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OPTIMIZATION METHOD FOR IDENTIFYING THE SOURCE TERM IN AN INVERSE WAVE EQUATION

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Abstract. In this work, we investigate the inverse problem of identifying a space-wise dependent source term of wave equation from the measurement on the boundary. On the basis of the optimal control framework, the inverse problem is transformed into an optimization problem. The existence and necessary condition of the minimizer for the cost functional are obtained. The projected gradient method and two-parameter model function method are applied to the minimization problem and numerical results are illustrated.

1. Introduction

We consider an initial-boundary value problem for wave equation in the form

$$\begin{gathered} u_{tt}(x,t) = \Delta u(x,t) + \sigma(t) f(x), \quad x \in \Omega,\ 0 < t < T, \\ u(x,0) = u_t(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ 0 < t < T, \end{gathered} \tag{1.1}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, σ is a known non-zero function and independent of the space variable x, $f \in L^2(\Omega)$ is unknown and Δ is the Laplacian operator. An additional condition is assumed in the form

$$\frac{\partial u}{\partial n}(x,t) = g(x,t), \quad x \in \partial\Omega,\ 0 < t < T, \tag{1.2}$$

where g is a known function and

$$\frac{\partial u}{\partial n}(x,t) = \sum_{i=1}^{N} \gamma_i(x) \frac{\partial u}{\partial x_i}(x,t), \quad x \in \partial\Omega,\ 0 < t < T,\ i = 1, 2, \dots, N,$$

where $\gamma_i(x) = (\gamma_1(x), \dots, \gamma_N(x))$ is the outward unit normal to $\partial\Omega$ at x.

We set $d = \sup\{|x_1 - x_2| : x_1, x_2 \in \Omega\}$ is the diameter of Ω. Henceforth we assume

$$T > d, \tag{1.3}$$

$$\sigma(0) \neq 0, \sigma \in C^1[0,T]. \tag{1.4}$$

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For uniqueness and stability of our governing equation we have to choose a large observation time T. The external forces $\sigma(t)f(x)$ in the form of separation of variables are important in modelling vibrations. For example, if we set $\sigma(t) = \cos \omega t$ ($\omega \in R$), then it describes a spatial force which varies harmonically. Moreover the system (1.1) is regarded as an approximation to a model for elastic waves from a point dislocation source. For instance, this kind of point source can be related to models in reflection seismology, oil and gas exploration, ground-penetrating radar and many other physical problems [1]. According to the Hadamard requirements (existence, uniqueness and stability of the solution), the inverse problem is ill-posed mathematically [15, 19].

For an inverse problem with a single measurement, the main methodology is based on an L^2-weighted inequality called a Carleman estimate. Bellassoued [4], Imanuvilov and Yamamoto [14], Klibanov and Timonov [20] discussed the applications of Carleman estimates to inverse problems.

Yamamoto [34] studied the uniqueness and stability result for reconstruction algorithm using exact controllability for an inverse problem described by the wave equation. Nicaise and Zair [25] identified the source term from interior measurements by using some observability estimates and controllability results by using multiplier and Hilbert uniqueness method.

Bellassoued et al. [5], Cipolatti and Lopez [7] and Rakesh [27] obtained uniqueness and stability of inverse problem for the wave equation by using Dirichlet to Neumann map. Stability estimate was established for inverse problem for the wave equation by using Neumann to Dirichlet map in [2].

Mordukhovich and Raymond [24], Lagnese et al. [22] proved the optimal control problems for hyperbolic equations with boundary control. In [3], Barbu and Pavel had considered coefficient optimal control problem for 1-D wave equation with nonhomogeneous boundary periodic inputs. Liang [23] studied the bilinear optimal control problem of the wave equation. Ton [30] used optimal techniques and established feedback laws to identify the surface of the unknown source and its intensity from the observed values of the solution of the wave equation on a portion of fixed closed surface.

For stable reconstruction, we have some regularization techniques [8]. Engl et al. [9] established the uniqueness of inverse source problem of parabolic and hyperbolic equations and analyzed the convergence rate of the regularized solution. In [35], Yamamoto derived the convergence rate of Tikhonov regularization scheme for multidimensional inverse hyperbolic problem. Cheng et al. [6] employed a new strategy for a priori choice of regularizing parameter in Tikhonov's regularization. Feng et al. [10] solved the identification problem of the wave equation by using optimal control method. In [36], Yang obtained the idea to use the techniques of optimal control framework to the inverse problem of recovering the source term in a parabolic equation. Gnanavel et al. [11] studied an inverse problem of reconstructing two time independent coefficients and the initial data in the linear reaction diffusion system from the arbitrary sub-domain measurement and final measurement. Tröltzsch [31] analyzed the existence of optimal solutions, necessary optimality conditions on optimal control problems of partial differential equation and main principles of selected numerical techniques. Hasanov [12] applied conjugate gradient method to identify the unknown spacewise and time dependent heat sources of the variable coefficient heat conduction equation. In [13], Hasanov et al.

established the direct relationship between two widely used methods, least square method and singular value expansion, in inverse source and backward problems with final overdetermination for parabolic and hyperbolic equations. Kabanikhin et al. [16, 17] obtained the iteration methods for solve a parameter identification problem in a one and two dimensional hyperbolic equation of second order respectively. Kabanikhin et al. [18] analyzed a numerical method for inverse problem in hyperbolic equation.

