset of $\mathbb{R}^n$ and $g : U \to GL_n(\mathbb{R})$.*

1. *If g is locally Lipschitz and g^{-1} (the inverse matrix function) is locally bounded, then g^{-1} is locally Lipschitz.*

2. *If g is locally Lipschitz when fixing the first variable and g^{-1} is locally bounded, then g^{-1} is locally Lipschitz when fixing the first variable.*

Proof. 1. Let K be a compact subset of U, k_1 be a Lipschitz constant for g in K and k_2 a bound for g^{-1} in K. Then, for $x, y \in K$, using Lemma 3.1,

$$k_1\|x-y\| \geq \|g(x) - g(y)\| = \|g(x)(g(y)^{-1} - g(x)^{-1})g(y)\| \geq \frac{\|g(y)^{-1} - g(x)^{-1}\|}{k_2^2}.$$

Hence, $\|g(x)^{-1} - g(y)^{-1}\| \leq k_1k_2^2\|x-y\|$ in K and g^{-1} is locally Lipschitz.

2. We proceed as in 2. Let K be a compact subset of U, $(t,x),(t,y) \in K$, k_1 be a Lipschitz constant for g in K when fixing t and k_2 a bound for g^{-1} in K. Then,

$$\begin{aligned} k_1\|x-y\| \geq& \|g(t,x) - g(t,y)\| = \|g(t,x)(g(t,y)^{-1} - g(t,x)^{-1})g(t,y)\| \\ \geq& \frac{\|g(t,y)^{-1} - g(t,x)^{-1}\|}{k_2^2}. \end{aligned}$$

Hence, $\|g(t,x)^{-1} - g(t,y)^{-1}\| \leq k_1k_2^2\|x-y\|$ and g^{-1} is locally Lipschitz when fixing the first variable. $\square$

Corollary 3.3. *Let U be an open subset of $\mathbb{R}^n$, $f : U \to f(U) \subset \mathbb{R}^n$ be a diffeomorphism (notice that, in that case, $f' : U \to GL_n(\mathbb{R})$).*

1. *If f' is locally Lipschitz and $(f')^{-1}$ is locally bounded, then $(f')^{-1}$ is locally Lipschitz.*

2. *If f' is locally Lipschitz and $(f')^{-1}$ is locally bounded, then $(f^{-1})'$ is locally Lipschitz.*

3. *If f' is locally Lipschitz when fixing the first variable and $(f')^{-1}$ is locally bounded, then $(f')^{-1}$ is locally Lipschitz when fixing the first variable.*

Proof. 1. Just apply Lemma 3.2.1 to $g = f'$.

2. Notice that
$$(f^{-1})'(x) = (f')^{-1}(f^{-1}(x)),$$
and that $(f')^{-1}$ is locally Lipschitz by the previous claim. On the other hand, since f' is locally continuous we have that f is locally a $\mathcal{C}^1$-diffeomorphism, and thus f^{-1} is locally Lipschitz. Therefore $(f^{-1})'$ is locally Lipschitz since it is the composition of two locally Lipschitz functions.

3. Just apply Lemma 3.2.2 to $g = f'$. □

3.1 A modulus of continuity for Lipschitz functions along a hyperplane

Let U be an open subset of $\mathbb{R}^{n+1}$, $p_0 \in U$ and consider the tangent space of U at p, which can be identified with $\mathbb{R}^{n+1}$. Consider now the real Grassmannian $\mathrm{Gr}(n, n+1)$, that is, the manifold of hyperplanes of $\mathbb{R}^{n+1}$. We know that $\mathrm{Gr}(n, n+1) \cong \mathrm{Gr}(1, n+1) = \mathbb{P}^n$, that is, we can identify unequivocally each hyperplane with their perpendicular lines, which are elements of the projective space $\mathbb{P}^n$.

