CENTRO INTERNACIONAL DE ESTUDOS DE DOUTORAMENTO E AVANZADOS DA USC (CIEDUS)

Tese de Doutoramento

Higman-Neumann-Neumann extension and embedding theorems for Leibniz algebras

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ESCOLA DE DOUTORAMENTO INTERNACIONAL

PROGRAMA DE DOUTORAMENTO EN MATEMÁTICAS

SANTIAGO DE COMPOSTELA

2018

UNIVERSIDADE DE SANTIAGO DE COMPOSTELA

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Higman-Neumann-Neumann extension and embedding theorems for Leibniz algebras

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DEPARTAMENTO DE MATEMÁTICAS

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by

CHIA ZARGEH

DISSERTATION

Submitted for the degree of

DOCTOR EN MATEMÁTICAS

UNIVERSIDAD DE SANTIAGO DE COMPOSTELA

Santiago de Compostela, 2018

Acknowledgements

My Ph.D. study could not be accomplished without the meticulous guidance, generous assistance and constant support from my advisor, Prof. Manuel Ladra and my co-advisor, Prof. Mohammad Shahryari. Their unconditional support and kind consideration were an inspiration to me.

I would like to stress the help over the years with the research underlying this thesis and personal support provided by Prof. Manuel Ladra, which was invaluable to finish this work.

I would also like to appreciate Professor Mohammad Shahryari, for useful discussions and insightful comments. His words of encouragement and help with the proofs and the underlying concepts was essential during working on the thesis.

I thank Xabier García-Martínez, Rustam Turdibaev and María Pilar Páez-Guillán for making friendly atmosphere and their help and support.

Particular thanks go to the Mathematics Department at the University of Santiago de Compostela for providing a great environment to work in. The support from Mathematics Department's staff has been priceless.

Very warm thanks go to my family and my wife for supporting me during the time of my studies. Furthermore, I would like to thank my friends, Asef Hemmati and Morad Ebrahimkhas for encouragement.

Contents

Introduction

Aims and background

Two of the most important constructions in combinatorial group theory are the coproduct with amalgamation (pushout or amalgamated free product) and the construction introduced by Higman, Neumann and Neumann [18] and called the HNN-extension; these two constructions are closely related to each other (see [32]).

If A is a subgroup of a group G and $t \in G$, then the mapping $a \mapsto t^{-1}at$ is an isomorphism between the two subgroups A and $t^{-1}At$ of G. The HNN construction tries to reverse the viewpoint. For a group G with an isomorphism ϕ between two of its subgroups A and B, H is an extension of G with an element $t \in H$ such that $t^{-1}at = \phi(a)$ for every $a \in A$. The group H is presented by

$$H = \langle G, t \mid t^{-1}at = \phi(a), \text{ for all } a \in A \rangle$$

and it implies that G is embedded in H. The HNN-extension of a group possesses an important position in algorithmic group theory which has been used for the proof of the embedding theorem, namely, that every countable group is embeddable into a group with two generators.

The HNN-extension of a group has a topological interpretation described in [9, 32], which is used as a motivation for its study. Suppose that U and V are both subspaces of an arcwise connected topological space X such that there is a homeomorphism $h\colon U \to V$. Choose a basepoint $u \in U$ for the fundamental groups of U and X. There is a homomorphism $\eta\colon \pi_1(U) \to \pi_1(X)$, defined by considering a loop in U

as a loop in X, which is injective. The homeomorphism h induces an isomorphism $h^*\colon \pi_1(U) \to \pi_1(V)$. Suppose we identify U and V by the homeomorphism h to obtain a new space Z. Under these assumptions the Seifert-van Kampen Theorem says

$$\pi_1(Z) = \langle \pi_1(X) * \pi_1(Y) \mid \pi_1(U) = \pi_1(V),\ h^* \rangle.$$

The HNN-extension of a group has a similar topological interpretation. Let I be the unit interval, and $C = U \times I$. Identify $U \times \{0\}$ with U and $U \times I$ with V by the homeomorphism h. Let Z be the resulting space. The Seifert-van Kampen Theorem can be used to show that

$$\pi_1(Z) = \langle \pi_1(X), t \mid t^{-1}\pi_1(U)t = \pi_1(V) \rangle.$$

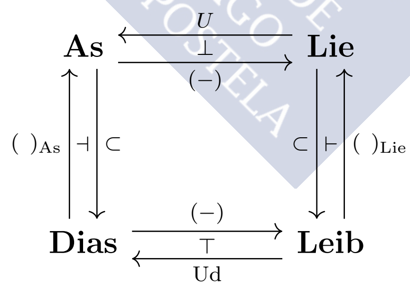
The idea of the construction of HNN-extension has been spread in several ways to other algebraic structures such as semigroups, rings and Lie algebras which are areas of independent importance.

HNN-extension of semigroups was introduced in the works of Ash [1], Jackson's PhD thesis [21] and Howie [19]. For instance, Howie used a similar approach to the case of groups, which depends on the properties of the amalgamated free product of semigroups. He constructed HNN-extension for a semigroup S with unitary subsemigroups S_1 and S_2. This concept was also generalized to other classes of semigroups such as inverse semigroups by Gilbert [17] and Yamamura [40].

The HNN-extension for associative rings was defined by Dicks [15] and Lichtman and Shirvani [25]. Dicks presented the concept of HNN-extension of rings based on the analogy with the notion of coproduct. The approach of Lichtman and Shirvani to define differential HNN-extension of associative rings lies in a more general context. They considered an associative ring R and a family of subrings of R, together with injective homomorphisms and derivations. The construction of differential HNN-extensions of rings is in conformity with the construction of this concept for the universal enveloping algebras of Lie algebras.

HNN-extension of Lie algebras was constructed by Lichtman and Shirvani [25] and Wasserman [39]. Lichtman and Shirvani defined HNN-extension for restricted Lie algebras and applied it to give a new proof of Shirshov's theorem [35], namely, a Lie algebra of finite or countable dimension can be embedded into a 2-generator Lie algebra. They also gave a proof for the fact that every Lie algebra can be embedded into a simple Lie algebra. Wasserman followed a different approach based on Shirshov's composition lemma which was the method that had been described in [9] until the time of his work: to construct HNN-extension for Lie algebras and use it to obtain some analogous embedding results to group theory similar to Lichtman and Shirvani's theorems. In addition, Wasserman proved that Markov properties of finitely presented Lie algebras are undecidable.

Leibniz algebras are a non-antisymmetric generalization of the Lie algebras introduced by Bloh [4] and Loday [27]. The theory of Leibniz algebras has been actively investigated and many results of the theory of Lie algebras have been transferred to Leibniz algebras. Additionally, Loday [29] introduced the concept of an associative dialgebra (or dialgebra, for short); the generalization of the Lie bracket produces Lie dialgebras, also called Leibniz algebras. In fact, the dialgebras are closely connected to the notion of Leibniz algebras in the same way as the associative algebras are connected to Lie algebras. The following diagram which was described in [14] illustrates an adjunction between the functor that allocates to dialgebra D the Leibniz algebra given by $[x, y] = x \dashv y - y \vdash x$, for all $x, y \in D$, and the universal enveloping dialgebra functor,



where the inner square is commutative and the outer square commutes up to isomorphism. Our approach to construct HNN-extension for Leibniz algebras is related to the construction of HNN-extension for dialgebras. In order to define HNN-extension for dialgebras, we employ

Groebner-Shirshov bases theory and Composition-Diamond Lemma as effective tools to determine Groebner-Shirshov basis for the recently defined presentation of HNN-extension.

As an explanation of Groebner-Shirshov bases, we recall that the idea of the formulation of Groebner-Shirshov bases is rooted in the Euclidean algorithm for polynomials in one variable, and in the Gauss Elimination algorithm for linear (in degree 1) polynomials in several variables (see [11]).

For the case of free Lie algebras, the method of Groebner-Shirshov bases was developed by Shirshov [36] in 1958 when he introduced a new basis known as Lyndon-Shirshov basis. In addition, there exists a parallel concept to the Groebner-Shirshov bases in commutative algebra called Groebner bases and developed by Buchberger in 1965 (see [12,13]). Groebner-Shirshov bases theory is applied to algebras which are presented as free algebras modulo some relations and an algorithmic procedure is used to reduce a general element to its normal form [10]. In each relation of a given free algebra we can determine a leading monomial with respect to a monomial ordering and then replace each leading monomial by the corresponding lower terms. Then we say that the element is in normal form or it is a reduced element. Shirshov in [37] introduced the notion of *composition* of polynomials in a free Lie algebra and Bergman [3] used the notion of *ambiguity* instead of *composition* for the case of associative algebras. If the composition of two relations is not an immediate consequence of the original relations, the composition will be added to the set of relations. If all compositions are immediate results, the relation set is called a Groebner-Shirshov basis.

Composition-Diamond Lemma is a powerful theorem in algorithmic and combinatorial algebra which provides linear bases and normal forms of the elements for an algebra presented by generators and defining relations. In fact, Composition-Diamond Lemma states that if S is a Groebner-Shirshov basis, then the leading term of any polynomial in an ideal I of the free algebra $F(X)$ generated by S contains $\bar{s}$ as a subword where $\bar{s}$ is the leading term of $s \in S$. The main consequence is that the normal form of the elements in $F(X)/\operatorname{Id}(S)$ consists of $\mathbb{K}$-linear combinations of words without any subword $\bar{s}$. Bokut in [6] reformulated Composition-Diamond Lemma for the case of Lie algebras in a modern way. In the last two decades, the Groebner-Shirshov bases theory has been defined for different classes of algebras [7] such as dialgebras by Bokut, Chen and Liu [8] or replicated algebras [23].

The validity of Poincaré-Birkhoff-Witt theorem for the case of Leibniz algebras (see [2], [22] and [29]) justifies the relation between the construction of HNN-extension for Leibniz algebras and the HNN-extension for dialgebras. Therefore, using the functor of universal enveloping dialgebra we transfer our results to the case of Leibniz algebras.

Outline of the thesis

In Chapter 1, we provide essential definitions and theorems on free Leibniz algebras and free dialgebras with some examples. Section 1.3 is devoted to the concept of the universal enveloping dialgebra of Leibniz algebras which will be applied to construct HNN-extension of Leibniz algebras in Section 3.4. As it was mentioned in the previous section, due to the importance of connection of dialgebras and Leibniz algebras and the relationship between associative algebras and Lie algebras, in Chapter 2, Section 2.1, we recall the theory of Groebner-Shirshov bases, and in Section 2.2, the Composition-Diamond Lemma (for associative algebras in Subsection 2.2.1 and for Lie algebras in Subsection 2.2.2). The last subsection of Section 2.2 contains Composition-Diamond Lemma for dialgebras as our main tool to introduce the HNN-extension for dialgebras. Chapter 3 is devoted to the HNN-extension constructions. Firstly, we remind the reader the HNN-extension for groups in Section 3.1 and recall the case of Lie algebras in Section 3.2. Then we introduce the HNN-extension for dialgebras and Leibniz algebras in Section 3.3 and Section 3.4, respectively. The concept of HNN-extension for dialgebras is constructed by defining a new presentation on a generating set X with a set of relations S. The complete computations of the possible compositions of the dialgebra polynomials of newly defined HNN-extension is provided in Subsection 3.3.1. As an application of HNN-extension of Leibniz algebras, we provide an embedding theorem for Leibniz algebras in the last Subsection 3.4.1 of this thesis.

Chapter 1

Preliminaries

We recall the necessary background in order to make the comprehensive thesis. Since all of the results are well known, we shall omit some proofs and instead provide references where interested readers can find the details. We state helpful facts and give some examples.

1.1 Leibniz algebras

Definition 1.1.1. A *(right) Leibniz algebra* (or *Leibniz algebra*) L is a vector space over $\mathbb{K}$ equipped with a bilinear map (multiplication)

$$[-,-]\colon L \times L \to L$$

satisfying the Leibniz identity

$$[[a,b],c] = [[a,c],b] + [a,[b,c]], \tag{1.1.1}$$

for all $a,b,c \in L$.

For $a \in L$, the left and right multiplication operators $L_a\colon L \to L$, $R_a\colon L \to L$ are defined as $L_a(b) = [a,b]$ and $R_a(b) = [b,a]$, respectively.

Note that the operator R_a is a derivation but L_a is not a derivation. A right Leibniz algebra is defined as a vector space with a bilinear multiplication such that the right multiplication is s derivation. The following example shows a (left) Leibniz algebra is not necessarily a (right) Leibniz algebra.

Example 1.1.2. Let L be a 2-dimensional algebra with the following multiplications.

$$[x, x] = 0, \quad [x, y] = 0, \quad [y, x] = x, \quad [y, y] = x.$$

L is a (left) Leibniz algebra, but it is not a (right) Leibniz algebra, since $[[y, y], y] \neq [y, [y, y]] + [[y, y], y]$.

Any Lie algebra is a Leibniz algebra. A Leibniz algebra satisfying the condition $[a, a] = a^2 = 0$ for all $a \in L$ is a Lie algebra and the Leibniz identity becomes the Jacobi identity. The linear subspace generated by the subset $[a, a]$ is left and right ideal denoted by $\mathrm{Ker}(L)$.

Remark 1.1.3. Quotiening the Leibniz algebra L by the ideal generated by the elements $[x, x]$ for all $x \in L$, i.e. the factor algebra $L/\mathrm{Ker}(L)$, gives a non-zero algebra which is a Lie algebra that we denote by L_{Lie}. In some literature $L_{\mathrm{Lie}} = L/\mathrm{Ker}(L)$ is called the "Liezation" of L.

The Leibniz identity (1.1.1) allows us to reduce every commutator to a linear combination of left normed commutators:

$$[x, y] = x(\mathrm{ad}\, y), \qquad [x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

The following proposition summarizes the description of the low-dimensional Leibniz algebras given in [27].

Proposition 1.1.4 ([16]). *Let L be a finite-dimensional Leibniz algebra.*

(1) If $\dim L = 1$, then L is an abelian Lie algebra.

(2) If $\dim L = 2$, then L is isomorphic to one of the following algebras:

(i) one of the two two-dimensional Lie algebras; the abelian Lie algebra and the nonabelian Lie algebra defined by the multiplication $[b, a] = -[a, b] = b$.

(ii) The algebra with basis $\{a, [a, a]\}$ *and with defining relations* $[a, a, a] = 0$. *It is nilpotent of class two, i.e. satisfying the identity* $[x, y, z] = 0$.

(iii) The algebra generated by a, b *and with defining relations* $[b, a] = [a, a] = b$*;* $\mathbb{K}b$ *is an ideal in* L *and* L *satisfies the identity of left nilpotency* $[x, [y, z]] = 0$.

Proof. The one-dimensional case is trivial because if $[a, a] = \alpha a$ for some $\alpha \in \mathbb{K}$, then $0 = [a, [a, a]] = \alpha^2 a$ and $\alpha = 0$. Let $\dim L = 2$ and let $\{a, b\}$ be a basis for L. If $[c, c] = 0$ for all $c \in L$, then L is a Lie algebra and this is the case (i). Let $[a, a] \neq 0$. Since $[a, a]$ cannot be equal to a, we can choose the basis $\{a, b\}$ in such a way that $[a, a] = b$. Repeated use of the Leibniz identity yields $[a, a] = b$, $[b, a] = \beta b$ for some $\beta \in \mathbb{K}$ and $[L, b] = 0$. The case $\beta = 0$ gives the algebra from (ii). On the other hand, replacing a with $\beta^{-1}a$, (iii) will be obtained. □

In the sequence we recall several important notions related to free Leibniz algebras with examples. These notions and examples have been provided by Loday and Pirashvili [30].

Definition 1.1.5 ([30]). The Leibniz algebra $Leib(X)$ is called *free Leibniz algebra* with a set of generators X if, for any Leibniz algebra L, an arbitrary map $X \to L$ can be extended to an algebra homomorphism $Leib(X) \to L$. Then X is called the set of free generators of $Leib(X)$.

The free Leibniz algebra $Leib(V)$ for the vector space V can be constructed as a quotient of the free non-associative algebra over V as in the following lemma.

Lemma 1.1.6 ([30]). *The tensor module* $T(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \cdots$ *equipped with the bracket defined inductively by*

- $[x, v] = x \otimes v$, *for* $x \in T(V)$ *and* $v \in V$,
- $[x, y \otimes v] = [x, y] \otimes v - [x \otimes v, y]$, *for* $x, y \in T(V)$, $v \in V$,

is the free Leibniz algebra over V.

Remark 1.1.7 ([30]). Let V be a one-dimensional vector space over field $\mathbb{K}$ generated by x, then $T(V) = \mathbb{K}x \oplus \mathbb{K}x^2 \oplus \cdots \oplus \mathbb{K}x^n \oplus \cdots$ and the Leibniz structure is given by

$$[x^i, x^j] = \begin{cases} x^{i+1}, & \text{if } j = 1, \\ 0, & \text{if } j \geq 2. \end{cases}$$

For any V the Lie algebra associated to $Leib(V)$ is the free Lie algebra $Lie(V)$, which can be identified with the primitive part of the tensor Hopf algebra $T(V) = \mathbb{K} \oplus \bar{T}(V)$.

Example 1.1.8 ([30]). Let A be an associative algebra over $\mathbb{K}$ equipped with a $\mathbb{K}$-module map $T \colon A \to A$, such that $T(ab) = (Ta)b + a(Tb)$ and $T^2(a) = 0$ satisfying the condition

$$T(a(Tb)) = TaTb = T((Ta)b) \text{ for any } a, b \in A.$$

Define a bilinear map on A by

$$[x, y] = x(Ty) - (Ty)x.$$

Then this bracket satisfies the Leibniz identity (1.1.1), and so A becomes a Leibniz algebra. Note that if $T = \mathrm{id}$ then it is a Lie algebra.

Example 1.1.9 ([30])**.** Let A be an associative algebra with unit over $\mathbb{K}$. For $n \geq 3$ the noncommutative Steinberg algebra $\mathfrak{stl}_n(A)$ is the Leibniz algebra defined by generators $v_{ij}(a)$, $a \in A$, $1 \leq i \neq j \leq n$, subject to the relations

$$v_{ij}(\lambda a + \mu b) = \lambda v_{ij}(a) + \mu v_{ij}(b), \text{ for } \lambda, \mu \in \mathbb{K} \text{ and } a, b \in A,$$

$$[v_{ij}(a), v_{ml}(b)] = \begin{cases} 0, & \text{if } i \neq l \text{ and } j \neq m, \\ -v_{mj}(ba), & \text{if } i = 1 \text{ and } j \neq m, \\ v_{ii}(ab), & \text{if } i \neq l \text{ and } j = m. \end{cases}$$

Definition 1.1.10. A *derivation of Leibniz algebras* is defined in a similar way to the derivation of Lie algebras, that is, a linear map $d\colon L \to L$ satisfying

$$d([x, y]) = [d(x), y] + [x, d(y)],$$

for all $x, y \in L$.

Definition 1.1.11 ([14])**.** Let P and M be two Leibniz algebras. An *action of P on M* consists of a pair of bilinear maps, $P \times M \to M$, $(p, m) \mapsto [p, m]$ and $M \times P \to M$, $(m, p) \mapsto [m, p]$, such that

(1) $[p, [m_1, m_2]] = [[p, m_1], m_2] - [[p, m_2], m_1]$,

(2) $[m_1, [p, m_2]] = [[m_1, p], m_2] - [[m_1, m_2], p]$,

(3) $[m_1, [m_2, p]] = [[m_1, m_2], p] - [[m_1, p], m_2]$,

(4) $[m, [p_1, p_2]] = [[m, p_1], p_2] - [[m, p_2], p_1]$,

(5) $[p_1, [m, p_2]] = [[p_1, m], p_2] - [[p_1, p_2], m]$,

(6) $[p_1, [p_2, m]] = [[p_1, p_2], m] - [[p_1, m], p_2]$,

for all $m, m_1, m_2 \in M$ and $p, p_1, p_2 \in P$.

Note that, as an immediate consequence of (2) and (3),

$$[m_1, [p, m_2]] + [m_1, [m_2, p]] = 0$$

for all $m_1, m_2 \in M$ and $p \in P$. On the other hand, from (5) and (6),

$$[p_1, [m, p_2]] + [p_1, [p_2, m]] = 0$$

for all $m \in M$ and $p_1, p_2 \in P$.

Observe that we denote the action by the same symbol used for the multiplication in M and P. Note that the six identities in the definition of a Leibniz action can be obtained from the Leibniz identity by taking two elements in P and one in M (three identities), and two elements in M and one in P (the other three identities).

For example, if P is a Leibniz subalgebra of some Leibniz algebra Q, and if M is an ideal in Q, then the Leibniz bracket in Q yields an action of P on M.

Definition 1.1.12. Given a Leibniz action of P on M the *semidirect product Leibniz algebra* is denoted by $M \rtimes P$ and consists of the $\mathbb{K}$-vector space $M \oplus P$ together with the Leibniz bracket given by

$$[(m_1, p_1), (m_2, p_2)] = ([m_1, m_2] + [p_1, m_2] + [m_1, p_2], [p_1, p_2])$$

for all $(m_1, p_1),\ (m_2, p_2) \in M \oplus P$.