From the Theorem (2.3) in Section 2 we have observed the time derivative of $\frac{\partial u}{\partial n}$ as well as $\frac{\partial u}{\partial n}$ itself for stable construction of $f \in L^2(\Omega)$. However, from a practical point of view, the observation of the time derivative is not desirable and frequently we are obliged to construct $f \in L^2(\Omega)$ only on the basis of $\frac{\partial u}{\partial n}$ itself which is polluted with L^2-errors. Thus the problem of determining $f \in L^2(\Omega)$ from $\frac{\partial u}{\partial n} \in L^2\big(0,T;L^2(\partial\Omega)\big)$ is ill-posed in the sense of Hadamard. For stable construction of f we apply Tikhonov regularization. To solve the inverse problem, we consider the following optimal control problem for $\beta > 0$

$$\min_{f\in A} J_\beta(f), \tag{1.5}$$

where

$$J_\beta(f) = \frac{1}{2}\int_0^T \int_{\partial\Omega} \big|\frac{\partial u}{\partial n}(x,t,f) - g(x,t)\big|^2\,dx\,dt + \frac{\beta}{2}\int_\Omega |f|^2\,dx,$$
$$A = \{f \in L^2(\Omega) : |f| \le a\},$$

$J_\beta : A \subseteq L^2(\Omega) \to R^+$, J_β depends on $a > 0$, u and β is a regularization parameter. For each $\beta > 0$, the source term f is viewed as a control and is adjusted to get the corresponding $\frac{\partial u}{\partial n}$, close to the observations g. In the optimal control problem, the second integration in $J_\beta(f)$ is called the penalty term, which is used to stabilize the minimizer.

This article is organized as follows: In Section 2, we give some preliminaries. In Section 3, we consider the given inverse problem as a optimal control problem and prove the existence of the minimizer, the necessary optimality condition which has to be satisfied by each optimal control is deduced. The projected gradient method and two-parameter model function method are applied to the inverse problem and numerical examples are given in Section 4.

2. Preliminaries

Weak solution: Given $\sigma f \in L^1(0,T;L^2(\Omega))$, we say that a function $u \in C([0,T];H_0^1(\Omega))$ with $u_t \in C([0,T];L^2(\Omega)), u_{tt} \in C([0,T];H^{-1}(\Omega))$ is a weak solution of the problem (1.1) and (1.2) provided

(1) $\langle u_{tt}, \phi\rangle + B[u,\phi;t] = \sigma\int_\Omega f\phi\,dx$, for any $\phi \in H_0^1(\Omega)$ and a.e. $0 \le t \le T$;
(2) $u(\cdot,0) = 0$;
(3) $u_t(\cdot,0) = 0$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ and $B[u,\phi;t] = \int_\Omega \nabla u\nabla\phi\,dx$.

Lemma 2.1 ([26])**.** *If $\sigma f \in L^1(0,T;L^2(\Omega))$, then there exists a unique solution u to* (1.1) *such that $u \in C\big([0,T];H_0^1(\Omega)\big) \cap C^1\big([0,T];L^2(\Omega)\big)$ and*

$$\frac{\partial u}{\partial n} \in L^2\left(\partial\Omega \times (0,T)\right). \tag{2.1}$$

Lemma 2.2 ([26]). *If $\sigma f \in L^1(0,T;L^2(\Omega)), \partial\Omega$ is C^2, then the weak solution $u = u(f)$ satisfies*

$$\sup_{0\le t\le T}\Big(\|u\|_{H_0^1(\Omega)} + \|u_t\|_{L^2(\Omega)}\Big) + \|u_{tt}\|_{L^2(0,T;H^{-1}(\Omega))} \le C\|f\|_{L^2(\Omega)}, \tag{2.2}$$

$$\|\frac{\partial u}{\partial n}\|_{L^2(\partial\Omega\times(0,T))} \le C_0\|f\|_{L^2(\Omega)}, \tag{2.3}$$

where C and C_0 are constants depending only on Ω, T and σ.

Theorem 2.3 ([33]). *Under assumptions* (1.3) *and* (1.4) *we have:*

(1) (Uniqueness) If the solution $u(f)$ to (1.1) *satisfies*

$$\frac{\partial u}{\partial n}(x,t) = 0, \quad x \in \partial\Omega,\ 0 < t < T,$$

then $f(x) = 0$ for almost all $x \in \Omega$.

(2) (Continuity) There exists a constant $C = C(\Omega, T)$ such that

$$C^{-1}\|\frac{\partial u}{\partial n}(f)\|_{H^1(0,T:L^2(\partial\Omega))} \le \|f\|_{L^2(\Omega)} \le C\|\frac{\partial u}{\partial n}(f)\|_{H^1(0,T:L^2(\partial\Omega))} \tag{2.4}$$

for any $f \in L^2(\Omega)$.