Definition 3.4. Consider $B_{n+1}(p, \delta) \subset \mathbb{R}^{n+1}$ to be the open ball of center p and radius δ. Then, for a function $F : U \to \mathbb{R}^{n+1}$ and every $p \in U$, $v \in \mathbb{P}^n$ and $\delta \in \mathbb{R}^+$ we define the *modulus of continuity*
$$\omega_F(p, v, \delta) := \sup_{\substack{x,y \in B_{n+1}(p,\delta) \\ x-p,y-p \perp v \\ x \neq y}} \frac{\|F(x) - F(y)\|}{\|x - y\|} \in [0, +\infty].$$

We also define
$$\omega_F(p, v) := \lim_{\delta \to 0} \omega_F(p, v, \delta) = \lim_{\delta \to 0} \sup_{\substack{x,y \in B_{n+1}(p,\delta) \\ x-p,y-p \perp v \\ x \neq y}} \frac{\|F(x) - F(y)\|}{\|x - y\|}$$
$$= \overline{\lim_{\substack{(x,y) \to (p,p) \\ x-p,y-p \perp v \\ x \neq y}}} \frac{\|F(x) - F(y)\|}{\|x - y\|} \in [0, +\infty].$$

Remark 3.5. If $\omega_F(p, v) < +\infty$, then there exist $\delta, \epsilon \in \mathbb{R}^+$ such that
$$\|F(x) - F(y)\| \leq (\omega_F(p, v) + \epsilon)\|x - y\|, \qquad x, y \in B_{n+1}(p, \delta),\ x - p, y - p \perp v.$$

Equivalently,
$$\|F(x + p) - F(y + p)\| \leq (\omega_F(p, v) + \epsilon)\|x - y\|, \qquad x, y \in B_{n+1}(0, \delta),\ x, y \perp v.$$

Let A be a orthonormal matrix such that its first column is parallel to v. In that case, since A is orthogonal, $x \perp e_1$ implies that $Ax \perp v$. Then,
$$\|F(Ax + p) - F(Ay + p)\| \leq (\omega_F(p, v) + \epsilon)\|A(x - y)\|, \qquad x, y \in B_{n+1}(0, \delta),\ x, y \perp e_1.$$

That is, taking into account that $\|A\| = 1$,
$$\|F(A(0, x) + p) - F(A(0, y) + p)\| \leq (\omega_F(p, v) + \epsilon)\|x - y\|, \qquad x, y \in B_n(0, \delta).$$

Hence, if $\varphi(x) = Ax + p$ then $F \circ \varphi$ is locally Lipschitz in an neighborhood of the origin *when the first variable is equal to zero.*

The following lemma illustrates the relation between the modulus of continuity ω_F and the partial derivatives of F.

Lemma 3.6. *Assume F is continuously differentiable in a neighborhood N of p. Then*

$$\omega_F(p,v) = \sup_{\substack{w\perp v\\ \|w\|=1}} \|D_w F(p)\|.$$

Proof. Since $F'(z)$ is continuous at p, for $\{\epsilon_n\}\to 0$ there exists $\{\delta_n\}\to 0$ such that if $z\in B_{n+1}(p,\delta_n)$ and $\|w\|=1$ then $\|F'(z)(w)\|\le\|F'(p)(w)\|+\epsilon_n$. Hence, using the mean value theorem,

$$\begin{aligned}
\sup_{\substack{x,y\in B_{n+1}(p,\delta_n)\\ x-p,y-p\perp v\\ x\neq y}} \frac{\|F(x)-F(y)\|}{\|x-y\|} &\le \sup_{\substack{x,y,z\in B_{n+1}(p,\delta_n)\\ x-p,y-p\perp v\\ x\neq y}} \frac{\|F'(z)(x-y)\|}{\|x-y\|} \le \sup_{\substack{z\in B_{n+1}(p,\delta_n)\\ u\in B_{n+1}(0,2\delta_n)\\ u\perp v\\ u\neq 0}} \frac{\|F'(z)(u)\|}{\|u\|} \\
&= \sup_{\substack{z\in B_{n+1}(p,\delta_n)\\ d\in(0,2\delta_n)\\ w\perp v\\ \|w\|=1}} \frac{\|F'(z)(dw)\|}{\|dw\|} = \sup_{\substack{z\in B_{n+1}(p,\delta_n)\\ w\perp v\\ \|w\|=1}} \|F'(z)(w)\| \\
&\le \sup_{\substack{w\perp v\\ \|w\|=1}} \|F'(p)(w)\|+\epsilon_n = \sup_{\substack{w\perp v\\ \|w\|=1}} \|D_w F(p)\|+\epsilon_n.
\end{aligned}$$

