Primitive ideals of $U_q(\mathfrak{sl}_n^+)$

Samuel A. Lopes *
Centro de Matemática da Universidade do Porto,
R. Campo Alegre 687, 4169-007 Porto, Portugal
E-mail: slopes@fc.up.pt

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Abstract

Let $U_q(\mathfrak{g}^+)$ be the quantized enveloping algebra of the nilpotent Lie algebra $\mathfrak{g}^+ = \mathfrak{sl}_{n+1}^+$ which occurs as the positive part in the triangular decomposition of the simple Lie algebra $\mathfrak{sl}_{n+1}$ of type A_n. Assuming the base field $\mathbb{K}$ is algebraically closed and of characteristic 0, and that the parameter $q \in \mathbb{K}^*$ is not a root of unity, we define and study certain quotients of $U_q(\mathfrak{g}^+)$ which coincide with the Weyl-Hayashi algebra when $n = 2$ (see [16], [2] and [15]). We show that these are simple Noetherian domains, with a trivial center and even Gelfand-Kirillov dimension. Hence, they play a role analogous to that played by the Weyl algebras in the classical case. In the remainder of the paper we study the primitive spectrum of $U_q(\mathfrak{sl}_4^+)$ in detail, somewhat in the spirit of [17]. We determine all primitive ideals of $U_q(\mathfrak{sl}_4^+)$, find a set of generators for each one, compute their heights and find a simple $U_q(\mathfrak{sl}_4^+)$-module corresponding to each primitive ideal of $U_q(\mathfrak{sl}_4^+)$.

1 Introduction

This paper is concerned with the primitive ideals of the quantized enveloping algebra $U_q(\mathfrak{g}^+)$ of the nilpotent Lie algebra $\mathfrak{g}^+ = \mathfrak{sl}_{n+1}^+$ of strictly upper triangular matrices of size $n+1$, with an emphasis on $U_q(\mathfrak{sl}_4^+)$. In the classical case, the primitive factors of the enveloping algebra $U(\mathfrak{g}^+)$ of $\mathfrak{g}^+$ are isomorphic to Weyl algebras, and consequently the primitive ideals of $U(\mathfrak{g}^+)$ are simply its maximal ideals. For example, if $n = 2$ then $U(\mathfrak{sl}_3^+)$ admits generators x, y, z, satisfying the relations:

$$xz = zx, \qquad yz = zy, \qquad xy - yx = z.$$

The center of $U(\mathfrak{sl}_3^+)$ is the polynomial algebra in the central variable z, and $U(\mathfrak{sl}_3^+)/(z-1)$ is isomorphic to the first Weyl algebra over the ground field. Here, the quantum scenario differs from the classical one: it is well-known that there

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are primitive ideals of $U_q(\mathfrak{sl}_3^+)$ which are not maximal (see [20], for example), and consequently $U_q(\mathfrak{g}^+)$ has non-simple primitive quotients, in general.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and assume $q \in \mathbb{K}^*$ is not a root of unity. Then, $U_q(\mathfrak{g}^+)$ is the $\mathbb{K}$-algebra with generators $e_1, \dots, e_n$, which satisfy the so-called quantum Serre relations:

$$\begin{aligned} e_ie_j - e_je_i =& 0 \qquad \text{if } |i-j| \neq 1, \\ e_i^2e_j - (q+q^{-1})e_ie_je_i + e_je_i^2 =& 0 \qquad \text{if } |i-j| = 1. \end{aligned}$$

The center of $U_q(\mathfrak{g}^+)$ was computed by Alev and Dumas [1], and by Caldero [6, 7]. It is the polynomial algebra over $\mathbb{K}$ in the central indeterminates $z_1, \dots, z_l$, where $l = \lfloor \frac{n+1}{2} \rfloor$, the greatest integer not exceeding $\frac{n+1}{2}$. If $n = 2$, then $U_q(\mathfrak{sl}_3^+)$ can be presented by generators X, Y, Z, satisfying the relations:

$$ZX = q^{-1}XZ, \qquad ZY = qYZ, \qquad XY - q^{-1}YX = Z.$$

The center of $U_q(\mathfrak{sl}_3^+)$ is the polynomial algebra in the variable $z_1 = (XY - qYX)Z$.

In [16], Kirkman and Small showed that $(z_1 - 1)$ is a maximal ideal of $U_q(\mathfrak{sl}_3^+)$ and that the factor algebra $A_q = U_q(\mathfrak{sl}_3^+)/(z_1 - 1)$, in spite of not being isomorphic to the Weyl algebra $\mathbb{A}_1(\mathbb{K})$, shares a number of ring theoretical properties with it: it is a simple Noetherian domain with trivial center, Gelfand-Kirillov dimension 2 and Krull dimension 1. In the first part of this paper, we generalize these results of Kirkman and Small to $U_q(\mathfrak{g}^+)$, for all $n \geq 2$. Specifically, we show that $(z_1 - \alpha_1, \dots, z_l - \alpha_l)$ is a maximal ideal of $U_q(\mathfrak{g}^+)$ for any $\alpha_1, \dots, \alpha_l \in \mathbb{K}^*$, and conclude that the corresponding factor algebra is a simple Noetherian domain with trivial center and even Gelfand-Kirillov dimension, which is not isomorphic to a Weyl algebra over $\mathbb{K}$.

