TOPOLOGICAL ENTROPY OF CONTINUOUS SELF–MAPS ON A GRAPH

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ABSTRACT. Let G be a graph and f be a continuous self–map on G. We provide sufficient conditions based on the Lefschetz zeta function in order that f has positive topological entropy. Moreover, for the particular graphs: p–flower graph, n-lips graph and $(p+r_1L^1+...+r_sL^s)$–graph we are able to go further and state more precise conditions for having positive topological entropy.

1. Introduction and statement of the main results

Along this work a *graph* G will be a compact connected space containing a finite set V such that $G \setminus V$ has finitely many open connected components, each one of them homeomorphic to the interval $(0, 1)$, called *edges* of G, and the points of V are called the *vertexes* of G. The edges are disjoint from the vertexes, and the vertexes are at the boundary of the edges.

For a graph G, the *degree* of a vertex $\mathcal{V}$ is the number of edges having $\mathcal{V}$ in its boundary, if an edge has both boundaries in $\mathcal{V}$ then we compute this edge twice. An *endpoint* of a graph G is a vertex of degree one. A *branching point* of a graph G is a vertex of degree at least three.

In this paper a *discrete dynamical system* (G, f) is formed by a continuous map $f : G \to G$ where G is a graph.

A point $x \in G$ is *periodic* of *period* k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 < i < k$. If $k = 1$, then x is called a *fixed point*. $\mathrm{Per}(f)$ denotes the *set of periods* of all the periodic points of f.

The set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$, where by f^n we denote the composition of f with itself n times, is called the *orbit* of the point $x \in G$. To understand the behaviour of all different kind of orbits of f is to study *the topological dynamics of the map* f.

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Roughly speaking the *topological entropy* $h(f)$ of a discrete dynamical system (G, f) is a non–negative real number (possibly infinite) which measures how much f mixes up the phase space of G. When $h(f)$ is positive the dynamics of the system is said to be *complicated* and the positivity of $h(f)$ is used as a measure of the so called *topological chaos*.

Here we introduce the topological entropy using the definition of Bowen [4].

Since it is possible to embedded a graph G in $\mathbb{R}^3$, we consider the distance between two points of G as the distance of these two points in $\mathbb{R}^3$. Now, we define the distance d_n on G by

$$d_n(x, y) = \max_{0\leq i\leq n} d(f^i(x), f^i(y)), \quad \forall x, y \in G.$$

A finite set S is called (n, ε)*–separated with respect to* f if for different points $x, y \in S$ we have $d_n(x, y) > \varepsilon$. We denote by S_n the maximal cardinality of an (n, ε)–separated set. Define

$$h(f, \varepsilon) = \limsup_{n\to\infty} \frac{1}{n} \log S_n.$$

Then

$$h(f) = \lim_{\varepsilon\to 0} h(f, \varepsilon)$$

is the *topological entropy* of f.

We have chosen the definition by Bowen because, probably it is the shorter one. The classical definition was due to Adler, Konheim and McAndrew [1]. See for instance the book of Hasselblatt and Katok [16] and [3] for other equivalent definitions and properties of the topological entropy. See [1, 2, 9, 22] for more details on the topological entropy.

The homological spaces of G with coefficients in $\mathbb{Q}$ are denoted by $H_k(G, \mathbb{Q})$. Since G is a graph $k = 0, 1$. A continuous map $f : G \to G$ induces linear maps $f_{*k} : H_k(G, \mathbb{Q}) \to H_k(G, \mathbb{Q})$. $H_0(G, \mathbb{Q}) \approx \mathbb{Q}$ and we have that f_{*0} is the identity map because G is connected. A subset of G homeomorphic to a circle is a *circuit*. It is known that $H_1(G, \mathbb{Q}) \approx \mathbb{Q}^m$ being m the number of the independent circuits of G in the sense of the homology. Here f_{*1} is a $m \times m$ matrix A with integer entries. For more details on this homology see for instance [21].