Then, taking the limit when $n\to\infty$, we obtain

$$\omega_F(p,v) \le \sup_{\substack{w\perp v\\ \|w\|=1}} \|D_w F(p)\|.$$

On the other hand, assume $w\in\mathbb{S}^n$ and $w\perp v$. Then $F(p+tw)=F(p)+t(D_wF(p)+g(t))$ where g is continuous and $\lim_{t\to 0} g(t)=0$. Therefore,

$$\|D_w F(p)\| = \left\|\frac{F(p+tw)-F(p)}{t}-g(t)\right\| \le \sup_{\substack{x,y\in B_{n+1}(p,t)\\ x-p,y-p\perp v\\ x\neq y}} \left[\frac{\|F(x)-F(y)\|}{\|x-y\|}+|g(t)|\right].$$

Taking the limit when t tends to zero, $\|D_wF(p)\|\le\omega_F(p,v)$, which ends the proof. □

Remark 3.7. This definition of the modulus of continuity $\omega_F(\cdot,\cdot)$ is somewhat similar to the definition of strong absolute differentiation which appears in [2, expression (1)]:

Let (X,d_X) and (Y,d_Y) be two metric spaces and consider $F: X\to Y$ and $p\in X$. We say F is *strongly absolutely differentiable at p* if and only if the following limit exists:

$$F^{|\prime|}(p) := \lim_{\substack{(x,y)\to(p,p)\\ x\neq y}} \frac{d_Y(F(x),F(y))}{d_X(x,y)}.$$

However, notice that there some important differences between $\omega_F(\cdot,\cdot)$ and $F^{|\prime|}$ when $X=\mathbb{R}^n$ and $Y=\mathbb{R}^m$. First, since $\omega(\cdot,\cdot)$ is defined with a supremum, $\omega(\cdot,\cdot)$ is well defined in more cases than $F^{|\prime|}$. Also, in the definition of $\omega_F(\cdot,v)$, we are avoiding the direction of a certain vector v. This means that, while strong absolute differentiation implies continuity at the point (see [2, Theorem 3.1]), $\omega(\cdot,\cdot)$ does not.

Regarding the similarities, when the partial derivatives of F exist, $F^{|\prime|} = \|\sum_{k=1}^{n}\frac{\partial F}{\partial x_k}\|$ (see [2, Theorem 3.6]).

Example 3.8. Consider again $F(x,y) := 1 + (y - x^2)^{\frac{2}{3}}$ and $S = \{(x,y) \in \mathbb{R}^2 \ : y = x^2\}$. As was stated in Example 2.5, we have that $F|_{\mathbb{R}^2\setminus S} \in \mathcal{C}^\infty(\mathbb{R}^2\setminus S)$ and

$$\nabla F(x,y) = \frac{2}{3}(y-x^2)^{-\frac{1}{3}}(-2x,1), \quad \text{for } (x,y) \in \mathbb{R}^2\setminus S.$$

Therefore, $\omega(p,v) < +\infty$ for every $(p,v) \in (\mathbb{R}^2\setminus S) \times \mathbb{P}^1$.