3. Optimal control problem

For a fixed β, we consider the functional $J_\beta(f)$ as $J(f)$ and

$$J(\bar f) = \min_{f\in A} J(f), \tag{3.1}$$

3.1. Existence of minimizer.

Theorem 3.1. *There exists a unique minimizer $\bar f \in A$ of J, that is, $J(\bar f) = \min_{f\in A} J(f)$.*

Proof. It can be easily seen that $J(f)$ is nonnegative and thus $J(f)$ has greatest lower bound $\inf_{f\in A} J(f)$. Let $\{f_k\}$ be a minimizing sequence, for example,

$$\inf_{f\in A} J(f) \le J(f_k) \le \inf_{f\in A} J(f) + \frac{1}{k}, \quad k = 1, 2, \dots.$$

Since $J(f_k) \le C_1$ and from the structure of J we easily deduce that $\|f_k\|_{L^2(\Omega)} \le C_1$, where C_1 is independent of k. Let $\{u_k\}$ be the solution of (1.1) corresponding to $\{f_k\}$. By Lemma 2.2, we have

$$\sup_{0\le t\le T}\Big(\|u_k\|_{H_0^1(\Omega)} + \|(u_k)_t\|_{L^2(\Omega)}\Big) + \|(u_k)_{tt}\|_{L^2(0,T;H^{-1}(\Omega))} \le C\|f_k\|_{L^2(\Omega)}.$$

This means that we have uniform bounds for $u_k \in L^\infty(0,T;H_0^1(\Omega))$ and $(u_k)_t \in L^\infty(0,T;L^2(\Omega))$. On a subsequence of f_k and u_k, by weak compactness, there exists $\bar u$ in $C([0,T];H_0^1(\Omega))$ such that

$$\begin{gathered}
f_k \rightharpoonup \bar f \quad \text{weakly in } L^2(\Omega),\\
u_k \rightharpoonup \bar u \quad \text{weak* in } L^\infty(0,T;H_0^1(\Omega)),\\
(u_k)_t \rightharpoonup \bar u_t \quad \text{weak* in } L^\infty(0,T;L^2(\Omega)),\\
(u_k)_{tt} \rightharpoonup \bar u_{tt} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)),\\
\frac{\partial u_k}{\partial n} \rightharpoonup \frac{\partial \bar u}{\partial n} \quad \text{weakly in } L^2\left(\partial\Omega\times(0,T)\right).
\end{gathered}$$

Using a compactness result from [29], we have $u_k \to \bar{u}$ strongly in $L^\infty(0,T;L^2(\Omega))$. By the definition of weak solution, we have

$$\langle (u_k)_{tt}, \phi\rangle = -\int_\Omega [\nabla u_k \nabla \phi - \sigma f_k \phi]\, dx$$

for any $\phi \in H_0^1(\Omega)$ and *a.e.* $0 \le t \le T$. If we pass to the limit as $k \to \infty$ in the weak formulation of u_k, we obtain

$$\langle \bar{u}_{tt}, \phi\rangle = -\int_\Omega [\nabla \bar{u} \nabla \phi - \sigma \bar{f} \phi]\, dx.$$

Thus $\big(\bar{f}(x), \bar{u}(x,t)\big)$ satisfies (1.1). Moreover, using (2.3) and the lower-semicontinuity of the L^2 norm with respect to weak convergence, we obtain

$$J(\bar{f}) \le \liminf_{k\to\infty} J(f_k) = \min_{f\in A} J(f).$$

Hence $J(\bar{f}) = \min_{f\in A} J(f)$. We can easily know that $\frac{\partial u}{\partial n}(x,t,f)$ has the linearity and convexity with respect to f; that is,

$$\frac{\partial u}{\partial n}(x,t,\epsilon f_1 + (1-\epsilon) f_2) = \epsilon \frac{\partial u}{\partial n}(x,t,f_1) + (1-\epsilon)\frac{\partial u}{\partial n}(x,t,f_2), \quad \forall \epsilon \in [0,1].$$

Then the strict convexity of L^2-norm naturally leads to the strict convexity of $J(f)$ which implies that the minimizer $\bar{f}$ is unique. This completes the proof. □

3.2. **Necessary condition.** We are now in a position to state the necessary (and, owing to the convexity, also sufficient) optimality conditions.

Theorem 3.2. *Let f be the solution of the optimal control problem* (3.1). *Then there exists a triple of functions (u, v, f) satisfying the system*

$$\begin{gathered} u_{tt}(x,t) = \Delta u(x,t) + \sigma(t) f(x), \quad x \in \Omega,\ 0 < t < T, \\ u(x,0) = u_t(x,0) = 0, \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega,\ 0 < t < T. \end{gathered} \tag{3.2}$$

and

$$\begin{gathered} v_{tt}(x,t) = \Delta v(x,t), \quad x \in \Omega, \quad 0 < t < T, \\ v(x,0) = v_t(x,0) = 0, \quad x \in \Omega, \\ v(x,t) = \frac{\partial u}{\partial n} - g(x,t), \quad x \in \partial\Omega,\ 0 < t < T. \end{gathered} \tag{3.3}$$

Moreover

$$\beta \int_\Omega f(h-f)\, dx - \int_0^T \int_\Omega (v\sigma(t)(h-f))\, dx\, dt \ge 0. \tag{3.4}$$

for any $h \in A$.