We then proceed to the second part of the paper, where we study the (left) primitive ideals of $U_q(\mathfrak{sl}_4^+)$ in full detail. To make use of the stratification theory of Goodearl and Letzter [12], we consider the natural action of the 3-torus $\mathcal{H} = (\mathbb{K}^*)^3$ on $U_q(\mathfrak{sl}_4^+)$. Relative to this action, the prime spectrum of $U_q(\mathfrak{sl}_4^+)$ is partitioned into $4! = 24$ strata (as shown, in a much more general context, in [14]), given in Proposition 4.1. By analyzing the maximal portion of each stratum, we obtain all primitive ideals of $U_q(\mathfrak{sl}_4^+)$, and we also compute their heights using the catenarity of $U_q(\mathfrak{sl}_4^+)$ and Tauvel's height formula (see [11]). This was achieved by Malliavin [20] for $U_q(\mathfrak{sl}_3^+)$ and, recently, by Launois [17] for $U_q(\mathfrak{so}_5^+)$. In the latter case, Launois succeeded in computing the automorphism group of $U_q(\mathfrak{so}_5^+)$, using partial results of Andruskiewitsch and Dumas [3]. A noteworthy by-product of our study is that the Gelfand-Kirillov dimension of the primitive factors of $U_q(\mathfrak{sl}_4^+)$ is one of 0, 2 or 4. In particular, as in the classical case, this dimension is always even. Note that this is also the case with $U_q(\mathfrak{sl}_3^+)$ and $U_q(\mathfrak{so}_5^+)$ (see [17]), so it is natural to expect that, in general, the primitive factor algebras of $U_q(\mathfrak{L}^+)$, where $\mathfrak{L}$ is a finite-dimensional complex semisimple Lie algebra, have even Gelfand-Kirillov dimension. We are unaware of a general result of this kind in the literature.

Finally, we construct a simple $U_q(\mathfrak{sl}_4^+)$-module with annihilator P, for any primitive ideal P of $U_q(\mathfrak{sl}_4^+)$. This, of course, doesn't exhaust the simple $U_q(\mathfrak{sl}_4^+)$-modules, but it solves the problem of deciding whether or not an element belongs to a specified primitive ideal of $U_q(\mathfrak{sl}_4^+)$, and thus makes it quite easy to distinguish between the primitive ideals of this algebra.

2 Basic set-up

We work over an algebraically closed field $\mathbb{K}$ of characteristic 0 and fix a parameter $q \in \mathbb{K}^*$ which is not a root of unity. Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ be the complex semisimple Lie algebra of traceless $(n+1) \times (n+1)$ matrices and consider its maximal nilpotent subalgebra $\mathfrak{g}^+ = \mathfrak{sl}_{n+1}^+$ consisting of the strictly upper triangular matrices in $\mathfrak{g}$. As usual, $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ is the q-version of the integer $k \in \mathbb{Z}$.

2.1 The algebra $U_q(\mathfrak{g}^+)$

The quantized enveloping algebra $U_q(\mathfrak{g}^+)$ is the associative unital $\mathbb{K}$-algebra given by the Chevalley generators $e_1, \dots, e_n$, subject to the quantum Serre relations

$$e_i e_j - e_j e_i = 0 \qquad \text{if } |i-j| \neq 1, \tag{1}$$

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \qquad \text{if } |i-j| = 1. \tag{2}$$

Let $Q = \mathbb{Z}^n$ be the free abelian group of rank n with canonical basis $\{\alpha_1, \dots, \alpha_n\}$, and $Q^+ = \mathbb{N}^n$ be its submonoid. There is a nondegenerate bilinear form on $Q \times Q$ determined by $(\alpha_i, \alpha_j) = 2, -1$ or 0 if $i = j$, $|i-j| = 1$ or $|i-j| > 1$, respectively. By the homogeneity of the quantum Serre relations it follows that $U_q(\mathfrak{g}^+)$ has a Q^+-grading given by assigning degree α_i to the generator e_i. We use the terminology *weight* instead of degree for this grading and write $wt(u) = \beta$ if $u \in U_q(\mathfrak{g}^+)$ has weight $\beta \in Q^+$.

2.2 PBW basis

As in [22, App. 2], we recursively define weight elements X_{ij}, $1 \leq i < j \leq n+1$, by setting $X_{i,i+1} = e_i$ and $X_{ij} = X_{ik} X_{kj} - q^{-1} X_{kj} X_{ik}$, for $1 \leq i < k < j \leq n+1$ (this is independent of the choice of k). Note that $wt(X_{ij}) = \alpha_i + \cdots + \alpha_{j-1}$, for $i < j$. The set $\{X_{ij}\}$ can be linearly ordered using the rule

$$X_{ij} < X_{kl} \quad \iff \quad (k < i) \quad \text{or} \quad (k = i \quad \text{and} \quad l < j),$$

and we use the alternative notation X_k for the kth element in this increasing chain, so that $\{X_{ij}\}_{1 \leq i < j \leq n+1} = \{X_k\}_{1 \leq k \leq m}$, where $m = \frac{1}{2} n(n+1)$. We also write $X^{\mathbf{b}} = X_1^{b_1} \cdots X_m^{b_m}$, for $\mathbf{b} = (b_1, \dots, b_m) \in \mathbb{N}^m$.

The following results of Ringel are well-known.

Theorem 2.1 ([22, Thm. 2, Cor.]). *Let $v_{ji} = (wt(X_i), wt(X_j))$.*

(a) *The algebra $U_q(\mathfrak{g}^+)$ is an iterated skew polynomial ring of the form*

$$\mathbb{K}[X_1][X_2; \tau_2, \delta_2] \cdots [X_m; \tau_m, \delta_m]$$

where, for $i < j$, τ_j is the algebra automorphism given by $\tau_j(X_i) = q^{v_{ji}} X_i$ and δ_j is a $\mathbb{K}$-linear τ_j-derivation such that $\delta_j(X_i)$ is a linear combination of weight $wt(X_i) + wt(X_j)$ of monomials in $X_{i+1}, \dots, X_{j-1}$;

(b) *The monomials in $\{X^{\mathbf{b}} \mid \mathbf{b} \in \mathbb{N}^m\}$ form a basis of $U_q(\mathfrak{g}^+)$;*

(c) The prime ideals of $U_q(\mathfrak{g}^+)$ are completely prime.