On the other hand, for $p = (x_0, x_0^2) \in S$ and $v = (v_1 : v_2) \in \mathbb{P}^1$, if $x = (x_1, y_1) - p \perp v$ then $x = \lambda(-v_2, v_1) + p$ for some $\lambda \in \mathbb{R}$. Analogously, we take $y = \mu(-v_2, v_1) + p$ for some $\mu \in \mathbb{R}$. Hence,

$$\begin{aligned}\omega_F(p,v) &= \overline{\lim_{\substack{(x,y)\to(p,p)\\ x-p,y-p\perp v\\ x\neq y}}} \frac{\|F(x)-F(y)\|}{\|x-y\|} = \overline{\lim_{\substack{(\lambda,\mu)\to(0,0)\\ \lambda\neq\mu}}} \frac{|F(\lambda(-v_2,v_1)+p) - F(\mu(-v_2,v_1)+p)|}{\|(\lambda-\mu)(-v_2,v_1)\|}\\ &= \overline{\lim_{\substack{(\lambda,\mu)\to(0,0)\\ \lambda\neq\mu}}} \frac{|[\lambda(2x_0v_2+v_1) - \lambda^2v_2^2]^{\frac{2}{3}} - [\mu(2x_0v_2+v_1) - \mu^2v_2^2]^{\frac{2}{3}}|}{|\lambda-\mu|}.\end{aligned}$$

We now can consider two cases: $(v_1 : v_2) = (-2x_0 : 1)$ and $(v_1 : v_2) \neq (-2x_0 : 1)$. In the first case, taking into account that $z^2 + z + 1 \geq 3/4$ for every $z \in \mathbb{R}$,

$$\begin{aligned}\omega_F(p,v) &= \overline{\lim_{\substack{(\lambda,\mu)\to(0,0)\\ \lambda\neq\mu}}} \frac{|(-\lambda^2v_2^2)^{\frac{2}{3}} - (-\mu^2v_2^2)^{\frac{2}{3}}|}{|\lambda-\mu|} = \overline{\lim_{\substack{(\lambda,\mu)\to(0,0)\\ \lambda\neq\mu}}} \frac{|\mu^{\frac{4}{3}} - \lambda^{\frac{4}{3}}||v_2|^{\frac{2}{3}}}{|\lambda-\mu|}\\ &= |v_2|^{\frac{2}{3}} \overline{\lim_{\substack{(\lambda,\mu)\to(0,0)\\ \lambda\neq\mu}}} \left|\mu^{\frac{1}{3}} + \frac{\lambda}{\mu^{\frac{2}{3}} + \mu^{\frac{1}{3}}\lambda^{\frac{1}{3}} + \lambda^{\frac{2}{3}}}\right| = |v_2|^{\frac{2}{3}} \overline{\lim_{\substack{(\lambda,\mu)\to(0,0)\\ \lambda\neq\mu}}} \left|\mu^{\frac{1}{3}} + \lambda^{\frac{1}{3}} \frac{1}{\left(\frac{\mu}{\lambda}\right)^{\frac{2}{3}} + \left(\frac{\mu}{\lambda}\right)^{\frac{1}{3}} + 1}\right|\\ &\leq |v_2|^{\frac{2}{3}} \overline{\lim_{\substack{(\lambda,\mu)\to(0,0)\\ \lambda\neq\mu}}} \left|\mu^{\frac{1}{3}} + \frac{4}{3}\lambda^{\frac{1}{3}}\right| = 0.\end{aligned}$$

Observe that in this deduction we have assumed $\lambda \neq 0$. It is clear that, when $\lambda = 0$, the limit is zero as well.

In the case $(v_1 : v_2) \neq (-2x_0 : 1)$ the quotient inside the limit is not bounded and $\omega_F(p,v) = +\infty$. Therefore,

$$\omega_F^{-1}([0,+\infty)) = (\mathbb{R}^2\setminus S) \times \mathbb{P}^1 \cup \{((x,x^2),(-2x:1)) \in \mathbb{R}^2 \times \mathbb{P}^1 \ : \ x \in \mathbb{R}\}.$$

4 How to get a Lipschitz condition along a foliation

The next lemma is a key ingredient in the main result of this section. It gives an alternative expression to the rotation matrix provided by Rodrigues' rotation formula and generalizes it for n-dimensional vector spaces.