Independently of the fact that to study the dynamical complexity via the topological entropy of these kind of graph maps is relevant by itself for understanding their dynamics, the graph maps are relevant for studying the dynamics of some different kind of surface maps, see for instance [15, 19].

For a polynomial $H(t)$ we define $H^*(t)$ by

$$H(t) = (1-t)^\alpha(1+t)^\beta t^\gamma H^*(t),$$

where α, β and γ are non-negative integers such that $1-t$, $1+t$ and t do not divide $H^*(t)$. We also define $H^{**}(t)$ by

$$H(t) = (1-t)^\alpha(1+t)^\beta H^{**}(t),$$

where α and β are non-negative integers such that $1-t$ and $1+t$ do not divide $H^{**}(t)$.

Inspired in the apparently relation between the topological entropy and the periodic orbit structure and using as precedents the results of the papers [6, 12, 18, 20] we present our main results which include applications to some particular graphs where we are able to provide more precise information on when they admits positive topological entropy systems.

Our main results are the following four theorems, in them it appears the notion of Lefschetz zeta function $\mathcal{Z}_f(t)$ for a map f, for its definition see subsection 2.1.

Theorem 1. *Let (G, f) be a discrete dynamical system induced by a continuous self–map f defined on a graph G, and let $\mathcal{Z}_f(t) = P(t)/Q(t)$ be its Lefschetz zeta function.*

(a) *Assume that $P^*(t)$ or $Q^*(t)$ has odd degree, then the topological entropy of f is positive.*
(b) *Assume that $P^{**}(t)$ or $Q^{**}(t)$ has odd degree, G is either $\mathbb{R}$ or $\mathbb{S}^1$ and f is a $\mathcal{C}^1$ map, then f has infinitely many periodic points.*

Statement (a) of Theorem 1 was proved by continuous self–maps on connected surfaces in [6]. Statement (b) of Theorem 1 was already known, see statement (c) of Theorem 1 of [11].

In the next corollary, statement (b) of Theorem 1 allows to re-proved in a different way a known result for continuous circle maps in the smooth case with a different approach from the classical one that can be read in [2].

Corollary 2. *Let $(\mathbb{S}^1, f)$ be a discrete dynamical system induced by a $\mathcal{C}^1$ map of degree d, then if $d \notin \{-1, 0, 1\}$ the map f has infinitely many periodic points.*

A *p-flower graph* is a graph with a unique branching point z and $p > 1$ edges all having a unique endpoint, the point z, equal for all of them.

So, this graph has p independent loops, each one is called a petal. See a 5-flower graph in Figure 1.

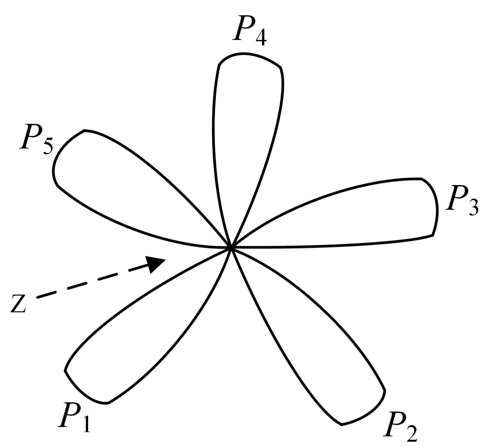

FIGURE 1. A 5-flower graph.

Theorem 3 (p-flower graph)**.** *Let (G, f) be a discrete dynamical system induced by a continuous self–map f being G a p–flower graph. Then the following conditions hold.*

(1) *If p is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*
(2) *If p is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd, then the topological entropy of f is positive.*

A graph with only two vertices z and w and $n \geq 1$ edges having every edge the vertices z and w as endpoints is called an *n-lips* graph and denoted by L^n. See a 7-lips graph L^7 in Figure 2.