2.3 The degree of an element of $U_q(\mathfrak{g}^+)$

Define an order relation on $\mathbb{N}^m$ by $\mathbf{b} < \mathbf{c} \iff$ there is $1 \leq k \leq m$ such that $b_k < c_k$ and $b_t = c_t$ for all $t > k$. Along with Theorem 2.1(b), this determines an increasing filtration $\{\mathcal{F}_\mathbf{a}\}_{\mathbf{a} \in \mathbb{N}^m}$ of $U_q(\mathfrak{g}^+)$ given by

$$\mathcal{F}_\mathbf{a} = \bigoplus_{\mathbf{b} \leq \mathbf{a}} \mathbb{K}.X^\mathbf{b},$$

and the corresponding graded algebra is the quantum affine space with generators $\theta_1, \ldots, \theta_m$ and relations $\theta_j \theta_i = q^{v_{ji}} \theta_i \theta_j$ for $i < j$, where $\theta_i = gr\, X_i$ and $v_{ji} = (wt(X_i), wt(X_j))$.

For $u \in U_q(\mathfrak{g}^+)$, set $deg(u) = \mathbf{a} \in \mathbb{N}^m$ if $u \neq 0$ and $\mathbf{a}$ is the unique element of $\mathbb{N}^m$ that satisfies $u \in \mathcal{F}_\mathbf{a}$ and $u \notin \mathcal{F}_\mathbf{b}$ for any $\mathbf{b} < \mathbf{a}$. We say that u has degree $\mathbf{a}$. Note that $deg(uv) = deg(u) + deg(v)$ for all nonzero $u, v \in U_q(\mathfrak{g}^+)$.

2.4 Normal elements of $U_q(\mathfrak{g}^+)$

According to work of Alev and Dumas [1], and Caldero [7, 8], there exist weight elements $\Delta_1, \ldots, \Delta_n$ of $U_q(\mathfrak{g}^+)$ such that the following theorem holds.

Theorem 2.2 ([7, 8]). *For $1 \leq i, j \leq n$, we have:*

(a) $e_i \Delta_j = q^{\delta_{ij} - \delta_{i,n+1-j}} \Delta_j e_i$;

(b) *The subalgebra of $U_q(\mathfrak{g}^+)$ generated by the Δ_i is a (commutative) polynomial algebra $\mathbb{K}[\Delta_1, \ldots, \Delta_n]$ in n variables;*

(c) *The center $Z_q(\mathfrak{g}^+)$ of $U_q(\mathfrak{g}^+)$ is the polynomial algebra in the variables $\{\Delta_k \Delta_{n+1-k} \mid 1 \leq k \leq n/2\}$ if n is even and $\{\Delta_k \Delta_{n+1-k} \mid 1 \leq k \leq (n-1)/2\} \cup \{\Delta_{(n+1)/2}\}$ if n is odd.*

It was noted in [19, 4.4] that

$$gr(\Delta_i) = gr(X_{i,n+1})\, gr(X_{i-1,n}) \cdots gr(X_{2,n+3-i})\, gr(X_{1,n+2-i}), \tag{3}$$

in the graded algebra of 2.3.

We fix some more notation for the entire paper. The center of $U_q(\mathfrak{g}^+)$ is denoted by $Z_q(\mathfrak{g}^+)$ and $l = \lfloor \frac{n+1}{2} \rfloor$, where $\lfloor k \rfloor$ is the greatest integer that doesn't exceed k. The elements $z_1, \ldots, z_l$ are defined by

$$z_i = \begin{cases} \Delta_i \Delta_{n+1-i} & \text{if} \quad i < l, \\ \Delta_l \Delta_{l+1} & \text{if} \quad i = l \text{ and } n \text{ is even}, \\ \Delta_l & \text{if} \quad i = l \text{ and } n \text{ is odd}, \end{cases}$$

so that $Z_q(\mathfrak{g}^+) = \mathbb{K}[z_1, \ldots, z_l]$. The integer m is the number of positive roots of the Lie algebra $\mathfrak{sl}_{n+1}$, i.e., $m = \frac{1}{2}n(n+1)$.

2.5 The prime and primitive ideals of $U_q(\mathfrak{g}^+)$

In this brief paragraph we summarize a portion of stratification theory of Goodearl and Letzter that is essential for our study. The reader should refer to [12] and [4] for further details, all proofs, and an explanation of the terminology.

Only left primitive ideals are considered; in view of the antiautomorphism $e_i \mapsto e_i$ of $U_q(\mathfrak{g}^+)$, this is not a serious restriction.

Let $\mathcal{H}$ be the n-torus $(\mathbb{K}^*)^n$. Corresponding to each $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathcal{H}$ there is an automorphism $\sigma_{\bar{\lambda}}$ of $U_q(\mathfrak{g}^+)$ given by $\sigma_{\bar{\lambda}}(e_i) = \lambda_i e_i$, for all $1 \leq i \leq n$. This defines a rational action of $\mathcal{H}$ on $U_q(\mathfrak{g}^+)$ such that the induced grading of $U_q(\mathfrak{g}^+)$ by the character group of $\mathcal{H}$ coincides with the weight space decomposition of 2.1 (identifying the ith projection map $\bar{\lambda} \mapsto \lambda_i$ with the simple root α_i). Since $\mathcal{H}$ acts by automorphisms, the action carries over to the spaces $\operatorname{Spec} U_q(\mathfrak{g}^+)$ and $\operatorname{Prim} U_q(\mathfrak{g}^+)$ of prime and primitive ideals of $U_q(\mathfrak{g}^+)$, respectively, equipped with the Jacobson topology. Let $\mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{g}^+) \subseteq \operatorname{Spec} U_q(\mathfrak{g}^+)$ be the subspace of $\mathcal{H}$-invariant prime ideals, that is, $\mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{g}^+)$ consists of the prime ideals J of $U_q(\mathfrak{g}^+)$ that are generated by weight elements of $U_q(\mathfrak{g}^+)$. By [12, Prop. 4.2] and Theorem 2.1, $\mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{g}^+)$ is a finite set consisting of the ideals of the form $(P : \mathcal{H}) := \bigcap_{h \in \mathcal{H}} h.P$, for $P \in \operatorname{Spec} U_q(\mathfrak{g}^+)$.