Proof. For any $h \in A, 0 \le \delta \le 1$, we have $f_\delta = (1-\delta) f + \delta h \in A$. Then

$$J_\delta = J(f_\delta) = \frac{1}{2}\int_0^T \int_{\partial\Omega} \big|\frac{\partial u}{\partial n}(x,t,f_\delta) - g(x,t)\big|^2\, dx\, dt + \frac{\beta}{2}\int_\Omega |f_\delta|^2 dx. \tag{3.5}$$

Let u_δ be the solution of (3.2) with given $f = f_\delta$. Since f is an optimal solution,

$$\begin{aligned} \frac{dJ_\delta}{d\delta}\big|_{\delta=0} &= \int_0^T \int_{\partial\Omega} [\frac{\partial u}{\partial n}(x,t,f_\delta) - g(x,t)] \frac{\partial}{\partial n}\big(\frac{\partial u_\delta}{\partial \delta}\big)\big|_{\delta=0}\, dx\, dt \\ &\quad + \beta \int_\Omega f(h-f) dx \ge 0. \end{aligned} \tag{3.6}$$

Let $\tilde{u}_\delta = (\frac{\partial u_\delta}{\partial \delta})$, direct calculations lead to the equation

$$\begin{gathered}\frac{\partial^2 \tilde{u}_\delta}{\partial t^2} = \Delta \tilde{u}_\delta + \sigma(h - f), \quad x \in \Omega, \quad 0 < t < T, \\ \tilde{u}_\delta(x, 0) = \frac{\partial \tilde{u}_\delta}{\partial t}(x, 0) = 0, \quad x \in \Omega, \\ \tilde{u}_\delta(x, t) = 0, \quad x \in \partial\Omega, \quad 0 < t < T.\end{gathered} \tag{3.7}$$

Let $\xi = \tilde{u}_\delta$ at $\delta = 0$. Then ξ satisfies the below equation

$$\begin{gathered}\xi_{tt} = \Delta \xi + \sigma(h - f), \quad x \in \Omega,\ 0 < t < T, \\ \xi(x, 0) = \xi_t(x, 0) = 0, \quad x \in \Omega, \\ \xi(x, t) = 0, \quad x \in \partial\Omega,\ 0 < t < T.\end{gathered} \tag{3.8}$$

From (3.6), we have

$$\int_0^T \int_{\partial\Omega} (\frac{\partial u}{\partial n} - g(x, t)) \frac{\partial \xi}{\partial n}(x, t)\, dx\, dt + \beta \int_\Omega f(h - f)\, dx \geq 0. \tag{3.9}$$

Let $L\xi = \xi_{tt} - \Delta\xi$ and v be the solution of the following problem

$$\begin{gathered}L^* v = v_{tt} - \Delta v = 0, \\ v(x, T) = v_t(x, T) = 0, \\ v(x, t) = \frac{\partial u}{\partial n} - g(x, t),\end{gathered}$$

where L^* is the adjoint operator of the operator L. From the above equation we have

$$\begin{aligned}0 &= \int_0^T \int_\Omega (\xi L^* v) dx\, dt \\ &= \int_0^T \int_\Omega v(\xi_{tt} - \Delta\xi) dx\, dt + \int_0^T \int_{\partial\Omega} \frac{\partial \xi}{\partial n}(\frac{\partial u}{\partial n} - g) dx\, dt.\end{aligned} \tag{3.10}$$

Combining (3.9) with (3.10) we have

$$\beta \int_\Omega f(h - f)\, dx - \int_0^T \int_\Omega (v\sigma(t)(h - f))\, dx\, dt \geq 0. \tag{3.11}$$

This completes the proof. □

4. Numerical examples

After obtaining the theoretical results, we propose the numerical schemes for the inverse problem. We solve the control problem (3.1) directly from the cost functional; but the regularization parameter plays a major role in the numerical simulation. In fact, the effectiveness of a regularization method depends strongly on the choice of the regularization parameter. Kunisch and Zou [21] proposed a two parameter algorithm to choose some reasonable regularization parameters in an efficient manner. The basic tool is to use the well known Morozov discrepancy principle [8, 19] and the damped Morozov discrepancy principle [21].

We consider the inverse problem of the form

$$\begin{gathered}P : L^2(\Omega) \to L^2(\partial\Omega \times (0, T)), \\ Pf = \frac{\partial u}{\partial n} = g(x, t),\end{gathered}$$

where P is a linear bounded operator, g is the observation data and $\frac{\partial u}{\partial n}$ satisfies the equation (1.1). In applications, g is often corrupted by some error and the noise data of g with noise level δ are denoted by g^δ.