1.2 Dialgebras

In this section we recall preliminary definitions for dialgebras. The concept of an associative dialgebra was introduced by Loday (see [28, 29]); the generalization of the Lie bracket produces Leibniz algebras.

An associative dimonoid (or simply a dimonoid) is a set X equipped with two associative products called left and right product, respectively

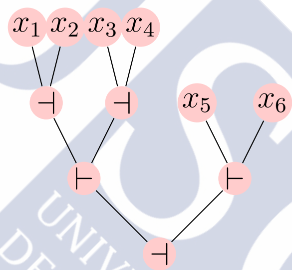
$$\dashv : X \times X \to X \quad \text{and} \quad \vdash : X \times X \to X,$$

with the following relations which are referred to as diassociativity:

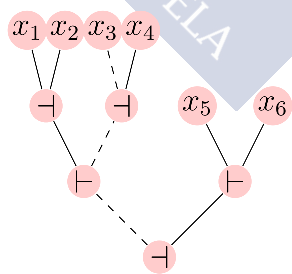
$$x \dashv (y \dashv z) = x \dashv (y \vdash z) \quad \text{and} \quad (x \dashv y) \vdash z = (x \vdash y) \vdash z.$$

In the notation $x \dashv y$, $y \vdash x$, the element x is said to be on the *pointer* side and the element y is said to be on the *bar* side.

A monomial in dimonoid X is a parenthesizing together with product signs, for example $((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6)$. Any such word can be represented by a rooted planar tree whose nodes are indexed by one of the symbols $\vdash$ or $\dashv$. In the case of the above word the corresponding tree is the following:



If we move from the root toward leafs and in any node following the directions indicated by the pointer, then we arrive a leaf x_j which is called the *center* element of the monomial. In our example the center element is x_3:



The next theorem provides useful calculations for monomials (see [29]).

Theorem 1.2.1. *Let $x_i : -n \leq i \leq m$ be elements in a dimonoid X. Then*

(a) Any parenthesizing of $x_{-n} \vdash \cdots \vdash x_0 \dashv \cdots \dashv x_m$ gives the same element in X which we denote it by $x_{-n} \dots \dot{x}_0 \dots x_m$.

(b) Let $m = x_1 \dots x_k$ be a monomial in X with center element x_i. Then

$$m = x_1 \dots \dot{x}_i \dots x_k.$$

(c) The following formulas is valid in D:

$$(x_1 \dots \dot{x}_i \dots x_k) \dashv (x_{k+1} \dots \dot{x}_j \dots x_l) = x_1 \dots \dot{x}_i \dots x_k x_{k+1} \dots x_j \dots x_l,$$

$$(x_1 \dots \dot{x}_i \dots x_k) \vdash (x_{k+1} \dots \dot{x}_j \dots x_l) = x_1 \dots x_i \dots x_k x_{k+1} \dots \dot{x}_j \dots x_l.$$

For instance, we have

$$((x_1 \dashv x_2) \vdash (x_3 \dashv x_4)) \dashv (x_5 \vdash x_6) = x_1 x_2 \dot{x}_3 x_4 x_5 x_6.$$

In the sequel, we fix a field $\mathbb{K}$ and for a nonempty set X, we denote by X^* and $[X^*]$ the free monoid and free dimonoid on X. Now we recall the definition of dialgebras with respect to [29].

Definition 1.2.2. A *diassociative algebra* (or *dialgebra*) is a $\mathbb{K}$-linear space, equipped with two $\mathbb{K}$-linear maps $\dashv, \vdash : D \times D \to D$, called respectively the left product and the right product such that the products $\dashv$ and $\vdash$ are associative and satisfy the following properties:

1. $x \dashv (y \vdash z) = x \dashv (y \dashv z)$,
2. $(x \dashv y) \vdash z = x \vdash (y \vdash z)$,
3. $x \vdash (y \dashv z) = (x \vdash y) \dashv z$.

The dialgebra D is called *abelian* if both the left and the right products are trivial, that is $x \dashv y = x \vdash y = 0$ for all $x, y \in D$.

A diassociative algebra structure on an n-dimensional vector space V with a basis $\{e_1, \ldots, e_n\}$ can be given by defining the products of the basis vectors.

Example 1.2.3. The products

$$e_1 \vdash e_1 = e_2, e_1 \vdash e_2 = e_2, e_1 \dashv e_1 = e_1, e_2 \dashv e_1 = e_2$$

on a two-dimensional vector space define a diassociative algebra structure.

The next example defines a dialgebra structure on the polynomial algebra with two variables was provided in the work of Lin and Zhang [26]. It is worth noting that, this structure can be extended to polynomial algebras with n variables.

Example 1.2.4. As associative dialgebra structure over polynomial algebra $F[x, y]$ with two variables x and y is determined as follows. Let define two multiplications $\dashv$ and $\vdash$ on $F[x, y]$ as

$$f(x, y) \dashv g(x, y) = f(x, y)g(y, y),$$
$$f(x, y) \vdash g(x, y) = f(x, x)g(x, y),$$

then $(F[x, y], \dashv, \vdash)$ has a dialgebra structure. The basis of $F[x, y]$ is defined as $\{x^m y^n \mid m, n \in \mathbb{Z}\}$ and it is sufficient to prove triple axioms of dialgebras. The following relations show the associativity of $\dashv$:

$$x^m y^n \dashv (x^s y^t \dashv x^i y^j) = x^m y^{n+s+t+i+j}$$
$$(x^m y^n \dashv x^s y^t) \dashv x^i y^j = x^m y^{n+s+t+i+j}.$$

Furthermore, we have

$$x^m y^n \dashv (x^s y^t \dashv x^i y^j) = x^m y^{n+s+t+i+j},$$
$$x^m y^n \dashv (x^s y^t \vdash x^i y^j) = x^m y^{n+s+t+i+j},$$

and so axiom (1) of Definition 1.2.2 holds. Similarly, axioms (2) and (3) hold. Thus $(F[x, y], \dashv, \vdash)$ is an associative dialgebra.

Definition 1.2.5. A subspace D_0 of a diassociative algebra D is called a *subalgebra* if $x \dashv y$ and $x \vdash y$ are in D_0 for $x, y \in D_0$.

Definition 1.2.6. A *morphism of dialgebras* from D to D' is a $\mathbb{K}$-linear map $f\colon D \to D'$ such that

$$f(x \dashv y) = f(x) \dashv f(y),$$
$$f(x \vdash y) = f(x) \vdash f(y),$$

for all $x, y \in D$.

Definition 1.2.7. For a dialgebra D, a *derivation* $d\colon D \to D$ is defined as a linear map satisfying the following rules:

$$d(x \dashv y) = d(x) \dashv y + x \dashv d(y), \tag{1.2.1}$$

$$d(x \vdash y) = d(x) \vdash y + x \vdash d(y), \tag{1.2.2}$$

for all $x, y \in D$.

Lemma 1.2.8. *For a dialgebra D, the map $Ad_a\colon D \to D$ defined by $Ad_a(x) \coloneqq x \dashv a - a \vdash x$ is a derivation of dialgebras.*

Proof. We have

$$Ad_a(x \dashv y) = (x \dashv y) \dashv a - a \vdash (x \dashv y),$$

and consider the following equation

$$\begin{aligned} Ad_a(x) \dashv y + x \dashv Ad_a(y) &= (x \dashv a - a \vdash x) \dashv y + x \dashv (y \dashv a - a \vdash y) \\ &= (x \dashv a) \dashv y - (a \vdash x) \dashv y + x \dashv (y \dashv a) \\ &- x \dashv (a \vdash y) = (x \dashv y) \dashv a - a \vdash (x \dashv y) \\ &= Ad_a(x \dashv y), \end{aligned}$$

where $a \in D_0$ and $x, y \in D$. Let check that the condition of derivation map for $Ad_a(x \vdash y) = (x \vdash y) \dashv a - a \vdash (x \vdash y)$, then

$$\begin{aligned} Ad_a(x) \vdash y + x \vdash Ad_a(y) &= (x \dashv a - a \vdash x) \vdash y + x \vdash (y \dashv a - a \vdash y) \\ &= (x \dashv a) \vdash y - (a \vdash x) \vdash y + x \vdash (y \dashv a) \\ &- x \vdash (a \vdash y) = Ad_a(x \vdash y). \end{aligned}$$

Therefore Ad_a satisfies the derivation rules (1.2.1) and (1.2.2), and so it is a derivation for dialgebras. □

A left module over a dialgebra D is a $\mathbb{K}$-module M equipped with two linear maps

$$\dashv : D \otimes M \to M,$$
$$\vdash : D \otimes M \to M,$$

satisfying the axioms of Definition 1.2.2 whenever make sense. There is a similar definition for right module over a dialgebra D.

Definition 1.2.9. A *bimodule over a dialgebra* D is a $\mathbb{K}$-module M equipped with four linear maps

$$\dashv, \vdash : M \otimes D \to M,$$

$$\dashv, \vdash : D \otimes M \to M,$$

satisfying the axioms of the definition of dialgebra (see Definition 1.2.2).

We use the notion of action and semidirect product of dialgebras in this thesis, so we recall these notions in full details with respect to [14].

Definition 1.2.10 ([14])**.** Let D and L be dialgebras. An *action of* D *on* L consists of four linear maps which are denoted by $\dashv$ and $\vdash$,

$$\dashv\colon D \otimes L \to L, \quad \dashv\colon L \otimes D \to L,$$

$$\vdash\colon D \otimes L \to L, \quad \vdash\colon L \otimes D \to L$$

such that the following equalities hold

(1) $(x \dashv a) \dashv b = x \dashv (a \vdash b)$,

(2) $(x \dashv a) \dashv b = x \dashv (a \dashv b)$,

(3) $(x \vdash a) \dashv b = x \vdash (a \dashv b)$,

(4) $(x \dashv a) \vdash b = x \vdash (a \vdash b)$,

(5) $(x \vdash a) \vdash b = x \vdash (a \vdash b)$,

(6) $(a \dashv x) \dashv b = a \dashv (x \vdash b)$,

(7) $(a \dashv x) \dashv b = a \dashv (x \dashv b)$,

(8) $(a \vdash x) \dashv b = a \vdash (x \dashv b)$,

(9) $(a \dashv x) \vdash b = a \vdash (x \vdash b)$,

(10) $(a \vdash x) \vdash b = a \vdash (x \vdash b)$,

(11) $(a \dashv b) \dashv x = a \dashv (b \vdash x)$,

(12) $(a \dashv b) \dashv x = a \dashv (b \dashv x)$,

(13) $(a \vdash b) \dashv x = a \vdash (b \dashv x)$,

(14) $(a \dashv b) \vdash x = a \vdash (b \vdash x)$,

(15) $(a \vdash b) \vdash x = a \vdash (b \vdash x)$,

(16) $(a \dashv x) \dashv y = a \dashv (x \vdash y)$,

(17) $(a \dashv x) \dashv y = a \dashv (x \dashv y)$,

(18) $(a \vdash x) \dashv y = a \vdash (x \dashv y)$,

(19) $(a \dashv x) \vdash y = a \vdash (x \vdash y)$,

(20) $(a \vdash x) \vdash y = a \vdash (x \vdash y)$,

(21) $(x \dashv a) \dashv y = x \dashv (a \vdash y)$,

(22) $(x \dashv a) \dashv y = x \dashv (a \dashv y)$,

(23) $(x \vdash a) \dashv y = x \vdash (a \dashv y)$,

(24) $(x \dashv a) \vdash y = x \vdash (a \vdash y)$,

(25) $(x \vdash a) \vdash y = x \vdash (a \vdash y)$, (28) $(x \vdash y) \dashv a = x \vdash (y \dashv a)$,

(26) $(x \dashv y) \dashv a = x \dashv (y \vdash a)$, (29) $(x \dashv y) \vdash a = x \vdash (y \vdash a)$,

(27) $(x \dashv y) \dashv a = x \dashv (y \dashv a)$, (30) $(x \vdash y) \vdash a = x \vdash (y \vdash a)$,

for all $x, y \in D$; $a, b \in L$.

The action is called *trivial* if these four maps are trivial. Note that the previous identities are obtained from the axioms that satisfy $\vdash, \dashv$ by taking one variable in D and two variables in L (15 equalities), and one variable in L and two variables in D (15 equalities). Observe that we denote the action by the same symbol used for the left and the right products in D and L.

Let us show some examples of actions.

Example 1.2.11.
(*i*) If $0 \to L \xrightarrow{\iota} E \xrightarrow{\sigma} D \to 0$ is a split short exact sequence of dialgebras, i.e. there exists a homomorphism of dialgebras $\varphi \colon D \to E$ (see Definition1.2.6) such that $\sigma\varphi = \mathrm{id}_D$, then there is an action of the dialgebra D on L defined in the standard way by taking the left and the right products in the dialgebra E:

$$x * a = \varphi(x) * \iota(a) \quad \text{and} \quad a * x = \iota(a) * \varphi(x),$$

for any $x \in D$, $a \in L$ Here, we use $*$ to denote both $\vdash$ and $\dashv$.

(*ii*) If D is a subdialgebra of a dialgebra E (see Definition 1.2.5) (maybe $D = E$) and I is an ideal in E, then the left and the right products in E yield an action of D on I.

(*iii*) Any morphism of dialgebras (see Definition 1.2.6) $D \to L$ induces an action of D on L in the standard way by taking images of the elements of D and the left and the right products in L.

(iv) If L is a bimodule over a dialgebra D (see Definition 1.2.9), thought as an abelian dialgebra, then the bimodule structure defines an action of D on the (abelian) dialgebra L.

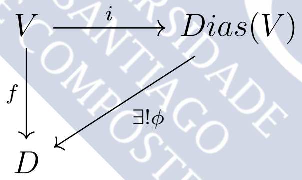
Note that if a dialgebra D acts on a dialgebra L, then L, as a $\mathbb{K}$-vector space, has a structure of bimodule over the dialgebra D.

Definition 1.2.12 ([14]). Given an action of a dialgebra D on a dialgebra L the *semidirect product dialgebra*, $L \rtimes D$, with the underlying $\mathbb{K}$-vector space $L \oplus D$ endowed with the left and the right products given by

$$(a_1, x_1) * (a_2, x_2) = (a_1 * a_2 + x_1 * a_2 + a_1 * x_2, x_1 * x_2),$$

for all $x_1, x_2 \in D$, $a_1, a_2 \in L$.

Definition 1.2.13 ([29]). Let V be a $\mathbb{K}$-linear space. The *free dialgebra* on V is the dialgebra $Dias(V)$ equipped with a $\mathbb{K}$-linear map $i \colon V \to Dias(V)$ such that for any $\mathbb{K}$-linear space map $f \colon V \to D$, there is a unique factorization



where $\phi \colon Dias(V) \to D$ is a dialgebra morphism.

The following theorem gives a description of free dialgebras in terms of the tensor module $T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \cdots$

Theorem 1.2.14 ([29]). *The free dialgebra on V is*

$$Dias(V) = T(V) \otimes V \otimes T(V)$$

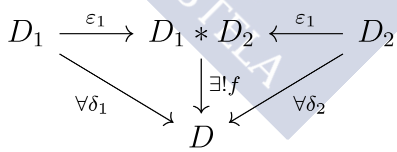
equipped with the two products induced by:

$$(v_{-n} \dots v_{-1} \otimes v_0 \otimes v_1 \dots v_m) \dashv (w_{-p} \dots w_{-1} \otimes w_0 \otimes w_1 \dots w_q)$$
$$= v_{-n} \dots v_{-1} \otimes v_0 \otimes v_1 \dots v_m w_{-p} \dots w_q$$

$$(v_{-n} \dots v_{-1} \otimes v_0 \otimes v_1 \dots v_m) \vdash (w_{-p} \dots w_{-1} \otimes w_0 \otimes w_1 \dots w_q)$$
$$= v_{-n} \dots v_m w_{-p} \dots w_{-1} \otimes w_0 \otimes w_1 \dots w_q.$$

Free product of dialgebras is applied for the description of HNN-extension of dialgebra through some examples, therefore, we recall that in the sequel with respect to [8].

Definition 1.2.15. Let D_1 and D_2 be dialgebras over $\mathbb{K}$. The dialgebra $D_1 * D_2$ with two dialgebra homomorphisms $\varepsilon_1 \colon D_1 \to D_1 * D_2$, $\varepsilon_2 \colon D_2 \to D_1 * D_2$ is called the *free product* of D_1 and D_2, if the following diagram commutes:



where D is a dialgebra, δ_1, δ_2 are dialgebra homomorphisms and $f \colon D_1 * D_2 \to D$ is a dialgebra homomorphism such that $f\varepsilon_1 = \delta_1$, $f\varepsilon_2 = \delta_2$.

To end the chapter we give some more terminology to precisely describe universal enveloping dialgebra of Leibniz algebras along with some important theorems in the next section.

1.3 Universal enveloping associative dialgebra of a Leibniz algebra

Dialgebras are closely connected to the notion of Leibniz algebras in the same way as associative algebras are connected to Lie algebras. This section is devoted to the relationship between Leibniz algebras and dialgebras.

It is customary to denote the category of algebras with a blackboard bold font, therefore, we consider these categories: **Dias**, **Leib**, **As** and **Lie**.

Proposition 1.3.1. *Let D be a dialgebra. Then the bracket*

$$[x,y] := x \dashv y - y \vdash x$$

makes to $(D,[-,-])$ into a Leibniz algebra, denoted by D_{Leib} or $D^{(-)}$.

Proof. By checking di-associativity axioms, we have

$$[x,[y,z]] = x \dashv (y \dashv z) - x \dashv (z \vdash y) - (y \dashv z) \vdash x + (z \vdash y) \vdash x,$$

$$-[[x,y],z] = -(x \dashv y) \dashv z + (y \vdash x) \dashv z + z \vdash (x \dashv y) - z \vdash (y \vdash x),$$

$$[[x,z],y] = (x \dashv z) \dashv y - (z \vdash x) \dashv y - y \vdash (x \dashv z) + y \vdash (z \vdash x).$$

□

This construction defines a functor $(-)$:

$$\mathbf{Dias} \xrightarrow{-} \mathbf{Leib}$$

from the category **Dias** of dialgebras to the category **Leib** of Leibniz algebras.

Definition 1.3.2 ([8])**.** Let L be a Leibniz algebra. A dialgebra $\mathrm{Ud}(L)$ together with a Leibniz homomorphism $\varepsilon\colon L \to \mathrm{Ud}(L)$ is called the *universal enveloping dialgebra* for L, if the following diagram commutes:

$$\begin{array}{ccc} L & \xrightarrow{\ \varepsilon\ } & \mathrm{Ud}(L) \\ {\scriptstyle \forall\delta}\downarrow & \swarrow{\scriptstyle \exists! f} & \\ D & & \end{array}$$

where D is a dialgebra, δ is a Leibniz algebra homomorphism and

$$f\colon \mathrm{Ud}(L) \to D$$

is a dialgebra homomorphism such that $f\varepsilon = \delta$, i.e. $\varepsilon\colon L \to \mathrm{Ud}(L)$ is a universal arrow.

The functor $(-)\colon \mathbf{As} \to \mathbf{Lie}$ has a left adjoint which is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$:

$$U(\mathfrak{g}) := T(\mathfrak{g})/\left\langle\{[x, y] - xy + yx \mid x, y \in L\}\right\rangle.$$

Similarly, the universal enveloping dialgebra of a Leibniz algebra L is defined as the following quotient of the free dialgebra on L:

$$\mathrm{Ud}(L) := T(L) \otimes L \otimes T(L)/\left\langle\{[x, y] - x \dashv y + y \vdash x \mid x, y \in L\}\right\rangle.$$

Proposition 1.3.3 ([29])**.** *The functor* $\mathrm{Ud}\colon$ ***Leib*** $\to$ ***Dias*** *is left adjoint to the functor* $(-)\colon$ ***Dias*** $\to$ ***Leib***.

Proof. Let $f\colon L \to D_{\mathrm{Leib}}$ be a morphism of Leibniz algebras. There is a unique extension of f as a morphism of dialgebras from $T(L) \otimes L \otimes T(L)$ to D. Since the image of $[x, y] - x \dashv y + y \vdash x$ under this morphism is 0, it defines a morphism from $\mathrm{Ud}(L)$ to D.

On the other hand the restriction of the morphism of dialgebras $g\colon \mathrm{Ud}(L) \to D$ to $L = \mathbb{K} \otimes L \otimes \mathbb{K}$ yields a morphism of Leibniz algebras $L \to D_{\mathrm{Leib}}$. Then these constructions give rise to isomorphisms

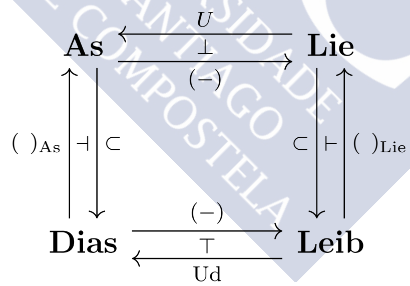
$$\mathrm{Hom}_{\mathbf{Dias}}(\mathrm{Ud}(L), D) \cong \mathrm{Hom}_{\mathbf{Leib}}(L, D_{\mathrm{Leib}}).$$

□

Lemma 1.3.4 ([29]). *For any Leibniz algebra L, we have* $\mathrm{Ud}(L)_{As} = U(L_{\mathrm{Lie}})$.