The above determines a decomposition of $\operatorname{Spec} U_q(\mathfrak{g}^+)$ into $\mathcal{H}$-strata:

$$\operatorname{Spec} U_q(\mathfrak{g}^+) = \bigcup_{J \in \mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{g}^+)} \operatorname{Spec}_J U_q(\mathfrak{g}^+),$$

where $\operatorname{Spec}_J U_q(\mathfrak{g}^+) = \{P \in \operatorname{Spec} U_q(\mathfrak{g}^+) \mid (P : \mathcal{H}) = J\}$ is the $\mathcal{H}$-stratum of J in $\operatorname{Spec} U_q(\mathfrak{g}^+)$, and similarly for $\operatorname{Prim} U_q(\mathfrak{g}^+)$. It follows from [12, Thm. 4.4] that the primitive ideals of $U_q(\mathfrak{g}^+)$ are the maximal elements of $\operatorname{Spec}_J U_q(\mathfrak{g}^+)$, for each $J \in \mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{g}^+)$. Furthermore, since $\mathbb{K}$ is algebraically closed, [12, Thm. 2.6 or Thm. 6.8] imply that $\mathcal{H}$ acts transitively on each of the sets $\operatorname{Prim}_J U_q(\mathfrak{g}^+)$. Therefore, the $\mathcal{H}$-orbits of primitive ideals of $U_q(\mathfrak{g}^+)$ are parametrized by the elements of the set $\mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{g}^+)$, which has cardinality $(n+1)!$ by work of Gorelik [14, Prop. 5.3.3] (see also [3, 3.4.1]).

Given $J \in \mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{g}^+)$, let Ξ_J be the set of nonzero weight elements of $U_q(\mathfrak{g}^+)/J$, with respect to the Q^+-grading inherited from $U_q(\mathfrak{g}^+)$. The following result of Goodearl and Letzter describes the $\mathcal{H}$-strata of $U_q(\mathfrak{g}^+)$.

Theorem 2.3 ([12, Thm. 6.6]). *Let J, Ξ_J and $\operatorname{Spec}_J U_q(\mathfrak{g}^+)$ be as above. Then Ξ_J is an Ore set in $U_q(\mathfrak{g}^+)/J$. If $U_q(\mathfrak{g}^+)_J$ denotes the localization of $U_q(\mathfrak{g}^+)/J$ at Ξ_J, we have:*

(a) *The localization map $U_q(\mathfrak{g}^+) \to U_q(\mathfrak{g}^+)/J \to U_q(\mathfrak{g}^+)_J$ induces a homeomorphism of $\operatorname{Spec}_J U_q(\mathfrak{g}^+)$ onto $\operatorname{Spec} U_q(\mathfrak{g}^+)_J$.*

(b) *Contraction and extension induce mutually inverse homeomorphisms between $\operatorname{Spec} U_q(\mathfrak{g}^+)_J$ and $\operatorname{Spec} Z(U_q(\mathfrak{g}^+)_J)$, where $Z(U_q(\mathfrak{g}^+)_J)$ is the center of $U_q(\mathfrak{g}^+)_J$.*

3 Generalizations of the Weyl-Hayashi algebra

If $n = 2$ then $Z_q(\mathfrak{sl}_3^+)$ is a polynomial algebra in the central variable z_1. The factor algebra $U_q(\mathfrak{sl}_3^+)/(z_1 - 1)$ was introduced by Hayashi in [15], in connection with oscillator representations of quantized enveloping algebras of semisimple Lie algebras of types A and C. In [16], Kirkman and Small showed that $U_q(\mathfrak{sl}_3^+)/(z_1 - 1)$ is a simple Noetherian domain of Gelfand-Kirillov dimension 2, which is not isomorphic to the Weyl algebra $\mathbb{A}_1(\mathbb{K})$ (see also [2] and [20]). The relevance of their result is that the primitive factor algebras of the enveloping algebra of a finite-dimensional nilpotent Lie algebra over a field of characteristic 0 are isomorphic to Weyl algebras over the base field (see [10, Thm. 4.7.9]). In particular, those primitive factor algebras of Gelfand-Kirillov dimension 2 must be isomorphic to $\mathbb{A}_1(\mathbb{K})$. For these reasons, $U_q(\mathfrak{sl}_3^+)/(z_1 - 1)$ is known as the Weyl-Hayashi algebra. In this section we generalize the result of Kirkman and Small to $U_q(\mathfrak{g}^+)$, thus proposing other analogues of the Weyl algebras $\mathbb{A}_k(\mathbb{K})$.