Lemma 4.1 (Codesido's rotation formula)**.** *Let $x, y \in \mathbb{R}^{n+1}$ and define $K_x^y \in \mathcal{M}_{n+1}(\mathbb{R})$ as*

$$K_x^y := yx^T - xy^T.$$

Now, let $u, v \in \mathbb{S}^n$, $v \neq -u$, and define $R_u^v \in \mathcal{M}_{n+1}(\mathbb{R})$ as

$$R_u^v := \operatorname{Id} + K_u^v + \frac{1}{1+\langle u, v\rangle}(K_u^v)^2, \tag{4.1}$$

where Id *is the identity matrix of order $n+1$.*

Then, $R_u^v \in \mathrm{SO}(n+1)$ and $R_u^v u = v$, that is, R_u^v is a rotation in $\mathbb{R}^{n+1}$ that sends the unitary vector u to v. Furthermore, the function $R : \{(u,v) \in \mathbb{S}^n \times \mathbb{S}^n \ : \ u \neq -v\} \to \mathrm{SO}(n+1)$, defined by $R(u,v) := R_u^v$, is analytic.

Proof. First, we show that $R_u^v \in \mathrm{O}(n+1)$, that is, $(R_u^v)^T = (R_u^v)^{-1}$. Observe that $(K_u^v)^T = -K_u^v$ and so $[(K_u^v)^2]^T = (K_u^v)^2$. That is, $(R_u^v)^T = \operatorname{Id} - K_u^v + \frac{1}{1+\langle u,v\rangle}(K_u^v)^2$. Therefore,

$$\begin{aligned}
(R_u^v)^T R_u^v =& \left[\operatorname{Id} - K_u^v + \frac{1}{1+\langle u,v\rangle}(K_u^v)^2\right]\left[\operatorname{Id} + K_u^v + \frac{1}{1+\langle u,v\rangle}(K_u^v)^2\right] \\
=& \operatorname{Id} + \frac{1-\langle u,v\rangle}{1+\langle u,v\rangle}(K_u^v)^2 + \frac{1}{(1+\langle u,v\rangle)^2}(K_u^v)^4.
\end{aligned}$$

Now,

$$\begin{aligned}
(K_u^v)^2 &= (vu^T - uv^T)^2 = vu^T vu^T + uv^T uv^T - vu^T uv^T - uv^T vu^T \\
&= \langle u,v\rangle\,(vu^T + uv^T) - (vv^T + uu^T), \\
(K_u^v)^4 &= \left[\langle u,v\rangle\,(vu^T + uv^T) - (vv^T + uu^T)\right]^2 = \left(\langle u,v\rangle^2 - 1\right)(K_u^v)^2.
\end{aligned}$$

Therefore,

$$(R_u^v)^T R_u^v = \operatorname{Id} + \frac{1-\langle u,v\rangle}{1+\langle u,v\rangle}(K_u^v)^2 - \frac{1-\langle u,v\rangle^2}{(1+\langle u,v\rangle)^2}(K_u^v)^2 = \operatorname{Id}.$$

Clearly, R_u^v is analytic on $S = \{(u,v) \in \mathbb{S}^n \times \mathbb{S}^n \ : \ u \neq -v\}$ and so is the determinant function. Now, we are going to prove that S is a connected set: firstly, define the linear subspaces

$$\begin{aligned}
V_1 &:= \{z \in \mathbb{R}^{2n+2} : \ z_i = -z_{n+1+i},\ i = 1, 2, \dots n+1\}, \\
V_2 &:= \{z \in \mathbb{R}^{2n+2} : \ z_i = 0,\ i = 1, 2, \dots n+1\}, \\
V_3 &:= \{z \in \mathbb{R}^{2n+2} : \ z_{n+1+i} = 0,\ i = 1, 2, \dots n+1\},
\end{aligned}$$