We rewrite the Tikhonov functional

$$\begin{gathered}\min_{f^\delta \in A} J_\beta(f^\delta) = \frac{1}{2}\int_0^T \int_{\partial\Omega} \left|Pf^\delta - g^\delta\right|^2 \,dx\,dt + \frac{\beta}{2}\int_\Omega \left|f^\delta\right|^2 \,dx, \\ A = \{f^\delta(x) : |f^\delta| \le a, f^\delta \in L^2(\Omega)\}.\end{gathered} \tag{4.1}$$

where f^δ is the corresponding regularization solution for g^δ. For fixed β, the problem (4.1) is solved by projected gradient method [31]. For this method, the derivative of J_β at an iterate f_n^δ is given by

$$J'_\beta(f_n^\delta)(h-f) = \int_\Omega \Big(-\int_0^T z_n\sigma(t)dt + \beta f_n^\delta\Big)(h-f)dx,$$

where z_n is the solution of the adjoint equation

$$\begin{gathered}z_{tt}(x,t) = \Delta z(x,t), \quad x \in \Omega,\ 0<t<T, \\ z(x,T) = z_t(x,T) = 0, \quad x \in \Omega, \\ z(x,t) = \frac{\partial u}{\partial n} - g(x,t), \quad x \in \partial\Omega,\ 0<t<T.\end{gathered} \tag{4.2}$$

By the Riesz representation theorem, we obtain the usual representation of the reduced gradient

$$w_n = J'_\beta(f_n^\delta) = -\int_0^T z_n(x,t)\sigma(t)dt + \beta f_n^\delta.$$

Set $f_{n+1}^\delta = \mathbb{P}_{[A]}\{f_n^\delta - sw_n\}$ for the iteration. where $\mathbb{P}$ denotes the projection onto A and s is optimal step size. The stopping criterion for the iteration is chosen as $\|f_{n+1}^\delta - f_n^\delta\|_{L^2(\Omega)} \le tol$.

The two equations (1.1) and (4.2) are solved by the implicit finite difference method [28]. They are discretized based on the difference approximation

$$\begin{gathered}u(x_i,y_j,t_k)_{tt} = \frac{u(x_i,y_j,t_{k+1}) - 2u(x_i,y_j,t_k) + u(x_i,y_j,t_{k-1})}{(\Delta t)^2}, \\ u(x_i,y_j,t_k)_{xx} = \frac{u(x_{i+1},y_j,t_k)^{1/4} - 2u(x_i,y_j,t_k)^{1/4} + u(x_{i-1},y_j,t_k)^{1/4}}{(\Delta x)^2}, \\ u(x_i,y_j,t_k)_{yy} = \frac{u(x_i,y_{j+1},t_k)^{1/4} - 2u(x_i,y_j,t_k)^{1/4} + u(x_i,y_{j-1},t_k)^{1/4}}{(\Delta y)^2}, \\ u(x_i,y_j,t_k)^{1/4} = \frac{1}{4}u(x_i,y_j,t_{k+1}) + \frac{1}{2}u(x_i,y_j,t_k) + \frac{1}{4}u(x_i,y_j,t_{k-1}),\end{gathered}$$

It is easy to check that all above approximation formulas are of second-order accuracy. The implicit schemes for (1.1) and (4.2) are obtained by approximating the derivatives using the above formulas.

The popular Morozov principle has received a considerable amount of attention in linear inverse problems and turns out to be very effective for many inverse problems. This principle suggests choosing the regularization parameter β in such a way that the error due to the regularization is equal to the error due to the observation data, that is, β is chosen according to

$$\int_0^T \int_{\partial\Omega} \left|Pf^\delta(\beta) - g^\delta\right|^2 \,dx\,dt + \beta^\gamma \int_\Omega \left|f^\delta(\beta)\right|^2 \,dx = \delta^2, \tag{4.3}$$

where $\gamma \in [1, \infty]$ and δ is the noise level defined by $\delta = \int_0^T \int_{\partial\Omega} |g - g^\delta|^2 \,dx\,dt$.

From [21], (4.1) has a unique minimizer for any fixed β, denoted as $f^\delta(\beta)$ and it can be characterized as the solution to the system

$$P^*Pf^\delta + \beta f^\delta = P^*g^\delta$$

or in variational form

$$(Pf^\delta, Pq)_{L^2(\partial\Omega\times(0,T))} + \beta(f^\delta, q)_{L^2(\Omega)} = (g^\delta, Pq)_{L^2(\partial\Omega\times(0,T))} \quad \text{for all } q \in L^2(\Omega).$$

It is obvious that the convergence rate of the damped Morozov discrepancy principle is quite important for the application of this strategy. For the linear operator P, this result can be stated as follows.