3.1 Gröbner bases

Let us introduce some basic techniques from the theory of Gröbner bases. We follow [5]. Recall the filtration, associated graded algebra and the notion of degree defined in 2.3 in terms of the PBW basis of $U_q(\mathfrak{g}^+)$. Given a subset F of $U_q(\mathfrak{g}^+)$, set $deg(F) = \{deg(f) \mid 0 \neq f \in F\} \subseteq \mathbb{N}^m$. It is clear that if L is a left, right, or two-sided ideal of $U_q(\mathfrak{g}^+)$, then $deg(L)$ is stable under translation by elements of $\mathbb{N}^m$; in other words, $deg(L)$ is a *monoideal* of $\mathbb{N}^m$. The set $\{f_1, \ldots, f_s\} \subseteq L$ is said to be a *Gröbner basis* for L if (see [5, Def. 2.8])

$$deg(L) = \bigcup_{j=1}^{s} \left(deg(f_j) + \mathbb{N}^m\right).$$

Recall also from 2.4 that the center of $U_q(\mathfrak{g}^+)$ is $Z_q(\mathfrak{g}^+) = \mathbb{K}[z_1, \ldots, z_l]$. Let $\mathbf{t} = (t_1, \ldots, t_l) \in \mathbb{K}^l$ and set $f_j^{\mathbf{t}} = z_j - t_j$, $I^{\mathbf{t}} = \sum_{j=1}^{l} U_q(\mathfrak{g}^+) f_j^{\mathbf{t}}$ and $\widetilde{I}^{\mathbf{t}} = \sum_{j=1}^{l} Z_q(\mathfrak{g}^+) f_j^{\mathbf{t}}$.

Proposition 3.1. *The set $\{f_1^{\mathbf{t}}, \ldots, f_l^{\mathbf{t}}\}$ is a Gröbner basis for $I^{\mathbf{t}}$.*

Proof. It is enough to show that

$$deg(I^{\mathbf{t}}) \subseteq \bigcup_{j=1}^{l} \left(deg(f_j^{\mathbf{t}}) + \mathbb{N}^m\right),$$

since the other inclusion follows from the fact that $deg(I^{\mathbf{t}})$ is a monoideal and $\{f_1^{\mathbf{t}}, \ldots, f_l^{\mathbf{t}}\} \subseteq I^{\mathbf{t}}$.

Recall that by our separation of variables results [19, Lem. 1, Thm. 2], there is a set $\mathcal{M}$ consisting of monomials $X^{\mathbf{a}}$ ($\mathbf{a} \in \mathbb{N}^m$) in the PBW basis elements $X_1, \ldots, X_m$ such that

$$U_q(\mathfrak{g}^+) = \bigoplus_{u \in \mathcal{M}} u Z_q(\mathfrak{g}^+).$$

Furthermore, by construction,

$$X^{\mathbf{a}} \in \mathcal{M} \Longrightarrow \mathbf{a} - deg(z_j) \notin \mathbb{N}^m, \quad \text{for all } 1 \leq j \leq l. \tag{4}$$

In the notation of [19, Sec. 5], $\mathcal{M}$ is obtained from $\mathcal{K} = \beta^{-1}(H)$, as defined in [19, 5.2, 5.3], by multiplying the monomials in $\mathcal{K}$ by the monomial basis of $\mathbb{K}[\Delta_1, \ldots, \Delta_n]$ over $Z_q(\mathfrak{g}^+)$ (see the paragraph preceding [19, Thm. 2]), constructed using the procedure described in [19, Lem. 1].

Thus,

$$I^{\mathbf{t}} = U_q(\mathfrak{g}^+)\widetilde{I}^{\mathbf{t}} = \bigoplus_{u \in \mathcal{M}} u \widetilde{I}^{\mathbf{t}}. \tag{5}$$

Assume $u, u' \in \mathcal{M}$, $p, p' \in \widetilde{I}^{\mathbf{t}}$ and $deg(up) = deg(u'p')$. By (3), the elements $deg(z_j)$, $1 \le j \le l$ are free generators of the monoid $deg(Z_q(\mathfrak{g}^+))$, and in particular $deg(\widetilde{I}^{\mathbf{t}}) \subseteq \bigoplus_{i=1}^{l} \mathbb{N} deg(z_i)$. Let $\mathbf{a}, \mathbf{b} \in \mathbb{N}^m$ be such that $u = X^{\mathbf{a}}$ and $u' = X^{\mathbf{b}}$. There exist $n_i, n_i' \in \mathbb{N}$, $1 \le i \le l$, with

$$\mathbf{a} + \sum_{i=1}^{l} n_i deg(z_i) = \mathbf{b} + \sum_{i=1}^{l} n_i' deg(z_i).$$

Assume $\mathbf{a} \neq \mathbf{b}$. Then there is $1 \le i \le l$ such that $n_i \neq n_i'$. Without loss of generality assume $n_1 < n_1'$. Then,

$$\mathbf{a} - deg(z_1) + \sum_{i=2}^{l} n_i deg(z_i) = \mathbf{b} + n_1'' deg(z_1) + \sum_{i=2}^{l} n_i' deg(z_i), \tag{6}$$

for $n_1'' = (n_1' - n_1 - 1) \in \mathbb{N}$. But (6) implies that $\mathbf{a} - deg(z_1) \in \mathbb{N}^m$ by (3), and this contradicts (4). Thus $\mathbf{a} = \mathbf{b}$ and $u = u'$.