Proof. For any dialgebra D let D_{As} be the quotient of D by the ideal generated by the elements $x \dashv y - x \vdash y$, for all $x, y \in D$. Clearly, D_{As} is an associative algebra.

Since the functor $(\ )_{\mathrm{As}}\colon \mathbf{Dias} \to \mathbf{As}$ is left adjoint to inc: $\mathbf{As} \to \mathbf{Dias}$ and since $(-)_{\mathrm{Lie}}\colon \mathbf{Leib} \to \mathbf{Lie}$ is left adjoint to inc: $\mathbf{Lie} \to \mathbf{Leib}$ both composites $U \circ (-)_{\mathrm{Lie}}$ and $(-)_{\mathrm{As}} \circ \mathrm{Ud}$ are left adjoints to the composite $\mathbf{Leib} \to \mathbf{Lie} \to \mathbf{As}$, and so are equal.



□

Theorem 1.3.5 ([29]). *The universal enveloping dialgebra* $\mathrm{Ud}(L)$ *is isomorphic to* $U(L_{\mathrm{Lie}}) \otimes L$*, equipped with the dialgebra structure issued from a* $U(L_{\mathrm{Lie}})$*-bimodule structure and the bimodule map* $U(L_{\mathrm{Lie}}) \otimes L \to U(L_{\mathrm{Lie}})$.

Proof. We define a $U(L_{\mathrm{Lie}})$-bimodule structure on $U(L_{\mathrm{Lie}}) \otimes L$. The left module structure is given by multiplication in the left factor. The right module structure is induced by

$$(\omega \otimes x).\bar{y} := \omega \otimes [x, y] + \omega \bar{y} \otimes x,$$

where $\omega \in U(L_{\mathrm{Lie}})$, $x \in L$, $\bar{y} \in L_{\mathrm{Lie}}$ and $y \in L$ is a lifting of $\bar{y} \in L_{\mathrm{Lie}}$. The bracket $[x, y]$ in the Leibniz algebra L depends only on the class of y in L_{Lie}. We show that this formula provides a representation of L_{Lie}. Let $\bar{y}$ and $\bar{z}$ be elements in L_{Lie} and y, z be lifting in L. On one hand one gets

$$(\omega \otimes x.\bar{y}).\bar{z} = \omega \otimes [[x, y], z] + \omega \bar{z} \otimes [x, y] + \omega \bar{y} \otimes [x, z] + \omega \bar{y}\bar{z} \otimes x,$$

and

$$(\omega \otimes x.\bar{z}).\bar{y} = \omega \otimes [[x, z], y] + \omega \bar{y} \otimes [x, z] + \omega \bar{z} \otimes [x, y] + \omega \bar{z}\bar{y} \otimes x.$$

Hence one has

$$\begin{aligned}(\omega \otimes x.\bar{z}).\bar{y} - (\omega \otimes x.\bar{z}).\bar{y} &= \omega \otimes ([[x, y], z] - [[x, z], y]) - \omega(\bar{y}\bar{z} - \bar{z}\bar{y}) \otimes x \\ &= \omega \otimes [x, [y, z]] - \omega\overline{[y, z]} \otimes x \\ &= (\omega \otimes x).\overline{[y, z]}.\end{aligned}$$

The right and left module structure are immediately seen to be compatible, hence $U(L_{\mathrm{Lie}}) \otimes L$ is a $U(L_{\mathrm{Lie}})$-bimodule.

The linear map $U(L_{\mathrm{Lie}}) \otimes L \to U(L_{\mathrm{Lie}})$, $\omega \otimes x \to \omega\bar{x}$ is a $U(L_{\mathrm{Lie}})$-bimodule because $\omega\bar{x}\bar{y} = \omega\overline{[x, y]} + \omega\bar{y}\bar{x}$.

Therefore, $U(L_{\mathrm{Lie}}) \otimes L$ is equipped with a dialgebra structure. The nonunital associative algebra associative to this dialgebra is the augmentation ideal of $U(L_{\mathrm{Lie}})$. A well-defined dialgebra map

$$\mathrm{Ud}(L) \to U(L_{\mathrm{Lie}}) \otimes L$$

sends $\omega \otimes x \otimes 1$ to $\bar{\omega} \otimes x$, for $\omega \in T(L)$ and $\bar{\omega}$ its image in $U(L_{\mathrm{Lie}})$. Indeed, any element in $\mathrm{Ud}(L)$ can be written as a linear combination of the elements of the form $\omega \otimes x \otimes 1$ since

$$\omega \otimes x \otimes 1 = \omega \otimes [x, y] \otimes 1 + \omega y \otimes x \otimes 1.$$

Since by Lemma 1.3.4 one has $\mathrm{Ud}(L)_{As} = U(L_{\mathrm{Lie}})$, it follows that the element $\omega\dot{x}\omega'$ in $\mathrm{Ud}(L)$ depends only on the class of $\omega \in T(L)$ (resp. $\omega' \in T(L)$) in $U(L_{\mathrm{Lie}})$. So, one can define a dialgebra map

$$U(L_{\mathrm{Lie}}) \otimes L \to \mathrm{Ud}(L)$$

by sending $\bar{\omega} \otimes x$ to $\omega\dot{x}$, where ω is a lifting of $\bar{\omega}$. It is immediate to check both composites are the identity, whence the isomorphism. $\square$

Chapter 2

Composition-Diamond Lemma

We recall the theory of Groebner-Shirshov bases in Section 2.1 and Composition-Diamond Lemma for associative algebras, Lie algebras and dialgebras in Section 2.2. Within this chapter we use several concepts and notations which are used in different types of algebras commonly. In the sequel, we explain such these concepts and notations.

For X as a well-ordered set, we denote the set of all associative words or free monoid on X by X^*, the set of all non-associative words by X^{**}, the set of all diwords by $[X^*]$. The length of the word u is denoted by $|u|$. Let f be a polynomial in a variety of free algebras, it is called monic if the coefficient of its leading term which is denoted by $\bar{f}$ equals 1. Additional material which we use in this chapter is monomial ordering; that is, given a well-ordered set of monomials if it satisfies the condition $u_1 < u_2$ with $vu_1w < vu_2w$ for all monomials u_1, u_2, v, and w, then it is called a monomial ordering. It is worth pointing out that the property of being a Groebner-Shirshov basis is always relative to a specific monomial ordering.

In accordance with the comprehensive paper of Bokut [7] regarding definition of Composition-Diamond Lemma (or CD Lemma, for short) for different varieties of algebras, the following pattern must be structured for $F(X)$ as a free algebra generated by X.

1. Define appropriate linear basis called normal (regular) words of $F(X)$.
2. Define monomial ordering of normal words or some related words.
3. Define possible composition of the elements of subset S of $F(X)$.
4. Prove two key lemmas
   - Let S be a Groebner-Shirshov basis, i.e. any composition of polynomials from S is trivial. Then any S-word is a linear combination of normal S-words.
   - Let S be a Groebner-Shirshov basis, $[a_1 s_1 b_1]$ and $[a_2 s_2 b_2]$ normal S-words where $s_1, s_2 \in S$. If $w = a_1 \bar{s_1} b_1 = a_2 \bar{s_2} b_2$, then $[a_1 s_1 b_1] \equiv [a_2 s_2 b_2] \mod (S, (w))$.

2.1 Groebner-Shirshov bases

As prerequisite of the introduction to the theory of Groebner-Shirshov bases, we recall the notion of associative and non-associative Lyndon-Shirshov words in Subsection 2.1.1 and associative diwords in Subsection 2.1.2. Moreover, the composition of Lie polynomials and dialgebra polynomials are provided in Subsection 2.1.3. We note that the problem of triviality of composition of polynomials modulo a finite set S can be solved using Shirshov's Lie reduction algorithm [35].

The next definition can be considered as a compendious definition of Groebner-Shirshov basis for different types of free algebras.

Definition 2.1.1. A set S of relations in a free algebra is called a *Groebner-Shirshov basis* for an ideal generated by S, if every possible composition of elements is trivial.

2.1.1 Associative and non-associative Lyndon-Shirshov words

Associative Lyndon-Shirshov words are defined under numerous names such as *(Shirshov's) regular associative words* [24], *Lyndon words* [31, 39] and *Lyndon-Shirshov words* [7, 9, 38]. Associative Lyndon-Shirshov words play fundamental role in introduction of Composition-Diamond Lemma for Lie algebras. Therefore, the expression of the concept of the associative and non-associative Lyndon-Shirshov words reveals some similar and distinctive techniques among Composition-Diamond Lemma for associative algebras and Lie algebras in such a way benefits us to understand Composition-Diamond Lemma for the mentioned algebras. Furthermore, it is useful in understanding some other notations such as S-words and S-diwords which are appeared in compositions of polynomials during the construction of their HNN-extension in Chapter 3.

Denote by $<_{\texttt{deg-lex}}$ the degree-lexicographic ordering on X^* in which two words are compared first by the degree and then lexicographically. Degree-lexicographic ordering is a monomial ordering. It is because of its compatibility with the multiplication of words; that is, for all $u, v \in X^*$ we have

$$u <_{\texttt{deg-lex}} v \Rightarrow w_1 u w_2 <_{\texttt{deg-lex}} w_1 v w_2, \text{ for all } w_1, w_2 \in X^*.$$

There are several orderings of interest in computational algebra, but as an effective monomial ordering on X^* we mostly use `deg-lex` ordering in this subsection.

Definition 2.1.2 ([6]). Let $X = \{x_i \mid i \in I\}$ be a set with $x_i > x_j$ if $i > j$ for any $i, j \in I$ and $u \in X^*$. Then u is called an *associative Lyndon-Shirshov word* which is sometimes denoted by $ALSW$, if

$$(\forall\, v, w \in X^*,\ v, w \neq 0)\ u = vw \Rightarrow vw > wv.$$

In other words, an $ALSW$ word is a word greater than any cyclic permutation of itself. Associative Lyndon-Shirshov words have several useful properties (see [7, 34, 35]).

We can correspond to an associative Lyndon-Shirshov word, a non-associative word through three different bracketing methods namely *up-to-down*, *down-to-up* and *Shirshov's bracketing* (see [6]). The last one is used in Composition-Diamond Lemma for the case of Lie algebras (see Theorem 2.2.7) under the name *non-associative Lyndon-Shirshov words* (see Definition 2.1.4).

Let briefly recall up-to-down bracketing which is defined inductively by

$$(x_i) = x_i, \ (u) = ((v)(w)),$$

where $u = vw$ and w is the longest associative Lyndon-Shirshov proper end of u.

Example 2.1.3. Let $u = x_2x_2x_1x_1x_2x_1$. Then

$$u \to ((x_2x_2x_1x_1)(x_2x_1)) \to ((x_2(x_2x_1x_1))(x_2x_1)) \to ((x_2((x_2x_1)x_1))(x_2x_1)).$$

The next definition describes non-associative Lyndon-Shirshov words. For more details see [6].

Definition 2.1.4. Let $<$ be the ordering on X^* and (u) be a non-associative word. Then (u) is called *non-associative Lyndon-Shirshov word*, denoted by *NLSW*, if

1. u is an ALSW,
2. if $(u) = ((v)(w))$, then both (u) and (w) are NLSW's,
3. in (2) if $(v) = ((v_1)(v_2))$, then $v_2 \leq w$ in X^*.

We note that for an associative Lyndon-Shirshov word, there is a unique bracketing method in such a way a non-associative Lyndon-Shirshov word is obtained. If non-associative word (u) is a non-associative Lyndon-Shirshov word, then we denote it by $[u]$.

Let $\mathbb{K}\langle X\rangle$ be the free associative algebra generated by X and $[-,-]$ the Lie bracket, i.e. for any $a, b \in \mathbb{K}\langle X\rangle$, $[a,b] = ab - ba$. In fact, $Lie(X)$

is a free Lie algebra which is considered as subalgebra of Lie of $\mathbb{K}\langle X\rangle$ generated by X. Given a polynomial $f \in \mathbb{K}\langle X\rangle$, it has the leading word $\bar{f} \in X^*$ with respect to the `deg-lex` ordering on X^* such that

$$f = \alpha\bar{f} + \sum \alpha_i u_i, \tag{2.1.1}$$

where $\bar{f}, u_i \in X^*$, $\bar{f} > u_i$, $\alpha, \alpha_i \in \mathbb{K}$.

The following theorem states that every non-associative word in X^{**} can be represented as linear combination of NLSW's.

Theorem 2.1.5 ([6]). *Let the ordering $<$ be as before. Then for any $(u) \in X^{**}$, (u) has a representation*

$$(u) = \sum \alpha_i[u_i]$$

where each $\alpha_i \in \mathbb{K}, [u_i]$ is a NLSW and $|u_i| = |u|$. Moreover, if $(u) = ([v][w])$ then $u_i > \min\{v, w\}$.

Proof. Induction on $|u|$. If $|u| = 1$, then $(u) = [u]$ and the result holds. Suppose that $|u| > 1$ and $(u) = ((v)(w))$. Then, by induction,

$$(v) = \sum \alpha_i[v_i] \quad \text{and} \quad (w) = \sum \beta_j[w_j],$$

where $\alpha_i, \beta_j \in \mathbb{K}$, $[v_i], [w_j]$ are NLSW's, $|v_i| = |v|$ and $|w_j| = |w|$. We may assume that $(u) = ([v][w])$ with $v > w$ because of $([v][w]) = -([w][v])$. If $|v| = 1$, then

$$(u) = ([v][w])$$

is a NLSW. Suppose that $|v| > 1$ and $[v] = [[v_1][v_1]]$. There are two subcases

- If $v_2 \le w$, then $(u) = (([v_1][v_2])[w])$ is a NLSW.

- If $v_2 > w$, then

$$(u) = (([v_1][v_2])[w]) = (([v_1][w])[v_2]) + ([v_1]([v_2][w])).$$

By induction,

$$([v_1][w]) = \sum \eta_i[t_i],\ t_i > \min\{v_1, w\} = w$$

$$([v_2][w]) = \sum \eta'_j[t'_j],\ t'_j > \min\{v_2, w\} = w.$$

Then,

$$(u) - \sum \eta_i([t_i][v_2]) + \sum \eta'_j([v_1][t'_j]).$$

Note that

$$\min\{t_i, v_2\} \text{ and } \min\{t'_j, v_1\} > \min\{v, w\} = w,$$

and the result follows from the inverse induction on $\min\{v, w\}$. □

Example 2.1.6. Let $(u) = (((x_3x_2)(x_2x_1))(x_2x_1x_1))$. Then we show that (u) is written as a linear combination of non-associative Lyndon-Shirshov words.

$$(u) = (((x_3(x_2x_1))x_2)(x_2x_1x_1)) + ((x_3(x_2(x_2x_1)))(x_2x_1x_1)),$$

$$\begin{aligned}
&(((x_3(x_2x_1))x_2)(x_2x_1x_1))\\
&= (((x_3(x_2x_1))(x_2x_1x_1))x_2) + ((x_3(x_2x_1))(x_2(x_2x_1x_1)))\\
&= (((x_3(x_2x_1x_1))(x_2x_1))x_2) + ((x_3((x_2x_1)(x_2x_1x_1)))x_2)\\
&\ + ((x_3(x_2x_1))(x_2(x_2x_1x_1))),
\end{aligned}$$

$$\begin{aligned}
&((x_3(x_2(x_2x_1)))(x_2x_1x_1))\\
&= ((x_3(x_2x_1x_1))(x_2(x_2x_1))) + (x_3((x_2(x_2x_1))(x_2x_1x_1)))\\
&= ((x_3(x_2x_1x_1))(x_2(x_2x_1))) + (x_3((x_2(x_2x_1x_1))(x_2x_1)))\\
&\ + (x_3(x_2((x_2x_1)(x_2x_1x_1))))
\end{aligned}$$

and hence,

$$\begin{aligned}(u) = &(((x_3(x_2x_1x_1))(x_2x_1))x_2) + ((x_3((x_2x_1)(x_2x_1x_1)))x_2)\\ &+ ((x_3(x_2x_1))(x_2(x_2x_1x_1))) + ((x_3(x_2x_1x_1))(x_2(x_2x_1)))\\ &+ (x_3((x_2(x_2x_1x_1))(x_2x_1))) + (x_3(x_2((x_2x_1)(x_2x_1x_1))))\end{aligned}$$

(u) is a linear combination of NLSW's.

Lemma 2.1.7 ([6]). *Let $[u]$ be a non-associative Lyndon-Shirshov word. Then $\overline{[u]} = u$.*

Lemma 2.1.8 ([6]). *Non-associative Lyndon-Shirshov words are $\mathbb{K}$-independent.*

As a direct consequence of Theorem 2.1.5 and Lemma 2.1.8, the non-associative Lyndon-Shirshov words are a linear basis of the free Lie algebra $Lie(X)$. Moreover, Lemma 2.1.7 implies that for any Lie polynomial $f \in Lie(X)$, $\bar{f}$ is an associative Lyndon-Shirshov word (see Definition 2.1.2).

For a monic Lie polynomial f and $a \in X^*$, let $w = \bar{f}a$ be an associative Lyndon-Shirshov word. Denote by $[w]_{\bar{f}}$ the bracketing of the associative Lyndon-Shirshov word $w = \bar{f}a$ relative to $\bar{f}$. The notation $[w]_f$ indicates that a Lie polynomial obtained from $[w]_{\bar{f}}$ by replacing $\bar{f}$ by f. The above description is known as Shirshov's special bracketing method which plays a key role in Composition-Diamond Lemma for Lie algebras. The following theorem provides full details on Shirshov's bracketing method.

Theorem 2.1.9 ([6]). *Let u and v be associative Lyndon-Shirshov words, $u = avb$, $a, b \in X^*$. Then*

(i) $[u] = [a[vc]d]$*, where* $b = cd$ *for* $c, d \in X^*$*.*

(ii) Let

$$[u]_v = [u]|_{[vc]\mapsto[[[v][c_1]]\dots[c_k]]},$$

where $c = c_1 \cdots c_k$, c_j *is an associative Lyndon-Shirshov word and* $c_1 \leq c_2 \leq \cdots \leq c_k$. *Then, in* $\mathbb{K}\langle X \rangle$,

$$\overline{[u]_v} = u.$$

Moreover,

$$[u]_v = a[v]b + \sum_i \alpha_i a_i [v] b_i,$$

where each $\alpha_i \in \mathbb{K}$ *and* $a_i v b_i < avb$.

Proof. (i) Induction on $|u|$. If $|u| = 1$, then $u = v = x_i$ and the result holds. Assume that $|u| > 1$. If $v = x_i$, then $[u] = [a[x_i]d]$ and the result holds. Now, we consider the case of $|v| > 1$. Let $x_\beta = \min(u)$ and $b = x_\beta^e \tilde{b}$ for $e \geq 0$ and $\text{first}(\tilde{b}) \neq x_\beta$, where first denotes the first letter of each associative word. Then

$$u = avb = avx_\beta^e \tilde{b} = a\tilde{v}\tilde{b},$$

where $\tilde{v} = vx_\beta^e$ is also an associative Lyndon-Shirshov word. Then, by induction, for $u' = a'_u v'_u \tilde{b}'_u$, we have $[u'[\tilde{v}'_u \tilde{c}'_u] d'_u]$, $\tilde{b}'_u = \tilde{c}'_u d'_u$. By substitution $x_i^j \mapsto [[x_i, x_\beta], \ldots, x_\beta]$ we obtain

$$[u] = [a[\tilde{v}\tilde{c}]d] = [a[vx_\beta^e \tilde{c}]d] = [a[vc]d],$$

where $c = x_\beta^e \tilde{c}$.