Lemma 4.1 ([32]). *Let $Pf = g$ with noisy data g^δ such that $\|g - g^\delta\| \le \delta < \|g^\delta\|$. Let the Tikhonov solution f^δ satisfy the damped Morozov discrepancy principle* (4.3). *Assume that there exists $w \in L^2(\partial\Omega \times (0,T))$ such that $f = P^*w \in P^*(L^2(\partial\Omega \times (0,T)))$. Then*

$$\|f^\delta - f\|_{L^2(\Omega)} = O(\delta^{\min\{1/2, 2(\gamma-1)/\gamma\}}).$$

We frequently use the minimal cost functional of (4.1)

$$F(\beta) = \frac{1}{2}\int_0^T \int_{\partial\Omega} |Pf^\delta(\beta) - g^\delta|^2 \,dx\,dt + \frac{\beta}{2}\int_\Omega |f^\delta(\beta)|^2 \,dx. \tag{4.4}$$

It is known that both $f^\delta(\beta)$ and $F(\beta)$ are infinitely differentiable with respect to β. Moreover we have

$$F'(\beta) = \frac{1}{2}\int_\Omega |f^\delta(\beta)|^2 \,dx. \tag{4.5}$$

In terms of $F(\beta)$, the Morozov equation (4.3) can be written as

$$F(\beta) + (\beta^\gamma - \beta)F'(\beta) = \frac{1}{2}\delta^2. \tag{4.6}$$

Then the entire difficulty of choosing the regularization parameter β lies in solving the highly nonlinear equation (4.6) for β effectively.

Lemma 4.2 ([21]). *If $F(0) < \frac{1}{2}\delta^2 \le F(1)$, then there exists a unique solution $\beta^* \in (0,1]$ to the Morozov equation* (4.6).

To solve (4.6), we use model function approach. By a model function we mean a parametrized function which presserves the major properties of the non-negative function $F(\beta)$ and which approximates $F(\beta)$ in a manner to be specified below.

From [21] the two-parameter model function algorithm is based on the important identity

$$2F(\beta) + 2\beta F'(\beta) + \int_0^T \int_{\partial\Omega} |Pf^\delta(\beta)|^2 \,dx\,dt = 2\hat{C}, \tag{4.7}$$

where $\hat{C}$ is an integration constant. To derive the model function, we make the following approximation in the equation (4.7).

$$(Pf^\delta(\beta), Pq(\beta))_{L^2(\partial\Omega\times(0,T))} \approx \tilde{P}(f^\delta(\beta), q(\beta))_{L^2(\Omega)} \tag{4.8}$$

where $\tilde{P}$ is a positive constant. Then equation (4.7) reduces to

$$\beta m'(\beta) + m(\beta) + \tilde{P} m'(\beta) = \hat{C} \tag{4.9}$$

Solving the ordinary differential equation (4.9) we obtain

$$m(\beta) = \hat{C} + \frac{\tilde{C}}{\tilde{P} + \beta}.$$

where $\tilde{C}$ is an integration constant. Then, by assuming $F(0) = 0$ or $m(0) = 0$, one can remove the constant $\hat{C}$ and arrive at the two-parameter model function

$$m(\beta) = \bar{C} + \Big(1 - \frac{\bar{P}}{\bar{P} + \beta}\Big). \tag{4.10}$$

With this model function, the two-parameter algorithm is used to solve the Morozov equation (4.6).

Based on the analysis above, the procedure of the iteration can be stated as follows: Given $\beta_0 > 0$ and $\epsilon > 0$, set $k = 0$.

Step 1: Choose an initial value of iteration $f^\delta = f_0^\delta(x)$.

Step 2: Solve the optimal control problem (4.1) to obtain $f^\delta(\beta_k)$ and compute $F(\beta_k)$ and $F'(\beta_k)$. Then update $\bar{C}_k$ and $\bar{P}_k$ from

$$m(\beta_k) = \bar{C}_k + \Big(1 - \frac{\bar{P}_k}{\bar{P}_k + \beta_k}\Big) = F(\beta_k), \tag{4.11}$$

$$m'(\beta_k) = \frac{\bar{C}_k \bar{P}_k}{(\bar{P}_k + \beta_k)^2} = F'(\beta_k). \tag{4.12}$$

Step 3: Set the kth model function

$$m(\beta) = \bar{C}_k + \Big(1 - \frac{\bar{P}_k}{\bar{P}_k + \beta}\Big)$$

and solve for β_{k+1} the approximate Morozov's equation

$$m(\beta) + (\beta^\gamma - \beta) m'(\beta) = \frac{1}{2}\delta^2. \tag{4.13}$$

Step 4: Compare it with ϵ. If $\|\beta_{k+1} - \beta_k\| < \epsilon$, then stop the iteration; otherwise set $k = k + 1$ and go to step 1.

We have performed two numerical experiments to test the stability of our algorithm for different noise levels and initial data. The stopping criterion for the two-parameter iteration is chosen as $|\beta_{k+1} - \beta_k|/\beta_{k+1} \le 10^{-2}$. In all experiments, some basic parameters are $T = 1, \delta(t) = \cos t, s = 1$ and $\gamma = 1.4$. We apply the noise data generated in the form

$$g^\delta = g(1 + \hat{\delta} \times \text{random}(0,1)).$$

where $\hat{\delta}$ is a noise level.

In the first numerical experiment, we consider one dimensional problem ($N = 1$).