Let $0 \neq f \in I^{\mathbf{t}}$. Then by (5) we can write $f = \sum_{j=1}^{k} u_j p_j$ with $u_j \in \mathcal{M}$, $0 \neq p_j \in \widetilde{I}^{\mathbf{t}}$ and the u_j pairwise distinct, $1 \le j \le k$. The discussion above shows that $deg(f) = \max_j deg(u_j p_j)$, with respect to the total order on $\mathbb{N}^m$ defined in 2.3. Say $deg(f) = deg(u_1 p_1)$. We have

$$deg(p_1) \in deg(\widetilde{I}^{\mathbf{t}}) \subseteq \left(\bigoplus_{i=1}^{l} \mathbb{N} deg(z_i) \right) \setminus \{\bar{0}\} \subseteq \bigcup_{i=1}^{l} \left(deg(z_i) + \mathbb{N}^m \right),$$

since $\widetilde{I}^{\mathbf{t}}$ is a proper ideal of $Z_q(\mathfrak{g}^+)$. Finally, as $deg(f_i^{\mathbf{t}}) = deg(z_i)$ for all $1 \le i \le l$, we obtain the desired conclusion:

$$deg(f) = deg(u_1) + deg(p_1) \in \bigcup_{i=1}^{l} \left(deg(f_i^{\mathbf{t}}) + \mathbb{N}^m \right).$$

□

For $0 \neq f \in U_q(\mathfrak{g}^+)$ write $f = \sum_{\mathbf{a} \in \mathbb{N}^m} c_{\mathbf{a}} X^{\mathbf{a}}$, where $c_{\mathbf{a}} \in \mathbb{K}$ and the sum is finite. Define $\mathcal{N}(f) = \{\mathbf{a} \in \mathbb{N}^m \mid c_{\mathbf{a}} \neq 0\}$. We are going to use the division algorithm developed in [5, Thm. 2.1] to prove the next result.

Corollary 3.2. *For all $\mathbf{t} \in \mathbb{K}^l$, the ideal $I^{\mathbf{t}}$ of $U_q(\mathfrak{g}^+)$ is semiprime.*

Proof. Using for example [13, Thm. 2.7], it is enough to show $f^2 \in I^{\mathbf{t}} \implies f \in I^{\mathbf{t}}$, for all $f \in U_q(\mathfrak{g}^+)$. By the (left) division algorithm of [5, Thm. 2.1], there exist elements $g_1, \ldots, g_l, r \in U_q(\mathfrak{g}^+)$ (unique under certain conditions) such that

$$f = \sum_{i=1}^{l} g_i f_i^{\mathbf{t}} + r,$$

with either $r = 0$ or $\mathcal{N}(r) \subseteq \mathbb{N}^m \setminus \bigcup_{i=1}^{l} \left(deg(f_i^{\mathbf{t}}) + \mathbb{N}^m \right)$. If $r = 0$ there is nothing to prove, so assume $r \neq 0$. Since

$$I^{\mathbf{t}} \ni f^2 = \sum_{i,j=1}^{l} g_i f_i^{\mathbf{t}} g_j f_j^{\mathbf{t}} + \sum_{j=1}^{l} r g_j f_j^{\mathbf{t}} + \sum_{i=1}^{l} g_i f_i^{\mathbf{t}} r + r^2$$

and the $f_i^{\mathbf{t}}$ are central, it must be that $r^2 \in I^{\mathbf{t}}$. If $deg(r) = (a_1, \ldots, a_m)$, then $deg(r^2) = (2a_1, \ldots, 2a_m) \in deg(I^{\mathbf{t}})$. By Proposition 3.1, there are $1 \leq i \leq l$ and $\mathbf{b} \in \mathbb{N}^m$ such that $deg(r^2) = deg(f_i^{\mathbf{t}}) + \mathbf{b}$. But by (3), $deg(f_i^{\mathbf{t}})$ is a string of zeros and ones, so it follows that $deg(r) \in deg(f_i^{\mathbf{t}}) + \mathbb{N}^m$, which contradicts the assumption on $\mathcal{N}(r)$. Therefore $r = 0$ and $f \in I^{\mathbf{t}}$. $\square$

3.2 The $\mathcal{H}$-stratum of (0)

We want to describe the space of all primitive ideals of $U_q(\mathfrak{sl}_{n+1})^+$ that do not contain nonzero weight elements, i.e. $\mathrm{Prim}_{(0)} U_q(\mathfrak{g}^+)$. By [12, Thm. 4.4] and Theorem 2.3, this space is homeomorphic to the space of maximal ideals of $Z(\mathcal{Q})$, where $\mathcal{Q}$ is the localization of $U_q(\mathfrak{g}^+)$ at the Ore set $\Xi_{(0)}$ of nonzero weight elements of $U_q(\mathfrak{g}^+)$, and $Z(\mathcal{Q})$ is the center of $\mathcal{Q}$.

Lemma 3.3. *The center of $\mathcal{Q}$ is the commutative Laurent polynomial algebra*

$$Z(\mathcal{Q}) = \mathbb{K}[z_1^{\pm 1}, \ldots, z_l^{\pm 1}].$$

Proof. Let Λ be the set of nonzero weight elements of $Z_q(\mathfrak{g}^+)$. First we want to show that $Z(\mathcal{Q}) = Z_q(\mathfrak{g}^+)[\Lambda^{-1}]$, the localization of the center of $U_q(\mathfrak{g}^+)$ at Λ. Fix $b^{-1}a \in Z(\mathcal{Q})$, where $a, b \in U_q(\mathfrak{g}^+)$ with b a nonzero weight element. Without loss of generality, it can be assumed that a is a weight element, as $Z(\mathcal{Q})$ is graded by Q. Consider the set $L = \{u \in U_q(\mathfrak{g}^+) \mid ub^{-1}a \in U_q(\mathfrak{g}^+)\}$. As $b^{-1}a$ is central, L is a (nonzero) two-sided ideal of $U_q(\mathfrak{g}^+)$. We can use [7, Thm. 5.2] to conclude that there is a nonzero $x \in Z_q(\mathfrak{g}^+)$ such that $xb^{-1}a \in U_q(\mathfrak{g}^+)$. Once more, by weight considerations, x can be assumed to be a weight element, so that $x \in \Lambda$. Therefore, $b^{-1}a = x^{-1}(xb^{-1}a)$ with $x \in \Lambda$ and $xb^{-1}a \in Z_q(\mathfrak{g}^+)$. The above shows that $Z(\mathcal{Q}) \subseteq Z_q(\mathfrak{g}^+)[\Lambda^{-1}]$. The reverse inclusion is obvious.