(ii) If $c = 1$, then $[u]_v = [u]$ and the results hold. Otherwise, we consider a decomposition $c = x_\beta \cdots x_\beta c_{l+1} \cdots c_k$, where c_i is an associative Lyndon-Shirshov and $x_\beta < c_{l+1} \leq \cdots \leq c_k$. Then

$$[u]_v = [u]|_{[vx_\beta^e \tilde{c}] \mapsto [[[v], x_\beta], \ldots, x_\beta [c_{l+1}] \ldots [c_k]]}$$

and

$$[u]_{\tilde{v}} = [u]_{[\tilde{v}\tilde{c}] \mapsto [[[\tilde{v}][c_{l+1}]] \ldots [c_k]]}.$$

Now by induction on $|u|$, the proof will be completed. If $|u| = 1$, then this is a trivial case. Suppose that $|u| > 1$ and $|v| > 1$. Then by (i),

$$u = a\tilde{v}\tilde{c}d,\ u' = a'_u\tilde{v}'_u\tilde{c}'_u d'_u$$

and by induction,

$$[u']_{\tilde{v}'_u} = a'_u[\tilde{v}'_u]\tilde{c}'_u d'_u + \sum_{i\in I_1} \alpha_i a'_{i_u}[\tilde{v}'_u]b'_{i_u},$$

where each $a'_{i_u}\tilde{v}'_u b'_{i_u} < u'$. Now, we have

$$[[x_i, x_\beta], \dots, x_\beta] = \sum_{m\geq 0}(-1)^m \binom{j}{m} x_\beta^m x_i x_\beta^{j-m}$$

and

$$x_\beta^m x_i x_\beta^{j-m} < x_i x_\beta^j, \quad m > 0.$$

By making the substitution $x_i^j \mapsto [[x_i, x_\beta], \dots, x_\beta]$, we obtain $[u]_{\tilde{v}} = a[\tilde{v}]\tilde{c}d + \sum_{i\in I_2} \alpha_i a_i[\tilde{v}]b_i$, where each $a_i\tilde{v}b_i < a\tilde{v}\tilde{c}d$. Also, by substitution $[\tilde{v}] \mapsto [[[v], x_\beta], \dots, x_\beta]$, we have

$$[u]_v = a[v]x_\beta^e\tilde{c}d + \sum_{j\in I} \beta_j a_j[v]b_j = a[v]b + \sum_{j\in I} \beta_j a_j[v]b_j,$$

such that $a_j v b_j < avb$. □

2.1.2 Associative diwords

We recall that for a dialgebra D and a subset $B \subset D$, a diword of D in B is defined by induction on the length of diwords such that

- $b = (b), b \in B$ is a diword in B of length $|b| = 1$.
- (u) is called a diword in B of length $|(u)| = n$, if $(u) = ((v) \dashv (w))$ or $(u) = ((v) \vdash (w))$, where (v) and (w) are diwords in B of length k and l, respectively. Moreover, we have $k + l = n$.

Let $D(X)$ be the free di-algebra over $\mathbb{K}$ generated by a set X. Any element of $[X^*]$ is called a diword. The next proposition includes the concept of normal diwords. For its proof see [29].

Proposition 2.1.10. *Any diword in $D(X)$ is equal to a unique diword of the form*

$$[u] = x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \cdots \dashv x_n = x_{-m} \cdots \dot{x}_0 \cdots x_n$$

where $x_i \in X$, $m \geq 0$, $n \geq 0$, and x_0 is called the center of the diword $[u]$.

As mentioned before, the set of all normal diwords is denoted by $[X^*]$. Given two normal diwords $[u]$ and $[v]$ in $[X^*]$, the diword $[u] \dashv [v]$ (resp. $[u] \vdash [v]$) is again a normal diword which is denoted by $[uv]$ with the center at the center of $[v]$ (resp. $[u]$).

Now we recall a monomial ordering on $[X^*]$ which has been defined by Bokut in [8].

Definition 2.1.11. Let X be a totally ordered set. The *monomial ordering weight lexicographic* $<$ on normal diwords is defined as:

$$[u] < [v] \Leftrightarrow \mathrm{wt}([u]) <_{\texttt{lex}} \mathrm{wt}([v]) \text{ (lexicographically)},$$

where $\mathrm{wt}([u]) = (n + m + 1, m, x_{-m}, \ldots, x_0, \ldots, x_n)$ with

$$[u] = x_{-m} \cdots \dot{x}_0 \cdots x_n.$$

The ordering $<$ is compatible with the left and right products:

$$[u] < [v] \Rightarrow x \dashv [u] < x \dashv [v], \quad [u] \vdash x < [v] \vdash x, \text{ for any } x \in X.$$

The following equation shows that any polynomial f in the free dialgebra $D(X)$ can be written in a similar form to (2.1.1):

$$f = \sum_{[u_i] \in [X^*]} \alpha[\bar{f}] + \sum \alpha_i [u_i], \tag{2.1.2}$$

where $[u_i]$ are normal diwords (see Proposition 2.1.10), $\alpha_i \in \mathbb{K}$ and $[\bar{f}]$ is the leading monomial with respect to the monomial ordering described in Definition 2.1.11. f is called monic if $\alpha = 1$. The *degree* of f, $\deg(f)$ or $|\overline{[f]}|$, is the length of the leading monomial.

Definition 2.1.12 ([29]). We call f a *left (right) normed diword* if $f = \sum \alpha_i u_i \dot{x}_i$ ($f = \sum \alpha_i \dot{x}_i u_i$), where $\alpha_i \in \mathbb{K}, x_i \in X, u_i \in X^*$ and $\dot{x}_i$ is the center element.

Example 2.1.13. Let $X = \{x_1, x_2\}$, $x_1 > x_2$, and $f = x_1 \vdash x_2 \vdash x_1 - x_1 \vdash x_2 \dashv x_1 + x_1 \vdash x_2$. Then f is monic and we have $\overline{[f]} = x_1 \vdash x_2 \vdash x_1$ and $|\overline{[f]}| = 3$.

Definition 2.1.14. Let S be a subset of $D(X)$. An S-*diword* is a diword on $X \cup S$ with only one occurrence of $s \in S$. Therefore, Proposition 2.1.10 implies that any S-diword $g = [asb]$, for $a, b \in X^*$, $s \in S$, is equal to

$$[asb] = x_{-m} \vdash \cdots \vdash x_{-1} \vdash x_0 \dashv x_1 \dashv \cdots \dashv x_n \mid_{x_k \mapsto s},$$

where $-m \leq k \leq n$, $s \in S$, $x_i \in X$, $-m \leq i \leq n$, and $x_k \mapsto s$ means that we replace s into the variable x_k. We call g a *normal* S-*diword* if one of the following conditions holds:

(i) $k = 0$,

(ii) $k < 0$ and s is left normed,

(iii) $k > 0$ and s is right normed.

For a normal S-diword $[asb]$, the leading term of $[asb]$ is determined as $\overline{[asb]} = [a\bar{s}b]$.

Example 2.1.15. Let $X = \{x_1, x_2\}$, $x_1 > x_2$, and $S = \{s_1, s_2, s_3\} \subset D(X)$, where

$$\begin{aligned}
s_1 &= x_1 \vdash x_2 + x_1 \dashv x_2,\\
s_2 &= x_1 \vdash x_2 \vdash x_1 - x_1 \vdash x_2 \dashv x_1 + x_1 \dashv x_2 \dashv x_1,\\
s_3 &= x_1 \vdash x_2 \vdash x_1.
\end{aligned}$$

Then we have $\bar{s}_1 = x_1 \vdash x_2$, $\bar{s}_2 = x_1 \vdash x_2 \vdash x_1$. Let us for example, calculate $s_1 \vdash x_1$ and $s_3 \vdash x_2$. We have

$$\begin{aligned}
s_1 \vdash x_1 &= (x_1 \vdash x_2 + x_1 \dashv x_2) \vdash x_1 = 2x_1 \vdash x_2 \vdash x_1 = 2\bar{s}_2,\\
s_3 \vdash x_2 &= x_1 \vdash x_2 \vdash x_1 \vdash x_2,
\end{aligned}$$

therefore, $s_1 \vdash x_1$ is a S-diword and $s_3 \vdash x_2$ is a normal S-diword.

2.1.3 Composition of polynomials

We recall compositions for polynomials in associative algebras, Lie algebras and dialgebras. Let take two monic polynomials f and g in a given variety of a free algebra $F(X)$. Then compositions of f and g might be vary from free algebra to free algebra (see [5], [6], [7] and [8]). Besides we express the criteria of triviality of compositions for the mentioned algebras. It is worth noting that, the composition of f and g denoted by (f, g) lies in the ideal generated by f and g.

Free associative algebra $\mathbb{K}\langle X\rangle$

The ease of computations on associative algebras sharpens some notations and concepts related to Composition-Diamond Lemma. Let consider the free associative algebra $\mathbb{K}\langle X\rangle$ generated by X over $\mathbb{K}$ and X^* be the free monoid generated by X along with a monomial ordering $<$. Take $f \in \mathbb{K}\langle X\rangle$ with the leading term $\bar{f}$. Then in a similar way to (2.1.1) and (2.1.2) we have

$$f = \alpha\bar{f} - r_f,$$

where $\alpha \in \mathbb{K}$ and $\overline{r_f} < \bar{f}$. f is called monic if $\alpha = 1$. For monic polynomials $f, g \in \mathbb{K}\langle X\rangle$, we have two types of compositions of polynomials as follows.

1. If w is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $|\bar{f}| + |\bar{g}| > |w|$ then the polynomial $(f,g)_w = fb - ag$ is called the *intersection composition* of f and g with $\overline{(f,g)_w} < w$.

2. If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$ then the polynomial $(f,g)_w = f - agb$ is called the *inclusion composition* of f and g such that $\overline{(f,g)_w} < w$.

Let $h \in \mathbb{K}\langle X\rangle$ be a polynomial in $\mathbb{K}\langle X\rangle$. Then it is called *trivial modulo* (S, w), if

$$h = \sum \alpha_i a_i s_i b_i,$$

where $\alpha_i \in \mathbb{K}$, $a_i, b_i \in X^*$ and $s_i \in S$ with $a_i s_i b_i < w$.

Free Lie algebra $Lie(X)$

Given monic Lie polynomials $f, g \in Lie(X)$ then $\bar{f}$, $\bar{g}$ are associative Lyndon-Shirshov words due to Lemma 2.1.7. Having considered the Shirshov's special bracketing method (see Theorem 2.1.9), two possible compositions are defined:

1. If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$ then the polynomial $(f,g)_w = f - [agb]_{\bar{g}}$ is called the *inclusion composition* of f and g.

2. If $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$ then the polynomial $(f,g)_w = [fb]_{\bar{f}} - [ag]_{\bar{g}}$ is called the *intersection composition* of f and g.

   We note that $w \in X^*$ is an associative Lyndon-Shirshov word.

Let h be a Lie polynomial, then it is called *trivial modulo* $S \subset Lie(X)$ if

$$h = \sum_i \alpha_i [a_i s_i b_i]_{\bar{s}_i},$$

where $\alpha_i \in \mathbb{K}$, $a_i, b_i \in X^*$, $s_i \in S$ and $[a_i\bar{s}_i b_i]_{\bar{s}_i}$ is a non-associative Lyndon-Shirshov word with respect to the subword s_i such that $a_i\bar{s}_i b_i < w$.

Free dialgebra $Dias(X)$

In addition to the intersection and inclusion compositions which have already been described for associative algebras and free Lie algebras, Bokut [6] introduced left and right multiplication compositions in dialgebras as follows.

Left-mc *Left multiplication composition.*

The composition of left multiplication is denoted by $x \dashv f$, where f is not a right normed word (see Definition 2.1.12). It is called trivial modulo a monic subset $S \subset D(X)$, denoted by $\equiv 0 \pmod{S}$, if

$$x \dashv f = \sum \alpha_i [a_i s_i b_i],$$

where $\alpha_i \in \mathbb{K}, a_i, b_i \in X^*$, $s_i \in S$ and $[a_i s_i b_i]$ are right normed S-diwords, such that $|[a_i[\bar{s}_i]b_i]| \leq \deg(x \dashv f)$.

Right-mc *Right multiplication composition.*

The composition of right multiplication is defined for not left normed polynomials g denoted by $g \vdash x$ and it is called trivial modulo S, if

$$g \vdash x = \sum \alpha_i [a_i s_i b_i],$$

where $\alpha_i \in \mathbb{K}, a_i, b_i \in X^*$, $s_i \in S$ and $[a_i s_i b_i]$ are left normed S-diwords such that $|[a_i[\bar{s}_i]b_i]| \leq \deg(g \vdash x)$.

Inter-c *Intersection composition.*

Let $[w] = [[\bar{f}]b] = [a[\bar{g}]]$, where $[fb]$ and $[ag]$ are normal S-diwords (see Definition 2.1.14), and $|\bar{f}| + |\bar{g}| > |w|$. Then the intersection composition is defined as $(f, g)_{[w]} = [fb] - [ag]$. It is trivial modulo S if

$$(f, g)_{[w]} = \sum \alpha_i [a_i s_i b_i],$$

where $\alpha_i \in \mathbb{K}, a_i, b_i \in X^*, s_i \in S$, $[a_i s_i b_i]$ are normal S-diwords, $[a_i[\bar{s}_i]b_i] < [w]$ and each $[a_i s_i b_i]$ is a right (left) normed S-diword, whenever either both $[fb]$ and $[ag]$ are right (left) normed S-diwords.

Inc-c *Inclusion composition.*

Let $[w] = [\bar{f}] = [a[\bar{g}]b]$ where $[agb]$ is a normal S-diword. Then $(f, g)_{[w]} = [f] - [agb]$ is called inclusion composition. The triviality of inclusion composition is defined in a similar way to the intersection composition.

2.2 Composition-Diamond Lemma

As a main consequence of Composition-Diamond Lemma, the S-reduced monomials, denoted by $\mathrm{Irr}(S)$, constitute a linear basis of the quotient algebra of $F(X)$ by the ideal $\mathrm{Id}(S)$ which is generated by S. In the case that S is not a Groebner-Shirshov basis, we denote by S^{comp} the Groebner-Shirshov basis which is obtained from S by adding nontrivial compositions of polynomials.

2.2.1 The case of associative algebras

Given an associative algebra $\mathbb{K}\langle X\rangle$ and its subset S which is Groebner-Shirshov basis for ideal generated by S, i.e. every composition of polynomials in S is trivial modulo S (see Subsection 2.1.3 on page 32), then the following definition is provided to introduce normal forms.

Definition 2.2.1. An element f in the free associative algebra $\mathbb{K}\langle X\rangle$ is said to be S*-irreducible* if none of the elements of S is a subword of f. More precisely, we write it as

$$\mathrm{Irr}(S) = \{f \in X^* \mid f \neq a\bar{s}b, s \in S, a, b \in X^*\}.$$

The following two lemmas are essential for proving Composition-Diamond Lemma for associative algebras.

Lemma 2.2.2 ([7]). *If S is a Groebner-Shirshov basis in $\mathbb{K}\langle X\rangle$ and $w = a_1\bar{s_1}b_1 = a_2\bar{s_2}b_2$, where $a_1, b_1, a_2, b_2 \in X^*$ and $s_1, s_2 \in S$, then* $a_1s_1b_1 \equiv a_2s_2b_2 \bmod (S, w)$.

Proof. There are three cases to consider.

• Case 1. Assume that the subwords $\bar{s_1}$ and $\bar{s_2}$ of w are disjoint, say, $|a_2| \geq |a_1| + |\bar{s_1}|$. Then, $a_2 = a_1\bar{s_1}c$ and $b_1 = c\bar{s_2}b_2$ for some $c \in X^*$, and so $w_1 = a_1\bar{s_1}c\bar{s_2}b_2$. We write

$$a_1\bar{s_1}b_1 - a_2\bar{s_2}b_2 = a_1s_1c(\bar{s_2} - s_2)b_2 + a_1(s_1 - \bar{s_1})cs_2b_2.$$

Since $\overline{\bar{s_2} - s_2} < \bar{s_2}$ and $\overline{s_1 - \bar{s_1}} < \bar{s_1}$, we conclude that

$$a_1s_1b_1 - a_2s_2b_2 = \sum_i \alpha_i u_i s_1 v_i + \sum_j \beta_j u_j s_2 v_j,$$

where $\alpha_i, \beta_j \in k$ and $u_i\bar{s_1}v_j, u_j\bar{s_2}v_j < w$.

• Case 2. Suppose the subword $\bar{s_1}$ of w contains $\bar{s_2}$ as a subword. Then $\bar{s_1} = a\bar{s_2}b$ with $a_2 = a_1a$ and $b_2 = bb_1$, that is, $w = a_1a\bar{s_2}bb_1$ for some S-word as_2b. We have

$$a_1\bar{s_1}b_1 - a_2\bar{s_2}b_2 = a_1(s_1 - as_2b)b_1 = a_1(s_1, s_2)_{\bar{s_1}}b_1.$$

As the composition is trivial then we obtain $a_1s_1b_1 \equiv a_2s_2b_2 \bmod (S, w)$.

• Case 3. If the subwords $\bar{s_1}$ and $\bar{s_2}$ of w have a nonempty intersection then we assume that $a_2 = a_1a$ and $b_1 = bb_2$ with $w = \bar{s_1}b = a\bar{s_s}$ and $|w| < |\bar{s_1}| + |\bar{s_2}|$. Then $a_1s_1b_1 \equiv a_2s_2b_2 \bmod (S, w)$. □

Lemma 2.2.3 ([7]). *Consider a set $S \subset \mathbb{K}\langle X\rangle$ of monic polynomials. For every $f \in \mathbb{K}\langle X\rangle$ we have*

$$f = \sum_{u_i \leq \bar{f}} \alpha_i u_i + \sum_{a_j\bar{s_j}b_j} \beta_j a_j s_j b_j,$$

where $\alpha_i, \beta_i \in \mathbb{K}, u_i \in \mathrm{Irr}(S)$. Therefore $\mathrm{Irr}(S)$ is a set of linear generators of the algebra $\mathbb{K}\langle X \mid S\rangle$.

Proof. By induction on $\bar{f}$. □

Theorem 2.2.4 ([7]). *(Composition-Diamond Lemma for associative algebras) Fix a monomial ordering $<$ on X^*. Consider a monic set $S \subset \mathbb{K}\langle X\rangle$ and the ideal* $\mathrm{Id}(S)$ *generated by S. The following statements are equivalent.*

(i) S is a Groebner-Shirshov basis in $\mathbb{K}\langle X\rangle$;

(ii) $f \in \mathrm{Id}(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$;

(iii) $\mathrm{Irr}(S)$ *defined in Definition 2.2.1 is a linear basis of the algebra $\mathbb{K}\langle X \mid S\rangle$.*

2.2.2 The case of Lie algebras

The expansion of Lie bracket $[ab] = ab - ba$ helps us to consider it as a monic element in $\mathbb{K}\langle X\rangle$. The free Lie algebra $Lie(x)$ is the minimal subspace of $\mathbb{K}\langle X\rangle$ containing X. Since the non-associative Lyndon-Shirshov words are a linear basis of $Lie(X)$ (see Theorem 2.1.5 and Lemma 2.1.7), then every Lie polynomial can be represented as linear combination of non-associative Lyndon-Shirshov words. It follows from Lemma 2.1.7 that $\bar{f}$ is an associative Lyndon-Shirshov word. Moreover, if a is any ALSW subword of u then $\overline{[u]_a} = u$.

Remark 2.2.5. The notation $[w]_{\bar{f}}$ is a special type of bracketing with respect to $\bar{f}$ and $[w]_f$ is the Lie polynomial obtained from $[w]_{\bar{f}}$ by replacing $[\bar{f}]$ by f. (Shirshov's bracketing method, see Theorem 2.1.9).

Let S be a Groebner-Shirshov basis, then the S-irreducibility for the case of Lie algebras is defined as follows.

Definition 2.2.6. The *S-irreducible set* for the free Lie algebra $Lie(X)$ is defined as

$$\mathrm{Irr}(S) = \{[u] \mid [u] \text{ is a NLSW}, u \neq a\bar{s}b,\ s \in S,\ a, b \in X^*\}.$$

Theorem 2.2.7 ([7,9]). *(Composition-Diamond Lemma for Lie algebras) Let $S \subset Lie(X) \subset \mathbb{K}\langle X\rangle$ be a nonempty set of monic Lie polynomials. Let $\mathrm{Id}(S)$ be the ideal of $Lie(X)$ generated by S. Then the following statements are equivalent.*

1. *S is a Groebner-Shirshov basis in $Lie(X)$;*
2. *$f \in \mathrm{Id}(S) \Rightarrow \bar{f} = a\bar{s}b$, for some $s \in S$ and $a, b \in X^*$;*
3. *$\mathrm{Irr}(S)$ is a basis for $Lie(X \mid S)$.*

Example 2.2.8. The special linear Lie algebra $\mathfrak{sl}_{n+1}(k)$ over a field $\mathbb{K}$, where $\mathrm{char}(\mathbb{K}) \neq 2$, is isomorphic to the Lie algebra

$$\begin{aligned} Lie(A_n) = & Lie(h_i, x_i, y_i,\ 1 \leq i \leq n \mid [h_i, h_j] = 0, \\ & [x_i, y_j] = \delta_{ij} h_i,\ [h_i, x_j] = 2\delta_{ij} x_i,\ [h_i, y_j] = -2\delta_{ij} y_i, \\ & [x_{i+1}, [x_{i+1}, x_i]] = 0,\ [x_j, x_i] = 0, \\ & [y_{i+1}, [y_{i+1}, y_i]] = 0,\ [y_j, y_i] = 0, \quad j \neq i+1), \end{aligned}$$

with the isomorphism

$$h_i \to e_{ii} - e_{i+1i+1},\ x_i \to e_{ii+1},\ y_i \to e_{i+1i},\ 1 \leq i \leq n.$$

A Groebner-Shirshov basis of $Lie(A_n)$ consists of the initial relations together with

$$[[x_{i+j}, x_{i+j-1}, \ldots, x_{i-1}], x_{i+j+1}],$$

$$[[x_{i+j}, \ldots, x_i][x_{i+j}, \ldots, x_i][x_{i+j}, \ldots, x_{i-1}]],$$

for $j \geq 1$, $i \geq 2$, $i + j \leq n$, and the same relations for $y_1, \ldots, y_n$, where by $[z_1, z_2, \ldots, z_m]$ we mean $[z_1, [z_2, \ldots, z_m]]$. By Composition-Diamond Lemma 2.2.7 for Lie algebras a linear basis of $Lie(A_n)$ is

$$h_i, \quad [x_i, x_{i-1}, \ldots, x_j], \quad [y_i, y_{i-1}, \ldots, y_j], \quad 1 \leq i \leq n,\ j \leq i.$$

Analogous results are valid for other types of simple Lie algebras.