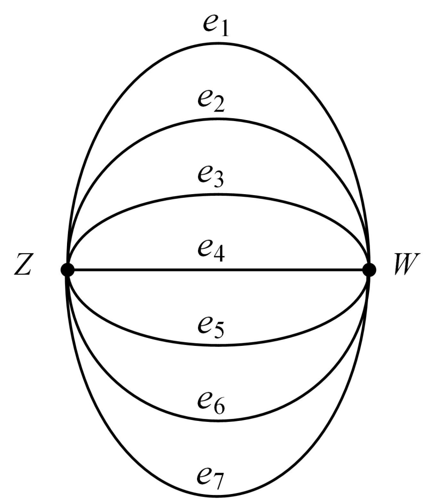

FIGURE 2. The 7-lips graphs.

Theorem 4 (n-lips graph)**.** *Let (G, f) be a discrete dynamical system induced by a continuous self–map f being G an n–lip graph, with $n > 1$. Then the following conditions hold.*

(1) *If $\frac{n(n-1)}{2}$ is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*
(2) *If $\frac{n(n-1)}{2}$ is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd, then the topological entropy of f is positive.*

A *graph* $p + r_1 L^1 + ... + r_s L^s$ is formed by p petals and $r_1 + ... + r_s$ lips where r_j lips are of type L^j for $j = 1, ..., s$. Note that a such graph has $p + \sum_{j=1}^{s} j r_j$ edges and

$$\mathcal{L}_{p,r_2,...,r_s} = p + r_2 + r_3 \binom{3}{2} + ... + r_s \binom{s}{2}$$

is the number of its independent circuits.

See a $(4 + 3L^1 + 2L^2 + 1L^3)$–graph in Figure 3, this graph has $p = 4$ and six lips, three lips L^1, two lips L^2 and one lip L^3.

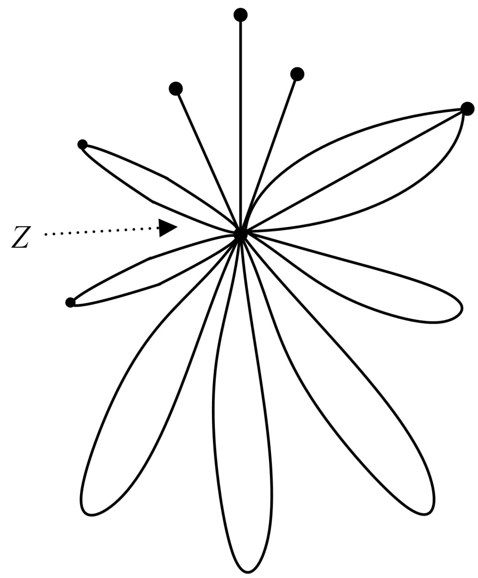

FIGURE 3. A $(4+3L^1+2L^2+1L^3)$–graph.

Theorem 5 ($(p + r_1L^1 + ... + r_sL^s)$–graph)**.** *Let (G, f) be a discrete dynamical system induced by a continuous self–map f being G a $(p + r_1L^1 + ... + r_sL^s)$–graph. Then the following conditions hold.*

(1) *If $\mathcal{L}_{p,r_2,...,r_s}$ is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then the topological entropy of f is positive.*
(2) *If $\mathcal{L}_{p,r_2,...,r_s}$ is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd, then the topological entropy of f is positive.*

2. Preliminary results

2.1. **Lefschetz zeta function.** Given a discrete dynamical system $(\mathbb{M}, f)$ where f is a continuous self–map defined on the compact n–dimensional topological space $\mathbb{M}$ the *Lefschetz number* is

$$L(f) = \sum_{k=0}^{n} (-1)^k \text{trace}(f_{*k}),$$

where the induced homomorphism by f on the k–th rational homology group of $\mathbb{M}$ is $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$. We note that $H_k(\mathbb{M}, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$, and that f_{*k} is a linear map given by a matrix with integer entries. The Lefschetz Fixed Point Theorem connects the fixed point theory with the algebraic topology via the following result.