and note that $\operatorname{codim}(V_i) = n+1 \geq 2$ for all $i \in \{1,2,3\}$. Then, it is known that $X := \mathbb{R}^{n+1} \setminus (V_1 \cup V_2 \cup V_3)$ is connected, see [11, Chapter V, Problem 5], and since the projection $\pi : X \to S$ defined as

$$\pi(z) = \left(\frac{(z_1, z_2, \dots, z_{n+1})}{\|(z_1 z_2, \dots, z_{n+1})\|}, \frac{(z_{n+2}, z_{n+3}, \dots, z_{2n+2})}{\|(z_{n+2}, z_{n+3}, \dots, z_{2n+2})\|}\right),$$

is continuous and onto, we have that S is connected too. Therefore, $|R_u^v|$ is continuous on the connected set S and takes values in $\{-1, 1\}$, so $|R_u^v|$ is constant. Since $|R_u^u| = |\operatorname{Id}| = 1$ we have that $|R_u^v| = 1$ on S, that is, $R_u^v \in \mathrm{SO}(n+1)$.

Last, observe that

$$\begin{aligned} R_u^v u &= u + (vu^T - uv^T)u + \frac{\langle u,v\rangle\,(vu^T + uv^T)u - (vv^T + uu^T)u}{1+\langle u,v\rangle} \\ &= u + v - uv^T u + \frac{\langle u,v\rangle\,(v + uv^T u) - (vv^T u + u)}{1+\langle u,v\rangle} \\ &= v + \frac{\langle u,v\rangle\,(v + uv^T u + u - uv^T u) - (vv^T u + u) + u - uv^T u}{1+\langle u,v\rangle} \\ &= v + \frac{\langle u,v\rangle\,(v+u) - vv^T u - uv^T u}{1+\langle u,v\rangle} \\ &= v + \frac{\langle u,v\rangle\,(v+u) - \langle u,v\rangle\, v - \langle u,v\rangle\, u}{1+\langle u,v\rangle} = v. \end{aligned}$$ □

Remark 4.2. For $n=1$ the function R admits a continuous extension to $\mathbb{S}^1 \times \mathbb{S}^1$. Indeed, let us consider $u, v \in \mathbb{S}^1$, $v \neq -u$. Then $u = (\cos(\alpha), \sin(\alpha))$ and $v = (\cos(\beta), \sin(\beta))$ for some $\alpha, \beta \in \mathbb{R}$, with $\beta \neq \alpha + (2k+1)\pi$, $k \in \mathbb{Z}$. Now, a direct computation shows that

$$R_u^v = \begin{pmatrix} \cos(\alpha-\beta) & \sin(\alpha-\beta) \\ -\sin(\alpha-\beta) & \cos(\alpha-\beta) \end{pmatrix}.$$

Therefore,

$$\lim_{v\to -u} R_u^v = \lim_{\beta\to\alpha+\pi} \begin{pmatrix} \cos(\alpha-\beta) & \sin(\alpha-\beta) \\ -\sin(\alpha-\beta) & \cos(\alpha-\beta) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

However, for $n \geq 2$ the function R does not admit a continuous extension to $\mathbb{S}^n \times \mathbb{S}^n$. To see this, consider $u \in \mathbb{S}^n$, $w \in \mathbb{R}^{n+1}$, $w \perp u$, $w \neq 0$ and define $v(w) = (w-u)/\|w-u\|$. Observe that $v(w) \in \mathbb{S}^n$, $v(w) \neq -u$, $\lim\limits_{\|w\|\to 0} v(w) = -u$ and

$$K_u^{v(w)} = \frac{1}{\|w-u\|} K_u^w.$$

Hence,

$$\begin{aligned} R_u^{v(w)} &= \mathrm{Id} + \frac{1}{\|w-u\|} K_u^w + \frac{\|w-u\|}{\|w-u\| + \langle u,w\rangle - 1} \frac{1}{\|w-u\|^2} (K_u^w)^2 \\ &= \mathrm{Id} + \frac{1}{\|w-u\|} K_u^w + \frac{-ww^T - \|w\|^2 uu^T}{\|w-u\|(\|w-u\|-1)}. \end{aligned}$$