Now we determine the set Λ. It is easy to see, for example from [7, 3.1], that $wt(z_1), \ldots, wt(z_l)$ are $\mathbb{Z}$-independent elements of Q^+; in fact, $wt(z_j) = 2(\varpi_j + \varpi_{n+1-j})$ for $1 \leq j \leq l$, where $\varpi_1, \ldots, \varpi_n$ are the fundamental weights of $\mathfrak{g}$. Since $Z_q(\mathfrak{g}^+)$ is the polynomial algebra $\mathbb{K}[z_1, \ldots, z_l]$, it follows that the weight elements of $Z_q(\mathfrak{g}^+)$ are precisely the scalar multiples of the monomials $z_1^{a_1} \cdots z_l^{a_l}$. Thus, $Z_q(\mathfrak{g}^+)[\Lambda^{-1}] = \mathbb{K}[z_1^{\pm 1}, \ldots, z_l^{\pm 1}]$, and the lemma is proved. $\square$

By the above lemma, the maximal ideals of $Z(\mathcal{Q})$ are those of the form

$$\sum_{j=1}^{l} Z(\mathcal{Q})\,(z_j - t_j)\,, \qquad \text{for } \mathbf{t} = (t_1, \ldots, t_l) \in (\mathbb{K}^*)^l. \tag{7}$$

Fix $\mathbf{t} \in (\mathbb{K}^*)^l$. The ideal (7) corresponds to the maximal ideal $\sum_{j=1}^{l} \mathcal{Q}\,(z_j - t_j)$ of $\mathcal{Q}$, by extension (see Theorem 2.3(b)), and the latter corresponds to the ideal

$$T^{\mathbf{t}} := \Big(\sum_{j=1}^{l} \mathcal{Q}\,(z_j - t_j)\Big) \bigcap U_q(\mathfrak{g}^+) \tag{8}$$

of $U_q(\mathfrak{g}^+)$ under the homeomorphism of Theorem 2.3(a) with $J = (0)$. Thus $T^{\mathbf{t}}$ is maximal within the $\mathcal{H}$-stratum of (0) in $\mathrm{Spec}\, U_q(\mathfrak{g}^+)$, and by [12, Thm. 4.4], $T^{\mathbf{t}}$ is a primitive ideal of $U_q(\mathfrak{g}^+)$ containing no nonzero weight elements. Furthermore, by construction, any primitive ideal of $U_q(\mathfrak{g}^+)$ with this property is of the form $T^{\mathbf{t}'}$, for some $\mathbf{t}' \in (\mathbb{K}^*)^l$.

Recall the definition of $I^{\mathbf{t}}$ in 3.1. The next proposition gives the ideal $T^{\mathbf{t}}$ a more familiar form.

Proposition 3.4. *Let $\mathbf{t} \in (\mathbb{K}^*)^l$. Then the ideal $I^{\mathbf{t}}$ is primitive. In fact,*

$$T^{\mathbf{t}} = I^{\mathbf{t}} = \sum_{j=1}^{l} U_q(\mathfrak{g}^+)\,(z_j - t_j)\,.$$

Proof. Assume $\mathbf{t} \in (\mathbb{K}^*)^l$. Note that the inclusion $I^{\mathbf{t}} \subseteq T^{\mathbf{t}}$ is clear. Let P be any prime ideal of $U_q(\mathfrak{g}^+)$ containing $I^{\mathbf{t}}$, and suppose P contains a nonzero weight element. Then $(P : \mathcal{H}) \neq (0)$, and hence there is a nonzero weight element $g \in (P : \mathcal{H}) \cap Z_q(\mathfrak{g}^+)$, by [7, Thm. 5.2] and the fact that $(P : \mathcal{H})$ is a graded ideal. In particular, $g \in P$. We have

$$P \cap Z_q(\mathfrak{g}^+) \supseteq I^{\mathbf{t}} \cap Z_q(\mathfrak{g}^+),$$

and $g \in P \cap Z_q(\mathfrak{g}^+) \setminus I^{\mathbf{t}} \cap Z_q(\mathfrak{g}^+)$, as $I^{\mathbf{t}}$ contains no nonzero weight elements since $I^{\mathbf{t}} \subseteq T^{\mathbf{t}}$. But

$$I^{\mathbf{t}} \cap Z_q(\mathfrak{g}^+) = \sum_{j=1}^{l} Z_q(\mathfrak{g}^+)\,(z_j - t_j)$$

is a maximal ideal of $Z_q(\mathfrak{g}^+)$, and therefore $P \cap Z_q(\mathfrak{g}^+) = Z_q(\mathfrak{g}^+)$. The latter is a contradiction since $1 \notin P$. This shows that $(P : \mathcal{H}) = (0)$ and thus $P \in \operatorname{Spec}_{(0)} U_q(\mathfrak{g}^+)$. Hence P extends to a prime ideal P^e of $\mathcal{Q}$ with

$$P^e \supseteq \sum_{j=1}^{l} \mathcal{Q}\,(z_j - t_j)\,, \tag{9}$$

as $P \supseteq I^{\mathbf{t}}$. The ideal on the right-hand side of (9) is a maximal ideal of $\mathcal{Q}$, and thus equality must hold in (9). Therefore, P is the contraction to $U_q(\mathfrak{g}^+)$ of $\sum_{j=1}^{l} \mathcal{Q}\,(z_j - t_j)$, which by definition is $T^{\mathbf{t}}$.