2.2.3 The case of associative dialgebras

The following essential lemmas are mentioned in [8] and applied to prove Composition-Diamond Lemma for the case of dialgebras.

Lemma 2.2.9 ([8]). *Let $S \subset D(X)$ and $[asb]$ an s-diword, $s \in S$ (see Definition 2.1.14). Assume that each composition of a right and left multiplication (see Right-mc and Left-mc) is trivial modulo S. Then $[asb]$ has a presentation:*

$$[asb] = \sum \alpha_i [a_i s_i b_i],$$

where each $\alpha_i \in \mathbb{K}$, $s_i \in S$, $a_i, b_i \in X^*$ and each $[a_i s_i b_i]$ is a normal s_i-diword (see Definition 2.1.14).

Lemma 2.2.10 ([8]). *Let $S \subset D(X)$ and each composition $(f,g)_{[w]}$ in S of inclusion (intersection) (see Inc-c and Inter-c) is trivial modulo $(S,[w])$. Let $[a_1 s_1 b_1]$ and $[a_2 s_2 b_2]$ be normal S-diwords such that $[w] = [a_1[\bar{s_1}]b_1] = [a_2[\bar{s_2}]b_2]$, where $s_1, s_2 \in S$, $a_1, a_2, b_1, b_2 \in X^*$. Then*

$$[a_1[\bar{s_1}]b_1] \equiv [a_2[\bar{s_2}]b_2] \bmod (S,[w]).$$

Definition 2.2.11 ([8]). Let $S \subset D(X)$. Then $\mathrm{Irr}(S)$ is defined as follows:

$$\mathrm{Irr}(S) = \{u \in [X^*] \mid u \neq [a[\bar{s}]b], [asb] \text{ is a normal } s\text{-diword}\},$$

where $s \in S$ and $a, b \in X^*$.

By using Definition 2.2.11 every $h \in D(X)$ has a representation

$$h = \sum_{I_1} \alpha_i [u_i] + \sum_{I_2} \beta_j [a_j s_j b_j] \qquad (2.2.1)$$

where $[u_i] \in \mathrm{Irr}(S)$, $i \in I_1$, $[a_j s_j b_j]$ are normal s_j-diwords, $s_j \in S$, $j \in I_2$ such that $[a_1[\bar{s_1}]b_1] > \cdots > [a_n[\bar{s_n}]b_n]$.

Theorem 2.2.12 ([8]). *(Composition-Diamond Lemma for dialgebras) Let $S \subset D(X)$ be a set of monic Lie polynomials, the ordering $<$ as defined for dialgebras, and $\mathrm{Id}(S)$ the ideal generated by S. Then the following statements are equivalent.*

(i) *S is a Groebner-Shirshov basis in $D(X)$;*

(ii) *$f \in \mathrm{Id}(S) \Rightarrow [\bar{f}] = [a[\bar{s}]b]$ for some $s \in S$, $a, b \in X^*$ and $[asb]$ a normal S-diword;*

(iii) *$\mathrm{Irr}(S) = \{u \in [X^*] \mid u \neq [a[\bar{s}]b],\ s \in S, a, b \in X^*, [asb]$ is normal s-diword$\}$ is a basis of the dialgebra $D(X \mid S)$.*

Proof. Let S be a Groebner-Shirshov basis and $0 \neq f \in \mathrm{Id}(S)$. We may assume, by Lemma 2.2.9, that

$$f = \sum_{i=1}^{n} \alpha_i [a_i s_i b_i],$$

where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $[a_i s_i b_i]$ normal S-diword. Let

$$[w_i] = [a_i[\bar{s_i}]b_i], \quad [w_1] = [w_2] = \cdots = [w_l] > [w_{l+1}] \geq \cdots \geq 1.$$

We will use induction on l and $[w_1]$ to prove that $[\bar{f}] = [a[\bar{s}]b]$ for some $s \in S$ and $a, b \in X^*$. If $l = 1$, then $[\bar{f}] = \overline{[a_1 s_1 b_1]} = [a_1[\bar{s_1}]b_1]$, and so the result holds. Assume that $l \geq 2$. Then, by Lemma 2.2.10 we have $[a_1 s_1 b_1] \equiv [a_2 s_2 b_2] \bmod (S, [w_1])$.

Then, if $\alpha_1 + \alpha_2 \neq 0$, then the result follows from induction on l. For the case $\alpha_1 + \alpha_2 = 0$ and $l = 2$, we use induction on $[w_1]$. So the result holds.

By (2.2.1) $\mathrm{Irr}(S)$ generates $D(X|S)$ as $\mathbb{K}$-space. Suppose that $0 \neq \sum \alpha_i [u_i] \in \mathrm{Id}(S)$, where $[u_1] > \cdots > [u_i] \in \mathrm{Irr}(S)$. Then by (ii), $[u_i] =$

$[a_1[\bar{s_1}]b_1]$, where $[a_1 s_1 b_1]$ is a normal S-diword which is a contradiction, so it proves (iii).

Let $0 \neq f \in \mathrm{Id}(S)$. Since the elements in $\mathrm{Irr}(S)$ are linearly independent in $D(X|S)$, by (2.2.1), $[\bar{f}] = [a[\bar{s}]b]$, where $[asb]$ is a normal S-diword. Thus, (ii) follows. □

In the following, by using Groebner-Shirshov bases for the universal enveloping dialgebra of a Leibniz algebra, a normal form for $\mathrm{Ud}(L)$ is obtained. Then as a corollary, Bokut [8] provided the Poincaré-Birkhoff-Witt theorem (see Theorem 1.3.5) using the Groebner-Shirshov bases theory.

Remark 2.2.13. Let f be a Leibniz polynomial in the variables X. Denote by $f^{(-)}$ the dialgebra polynomial in X obtained from f by the transformation $[a,b] \mapsto a \dashv b - b \vdash a$, which was expressed in Proposition 1.3.1.

If we denote the free Leibniz algebra generated by X and relations S by $L = Leib(X \mid S)$, then $\mathrm{Ud}(L) = D(X \mid S^{(-)})$ is the dialgebra with generators X and definition relations $S^{(-)} = \{s^{(-)} \mid s \in S\}$ obtained from S through Proposition 1.3.1. The following theorem defines a Groebner-Shirshov basis for the universal enveloping dialgebra of a Leibniz algebra (see Definition 1.3.2) which is applied for the introduction of the PBW theorem based on the notion of dialgebras. The proof of this theorem is essential to understand how to utilize the different compositions of polynomials in a free dialgebra which are described in Subsection 2.1.3 on page 34, and to determine whether a subset of a dialgebra such as S meets the triviality conditions to define a Groebner-Shirshov basis.

Theorem 2.2.14 ([8]). *Let L be a Leibniz algebra over $\mathbb{K}$ with the product $[\ ,\ ]$. Let L_0 be the subspace of L generated by the set $\{[a,a],[a,b]+[b,a] \mid a,b \in L\}$. Let $\{x_i \mid i \in I_0\}$ be a basis of L_0 and $X = \{x_i \mid i \in I\}$ a well-ordered basis of L such that $I_0 \subset I$. Denote by $\mathrm{Ud}(L) = D(X \mid x_i \dashv x_j - x_j \vdash x_i - [x_i, x_j])$ the universal enveloping dialgebra for L and the ordering $<$ on $[X^*]$ as mentioned before in Definition 2.1.11. Then* *(i)* $D(X \mid x_i \dashv x_j - x_j \vdash x_i - [x_i, x_j]) = D(X \mid S)$, *where* S *contains the following polynomials:*

(a) $f_{ij} = x_j \vdash x_i - x_i \dashv x_j + [x_i, x_j]$

(b) $f_{ji\vdash t} = x_j \vdash x_i \vdash x_t - x_i \vdash x_j \vdash x_t + [x_i, x_j] \vdash x_t$

(c) $h_{i_0\vdash t} = x_{i_0} \vdash x_t$

(d) $f_{t\dashv ji} = x_t \dashv x_j \dashv x_i - x_t \dashv x_i \dashv x_j + x_t \dashv [x_i, x_j]$

(e) $h_{t\dashv i_0} = x_t \dashv x_{i_0}$

(ii) S *is a Groebner-Shirshov basis in* $D(X)$.

(iii) *The set*

$$\{x_j \dashv x_{i_1} \dashv \cdots \dashv x_{i_k}\}$$

with $j \in I,\ i_p \in I \setminus I_0,\ 1 \le p \le k,\ i_1 \le \cdots \le i_k,\ k \ge 0$,

is a linear basis of the universal enveloping dialgebra $\mathrm{Ud}(L)$.

Proof. (i) Consider the following equations

$$f_{ji\vdash t} = f_{ji} \vdash x_t \quad \text{and} \quad f_{ji} \vdash x_t + f_{ij} \vdash x_t = ([x_i, x_j] + [x_j, x_i]) \vdash x_t,$$

then (b) and (c) are in the ideal generated by f_{ji}. Similarly, (d) and (e) are in the ideal generated by f_{ji}.

(ii) In order to show that S is a Groebner-Shirshov basis, we must prove that all compositions in S are trivial modulo S. We first extend linearly the functions $f_{ji}, f_{ji\vdash t}, h_{i_0\vdash t}, f_{t\dashv ij}$ and $h_{t\dashv i_0}$ respectively to $f_{j[p,q]}, f_{ji\vdash[p,q]}, h_{[p,q]\dashv i_0}$, etc.

For example, if $[p, q] = \sum \alpha_{pq}^s x_s$, then

$$f_{j[p,q]} = x_j \vdash [x_p, x_q] - [x_p, x_q] \dashv x_j + [[x_p, x_q].x_j] = \sum \alpha_{pq}^s f_{js},$$

$$f_{ji\vdash[p,q]} = \sum \alpha_{pq}^{s}(x_j \vdash x_i \vdash x_s - x_i \vdash x_j \vdash x_s + [x_i, x_j] \vdash x_s)$$
$$= f_{ji} \vdash [x_p, x_q],$$

and we have

$$h_{[p,q]\dashv i_0} = \sum \alpha_{pq}^{s} h_{s\dashv i_0}.$$

By using Leibniz identity, we have

$$[a, [b, b]] = 0 \text{ and } [a, [b, c] + [c, b]] = 0,$$

for all $a, b, c \in L$. Therefore, for any $i_0 \in I_0$ and $j \in I$,

$$[x_j, x_{i_0}] = 0.$$

Note that $[x_{i_0}, x_j] = [x_j, x_{i_0}] + [x_{i_0}, x_j]$, and so we have $[x_{i_0}, x_j] \in L_0$. This implies that L_0 is an ideal of L.

Let compute all possible compositions in S.

(1) Left and right multiplication compositions.

We compute the left multiplication composition in S which is related to $(a), (b)$ and (c). The right multiplication composition is computed for $(a), (d)$ and (e) in the same way.

$$x_s \dashv f_{ji} = f_{s\dashv ji} \ (j > i),$$
$$x_s \dashv f_{ji} = -f_{s\dashv ij} + x_s \dashv ([x_i, x_j] + [x_j, x_i]) \ (j < i),$$
$$x_s \dashv f_{ii} = x_s \dashv [x_i, x_i],$$
$$x_s \dashv f_{ji\vdash t} = f_{s\dashv ji} \dashv x_t \ (j > i)$$
$$x_s \dashv h_{i_0\vdash t} = h_{s\dashv i_0} \dashv x_t,$$

it is clear that all cases are trivial modulo S.

(2) Intersection and inclusion compositions.

Denote by $(a \wedge b)$ the composition of the polynomials of type (a) and (b). It is noted that since (b) and (c) are both left normed, the corresponding compositions of the cases of $(b \wedge b), (b \wedge c), (c \wedge c)$ and $(c \wedge b)$ must be a linear combination of left normed S-diwords in which the leading term of each S-diword is less that w. The same way is considered for (d) and (e).

In order to know how to use the compositions of the elements in S, we provide more details on the compositions of $(a \wedge c)$, $(a \wedge d)$, $(a \wedge e)$ and $(b \wedge a)$ as follows.

- $(a \wedge c)$, $[w] = x_{i_0} \vdash x_i$ $(i_0 \in I_0)$.

$$\begin{aligned}(f_{i_0 i} \wedge h_{i_0 \vdash i})_{[w]} &= x_i \dashv x_{i_0} + [x_i, x_{i_0}] = -h_{i \dashv i_0} \\ &\equiv 0 \bmod (S, [w]).\end{aligned}$$

- $(a \wedge d)$, $[w] = x_j \vdash x_i \dashv x_q \dashv x_p$ $(q > p)$.

$$\begin{aligned}(f_{ji} \wedge f_{i \dashv qp})_{[w]} &= -x_i \dashv x_j \dashv x_q \dashv x_p + [x_i, x_j] \dashv x_q \dashv x_p \\ &\quad + x_j v \vdash x_i \dashv x_p \dashv x_p - x_j \vdash x_i \dashv [x_p, x_q] \\ &= -x_i \dashv f_{j \dashv qp} + f_{\{i,j\} \dashv qp} + f_{ji} \dashv x_p \dashv x_q \\ &\quad - f_{ji} \dashv [x_p, x_q] \\ &\equiv 0 \bmod (S, [w]).\end{aligned}$$

- $(a \wedge e)$, $[w] = x_j \vdash x_i \dashv_{i_0}$,

$$\begin{aligned}(f_{ji} \wedge h_{i \dashv i_0})_{[w]} &= -x_i \dashv x_j \dashv x_{i_0} + [x_i, x_j] \dashv x_{i_0} \\ &= -x_i \dashv h_{j \dashv i_0} + h_{i,j \dashv i_0} \\ &\equiv 0 \bmod (S, [w]).\end{aligned}$$

- $(b \wedge a)$, then we must consider the two following cases: $[w_1] = x \dashv y \dashv z$ or $[w_2] = x_j \vdash x_i \vdash x_t \vdash x_p$

$$* \quad (f_{ji\vdash t} \wedge f_{it})_{[w_1]} = -x_i \vdash x_j \vdash x_t + [x_i, x_j] \vdash x_t + x_j \vdash x_t \dashv x_i - x_j \vdash [x_t, x_i]$$
$$= -x_i \vdash f_{jt} + f_{\{i,j\}t} + f_{jt} \dashv x_i - f_{j\{t,i\}} + f_{i\{t,j\}} - f_{it} \dashv x_j + f_{t\dashv ji}$$
$$\equiv 0 \bmod (S, [w]).$$

$$* \quad (f_{ji\vdash t} \wedge f_{tp})_{[w_2]} = -x_i \vdash x_j \vdash x_t \vdash x_p + [x_i, x_j] \vdash x_t \vdash x_p + x_j \vdash x_i \vdash x_p \dashv x_t - x_j \vdash x_i \vdash [x_p, x_t]$$
$$= -x_i \vdash x_j \vdash f_{tp} + [x_i, x_j] \vdash f_{tp} + f_{ji\vdash p} \dashv x_t - f_{ji\vdash\{p,t\}} - f_{it} \dashv x_j + f_{t\dashv ji}$$
$$\equiv 0 \bmod (S, [w]).$$

For an extensive proof of the theorem which includes all other possible compositions of the elements in S (see [8]). □

The following corollary states PBW-theorem for Leibniz algebras based on the Groebner-Shirshov bases for the universal enveloping dialgebra.

Corollary 2.2.15 ([2, 8]). *Let the notation be as in Theorem 2.2.14. Then as linear spaces,* $\mathrm{Ud}(L)$ *is isomorphic to* $L \otimes U(L_{\mathrm{Lie}})$*, where* $U(L_{\mathrm{Lie}})$ *is the universal enveloping algebra of the Lie algebra* L_{Lie}.

Proof. Clearly, $\{x_j \mid j \in I - I_0\}$ is a $\mathbb{K}$-basis of the Lie algebra $L_{\mathrm{Lie}} = L/L_0$. The universal enveloping algebra $U(L_{\mathrm{Lie}})$ has a basis

$$\{x_{i_1} x_{i_2} \ldots x_{i_k} \mid i_1 \leq \cdots \leq i_k,\ i_p \in I - I_0,\ 1 \leq p \leq k,\ k \geq 0\}.$$

By using (iii) in Theorem 2.2.14, the result holds. □

The following corollary holds for the universal enveloping algebra of Lie algebras (see [20]). We consider the analogous for the universal enveloping dialgebra of Leibniz algebras.

Corollary 2.2.16. *Let A be subalgebra of a Leibniz algebra L, and extend an ordered basis of A to an ordered basis of L. Then the homomorphism* $\mathrm{Ud}(A) \to \mathrm{Ud}(L)$ *induced by the injection* $A \to L \to \mathrm{Ud}(L)$ *is itself injective, and* $\mathrm{Ud}(L)$ *is a free* $\mathrm{Ud}(A)$*-bimodule with free basis.*

Chapter 3

HNN-extension and embedding theorems

In Section 3.1 we recall the description of HNN-extension for groups, the algebraic structure for which, the HNN-extension was originally introduced. We also recall the HNN-extension for semigroups and rings in Section 3.1. In Section 3.2 we mainly follow [25] and [39] for the construction of HNN-extension of Lie algebras. Section 3.3 is devoted to the construction of HNN-extension for dialgebras and Leibniz algebras. In Section 3.4 we define the HNN-extension for Leibniz algebras.

3.1 HNN-extension for groups

Let A_1 and A_2 are subgroups of a group G. Then it is possible to find a group H containing G such that A_1 and A_2 are conjugate to each other in H and such that G is embeddable in H. A necessary condition for this is that A_1 and A_2 are isomorphic. The following theorem shows that this condition is also enough; and indeed that any isomorphism between A_1 and A_2 can be realized by transformation by a suitable element in a group containing G.

Theorem 3.1.1 ([18]). *Let μ be an isomorphism of a subgroup A_1 of a group G onto a second subgroup A_2 of G. Then there exists a group H containing G, and an element t of H, such that the transform by t of any element of A_1 is its image under μ:*

$$t^{-1}at = \mu(a) \quad \textit{for all } a \in A_1. \tag{3.1.1}$$

Proof. Let form two free products

$$K = G * \{u\}, \quad L = G * \{v\},$$

where $\{u\}$ and $\{v\}$ are free cyclic groups. We consider the subgroup U of K generated by G and $u^{-1}A_1u$:

$$U = \{G, u^{-1}A_1u\}.$$

U is the free product of G and $u^{-1}A_1u$, $U = G * u^{-1}A_1u$. If $g_1, \dots, g_r$ are elements not equal to 1 in G, and

$$u^{-1}a_1u, \dots, u^{-1}a_ru$$

are elements not equal to 1 in $u^{-1}A_1u$, then

$$g_1u^{-1}a_1u \cdots g_ru^{-1}a_ru$$

is not equal to 1 in $K = G*\{u\}$. Similarly the subgroup V of L generated by G and vA_2v^{-1} is their free product:

$$V = G * vA_2v^{-1}.$$

U and V are therefore isomorphic; and specially there is an isomorphism of U onto V which maps G, as subgroup of U, identically onto G, as subgroup of V, and the element $u^{-1}au$ of $u^{-1}A_1u$ onto the element $v\mu(a)v^{-1}$

of vA_2v^{-1}. Finally, we form the free product H of K and L, amalgamating the subgroups U and V according to this isomorphism. Then H contains G as a subgroup; and if $a \in A_1$, then $u^{-1}au$ has been identified in H with $v\mu(a)v^{-1}$, so that

$$v^{-1}u^{-1}auv = \mu(a) \quad \text{for all } a \in A_1.$$

Therefore, the element $t = uv$ satisfies (3.1.1) and the theorem is proved. □

The construction of the HNN-extension for semigroups is similar to the case of groups (see [19]). Let S be a semigroup containing two unitary subsemigroups S_1 and S_2 and let μ be an isomorphism from S_1 onto S_2. Then there exists a semigroup T with identity containing S^{+1} as a unitary subsemigroup and containing also two elements t and t^{-1} such that

$$tt^{-1} = t^{-1}t = 1,$$

and for all $s \in S_1$,

$$t^{-1}st = \mu(s).$$

Note that S^{+1} is a semigroup with identity obtain from S simply by adjoining an identity element 1.

One of the applications of the HNN-extension is providing embeddability theorems. Accordingly, Higman, Neumann and Neumann proved the following theorem for groups.

Theorem 3.1.2 ([18]). *Any countable group G can be embedded in a group H generated by only two elements. If the number of defining relations for G is n, the number of defining relations for H can be taken to be n.*

For the case of an associative ring R with identity, we provide the notion of the HNN-extension with respect to [15]. Let R_1 and R_2 two rings, and $\mu_1\colon R_1 \to R_2$ and $\mu_2\colon R_1 \to R_2$ two ring homomorphisms.