Theorem 6. *Let* $(\mathbb{M}, f)$ *be a discrete dynamical system induced by a continuous self–map* f *on a compact topological space* $\mathbb{M}$ *and* $L(f)$ *be its Lefschetz number. If* $L(f) \neq 0$ *then* f *has a fixed point.*

For a proof of Theorem 6 see for instance [5].

Part of our interest in the present work is to provide information on the set of periodic points of f. To this objective we shall use the sequence of the Lefschetz numbers of all iterates of f denoted by $\{L(f^m)\}_{m=0}^{\infty}$. We remark that the Lefschetz *zeta function* of f

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right)$$

contains the information of all the sequence of the iterated Lefschetz numbers. Note that the function $\mathcal{Z}_f(t)$ can be computed also through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}}, \tag{1}$$

where $n = \dim \mathbb{M}$, $n_k = \dim H_k(\mathbb{M}, \mathbb{Q})$, I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - t f_{*k}) = 1$ if $n_k = 0$, for more details on the function $\mathcal{Z}_f(t)$ see [7].

From (1) the Lefschetz zeta function is a rational function and it contains the information of the infinite sequence of the iterated Lefschetz numbers. Note that this information is contained in two polynomials.

2.2. **Cyclotomic polynomials.**

The n–th cyclotomic polynomial is defined by

$$c_n(t) = \prod_k (w_k - t),$$

being $w_k = e^{2\pi i k/n}$ a primitive n–th root of unity and where k runs over all the relative primes $\leq n$. See [17] for the properties of these polynomials.

For a positive integer n the *Euler function* is $\varphi(n) = n \displaystyle\prod_{p|n, p \text{ prime}} \left(1 - \frac{1}{p}\right)$.

It is known that the degree of the polynomial $c_n(t)$ is $\varphi(n)$. Note that $\varphi(n)$ is even for $n > 2$.

A proof of the next result can be found in [17].

Proposition 7. *Let ξ be a primitive n–th root of the unity and $P(t)$ a polynomial with rational coefficients. If $P(\xi) = 0$ then $c_n(t)|P(t)$.*

Lemma 8. *If a polynomial has integer coefficients, constant term 1 and all of whose roots have modulus 1, then all of its roots are roots of unity.*

For a proof of Lemma 8 see [23].

2.3. **Topological entropy.** As we showed in subsection 2.1, given a discrete dynamical system $(\mathbb{M}, f)$ with f a continuous self–map defined on a compact n–dimensional topological space manifold $\mathbb{M}$, the map f induces an action on the homology groups of $\mathbb{M}$, which we denote $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$, for $k \in \{0, 1, \ldots, m\}$. The *spectral radii* of these maps are denoted $\mathrm{sp}(f_{*k})$, and they are equal to the largest modulus of all the eigenvalues of the linear map f_{*k}. The *spectral radius of* f_* is

$$\mathrm{sp}(f_*) = \max_{k=0,\ldots,m} \mathrm{sp}(f_{*k}).$$

The next result is due to Guaschi and Llibre [10] and Jiang [13, 14], for more details see Theorem 5.4.2 from [2].

Theorem 9. *Let $f : G \to G$ be a continuous map on the graph G. Then $\log \max\{1, \mathrm{sp}(f_{*1})\} \leq h(f)$.*

3. Auxiliary results

We need the following results for proving our theorems. The next result is Theorem 6 from [8].