Now, consider $\varpi \perp u$ with $\|\varpi\| = 1$.Therefore, if it exists,

$$\lim_{v\to -u} R_u^v = \lim_{t\to 0} R_u^{v(t\varpi)} = \mathrm{Id} + \lim_{t\to 0} \frac{-t^2(\varpi\varpi^T - uu^T)}{\sqrt{t^2+1}(\sqrt{t^2+1}-1)} = \mathrm{Id} - 2(\varpi\varpi^T - uu^T).$$

But in $\mathbb{R}^{n+1}$, with $n \geq 2$, there exist at least two independent unitary vectors ϖ_1 and ϖ_2 in $\langle u\rangle^\perp$, each of them leading to a different value of the right-hand side of the previous expression. Hence, the $\lim\limits_{v\to -u} R_u^v$ does not exist and thus R can not be continuously extended to $\mathbb{S}^n \times \mathbb{S}^n$.

The following is the main result in this section and gives sufficient conditions for the existence of a n-foliation which allows F to satisfy condition (C2) in Theorem 2.3.

Theorem 4.3. *Let U be an open subset of $\mathbb{R}^{n+1}$, $p_0 \in U$ and $F : U \to \mathbb{R}^{n+1}$ continuous. Assume there exists an open interval J with $0 \in J$ and a simple path $\gamma = (\gamma_1, \gamma_2) \in \mathcal{C}^1(J, U \times \mathbb{P}^n)$ such that the following conditions hold:*

(i) $\gamma_1(0) = p_0$.

(ii) There exist $\delta, M \in \mathbb{R}^+$, such that $\omega_F(\gamma_1(t), \gamma_2(t), \delta) < M$ for all $t \in J$.

(iii) $\gamma_1'(0) \not\angle \gamma_2(0)$.

Then, there exists an open neighborhood of zero $\hat{U} \subset U \subset \mathbb{R}^{n+1}$ such that $\Phi(s, y)$ is a local n-foliation of $\hat{U}$. Moreover, $F \circ \Phi$ and $(\Phi')^{-1}$ are Lipschitz in a neighborhood of zero when fixing the first variable.

Proof. Assume, without loss of generality, that γ_1 is parameterized by arc length, that is, $\|\gamma_1'(t)\| = 1$ for all $t \in J$. Consider $\mathbb{S}^n$ as covering space of $\mathbb{P}^n$ with the usual projection $\pi : \mathbb{S}^n \to \mathbb{P}^n$. Take $v_0 \in \pi^{-1}(\gamma_2(0))$, such that $v_0 \neq -e_1$ where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$, and consider the lift $\tilde{\gamma} = (\gamma_1, \tilde{\gamma}_2) : J \to V \times \mathbb{S}^n$ of γ such that $\tilde{\gamma}(0) = (p_0, v_0)$.

Now, $\tilde{\gamma}_2$ is continuous, and $\langle e_1, \tilde{\gamma}_2(0)\rangle = \langle e_1, v_0\rangle \neq -1$ so we can consider an open interval $\tilde{J} \subset J$ where $\langle e_1, \tilde{\gamma}_2(s)\rangle \neq -1$ (that is, $\tilde{\gamma}_2(s) \neq -e_1$) for $s \in \tilde{J}$. Since $\tilde{\gamma}$ is differentiable and $\|\tilde{\gamma}_2(s)\| = 1$ for every $s \in \tilde{J}$, we can consider the continuously differentiable function

$$\tilde{J} \xrightarrow{\ A\ } \mathrm{SO}(n+1)$$

$$s \longmapsto A(s) := R_{e_1}^{\tilde{\gamma}_2(s)}$$

where R_u^v is defined as in Lemma 4.1. Observe that denoting by $a_j(s)$ the columns of $A(s)$, that is,

$$A(s) = \left(\ a_1(s)\ \middle|\ a_2(s)\ \middle|\ \dots\ \middle|\ a_{n+1}(s)\ \right),$$

we have that $a_1(s) = \tilde{\gamma}_2(s)$ and $\{a_2(s), a_3(s), \dots, a_{n+1}(s)\}$ is an orthonormal basis of $\tilde{\gamma}_2(s)^\perp$, (remember that $A(s)e_1 = \tilde{\gamma}_2(s)$ and that $A(s)$ is an orthogonal matrix).