The HNN-extension is the ring $R = R_2[t, t^{-1}\,;\ \mu_1, \mu_2]$ presented on the generators and relations of R_2 together with two new generators t, t^{-1} and new relations saying that t, t^{-1} are mutually inverse and that $t^{-1}\mu_1(r)t = \mu_2(r)$ for all $r \in R_1$. With this presentation, R is an R_2-ring, i.e. there is a ring homomorphism $\eta\colon R_2 \to R$.

In the next sections we provide the construction of HNN-extension for Lie algebras, dialgebras and Leibniz algebras. The main difference between construction of HNN-extension for groups and algebras is that the concepts of subgroups and isomorphism are replaced by subalgebras and derivation, respectively. In other words, we use the derivation map defined on a subalgebra instead of isomorphism between subgroups.

3.2 Construction of the HNN-extension for Lie algebras

Let consider a subalgebra $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ and a derivation defined as $d\colon \mathfrak{a} \to \mathfrak{g}$. We define the HNN-extension of $\mathfrak{g}$ which is a Lie algebra given by the presentation

$$\mathfrak{h} = \langle \mathfrak{g}, t \mid [t, a] = d(a) \ \text{ for all } a \in \mathfrak{a} \rangle,$$

where t is a new generating letter with the relation $[t, a] = d(a)$ such that $a, d(a) \in \mathfrak{g}$. Let consider X be a $\mathbb{K}$-linear basis of $\mathfrak{g}$ including a linear basis of the subalgebra $\mathfrak{a}$ which is denoted by B. The structure constants of $\mathfrak{g}$ are as follows.

$$[x, y] = \sum_v \alpha_{xy}^v v$$

where $x, y \in X$. $\mathfrak{a}$ is a subalgebra then we have

$$\alpha_{ab}^v = 0 \ \text{ for } a, b \in B \ \text{ and } v \notin B.$$

Since $d(a) \in \mathfrak{g}$ then it is a linear combination of basis elements

$$d(a) = \sum_v \beta_a^v v$$

for $a \in B$. Therefore we obtain the following equivalent presentation for $\mathfrak{h}$:

$$\mathfrak{h} = \langle X, t \mid [x,y] = \sum_v \alpha_{xy}^v v,\ [t,a] = \sum \beta_a^v v,\ x,y \in X,\ a \in B\rangle. \quad (3.2.1)$$

3.2.1 Triviality of compositions

Now we investigate the possible compositions of polynomials of the presentation (3.2.1) and show that are trivial. At first, we find some relations between the structure constants. The Jacobi identity implies the following equation:

$$\sum_v (\alpha_{xy}^v \alpha_{vz}^u + \alpha_{yz}^v \alpha_{vx}^u + \alpha_{zx}^v \alpha_{vy}^u) = 0, \quad (3.2.2)$$

for every $x, y, z, u \in X$. With respect to the Lie bracket we have $\alpha_{xx}^u = 0$ and $\alpha_{xy}^u = -\alpha_{yx}^u$. The condition of derivation $d([a,b]) = [d(a), b] + [a, d(b)]$ is written as:

$$\sum_v \alpha_{ab}^v \beta_v^u = \sum_v (\beta_b^v \alpha_{av}^u + \beta_a^v \alpha_{vb}^u), \quad (3.2.3)$$

for every $a, b \in B$ and $u \in X$ such that if $v \notin B$ then $\alpha_{ab}^v = 0$. We write the relations of the elements of $\mathfrak{h}$ in the following ways:

$$f_{xy} = [x,y] - \sum_v \alpha_{xy}^v v$$

and

$$g_a = [t,a] - \sum_v \beta_a^v v.$$

If we order the basis of $\mathfrak{g}$ then the elements of the basis of its subalgebra is smaller than the elements of $X \setminus B$. In addition, the letter t is an element of $\mathfrak{g}$ that can be represented as linear combination of the ordered elements of the basis in $\mathfrak{g}$, therefore t is the largest letter. Let suppose an associative Lyndon-Shirshov word (see Definition 2.1.2), xyz with $x, y, z \in X$ and $x > y > z$ and the words tab with $a, b \in B$ and

$a > b$. Then it is obvious that the possible composition is the intersection composition. Therefore, we calculate the intersection composition corresponding to the word xyz with $x > y > z$ with respect to Subsection 2.1.3 (see page 33).

For Lie polynomials f_{xy}, f_{yz} and the defined ordering we have $\overline{f_{xy}} = xy$ and $\overline{f_{yz}} = yz$. We know that xyz is an associative Lyndon-Shirshov word, so replace xy with f_{xy}.

$$\begin{aligned}
(f_{xy}, f_{yz})_w &= [f_{xy}, z] - [x, f_{yz}] \\
&= \big[[x, y] - \sum \alpha_{xy}^v v, z\big] - \big[x, [y, z] - \sum \alpha_{yz}^v v\big] \\
&= \big[[x, y, z]\big] - \sum \alpha_{xy}^v [v, z] - \big[x, [y, z]\big] + \sum \alpha_{yz}^v [x, v] \\
&\quad \text{using Jacobi's identity this equals to} \\
&= \big[[x, z], y\big] - \sum \alpha_{xy}^v \big([v, z] - \sum \alpha_{vz}^u u\big) \\
&+ \sum \alpha_{yz}^v \big([x, v] - \sum \alpha_{xv}^u u\big) - \sum \alpha_{xy}^v \alpha_{vz}^u u \\
&+ \sum \alpha_{yz}^v \alpha_{xv}^u u \\
&= \big[[x, z] - \sum \alpha_{xz}^v v, y\big] + \sum \alpha_{xz}^v \big([v, y] - \sum \alpha_{vy}^u u\big) \\
&- \sum \alpha_{xy}^v f_{vz} + \sum \alpha_{yz}^v f_{xv} \\
&- \sum_u \Big(\sum \alpha_{xy}^v \alpha_{vz}^u + \sum \alpha_{yz}^v \alpha_{vx}^u + \sum \alpha_{zx}^v \alpha_{vy}^u\Big) u
\end{aligned}$$

and using (3.2.2) we have

$$(f_{xy}, f_{yz})_w = [f_{xz}, y] + \sum \alpha_{xz}^v f_{vy} - \sum \alpha_{xy}^v f_{vz} + \sum \alpha_{yz}^v f_{xv}.$$

As we have $\overline{(f_{xy}, f_{yz})_w} = xzy < xyz$, then the composition is trivial. Let consider g_a and f_{ab}. We have $\overline{g_a} = ta$ and $\overline{f_{ab}} = ab$, then for the associative Lyndon-Shirshov word $w = tab$ with $a, b \in B$ and $a > b$, we compute the intersection composition:

$$\begin{aligned}(g_a, f_{ab})_w &= [g_a, b] - [t, f_{ab}] \\ &= [[t,a] - \sum \beta_a^v v, b] - [t, [a,b] - \sum \alpha_{ab}^v v] \\ &= [[t,a],b] - \sum \beta_a^v [v,b] - [t,[a,b]] + \sum \alpha_{ab}^v [t,v]\end{aligned}$$

noting that only $v \in B$ appears in the second sum

$$\begin{aligned}&= [[t,b],a] - \sum \beta_a^v([v,b] - \sum \alpha_{vb}^u u) \\ &+ \sum \alpha_{ab}^v([t,v] - \sum \beta_v^u u) - \sum \beta_a^v \alpha_{vb}^u u + \sum \alpha_{ab}^v \beta_v^u u \\ &= [[t,b] - \sum \beta_b^v v, a] + \sum \beta_b^v([v,a] - \sum \alpha_{va}^u u) \\ &- \sum \beta_a^v f_{vb} + \sum \alpha_{ab}^v g_v \\ &+ \sum_u (\sum \alpha_{ab}^v \beta_v^u - \sum \beta_b^v \alpha_{av}^u + \sum \beta_a^v \alpha_{vb}^u) u\end{aligned}$$

and using (3.2.3) we obtain

$$(g_a, f_{ab})_w = [g_b, a] + \sum \beta_b^v f_{va} - \sum \beta_a^v f_{vb} + \sum \alpha_{ab}^v g_v,$$

and so, $\overline{(g_a, f_{ab})_w} = tba$, which is less than tab and the intersection composition is trivial. There is no inclusion composition of polynomials and the presentation (3.2.1) is trivial modulo S, and therefore

$$S = \{f_{xy}, g_a \mid \ x, y \in X,\ a \in B\} \tag{3.2.4}$$

is a Groebner-Shirshov basis.

3.2.2 Embedding of Lie algebras

In this subsection we recall important theorems regarding embedding of Lie algebras. We recall the approach of Wasserman [39] to prove that every Lie algebra of finite or countable dimension can be embedded in a Lie algebra with two generators and the same number of relations.

According to the triviality of all the compositions of polynomials in the presentation (3.2.1) and considering the definition of S-irreducibility for Lie algebras (see Definition 2.2.6), we obtain an S-irreducible set for the elements of the HNN-extension $\mathfrak{h}$ where S is Groebner-Shirshov basis determined in (3.2.4). To this end, we have

$$\mathrm{Irr}(S) = \{[f] \mid f \text{ is ALSW}, f \neq xy,\ f \neq ta,\ x, y \in X,\ a \in B\}.$$

In other words, a $\mathbb{K}$-linear basis for HNN-extension $\mathfrak{h}$ is given by all the associative Lyndon-Shirshov words on $X \cup \{t\}$ which do not contain subwords of the form $xy \in X$ such that $x > y$ or of the form ta with $a \in B$.

The following theorem shows that a Lie algebra $\mathfrak{g}$ is embedded in its HNN-extension.

Theorem 3.2.1. *$\mathfrak{g}$ is embedded in its HNN-extension $\mathfrak{h}$.*

Proof. Since all the elements of X do not belong to $\mathrm{Irr}(S)$, we have that $\mathfrak{g}$ is embedded in $\mathfrak{h}$. □

Wasserman [39] proved the following lemma regarding extension of arbitrary maps to a derivation map in order to prove the embedding theorem.

Lemma 3.2.2 ([39])**.** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ a subalgebra which is free on a subset X. Then every map $d\colon X \to \mathfrak{g}$ can be extended to a derivation from $\mathfrak{a}$ to $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}_1$ be an abelian Lie algebra with underlying linear space $\mathfrak{g}$. Define $h\colon \mathfrak{a} \to \mathrm{Der}(\mathfrak{g}_1)$ by $h(g)(a) = [g, a]$. It is clear that $h(g)$ is in $\mathrm{Der}(\mathfrak{g}_1)$. h is a Lie algebra homomorphism.

Define $B = \mathfrak{a} \ltimes \mathfrak{g}_1$ and a Lie algebra homomorphism $f\colon \mathfrak{a} \to B$ given by $f(x) = (x, d(x))$ for every $x \in X$. Write $f(a) = (u(a), D(a))$. Then

$$\begin{aligned}(u([a_1, a_2]), D([a_1, a_2])) &= f([a_1, a_2]) = [(u(a_1), D(a_1)), (u(a_2), D(a_2))] \\ &= ([u(a_1), u(a_2)], [u(a_1), D(a_2)] - [u(a_2), D(a_1)]).\end{aligned}$$

Therefore, $u([a_1, a_2]) = [u(a_1), u(a_2)]$ which implies $u(a) = a$ for every $a \in \mathfrak{a}$. Using this

$$D([a_1, a_2]) = [a_1, D(a_2)] - [a_2, D(a_1)] = [a_1, D(a_2)] + [D(a_1), a_2],$$

and so D is a derivation which extends d. □

The following theorem is an analogue of an embedding theorem known to hold in groups (see Theorem 3.1.2).

Theorem 3.2.3 ([39]). *Every finite or countable-dimensional Lie algebra $\mathfrak{g}$ can be embedded in a Lie algebra $\mathfrak{h}$ generated by two elements. If $\mathfrak{g}$ has a presentation with n relations, so does $\mathfrak{h}$.*

Proof. Let

$$\mathfrak{a} = \langle c_1, c_2, \dots \mid R \rangle$$

be a presentation for $\mathfrak{a}$ using countably many generators. Define $\mathfrak{a}_1 = \mathfrak{a} * \mathfrak{g}(a, b)$, the free product of $\mathfrak{a}$ with a free Lie algebra of rank 2. The elements $a, ba, b^2a, \dots, b^na, \dots$ (by b^na we mean $[b, \dots, [b, a], \dots]$ where b appears n times) are a basis for a free Lie subalgebra of $\mathfrak{g}(a, b)$ (see Lazard elimination method, [33, Theorem 0.6]). Therefore, using Lemma 3.2.2, there exists a derivation from that subalgebra to $\mathfrak{a}_1$ that maps $a \to b$ and $b^na \to c_n$. Consider the corresponding HNN-extension

$$\mathfrak{h} = \langle \mathfrak{a}_1, t \mid [t, b^na] = c_n,\ [t, a] = b \rangle.$$

Then $\mathfrak{a}$ is embedded in $\mathfrak{h}$ and $\mathfrak{h}$ is generated by t and a. Note that all the HNN relations can be eliminated by Tietze transformations, so that if R has n relations, we can get a presentation for $\mathfrak{h}$ with n relations. □

3.3 Construction of the HNN-extension for dialgebras

Let D be a dialgebra and A be a subalgebra of D. We assume a derivation $d\colon A \to D$. We define the associated HNN-extension to be a dialgebra given by the presentation:

$$D_d^* = \langle D, t \mid a \dashv t - t \vdash a = d(a),\ a \in A \rangle.$$

We give an equivalent presentation for HNN-extension through a special set of structural constants. Let Y be a totally well-ordered basis of A and extend it to a basis of D. Suppose this basis is X and with a total ordering defined in Definition 2.1.11 such that $Y < X \setminus Y$, i.e. every element of Y is smaller than the remaining elements of X. Assume further that $t < X$.

1) Let $x, y \in X$. We have $x \dashv y = \sum_v \alpha_{xy}^v v$ for some $\alpha_{xy}^v \in \mathbb{K}$. Similarly, we have $x \vdash y = \sum_v \beta_{xy}^v v$ for some scalars β_{xy}^v. According to the associativity laws of Definition 1.2.2, we have the following relations between the structural constants:

$$\sum_{v,u} \beta_{yz}^v \alpha_{xv}^u = \sum_{v,u} \alpha_{yz}^v \alpha_{xv}^u,$$

$$\sum_{v,u} \alpha_{xy}^v \beta_{vz}^u = \sum_{v,u} \beta_{yz}^v \alpha_{xv}^u,$$

$$\sum_{v,u} \alpha_{yz}^v \beta_{xv}^u = \sum_{v,u} \beta_{xy}^v \alpha_{vz}^u.$$

Note that, since A is a subalgebra, we have $\alpha_{xy}^v = \beta_{xy}^v = 0$, for $x, y \in Y$ and $v \in X \setminus Y$.

2) Consider the derivation $d\colon A \to D$. For any $x \in Y$, there are scalars δ_x^v such that

$$d(x) = \sum_v \delta_x^v v.$$

Now, let $X' := X \cup \{t\}$ and define the following elements of the free dialgebra $D(X')$:

- $f_1(x,y) = x \dashv y - \sum_v \alpha_{xy}^v v$
- $f_2(x,y) = x \vdash y - \sum_v \beta_{xy}^v v$
- $g_x = x \vdash t - t \dashv x + \sum_v \delta_x^v v.$

Then D has the following presentation

$$D = \langle X \mid f_1(x,y),\ f_2(x,y),\ \ x,y \in X\rangle.$$

We extend this presentation to obtain a presentation of the HNN-extension D_d^*

$$D_d^* = \langle X' \mid f_1(x,y),\ f_2(x,y)\ \ x,y \in X;\ g_x\ \ x \in Y\rangle. \tag{3.3.1}$$

3.3.1 Triviality of compositions

Let S to be the subset of $D(X')$ including the following polynomials

$$\{f_1(x,y),\ f_2(x,y) : x,y \in X\} \cup \{g_x : x \in Y\}.$$

If all the possible compositions of elements of S are trivial modulo S then we obtain a Groebner-Shirshov basis, otherwise we must reduce every nontrivial composition to a polynomial modulo S and add this polynomial to the S. Having repeated this process results a Groebner-Shirshov basis denoted by S^{comp}. This process is called Buchberger-Shirshov algorithm.

To this end, we first check triviality of the possible compositions of elements S using Left-mc, Right-mc, Inter-c and Inc-c.

1) Since the polynomial $f_1(x,y)$ is right normed, so no left multiplication composition is defined for it.

2) The polynomial $f_2(x,y)$ is not right normed, then for an arbitrary $z \in X'$, the left multiplication $z \dashv f_2(x,y)$ is defined. We have the trivial equality

$$z \dashv f_2(x,y) = [zs],$$

where $s = f_2(x, y) \in S$ is a normal S-diword and

$$\overline{|[zs]|} = |z \dashv (x \vdash y)| = 3 \leq \deg(z \dashv f_2(x, y)).$$

Hence $z \dashv f_2(x, y) \equiv 0 \pmod{S}$.

3) For $x \in Y$, the polynomial g_x is not right normed, so we have the left multiplication $z \dashv g_x$. We have

$$z \dashv g_x = [zs],$$

where $s = g_x \in S$ is normal S-diword and

$$|\overline{[zs]}| = |z \dashv x \dashv t| = 3 \leq \deg(z \dashv g_x).$$

4) Since the polynomial $f_2(x, y)$ is left normed, so no right multiplication composition is defined for it.

5) The polynomial $f_1(x, y)$ is not left normed, then for an arbitrary $z \in X'$, the right multiplication $f_1(x, y) \vdash z$ is defined. We have the trivial equality

$$f_1(x, y) \vdash z = [sz],$$

where $s = f_1(x, y) \in S$ is a normal S-diword and

$$\overline{|[sz]|} = |(x \dashv y) \vdash z| = 3 \leq \deg(f_1(x, y) \vdash z).$$

Hence $f_1(x, y) \vdash z \equiv 0 \pmod{S}$.

6) For $x \in Y$, the polynomial g_x is not left normed, so we have the right multiplication $g_x \vdash z$. We have

$$g_x \vdash z = [sz],$$

where $s = g_x \in S$ is normal S-diword and

$$|\overline{[sz]}| = |x \vdash t \vdash z| = 3 \leq \deg(g_x \vdash z).$$

7) We compute the intersection compositions. Suppose $f = f_1(x, y)$ and $g = f_1(y, z)$. Then $\bar{f} = x \dashv y$ and $\bar{g} = y \dashv z$. So, $[w] = x \dashv y \dashv z$ and clearly $|\bar{f}| + |\bar{g}| > |w|$. Hence

$$\begin{aligned}
(f,g)_{[w]} &= [fz] - [xg] \\
&= (x \dashv y \dashv z) - \sum_v \alpha^v_{xy} v \dashv z - (x \dashv y \dashv z) + \sum_v \alpha^v_{yz} x \dashv v \\
&= \sum_v \alpha^v_{yz} x \dashv v - \sum_v \alpha^v_{xy} v \dashv z \\
&= \sum_v \alpha^v_{yz} (x \dashv v - \sum_u \alpha^u_{xv} u + \sum_u \alpha^u_{xv} u) \\
&- \sum_v \alpha^v_{xy} (v \dashv z - \sum_u \alpha^u_{vz} u + \sum_u \alpha^u_{vz} u) \\
&= \sum_v \alpha^v_{yz} f_1(x, v) - \sum_v \alpha^v_{xy} f_1(v, z) + \sum_{v,u} \alpha^v_{yz} \alpha^u_{xv} u \\
&- \sum_{v,u} \alpha^v_{xy} \alpha^u_{vz} u \\
&= \sum_v \alpha^v_{yz} f_1(x, v) - \sum_v \alpha^v_{xy} f_1(v, z).
\end{aligned}$$

We have $f_1(x, v), f_1(v, z) \in S$ and

$$\overline{[f_1(x, v)]} = x \dashv v < [w], \quad \overline{[f_1(v, z)]} = v \dashv z < [w].$$

This shows that $(f, g)_{[w]} \equiv 0 \pmod S$.

8) Suppose $f = f_1(x, y)$ and $g = f_2(y, z)$. Then $\bar{f} = x \dashv y$ and $\bar{g} = y \vdash z$. So, $[w] = x \dashv y \dashv z$ and clearly $|\bar{f}| + |\bar{g}| > |w|$. Hence

$$
\begin{aligned}
(f,g)_{[w]} &= [fz] - [xg] \\
&= (x \dashv y \dashv z) - \sum_v \alpha^v_{xy} v \dashv z - (x \dashv y \dashv z) + \sum_v \beta^v_{yz} x \dashv v \\
&= \sum_v \beta^v_{yz} x \dashv v - \sum_v \alpha^v_{xy} v \dashv z \\
&= \sum_v \beta^v_{yz} (x \dashv v - \sum_u \alpha^u_{xv} u + \sum_u \alpha^u_{xv} u) \\
&- \sum_v \alpha^v_{xy} (v \dashv z - \sum_u \alpha^u_{vz} u + \sum_u \alpha^u_{vz} u) \\
&= \sum_v \beta^v_{yz} f_1(x, v) - \sum_v \alpha^v_{xy} f_1(v, z) + \sum_{v,u} \beta^v_{yz} \alpha^u_{xv} u \\
&- \sum_{v,u} \alpha^v_{xy} \alpha^u_{vz} u \\
&= \sum_v \alpha^v_{yz} f_1(x, v) - \sum_v \alpha^v_{xy} f_1(v, z).
\end{aligned}
$$

We have $f_1(x, v), f_2(v, z) \in S$ and

$$
\overline{[f_1(x, v)]} = x \dashv v < [w], \quad \overline{[f_2(v, z)]} = v \dashv z < [w].
$$

This shows that $(f, g)_{[w]} \equiv 0 \pmod{S}$.