Proposition 10. *Let $\mathbb{M}$ be a smooth compact manifold and let $(\mathbb{M}, f)$ be a discrete dynamical system induced by a $\mathcal{C}^1$ self–map f such that $f(\mathbb{M}) \subseteq \mathrm{Int}(\mathbb{M})$, and assume that f has finitely many periodic points. Then $\mathcal{Z}_f(t)$ has a finite factorization in terms of the form $(1 \pm t^r)^{\pm 1}$ with r a positive integer.*

Lemma 11. *Let (G, f) be a discrete dynamical system induced by a continuous self–map f defined on graph G. If the topological entropy of f is zero, then all the eigenvalues of the induced homomorphism f_{*1} are zero or root of unity.*

Proof. Since the topological entropy is zero, by Theorem 9 we have $\mathrm{sp}(f_{*1}) = 1$. So, all the eigenvalues of f_{*1} have modulus in the interval $[0, 1]$ and at least one of them is 1. Then the characteristic polynomial of f_{*1} is of the form $t^m p(t)$, where m is a non–negative integer, positive if the zero is an eigenvalue. And $p(t)$ is a polynomial with integer coefficients and whose independent term a_0 is non–zero. Since the product of all non-zeros eigenvalues of f_{*1} is the integer a_0 and, these eigenvalues have modulus in $(0, 1]$, we have that any of these eigenvalues can have modulus smaller than one, otherwise we are in contradiction with the fact a_0 is an integer. In short, all the non–zero eigenvalues have modulus one, and consequently $a_0 = 1$. By Lemma 8 all the roots of the polynomial $p(t)$ are roots of unity finishing the proof. □

Lemma 12. *Let $(\mathbb{M}, f)$ be a discrete dynamical system induced by a $\mathcal{C}^1$ self–map f defined on a smooth compact connected n–dimensional manifold $\mathbb{M}$. Assume that $f(\mathbb{M}) \subseteq \text{Int}(\mathbb{M})$. If f has finitely many periodic points, then all the eigenvalues of the induced homomorphisms f_{*k}'s are zero or root of unity.*

Proof. Since by Proposition 10 the Lefschetz zeta function (1) has a finite factorization in terms of the form $(1 \pm t^r)^{\pm 1}$ with r a positive integer, it follows that all the eigenvalues of f_{*1}'s are roots of unity. This completes the proof. □

4. Proof of Theorem 1

Proof of Theorem 1. From the definitions of a polynomial H^* and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a(1+t)^b t^c \frac{P^*(t)}{Q^*(t)},$$

where a, b and c are integers.

Assume now that the topological entropy $h(f) = 0$. Then by Lemma 11 all the eigenvalues of the induced homomorphisms f_{*1}'s are zero or roots of unity. Therefore, by (1) all the roots of the polynomials $P^*(t)$ and $Q^*(t)$ are roots of the unity different from ± 1 and zero. Hence, by Proposition 7 the polynomials $P^*(t)$ and $Q^*(t)$ are product of cyclotomic polynomials different from $c_1(t) = 1 - t$ and $c_2(t) = 1 + t$. Consequently $P^*(t)$ and $Q^*(t)$ have even degree because all the cyclotomic polynomials which appear in them have even degree due to the fact that the Euler function $\varphi(n)$ for $n > 2$ only takes even values. But this is a contradiction with the assumption that $P^*(t)$ or $Q^*(t)$ has odd degree. This completes the proof of statement (a).

For proving statement (b) we shall use as key point Proposition 10 taking account that the unique graphs admitting $\mathcal{C}^1$ maps are the ones which are manifolds, i.e. the real line and the circle. Note that under the hypothesis of statement (b) if we assume that f has finitely many periodic points, by Lemma 12 all the eigenvalues of f_{*1}'s are zero or root of unity. From the definition of the polynomial H^{**} and of the Lefschetz zeta function we have

$$\mathcal{Z}_f(t) = \frac{P(t)}{Q(t)} = (1-t)^a (1+t)^b \frac{P^{**}(t)}{Q^{**}(t)},$$

where a and b are integers. By Proposition 10 all the roots of the polynomials $P^{**}(t)$ and $Q^{**}(t)$ are roots of unity different from ± 1. Therefore, the rest of the proof of statement (b) follows as in the last part of the proof of statement (a). This completes the proof of the theorem. □