Example 4.3. Let $f(x) = \sin \pi x, x \in (0, 1)$. The exact solution of the forward problem for this $f(x)$ is

$$u(x,t) = \frac{1}{1 - \pi^2} \sin \pi x \left(\cos \pi t - \cos t\right), \quad (x,t) \in [0,1] \times [0,1],$$

$$\frac{\partial u}{\partial n} = -\frac{\pi}{1-\pi^2}\left(\cos \pi t - \cos t\right), \quad \text{at } x = 0, 1.$$

The source term $f(x)$ is to be recovered from the noise observation data g^δ. In our implementations, the mesh size and time step size are $\Delta x = \Delta t = 1/50$. The tolerance of the optimal control problem is taken as $tol = 10^{-4}$.

Table 1. β value, the errors in observation and source for the initial value $f_0^\delta = 0$.

$\hat{\delta}$	β	$\Vert g - g^\delta\Vert_{L^2(\partial\Omega\times(0,T))}$	$\Vert f - f^\delta\Vert_{L^2(\Omega)}$	iter(β)
0.01	0.000106318	0.00145371	0.0156492	6
0.05	0.00121214	0.00756438	0.0271989	5
0.1	0.00147077	0.0112825	0.0486434	8

Table 2. β value, the errors in observation and source for the initial value $f_0^\delta = -1$.

$\hat{\delta}$	β	$\Vert g - g^\delta\Vert_{L^2(\partial\Omega\times(0,T))}$	$\Vert f - f^\delta\Vert_{L^2(\Omega)}$	iter(β)
0.01	0.000110693	0.00147976	0.0113912	6
0.05	0.000819191	0.00597265	0.0234172	6
0.1	0.00308773	0.014366	0.0402498	5

Table 3. β value, the errors in observation and source for the initial value $f_0^\delta = x(1-x)$.

$\hat{\delta}$	β	$\Vert g - g^\delta\Vert_{L^2(\partial\Omega\times(0,T))}$	$\Vert f - f^\delta\Vert_{L^2(\Omega)}$	iter(β)
0.01	0.000112646	0.00148598	0.0126763	6
0.05	0.00140652	0.00827753	0.0213082	5
0.1	0.00202417	0.0123668	0.0377899	6

In Tables 1–3, we present some numerical results of Example 4.3 with different noise levels $\hat{\delta}$, different initial value of $f^\delta = f_0^\delta$ and $\beta_0 = 0.1$. The regularization parameter β obtained by two-parameter algorithm is given in the second column. The third and forth columns of the tables give the errors in observation data g and errors in computed source term respectively. The last column shows the number of iterations of the two-parameter algorithm.

Figure 1 shows the plot of the approximation of the unknown source function $f(x)$ for different noise levels $\hat{\delta}$ and the initial guesses $f_0 = 0$. From this, we can see that the efficiency of reconstruction of source term depends on the noise level. Figure 2 shows the plot of the approximation of the unknown source function $f(x)$ for different noise levels $\hat{\delta}$ and the initial guesses $f_0 = -1$. From this, we can see that the approximation of $f(x)$ converges even when the initial guess is negative.

Figure 3 shows the plot of the approximation of the unknown source function $f(x)$ for different noise levels $\hat{\delta}$ and the initial guesses $f_0^\delta = x(1-x)$. The initial guesses are similar in characteristics to the known source. In the second numerical experiment, we consider a two dimensional problem ($N = 2$).

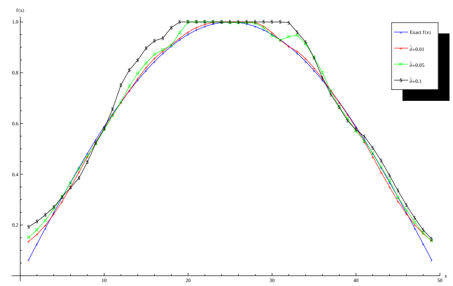

FIGURE 1. Exact and computed source term for different $\hat{\delta}$ and $f_0^\delta = 0$ in 1-D wave equation.

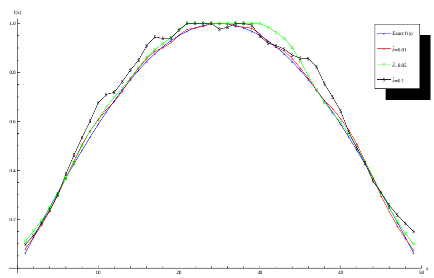

FIGURE 2. Exact and computed source term for different $\hat{\delta}$ and $f_0^\delta = -1$ in 1-D wave equation

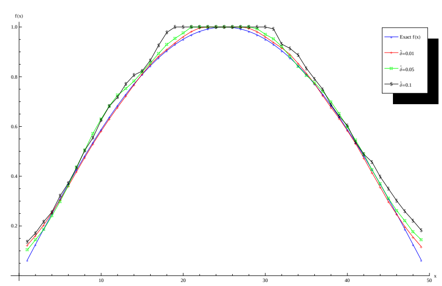

FIGURE 3. Exact and computed source term for different $\hat{\delta}$ and $f_0^\delta = x(1-x)$ in 1-D wave equation