9) Let $f = f_1(x, y)$ and $g = g_y$. We show that the composition $(f, g)_{[w]}$ may not be trivial so we must add it to S. Note that in this case $[w] = (x \dashv y) \vdash t = x \vdash y \vdash t$, so a composition shows that

$$
\begin{aligned}
(f,g)_{[w]} &= [ft] - [xg] \\
&= (x \vdash y \vdash t) - \sum_v \alpha^v_{xy} v \vdash t - (x \vdash y \vdash t) + (x \vdash t \dashv y) \\
&+ \sum_v \delta^v_y x \vdash v \\
&= x \vdash t \dashv y - \sum_v (\delta^v_y x \vdash v + \alpha^v_{xy} v \vdash t)
\end{aligned}
$$

According to the triviality conditions, $(f,g)_{[w]}$ is not trivial modulo S. Denote this new polynomial by $h^{\dashv}(x,y)$.

10) We also consider the intersection composition of $f = f_2(x,y)$ and $g = g_y$. Note that in this case $[w] = (x \vdash y) \vdash t = x \vdash y \vdash t$, so a composition shows that

$$\begin{aligned}(f,g)_{[w]} &= [ft] - [xg] \\ &= (x \vdash y \vdash t) - \sum_v \beta^v_{xy} v \vdash t - (x \vdash y \vdash t) + (x \vdash t \dashv y) \\ &+ \sum_v \delta^v_y x \vdash v \\ &= x \vdash t \dashv y - \sum_v (\delta^v_y x \vdash v + \beta^v_{xy} v \vdash t)\end{aligned}$$

Denote this new polynomial by $h^{\vdash}(x,y)$. So, we extend S to the larger set

$$S_1 = S \cup \{h^{\dashv}(x,y),\ h^{\vdash}(x,y) :\ x \in X,\ y \in Y\}.$$

There is no inclusion composition between the elements of S_1, but there are some intersection compositions as follows:

11) We compute the intersection composition of $f = f_1(x,y)$ and $h = h^{\dashv}(y,z)$. Note that $[w] = (x \dashv y) \vdash t \dashv z = x \vdash (y \vdash t \dashv z) = x \vdash y \vdash t \dashv z$. The intersection composition is

$$\begin{aligned}(f,h)_{[w]} &= [f(t \dashv z)] - [xh] \\ &= (x \vdash y \vdash t \dashv z) - \sum_v \alpha^v_{xy}(v \vdash t \dashv z) - (x \vdash y \vdash t \dashv z) \\ &+ \sum_v (\delta^v_z x \vdash y \vdash v + \alpha^v_{yz} x \vdash v \vdash t) \\ &= \sum_v (\delta^v_z x \vdash y \vdash v - \alpha^v_{xy} v \vdash t \dashv z + \alpha^v_{yz} x \vdash v \vdash t)\end{aligned}$$

12) We compute the intersection composition of $f = f_2(x,y)$ and $h = h^{\dashv}(y,z)$. Now $[w] = (x \dashv y) \vdash t \dashv z = x \vdash (y \vdash t \dashv z) = x \vdash y \vdash t \dashv z$.

The intersection composition is

$$\begin{aligned}(f,h)_{[w]} &= [f(t \dashv z)] - [xh] \\ &= (x \vdash y \vdash t \dashv z) - \sum_v \beta^v_{xy}(v \vdash t \dashv z) - (x \vdash y \vdash t \dashv z) \\ &+ \sum_v (\delta^v_z x \vdash y \vdash v + \alpha^v_{yz} x \vdash v \vdash t) \\ &= \sum_v (\delta^v_z x \vdash y \vdash v - \beta^v_{xy} v \vdash t \dashv z + \alpha^v_{yz} x \vdash v \vdash t)\end{aligned}$$

13) We compute the intersection composition of $f = f_1(x,y)$ and $h = h^{\vdash}(y,z)$. We have $[w] = (x \dashv y) \vdash t \dashv z = x \vdash (y \vdash t \dashv z) = x \vdash y \vdash t \dashv z$. The intersection composition is

$$(f,h)_{[w]} = \sum_v (\delta^v_z x \vdash y \vdash v - \alpha^v_{xy} v \vdash t \dashv z + \alpha^v_{yz} x \vdash v \vdash t).$$

14) The computation of the intersection composition of $f = f_2(x,y)$ and $h = h^{\vdash}(y,z)$ implies the following equation:

$$(f,h)_{[w]} = \sum_v (\delta^v_z x \vdash y \vdash v - \beta^v_{xy} v \vdash t \dashv z + \beta^v_{yz} x \vdash v \vdash t).$$

15) We compute the intersection composition of $h_1 = h^{\dashv}(x,y)$ and $h_2 = h^{\vdash}(x,y)$. Then we have

$$\begin{aligned}(h_1,h_2)_{[w]} &= [h_1] - [h_2] \\ &= (x \vdash t \dashv y) - \sum_v (\delta^v_y x \vdash v + \alpha^v_{xy} v \vdash t) - (x \vdash t \dashv y) \\ &+ \sum_v (\delta^v_y x \vdash y + \beta^v_{yz} v \vdash t) = \sum_v (\beta^v_{xy} - \alpha^v_{xy}) v \vdash t.\end{aligned}$$

This shows that, every time we extend S_i to a larger set S_{i+1}, we attach a new polynomials of degree at least two. Therefore, let S^{comp} be the Groebner-Shirshov extension of S. All the elements of S^{comp} have degree at least two, so all the elements of X are irreducible with respect to S^{comp}.

Example 3.3.1. There are two extreme cases of HNN-extensions of di-algebras.

- If $A = D$, then d is a derivation of D and D_d^* is then the semidirect product of D with a one-dimensional di-algebra which acts on D via d.

- If $A = 0$, then D_d^* is the free product of D with a one-dimensional di-algebra.

3.3.2 Embedding of dialgebras

As it was mentioned in the previous subsection, a Groebner-Shirshov basis for a HNN-extension of the dialgebra D_d^* with the presentation (3.3.1) is obtained by extending S to S^{comp} through adding nontrivial compositions of polynomials of (3.3.1) (see Shirshov's algorithm in [7]). In order to determine S^{comp}-irreducible set, we emphasize the degree of the elements in S^{comp} which have at least degree 2 and consider the following set according to Definition 2.2.11. Then we have

$$\text{Irr}(S^{\text{comp}}) = \{[u] \in [X^*] \mid [u] \neq [a[\bar{s}]b], [asb] \text{ is a normal } s\text{-diword}\},$$

such that $s \in S^{\text{comp}}$ and $a, b \in X^*$.

Theorem 3.3.2. *Every dialgebra embeds inside its HNN-extension.*

Proof. The elements of X are of degree 1, so they are S^{comp}-irreducible and it proves that D embeds inside D_d^*. □

3.4 Construction of the HNN-extension for Leibniz algebras

Let D be an arbitrary di-algebra over a field $\mathbb{K}$. We recall Proposition 1.3.1 which defines a Leibniz algebra $D^{(-)}$ by

$$[x, y] = x \dashv y - y \vdash x.$$

For L an arbitrary Leibniz algebra over $\mathbb{K}$, the universal enveloping di-algebra for L is defined (see Definition 1.3.2). If L has a presentation $\langle X \mid S\rangle$ in the variety of Leibniz algebras, then $\mathrm{Ud}(L)$ has the presentation $\langle X \mid S^{(-)}\rangle$ in the variety of di-algebras, where $S^{(-)}$ is the set of all di-polynomials (dialgebra polynomials) obtained from the elements of S by all the brackets $[x, y]$ to $x \dashv y - y \vdash x$.

Let us recall that for any Leibniz algebra L, the Poincaré-Birkhoff-Witt theorem is valid for it (see Corollary 2.2.15). This shows that every derivation of L (or every derivation of any subalgebra of L) extends to a derivation of $\mathrm{Ud}(L)$ (to a derivation of the corresponding subalgebra of $\mathrm{Ud}(L)$) (see Corollary 2.2.16). In the following definition, we introduce the HNN-extension for the case of Leibniz algebras.

Definition 3.4.1. Let L be a Leibniz algebra and A be a subalgebra. Assume that $d\colon A \to L$ is a derivation. We define the corresponding HNN-*extension* as the Leibniz algebra

$$L^*_d = \langle L, t \mid d(a) = [a, t],\ a \in A\rangle.$$

Here t is a new symbol not belonging to L.

Example 3.4.2. There are two extreme cases of HNN-extensions of Leibniz algebras.

- If $A = L$, then d is a derivation of L and L^*_d is then the semidirect product of L with a one-dimensional Leibniz algebra which acts on L via d. We note that the semidirect product was defined in Definition 1.1.12.

- If $A = 0$, then L_d^* is the free product of L with a one-dimensional Leibniz algebra.

3.4.1 Embedding of Leibniz algebras

Theorem 3.4.3. *Every Leibniz algebra embeds inside its any HNN-extension.*

Proof. We have $\mathrm{Ud}(A) \le Ud(L)$ and the derivation d extends to a di-algebra derivation $d\colon \mathrm{Ud}(A) \to \mathrm{Ud}(L)$. We prove that $U(L_d^*) = U(L)_d^*$. If we prove this, it implies that $U(L) \le U(L_d^*)$, by Theorem 3.3.2. But, then by PBW theorem L embeds in $\mathrm{Ud}(L_d^*)$ and then by the normal forms of the bases elements of $\mathrm{Ud}(L_d^*)$, we obtain $L \le L_d^*$.

Hence, we must prove that $\mathrm{Ud}(L_d^*) = \mathrm{Ud}(L)_d^*$. Suppose that $L * \langle t \rangle$ is the free product of the Leibniz algebra L and the free Leibniz algebra generated by t. Let R be the ideal generated by all the elements of the form $d(a) - [a, t]$, $a \in A$, and $\overline{R}$ be the corresponding ideal in the di-algebra $\mathrm{Ud}(L) * \langle t \rangle$ (this second $\langle t \rangle$ is the free di-algebra generated by t). Let $i\colon L \to \mathrm{Ud}(L)^{(-)}$ be the natural embedding coming from PBW theorem. By Theorem 3.3.2 we have $\mathrm{Ud}(L) \subseteq \mathrm{Ud}(L)_d^*$, and so in fact we get $i\colon L \to \mathrm{Ud}(L)_d^*$. Hence we have a Leibniz homomorphism $\varepsilon_0 = i * \mathrm{id}\colon L * \langle t \rangle \to (\mathrm{Ud}(L)_d^*)^{(-)}$. For any $a \in A$, we have

$$\begin{aligned}\varepsilon_0(d(a) - [a, t]) &= i(d(a)) - [i(a), t] \\ &= d(a) - a \dashv t + t \vdash a \\ &= 0.\end{aligned}$$

This shows that there exists a well-defined Leibniz homomorphism $\varepsilon\colon L_d^* \to (\mathrm{Ud}(L)_d^*)^{(-)}$ which sends $x + R$ to $x + \overline{R}$ and $t + R$ to $t + \overline{R}$.

Now, assume that D is an arbitrary di-algebra and let $\varphi\colon L_d^* \to D^{(-)}$ be a Leibniz homomorphism. Let $q\colon L \to L_d^*$ be the quotient map given by $q(x) = x + R$, and consider the map $\varphi_0 = \varphi \circ q\colon L \to D^{(-)}$. Then, there exists a di-algebra homomorphism $\overline{\varphi}_0\colon \mathrm{Ud}(L) \to D$, such that $\varphi_0 = \overline{\varphi}_0 \circ i$.

Let $\eta\colon \langle t\rangle \to D$ be defined by $\eta(t) = \varphi(t + R)$ and consider the map

$$\varphi^* = \overline{\varphi}_0 * \eta\colon \ \mathrm{Ud}(L) * \langle t\rangle \to D.$$

For every generator $a \in A \subseteq \mathrm{Ud}(A)$, we have

$$\begin{aligned}
\varphi^*(d(a) - a \dashv t + t \vdash a) &= \overline{\varphi}_0(d(a)) - \overline{\varphi}_0(a) \dashv \eta(t) + \eta(t) \vdash \overline{\varphi}_0(a) \\
&= \varphi_0(d(a)) - \varphi_0(a) \dashv \varphi(t + R) \\
&+ \varphi(t + R) \vdash \varphi_0(a) \\
&= \varphi(d(a) + R) - \varphi(a + R) \dashv \varphi(t + R) \\
&\quad + \varphi(t + R) \vdash \varphi(a + R) \\
&= \varphi(d(a) - a \dashv t + t \vdash a + R) \\
&= 0.
\end{aligned}$$

This proves that there exists a di-algebra homomorphism

$$\varphi'\colon \frac{\mathrm{Ud}(L) * \langle t\rangle}{\overline{R}} = \mathrm{Ud}(L)^*_d \to D,$$

and this shows that $\mathrm{Ud}(L^*_d) = \mathrm{Ud}(L)^*_d$. □

Resumen de la Tesis Doctoral (in Spanish)

Higman-Neumann-Neumann extension and embedding theorems for Leibniz algebras

Resumen abreviado: Extensión Higman-Neumann-Neumann y teoremas de encaje para álgebras de Leibniz

En el presente trabajo se introducen las extensiones Higman-Neumann-Neumann (HNN) y los adecuados teoremas de encaje para diálgebras y álgebras de Leibniz.

Debido a la importancia de la conexión entre las diálgebras y álgebras de Leibniz y a la relación entre las álgebras asociativas y las álgebras de Lie, se recuerda la teoría de las bases de Groebner-Shirshov, y el Lema de Composición-Diamante, en álgebras asociativas y álgebras de Lie, así como la teoría de las bases de Groebner-Shirshov para diálgebras.

Como aplicación de las extensiones HNN de diálgebras y álgebras de Leibniz, proporcionamos teoremas de encaje para diálgebras y álgebras de Leibniz, respectivamente: toda diálgebra se encaja en su extensión HNN y toda álgebra de Leibniz se encaja en su extensión HNN.

Dos de las construcciones más importantes en la teoría combinatoria de grupos son el coproducto con amalgamación (pushout o producto libre amalgamado) y la construcción introducida por Higman, Neumann y Neumann [18] y llamada la extensión HNN; estas dos construcciones están estrechamente relacionadas entre sí (véase [32]).

Si A es un subgrupo de un grupo G y $t \in G$, entonces la asignación $a \mapsto t^{-1}at$ es un isomorfismo entre los dos subgrupos A y $t^{-1}At$ de G. La construcción HNN intenta invertir el punto de vista. Para un grupo G con un isomorfismo ϕ entre dos de sus subgrupos A y B, H es una extensión de G con un elemento $t \in H$ tal que $t^{-1}at = \phi(a)$ para cada $a \in A$. El grupo H está presentado por

$$H = \langle G, t \mid t^{-1}at = \phi(a), \text{ para todo } a \in A\rangle$$

e implica que G está encajado en H. La extensión HNN de un grupo alcanza una posición importante en la teoría algorítmica de grupos que se ha utilizado para la prueba del teorema de encaje, a saber, que cada grupo numerable está encajado en un grupo con dos generadores.

La extensión HNN de un grupo tiene una interpretación topológica descrita en [9, 32], la cual se utiliza como motivación para su estudio. Supongamos que U y V son ambos subespacios de un espacio topológico conexo por arcos X de tal manera que hay un homeomorfismo $h\colon U \to V$. Elegimos un punto base $u \in U$ para los grupos fundamentales de U y X. Hay un homomorfismo $\eta\colon \pi_1(U) \to \pi_1(X)$, definido considerando un lazo en U como un lazo en X, el cual es inyectivo. El homeomorfismo h induce un isomorfismo $h^*\colon \pi_1(U) \to \pi_1(V)$. Supongamos que identificamos U y V por el homeomorfismo h para obtener un nuevo espacio Z. Bajo estos supuestos el Teorema de Seifert-van Kampen dice

$$\pi_1(Z) = \langle \pi_1(X) * \pi_1(Y) \mid \pi_1(U) = \pi_1(V),\ h^*\rangle.$$

La extensión HNN de un grupo tiene una interpretación topológica similar. Sea I el intervalo unidad, y $C = U \times I$. Identificamos $U \times \{0\}$ con U y $U \times I$ con V por el homeomorfismo h. Sea Z el espacio resultante. El Teorema de Seifert-van Kampen puede ser usado para mostrar que

$$\pi_1(Z) = \langle \pi_1(X), t \mid t^{-1}\pi_1(U)t = \pi_1(V)\rangle.$$

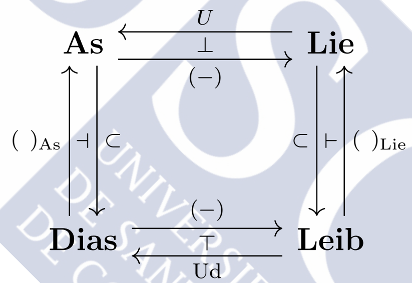
La idea de la construcción de la extensión HNN se ha extendido mediante diferentes vías a otras estructuras algebraicas como semigrupos, anillos y álgebras de Lie que son temas importantes independientes.

La extensión HNN de semigrupos fue introducido en los trabajos de Ash [1], tesis doctoral de Jackson [21] y Howie [19]. Por ejemplo, Howie utilizó un enfoque similar al caso de los grupos, que depende de las propiedades del producto libre amalgamado de semigrupos. Construyó la extensión HNN para un semigrupo S con subsemigrupos unitarios S_1 y S_2. Este concepto también se generalizó a otras clases de semigrupos, como los semigrupos inversos por Gilbert [17] y Yamamura [40].

La extensión HNN para anillos asociativos fue definida por Dicks [15] y Lichtman y Shirvani [25]. Dicks presentó el concepto de extensión HNN de anillos basado en la analogía con la noción del coproducto. El enfoque de Lichtman y Shirvani para definir la extensión HNN diferencial de anillos asociativos se encuentra en un contexto más general. Consideraron un anillo asociativo R y una familia de subanillos de R, junto con homomorfismos inyectivos y derivaciones. La construcción de las extensiones HNN diferenciales de los anillos está en concordancia con la construcción de este concepto para las álgebras envolventes universales de las álgebras de Lie.

La extensión HNN de álgebras de Lie fue construida por Lichtman y Shirvani [25] y Wasserman [39]. Lichtman y Shirvani definieron la extensión HNN para álgebras de Lie restringidas y la aplicaron para dar una nueva prueba del teorema de Shirshov, a saber, un álgebra de Lie de dimensión finita o numerable se puede encajar en un álgebra de Lie 2-generada. También dieron una prueba del hecho de que cada álgebra de Lie se puede encajar en un álgebra de Lie simple. Wasserman siguió un enfoque diferente basado en el lema de composición de Shirshov, que era el método que se había descrito en [9] hasta el momento de su trabajo: construir la extensión HNN para álgebras de Lie y utilizarla para obtener algunos resultados de encaje análogos a los de la teoría de grupos similares a los teoremas de Lichtman y Shirvani. Además, Wasserman demostró que propiedades de Markov de álgebras de Lie finitamente presentadas son indecidibles.

Las álgebras de Leibniz, introducidas por Bloh [4] y Loday [27], son una generalización no antisimétrica de las álgebras de Lie. La teoría de álgebras de Leibniz ha sido activamente investigada y muchos resultados de la teoría de álgebras de Lie han sido transferidos a las álgebras de Leibniz. Además, Loday introdujo el concepto de una diálgebra asociativa (o diálgebra, para abreviar); la generalización del corchete de Lie proporciona las diálgebras de Lie, también llamadas álgebras de Leibniz. De hecho, las diálgebras están estrechamente relacionadas con la noción de álgebras de Leibniz de la misma manera que las álgebras asociativas están relacionadas con las álgebras de Lie. El siguiente diagrama que fue descrito en [14] ilustra una adjunción entre el funtor que asigna a una diálgebra D el álgebra de Leibniz dada por $[x, y] = x \dashv y - y \vdash x$, para todo $x, y \in D$, y el funtor diálgebra envolvente universal,



donde el cuadrado interior es conmutativo y el cuadrado exterior conmuta salvo isomorfismo. Nuestro enfoque para construir la extensión HNN para álgebras de Leibniz está relacionado con la construcción de la extensión HNN para diálgebras. Para definir la extensión HNN para diálgebras, empleamos la teoría de las bases de Groebner-Shirshov y el Lema de Composición-Diamante como herramienta efectiva para determinar las bases de Groebner-Shirshov para la presentación recientemente definida de la extensión HNN.

Como explicación de las bases de Groebner-Shirshov, recordamos que la idea de la formulación de las bases de Groebner-Shirshov tiene sus raíces en el algoritmo euclidiano para polinomios en una variable, y en el algoritmo de eliminación de Gauss para polinomios lineales (de grados 1) en varias variables (véase [11]).