5. Proofs of Corollary 2 and Theorems 3, 4 and 5

Let $f : G \to G$ be a continuous map on the graph G. The homological spaces of G with coefficients in $\mathbb{Q}$ are denoted by $H_k(G, \mathbb{Q})$. Since G is a graph $k = 0, 1$ and f induces linear maps $f_{*k} : H_k(G, \mathbb{Q}) \to H_k(G, \mathbb{Q})$. Since G is a graph, then $H_0(G, \mathbb{Q}) \approx \mathbb{Q}$ and f_{*0} is the identity map. A subset of G homeomorphic to a circle is a *circuit*. It is known that $H_1(G, \mathbb{Q}) \approx \mathbb{Q}^m$ being m the number of the independent circuits of G in the sense of the homology. Here f_{*1} is a $m \times m$ matrix A with integer entries. For more details on this homology see for instance [21].

If A is a $m \times m$ matrix, then a submatrix lying in the same set of k rows and columns is a $k \times k$ *principal submatrix* of A. The determinant of a principal submatrix is a $k \times k$ *principal minor*. The sum of the $\binom{m}{k}$ different $k \times k$ principal minors of A is denoted by $E_k(A)$. Note that $E_m(A)$ is the determinant of A and $E_1(A)$ is the trace of A. Of course the characteristic polynomial of A is given by

$$\det(tI - A) = t^m - E_1(A)t^{m-1} + E_2(A)t^{m-2} - \ldots + (-1)^m E_m(A). \tag{2}$$

By (1), the form of the Lefschetz zeta function is the rational function

$$Z_f(t) = \frac{\det(I - tf_{*1})}{\det(I - tf_{*0})} = \frac{\det(I - tA)}{1 - t},$$

where A is the integer matrix defined by f_{*1}, for a proof see Franks [7].

Since $\det(I - tA) = t^m \det\left(\frac{1}{t} I - A\right)$, from (2) we get

$$\det(I - tA) = 1 - E_1(A)t + E_2(A)t^2 - \ldots + (-1)^m E_m(A)t^m.$$

Proof of Corollary 2. Since G is the circle, $H_1(G,\mathbb{Q}) \approx \mathbb{Q}$, so the Lefschetz zeta function is equal to

$$Z_f(t) = \frac{1-td}{1-t},$$

where d is the degree of f. Therefore the result follows directly from part (a) of Theorem 1 when $d \neq -1, 0, 1$. □

Proof of Theorem 3. Since G is a p-flower, which is a graph with p independent circuits, $H_1(G,\mathbb{Q}) \approx \mathbb{Q}^p$. Thus, the Lefschetz zeta function is equal to

$$Z_f(t) = \frac{\det(I-tA)}{1-t},$$

where $\det(I-tA)$ is a polynomial of degree p with integer coefficients and $f_{*1} = A$ is a $p \times p$ matrix with integer entries. Note that in this case $Q(t) = 1-t$ and $Q^*(t) = 1$. So, by Theorem 1 the main role will be play by the polynomial $P(t) = \det(I-tA)$ where $f_{*1} = A$. If p is even and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not even, then $P^*(t)$ has odd degree. Therefore, the result follows by the application of statement (a) of Theorem 1.

On the other hand, if p is odd and the number of roots of the characteristic polynomial of f_{*1} equal to ± 1 or 0 taking into account their multiplicities is not odd then $P^*(t)$ has odd degree and as before the proof of theorem follows. □

Proof of Theorem 4. The proof is the same than the proof of Theorem 3 taking account that an n–lip graph has $\binom{n}{2} = n(n-1)/2$ independent circuits and therefore f_{*1} is a polynomial of degree $n(n-1)/2$. □

Proof of Theorem 5. The proof follows from the arguments stated in the proof of Theorem 3 taking account that for a $(p + r_1 L^1 + ... + r_s L^s)$–graph, $\mathcal{L}_{p,r_2,\ldots,r_s} = p + r_2 + r_3\binom{3}{2} + ... + r_s\binom{s}{2}$ is the number of independent circuits. □

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