Example 4.4. Let $f(x,y) = \sin \pi x \sin \pi y, (x,y) \in (0,1) \times (0,1)$. The exact solution is

$$\begin{gathered}
u(x,y,t) = \frac{1}{1-2\pi^2} \sin \pi x \sin \pi y \big(\cos \pi \sqrt{2} t - \cos t \big), \\
\frac{\partial u}{\partial n} = -\frac{\pi}{1-2\pi^2} \sin \pi y \big(\cos \sqrt{2} \pi t - \cos t \big), \quad \text{on } x = 0,1, \\
\frac{\partial u}{\partial n} = -\frac{\pi}{1-2\pi^2} \sin \pi x \big(\cos \sqrt{2} \pi t - \cos t \big), \quad \text{on } y = 0,1.
\end{gathered}$$

The source term $f(x, y)$ is to be recovered from the noise observation data g^δ. In two dimensional case, the mesh sizes and time step size are $\Delta x = \Delta y = \Delta t = \frac{1}{10}$. The tolerance of the optimal control problem is taken as $tol = 10^{-4}$.

TABLE 4. β value, the errors in observation and source for the initial value $f_0^\delta = 0$.

$\hat{\delta}$	β	$\lVert g - g^\delta \rVert_{L^2(\partial\Omega\times(0,T))}$	$\lVert f - f^\delta \rVert_{L^2(\Omega)}$	iter(β)
0.05	0.00120084	0.00846277	0.020988	10
0.07	0.00260834	0.012032	0.0224599	7
0.1	0.0051492	0.017512	0.026386	5

TABLE 5. β value, the errors in observation and source for the initial value $f_0^\delta = -1$.

$\hat{\delta}$	β	$\lVert g - g^\delta \rVert_{L^2(\partial\Omega\times(0,T))}$	$\lVert f - f^\delta \rVert_{L^2(\Omega)}$	iter(β)
0.05	0.00118572	0.00816773	0.0235247	10
0.07	0.00272184	0.0120211	0.0245797	7
0.1	0.00505242	0.0172179	0.0251407	5

TABLE 6. β value, the errors in observation and source for the initial value $f_0^\delta = xy(1-x)(1-y)$.

$\hat{\delta}$	β	$\lVert g - g^\delta \rVert_{L^2(\partial\Omega\times(0,T))}$	$\lVert f - f^\delta \rVert_{L^2(\Omega)}$	iter(β)
0.05	0.00103108	0.00820616	0.0199434	12
0.07	0.00236611	0.0114796	0.0235887	7
0.1	0.00490916	0.0168844	0.0274358	5

In Tables 4–6, we present some numerical results of two dimensional equation as in Example 4.4 with different noise levels $\hat{\delta}$, different initial values of $f^\delta = f_0^\delta$ and $\beta_0 = 0.1$. From the result, we see that the source term f is recovered from the noise observation data g^δ stably by the different initial values.

Figures 4–6 we draw the computed source term $f(x, y)$ for the noise level $\hat{\delta} = 0.1$ and the different initial value of f^δ.

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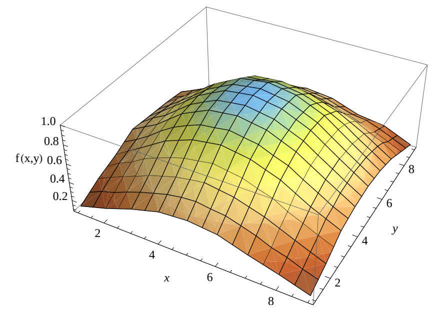

FIGURE 4. Computed source term in 2-D wave equation for $f_0^\delta = 0$ and $\hat{\delta} = 0.1$

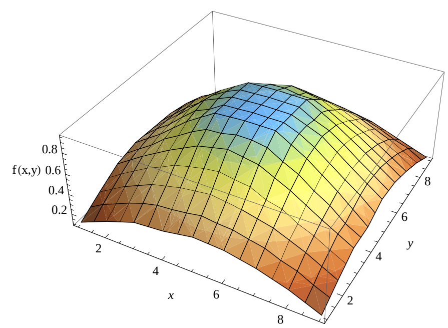

FIGURE 5. Computed source term in 2-D wave equation for $f_0^\delta = -1$ and $\hat{\delta} = 0.1$

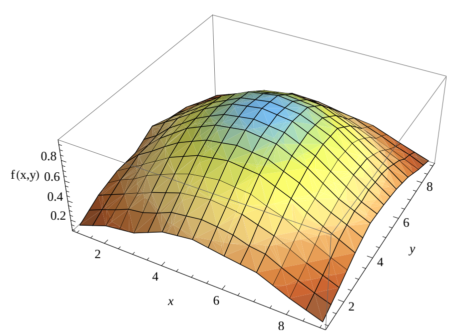

FIGURE 6. Computed source term in 2-D wave equation for $f_0^\delta = xy(1-x)(1-y)$ and $\hat{\delta} = 0.1$

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