Para el caso de las álgebras de Lie libres, el método de las bases Groebner-Shirshov fue desarrollado por Shirshov [36] en 1958 cuando introdujo una nueva base conocida como Lyndon-Shirshov. Asimismo, existe un concepto paralelo a las bases de Groebner-Shirshov en álgebra conmutativa llamado bases de Groebner y desarrollado por Buchberger en 1965 (véanse [12, 13]). La teoría de las bases de Groebner-Shirshov se aplica a las álgebras que se presentan como álgebras libres módulo algunas relaciones y se utiliza un procedimiento algorítmico para reducir un elemento general a su forma normal [10]. En cada relación de un álgebra libre dada podemos determinar un monomio principal con respecto a un orden monomial y luego reemplazar cada monomio principal por los términos inferiores correspondientes. Entonces decimos que el elemento está en forma normal o es un elemento reducido. Shirshov en [37] introdujo la noción de *composición* de polinomios en un álgebra de Lie libre y Bergman [3] usó la noción de *ambigüedad* en lugar de *composición* para el caso de las álgebras asociativas. Si la composición de dos relaciones no es una consecuencia inmediata de las relaciones originales, la composición se añadirá al conjunto de relaciones. Si todas las composiciones son resultados inmediatos, el conjunto de relaciones se denomina una base de Groebner-Shirshov.

El Lema de Composición-Diamante es un poderoso teorema en álgebra algorítmica y combinatoria que proporciona bases lineales y formas normales de los elementos para un álgebra presentada por generadores y relaciones definidas. De hecho, el Lema de Composición-Diamante afirma que si S es una base de Groebner-Shirshov, entonces el término principal de cualquier polinomio en un ideal I del álgebra libre $F(X)$ generada por S contiene a $\bar{s}$ como una subpalabra donde $\bar{s}$ es el término principal de $s \in S$. La principal consecuencia es que la forma normal de los elementos en $F(X)/\operatorname{Id}(S)$ consiste en $\mathbb{K}$-combinaciones lineales de palabras sin ninguna subpalabra $\bar{s}$. Bokut en [6] enunció el Lema de Composición-Diamante para el caso de álgebras de Lie en una forma más moderna. En las últimas dos décadas, la teoría de las bases de Groebner-Shirshov ha sido definida para diferentes clases de álgebras [7] tales como diálgebras por Bokut, Chen y Liu [8] o álgebras replicadas [23].

La validez del teorema de Poincaré-Birkhoff-Witt para el caso de las álgebras de Leibniz (véanse [2], [22] y [29]) justifica la relación entre la construcción de la extensión HNN para álgebras de Leibniz y la extensión HNN para diálgebras. Por lo tanto, usando el funtor diálgebra envolvente universal transferimos nuestros resultados al caso de álgebras de Leibniz.

En el Capítulo 1, proporcionamos definiciones y teoremas esenciales sobre las álgebras de Leibniz libres y las diálgebras libres con algunos ejemplos.

Definición 1.1.1. *Un álgebra de Leibniz por la derecha (o álgebra de Leibniz) L es un espacio vectorial sobre $\mathbb{K}$ equipado con un mapa bilineal (multiplicación)*

$$[-,-]\colon L \times L \to L$$

que verifica la identidad de Leibniz

$$[[a,b],c] = [[a,c],b] + [a,[b,c]],$$

para todo $a, b, c \in L$.

Lema 1.1.6 ([30]). *El módulo tensor $T(V) = V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes n} \cdots$ equipado con el corchete definido inductivamente por*

- $[x,v] = x \otimes v$, *para* $x \in T(V)$, $v \in V$,
- $[x, y \otimes v] = [x,y] \otimes v - [x \otimes v, y]$, *para* $x, y \in T(V)$, $v \in V$,

es el álgebra de Leibniz libre sobre V.

Definición 1.2.2. *Un álgebra diasociativa (o diálgebra) es un espacio $\mathbb{K}$-lineal, equipado con dos aplicaciones $\mathbb{K}$-lineales $\dashv$, $\vdash\colon D \times D \to D$, llamadas respectivamente producto a la izquierda y a la derecha, de tal manera que los productos $\dashv$ y $\vdash$ son asociativos y satisfacen las siguientes propiedades:*

1. $x \dashv (y \vdash z) = x \dashv (y \dashv z)$,

2. $(x \dashv y) \vdash z = x \vdash (y \vdash z)$,

3. $x \vdash (y \dashv z) = (x \vdash y) \dashv z$.

La diálgebra D se dice abeliana si tanto el producto a la izquierda como a la derecha son triviales, es decir $x \dashv y = x \vdash y = 0$, para todo $x, y \in D$.

La Sección 1.3 está dedicada al concepto de diálgebra envolvente universal de las álgebras de Leibniz que se aplicará para construir la extensión HNN de las álgebras de Leibniz en la Sección 3.4.

Proposición 1.3.1. *Sea D una diálgebra. Entonces el corchete*

$$[x, y] \coloneqq x \dashv y - y \vdash x$$

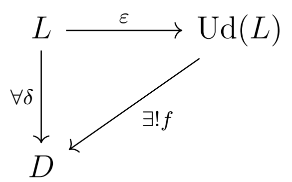
convierte a $(D, [-,-])$ en un álgebra de Leibniz, denotada por D_{Leib} o $D^{(-)}$.

Esta construcción define un funtor $(-)$:

$$\mathbf{Dias} \xrightarrow{-} \mathbf{Leib}$$

de la categoría **Dias** de diálgebras a la categoría **Leib** de álgebras de Leibniz.

Definición 1.3.2 ([8]). *Sea L un álgebra de Leibniz. Una diálgebra $\mathrm{Ud}(L)$ junto con un homomorfismo de álgebras de Leibniz $\varepsilon\colon L \to \mathrm{Ud}(L)$ se llama la diálgebra envolvente universal para L, si el siguiente diagrama conmuta:*



donde D es una diálgebra, δ es un homomorfismo de álgebras de Leibniz y

$$f\colon \operatorname{Ud}(L) \to D$$

es un homomorfismo de diálgebras tal que $f\varepsilon = \delta$, i.e. $\varepsilon\colon L \to \operatorname{Ud}(L)$ es una flecha universal.

La diálgebra envolvente universal de un álgebra de Leibniz L se define como el siguiente cociente de la diálgebra libre sobre L:

$$\operatorname{Ud}(L) \coloneqq T(L) \otimes L \otimes T(L) / \left\langle \{[x, y] - x \dashv y + y \vdash x \mid x, y \in L\} \right\rangle.$$

Proposición 1.3.3 ([29]). *El funtor* Ud: ***Leib*** $\to$ ***Dias*** *es adjunto por al izquierda del funtor* $(-)$: ***Dias*** $\to$ ***Leib***.

Debido a la importancia de la conexión entre las diálgebras y álgebras de Leibniz y la relación entre las álgebras asociativas y las álgebras de Lie, en el Capítulo 2, Sección 2.1, recordamos la teoría de las bases de Groebner-Shirshov.

Las palabras asociativas de Lyndon-Shirshov se definen bajo numerosos nombres tales como *palabras asociativas regulares (de Shirshov)* [24], *palabras de Lyndon* [31, 39] y *palabras de Lyndon-Shirshov* [7, 9, 38]. Las palabras asociativas de Lyndon-Shirshov juegan un papel fundamental en la introducción del Lema de Composición-Diamante para álgebras de Lie. Por lo tanto, la expresión del concepto de las palabras asociativas y no asociativas de Lyndon-Shirshov revela algunas similitudes y técnicas distintivas entre el Lema de Composición-Diamante para álgebras asociativas y para álgebras de Lie de tal manera que nos beneficia para entender el Lema de Composición-Diamante para las álgebras mencionadas. Además, es útil para entender otras notaciones como s-palabras y s-dipalabras que aparecen en las composiciones de polinomios durante la construcción de sus extensiones HNN en el Capítulo 3.

Denotamos por $<_{\texttt{deg-lex}}$ el orden graduado lexicográfico sobre X^* en el que se comparan dos palabras, primero por el grado y luego lexicográficamente. El orden graduado lexicográfico es un orden monomial. Esto es debido a su compatibilidad con la multiplicación de las palabras, es decir, para todo $u, v \in X^*$ tenemos

$$u <_{\texttt{deg-lex}} v \Rightarrow w_1 u w_2 <_{\texttt{deg-lex}} w_1 v w_2, \text{ para todo } w_1, w_2 \in X^*.$$

Hay varios órdenes interesantes en álgebra computacional, pero como orden monomial efectivo sobre X^* usamos principalmente el orden `deg-lex` en esta subsección.

Definición 2.1.2 ([6]). *Sea $X = \{x_i \mid i \in I\}$ un conjunto con $x_i > x_j$ si $i > j$, para cualquier $i, j \in I$ y $u \in X^*$. Entonces u se llama una palabra asociativa de Lyndon-Shirshov, que a veces es denotada por ALSW, si*

$$(\forall\, v, w \in X^*,\ v, w \neq 0)\ u = vw \Rightarrow vw > wv.$$

En otras palabras, una palabra *ALSW* es una palabra mayor que cualquier permutación cíclica de sí misma. Las palabras asociativas de Lyndon-Shirshov tienen varias propiedades útiles (véanse [7, 34, 35]).

Se puede vincular a una palabra asociativa de Lyndon-Shirshov, una palabra no asociativa a través de tres métodos de poner paréntesis diferentes, a saber, *de arriba hacia abajo*, *de abajo hacia arriba* y *poner paréntesis de Shirshov* (véase [6]). El último se utiliza en el Lema de Composición-Diamante para el caso de las álgebras de Lie (véase Teorema 2.2.7) bajo el nombre *palabras no asociativas de Lyndon-Shirshov* (véase Definición 2.1.4).

En la Sección 2.2 damos el Lema de Composición-Diamante (para álgebras asociativas en la Subsección 2.2.1 y para álgebras de Lie en la Subsección 2.2.2).

Teorema 2.2.4 ([7]). *(Lema de Composición-Diamante para álgebras asociativas)*

Sea un orden monomial $<$ sobre X^*. Consideremos un conjunto mónico $S \subset \mathbb{K}\langle X\rangle$ y el ideal $\mathrm{Id}(S)$ generado por S. Las siguientes afirmaciones son equivalentes.

(i) S *es una base de Groebner-Shirshov en* $\mathbb{K}\langle X\rangle$*;*

(ii) $f \in \mathrm{Id}(S) \Rightarrow \bar{f} = a\bar{s}b$ *para algún* $s \in S$ *y* $a, b \in X^*$*;*

(iii) $\mathrm{Irr}(S)$*, definido en la Definición 2.2.1, es una base lineal del álgebra* $\mathbb{K}\langle X \mid S\rangle$*.*

Teorema 2.2.7 ([7,9]). *(Lema de Composición-Diamante para álgebras de Lie)*

Sea $S \subset Lie(X) \subset \mathbb{K}\langle X\rangle$ *un subconjunto no vacío de polinomios mónicos de Lie. Sea* $\mathrm{Id}(S)$ *el ideal de* $Lie(X)$ *generado por* S*. Las siguientes afirmaciones son equivalentes.*

1. S *es una base de Groebner-Shirshov en* $Lie(X)$*;*
2. $f \in \mathrm{Id}(S) \Rightarrow \bar{f} = a\bar{s}b,$ *para algún* $s \in S$ *y* $a, b \in X^*$*;*
3. $\mathrm{Irr}(S)$ *es una base para* $Lie(X \mid S)$*.*

La última subsección de la Sección 2.2 contiene el Lema de Composición-Diamante para diálgebras como nuestra herramienta principal para introducir la extensión HNN para diálgebras.

Teorema 2.2.12 ([8]). *(Lema de Composición-Diamante para diálgebras)*

Sea $S \subset D(X)$ *un conjunto de polinomios mónicos de Lie, el orden* $<$ *definido para diálgebras, e* $\mathrm{Id}(S)$ *el ideal generado por* S*. Las siguientes afirmaciones son equivalentes.*

(i) S *es una base de Groebner-Shirshov en* $D(X)$*;*

(ii) $f \in \mathrm{Id}(S) \Rightarrow [\bar{f}] = [a[\bar{s}]b]$ *para algún* $s \in S,\ a, b \in X^*$ *y* $[asb]$ *es una* S*-dipalabra normal;*

(iii) $\mathrm{Irr}(S) = \{u \in [X^*] \mid u \neq [a[\bar{s}]b],\ s \in S, a, b \in X^*, [asb]$ *es una S-dipalabra normal*$\}$ *es una base de la diálgebra* $D(X \mid S)$.

El Capítulo 3 está dedicado a las construcciones de las extensiones HNN. En primer lugar, recordamos al lector la extensión HNN para grupos en la Sección 3.1.

Sean A_1 y A_2 subgrupos de un grupo G. Entonces es posible encontrar un grupo H que contenga a G de manera que A_1 y A_2 son conjugados entre sí en H y de manera que G se puede encajar en H. Una condición necesaria para esto es que A_1 y A_2 sean isomorfos. El siguiente teorema muestra que esta condición también es suficiente; y de hecho cualquier isomorfismo entre A_1 y A_2 se puede realizar por una transformación mediante un elemento adecuado en un grupo que contenga a G.

Teorema 3.1.1 ([18]). *Sea* μ *un isomorfismo de un subgrupo* A_1 *de un grupo* G *en un segundo subgrupo* A_2 *de* G. *Entonces existe un grupo* H *que contiene a* G, *y un elemento* t *de* H, *de tal forma que la transformación por* t *de cualquier elemento de* A_1 *es su imagen por* μ:

$$t^{-1}at = \mu(a) \ \ \text{para todo } a \in A_1.$$

Exponemos la construcción HNN para las álgebras de Lie en la Sección 3.2.

Consideremos una subálgebra $\mathfrak{a}$ de un álgebra de Lie $\mathfrak{g}$ y una derivación $d\colon \mathfrak{a} \to \mathfrak{g}$. Definimos la extensión HNN de $\mathfrak{g}$ la cual es un álgebra de Lie dada por la presentación

$$\mathfrak{h} = \langle \mathfrak{g}, t \mid [t, a] = d(a) \ \ \text{para todo } a \in \mathfrak{a} \rangle,$$

donde t es una nueva letra generadora con la relación $[t, a] = d(a)$ tal que $a, d(a) \in \mathfrak{g}$. Consideremos X como una base $\mathbb{K}$-lineal de $\mathfrak{g}$ incluyendo una base lineal de la subálgebra $\mathfrak{a}$ que es denotada por B. Las constantes de estructura de $\mathfrak{g}$ son las siguientes.

$$[x,y] = \sum_v \alpha_{xy}^v v,$$

donde $x, y \in X$. Ya que $\mathfrak{a}$ es una subálgebra entonces tenemos

$$\alpha_{ab}^v = 0 \quad \text{para } a, b \in B \quad \text{y } v \notin B.$$

Ya que $d(a) \in \mathfrak{g}$ entonces es una combinación lineal de los elementos de la base

$$d(a) = \sum_v \beta_a^v v$$

para $a \in B$. Por lo tanto, obtenemos la siguiente presentación equivalente para $\mathfrak{h}$:

$$\mathfrak{h} = \langle X, t \mid [x,y] = \sum_v \alpha_{xy}^v v,\ [t,a] = \sum \beta_a^v v,\ x, y \in X,\ a \in B \rangle.$$

A continuación introducimos la extensión HNN para las diálgebras en la Sección 3.3. El concepto de extensión HNN para diálgebras se construye definiendo una nueva presentación sobre un conjunto generador X con un conjunto de relaciones S. Los cálculos completos de las posibles composiciones de los polinomios diálgebra de la recién definida extensión HNN se presentan en la Subsección 3.3.1.

Sea D una diálgebra y A una subálgebra de D. Supongamos una derivación $d\colon A \to D$. Definimos la extensión HNN asociada como una diálgebra dada por la presentación:

$$D_d^* = \langle D, t \mid a \dashv t - t \vdash a = d(a),\ a \in A \rangle.$$

Damos una presentación equivalente para la extensión HNN a través de un conjunto especial de constantes estructurales. Sea Y una base totalmente bien ordenada de A y la extendemos a una base de D. Supongamos que esta base es X y con un orden total definido en la Definición 2.1.11 de modo que $Y < X \setminus Y$, es decir, cada elemento de Y es más pequeño que los elementos restantes de X. Supongamos además que $t < X$.

1) Sea $x, y \in X$. Tenemos $x \dashv y = \sum_v \alpha_{xy}^v v$ para algún $\alpha_{xy}^v \in \mathbb{K}$. De forma similar, tenemos $x \vdash y = \sum_v \beta_{xy}^v v$ para algunos escalares β_{xy}^v. De acuerdo con las leyes de asociatividad de la Definición 1.2.2, tenemos las siguientes relaciones entre las constantes estructurales:

$$\sum_{v,u} \beta_{yz}^v \alpha_{xv}^u = \sum_{v,u} \alpha_{yz}^v \alpha_{xv}^u,$$
$$\sum_{v,u} \alpha_{xy}^v \beta_{vz}^u = \sum_{v,u} \beta_{yz}^v \alpha_{xv}^u,$$
$$\sum_{v,u} \alpha_{yz}^v \beta_{xv}^u = \sum_{v,u} \beta_{xy}^v \alpha_{vz}^u.$$

Nótese que, ya que A es una subálgebra, tenemos $\alpha_{xy}^v = \beta_{xy}^v = 0$, para $x, y \in Y$ y $v \in X \setminus Y$.

2) Consideremos la derivación $d\colon A \to D$. Para cualquier $x \in Y$, existen escalares δ_x^v tal que

$$d(x) = \sum_v \delta_x^v v.$$

Ahora, sea $X' := X \cup \{t\}$, y definimos los siguientes elementos de la diálgebra libre $D(X')$:

- $f_1(x, y) = x \dashv y - \sum_v \alpha_{xy}^v v$
- $f_2(x, y) = x \vdash y - \sum_v \beta_{xy}^v v$
- $g_x = x \vdash t - t \dashv x + \sum_v \delta_x^v v$.

Entonces D tiene la siguiente presentación

$$D = \langle X \mid f_1(x, y),\ f_2(x, y),\ \ x, y \in X \rangle.$$

Extendemos esta presentación para obtener una presentación de la extensión HNN D_d^*

$$D_d^* = \langle X' \mid f_1(x, y),\ f_2(x, y)\ \ x, y \in X;\ g_x\ \ x \in Y \rangle.$$

Teorema 3.3.2. *Toda diálgebra se encaja en su extensión HNN.*

Finalmente, introducimos la extensión HNN para las álgebras de Leibniz en la Sección 3.4.

Sea D una diálgebra arbitraria sobre un cuerpo $\mathbb{K}$. Recordemos que la Proposición 1.3.1 define un álgebra de Leibniz $D^{(-)}$ por

$$[x, y] = x \dashv y - y \vdash x.$$

Para L un álgebra de Leibniz arbitraria sobre $\mathbb{K}$, se define la diálgebra envolvente universal para L (véase la Definición 1.3.2). Si L tiene una presentación $\langle X \mid S \rangle$ en la variedad de álgebras de Leibniz, entonces $\mathrm{Ud}(L)$ tiene la presentación $\langle X \mid S^{(-)} \rangle$ en la variedad de diálgebras, donde $S^{(-)}$ es el conjunto de todos los di-polinomios (polinomios diálgebra) obtenidos a partir de los elementos de S por todos los corchetes $[x, y]$ en $x \dashv y - y \vdash x$.

Recordemos que para cualquier álgebra de Leibniz L, el teorema de Poincaré-Birkhoff-Witt es válido (véase Corolario 2.2.15). Esto demuestra que cada derivación de L (o cada derivación de cualquier subálgebra de L) se extiende a una derivación de $\mathrm{Ud}(L)$ (a una derivación de la subálgebra correspondiente de $\mathrm{Ud}(L)$) (véase Corolario 2.2.16). En la siguiente definición, introducimos la extensión HNN para el caso de las álgebras de Leibniz.

Definición 3.4.1. *Sea L un álgebra de Leibniz y A una subálgebra de L. Supongamos una derivación $d\colon A \to L$. Definimos la correspondiente extensión HNN como el álgebra de Leibniz*

$$L_d^* = \langle L, t \mid d(a) = [a, t],\ a \in A \rangle.$$

Aquí t es el nuevo símbolo que no pertenece a L.

Ejemplo 3.4.2. *Hay dos casos extremos de extensiones HNN de álgebras de Leibniz.*

- *Si* $A = L$*, entonces* d *es una derivación de* L *y* L_d^* *es entonces el producto semidirecto de* L *con un álgebra de Leibniz de dimensión* 1 *que actúa sobre* L *vía* d*. Nótese que el producto semidirecto se introdujo en la Definición 1.1.12.*

- *Si* $A = 0$*, entonces* L_d^* *es el producto libre de* L *con un álgebra de Leibniz de dimensión* 1*.*

Como aplicación de la extensión HNN de las álgebras de Leibniz, proporcionamos un teorema de encaje para las álgebras de Leibniz en la última Subsección 3.4.1 de esta tesis.

Teorema 3.4.3. *Toda álgebra de Leibniz se encaja en su extensión HNN.*

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