Now, we can define the differentiable function $\Phi : \tilde{J} \times \mathbb{R}^n \to \mathbb{R}^{n+1}$ given by

$$\Phi(s, y) := \gamma_1(s) + A(s)(0, y).$$

Claim 1. $g_s(y) := \Phi(s, y)$ is a local n-foliation.

We easily compute

$$\frac{\partial \Phi}{\partial s}(s, y) = \gamma_1'(s) + A'(s)(0, y),$$

$$\frac{\partial \Phi}{\partial y}(s, y) = \left(\ a_2(s)\ \middle|\ a_3(s)\ \middle|\ \dots\ \middle|\ a_{n+1}(s)\ \right).$$

So

$$\Phi'(0, 0) = \left(\ \gamma_1'(0)\ \middle|\ a_2(0)\ \middle|\ a_3(0)\ \middle|\ \dots\ \middle|\ a_{n+1}(0)\ \right),$$

and since, by (iii), $\gamma_1'(0) \not\angle \tilde{\gamma}_2(0) = a_1(0)$ we have

$$J_\Phi(0, 0) = |\Phi'(0, 0)| \neq 0.$$

Then, by the inverse function theorem there exist open sets $\hat{J} \subset \tilde{J}, \hat{V} \subset V$ and $\hat{U} \subset U$ such that $\hat{J} \times \hat{V}$ contains the origin and $\Phi : \hat{J} \times \hat{V} \to \hat{U}$ is a diffeomorphism. Moreover, by (i), $\Phi(0, 0) = p_0$, so $\Phi(s, y)$, a local n-foliation of $\hat{U}$.

Claim 2. $F \circ \Phi$ is Lipschitz continuous in a neighborhood of zero when fixing the first variable.

Notice that, by construction, $\Phi(s,y) - \gamma_1(s) \in \langle \tilde{\gamma}_2(s) \rangle^{\perp}$. Now, condition (ii) implies that

$$\begin{aligned}
\|F \circ \Phi(s,y_1) - F \circ \Phi(s,y_2)\| &= \|F(\gamma_1(s) + A(s)(0,y_1)) - F(\gamma_1(s) + A(s)(0,y_2))\| \\
&\leq \omega_F(\gamma_1(s), \gamma_2(s), \delta) \|\gamma_1(s) + A(s)(0,y_1) - \gamma_1(s) + A(s)(0,y_2)\| \\
&\leq M \sup_{s \in \hat{J}} \|A(s)\| \|y_1 - y_2\|
\end{aligned}$$

for every $s \in \hat{J}$ and $y_1, y_2 \in B_n\left(0, \frac{\delta}{\sup_{s\in\hat{J}} \|A(s)\|}\right)$.

Claim 3. $(\Phi')^{-1}$ is Lipschitz continuous in a neighborhood of zero when fixing the first variable.

Fix $s \in \hat{J}$. We have that

$$\Phi'(s,y) = \left(\ \gamma_1'(s) + A'(s)(0,y) \ \middle| \ a_2(s) \ \middle| \ a_3(s) \ \middle| \ \dots \ \middle| \ a_{n+1}(s) \ \right).$$

Then,

$$\|\Phi'(s,x) - \Phi'(s,y)\| \leq \sup_{s \in \hat{J}} \|A(s)\| \|x - y\|,$$

so Φ' is Lipschitz continuous in a neighborhood of zero when fixing s.

On the other hand, $(\Phi')^{-1}$ is a continuous function, therefore locally bounded. Hence, by Corollary 3.3, $(\Phi')^{-1}$ is Lipschitz continuous in a neighborhood of zero when fixing the first variable. □

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