We have seen that $T^{\mathbf{t}}$ is the unique prime ideal of $U_q(\mathfrak{g}^+)$ containing $I^{\mathbf{t}}$. However, it was shown in Corollary 3.2 that $I^{\mathbf{t}}$ is an intersection of prime ideals, which forces $I^{\mathbf{t}} = T^{\mathbf{t}}$. $\square$

We remark that it is necessary to require that $\mathbf{t} \in (\mathbb{K}^*)^l$ for Proposition 3.4 to hold. If, for example, $t_1 = 0$ and $n \geq 2$, then $I^{\mathbf{t}}$ is not even a prime ideal (even though it remains semiprime by Corollary 3.2), since $\Delta_1 \Delta_n = z_1 \in I^{\mathbf{t}}$ and yet neither Δ_1 nor Δ_n is in $I^{\mathbf{t}}$, by Proposition 3.1. The only possible exception to this type of counterexample would be to have $t_l = 0$ and n odd, as in this case $z_l = \Delta_l$; we have not investigated this situation in general. However, in the case of $U_q(\mathfrak{sl}_4^+)$, we show in Proposition 4.6 that the ideal $I^{(\alpha,0)}$ is indeed primitive, for $\alpha \in \mathbb{K}^*$.

Theorem 3.5. *Assume $\mathbf{t} \in (\mathbb{K}^*)^l$. Then $I^{\mathbf{t}}$ is a maximal ideal of $U_q(\mathfrak{g}^+)$. It is also minimal among primitive ideals of $U_q(\mathfrak{g}^+)$.*

Proof. Assume $I^{\mathbf{t}}$ is not maximal. Then there is a maximal ideal P of $U_q(\mathfrak{g}^+)$ such that $I^{\mathbf{t}} \subsetneq P \subsetneq U_q(\mathfrak{g}^+)$. Since $\left(I^{\mathbf{t}} : \mathcal{H}\right) = (0)$ and $I^{\mathbf{t}}$ is primitive by

Proposition 3.4, it follows that $I^{\mathbf{t}}$ is a maximal element of $\operatorname{Spec}_{(0)} U_q(\mathfrak{g}^+)$, by [12, Thm. 4.4]. Hence $(P:\mathcal{H}) \neq (0)$. As in the proof of Proposition 3.4, this yields a contradiction. Thus $I^{\mathbf{t}}$ is indeed a maximal ideal.

The other statement of the theorem follows because $U_q(\mathfrak{g}^+)$ is an iterated skew polynomial ring over $\mathbb{K}$, hence a constructible $\mathbb{K}$-algebra (see [21, 9.4.12]), and by [21, Thm. 9.4.21] it satisfies the Nullstellensatz over the algebraically closed field $\mathbb{K}$. If M is a simple $U_q(\mathfrak{g}^+)$-module, this property implies that $\operatorname{End}_{U_q(\mathfrak{g}^+)}(M) \simeq \mathbb{K}$. Let $P = \operatorname{ann} M$ be the corresponding primitive ideal. The representation $\rho : U_q(\mathfrak{g}^+) \to \operatorname{End}_{\mathbb{K}}(M)$ restricts to an algebra homomorphism from $Z_q(\mathfrak{g}^+)$ to $\operatorname{End}_{U_q(\mathfrak{g}^+)}(M) \simeq \mathbb{K}$; thus, for every $i = 1, \ldots, l$, there exists $\alpha_i \in \mathbb{K}$ with $z_i - \alpha_i \in \ker \rho = P$. This shows that every primitive ideal of $U_q(\mathfrak{g}^+)$ contains an ideal of the form $I^{\mathbf{t}}$, for some unique $\mathbf{t} \in \mathbb{K}^l$, hence proving the minimality of $I^{\mathbf{t}}$ ($\mathbf{t} \in (\mathbb{K}^*)^l$) among the primitive ideals of $U_q(\mathfrak{g}^+)$. A direct proof of this fact can also be obtained by noting that if P is primitive and $P \subseteq I^{\mathbf{t}}$, for some $\mathbf{t} \in (\mathbb{K}^*)^l$, then $(P:\mathcal{H}) = (0)$ and as a consequence both P and $I^{\mathbf{t}}$ must be maximal in $\operatorname{Spec}_{(0)} U_q(\mathfrak{g}^+)$, implying that $P = I^{\mathbf{t}}$. □

3.3 Gelfand-Kirillov dimension of $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$

It is a simple matter to compute the GK (Gelfand-Kirillov) dimension of the factor algebra $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$. We can use Proposition 3.1 along with the techniques of [5, Sec. 4] for example, as we did in [18, 5.2.3]. Another approach is via [11, Thm. 4.8], where the authors show that $U_q(\mathfrak{g}^+)$ is catenary and that Tauvel's height formula holds in $U_q(\mathfrak{g}^+)$. By the latter property, given $P \in \operatorname{Spec} U_q(\mathfrak{g}^+)$,

$$\operatorname{GKdim}(U_q(\mathfrak{g}^+)/P) = \operatorname{GKdim}(U_q(\mathfrak{g}^+)) - \operatorname{height}(P) = m - \operatorname{height}(P). \quad (10)$$

If $\mathbf{t} \in (\mathbb{K}^*)^l$ and $J \subseteq I^{\mathbf{t}}$ is a prime ideal, then clearly $J \in \operatorname{Spec}_{(0)} U_q(\mathfrak{g}^+)$. Since the spaces $\operatorname{Spec}_{(0)} U_q(\mathfrak{g}^+)$ and $\operatorname{Spec} Z(\mathcal{Q})$ are homeomorphic, it follows that the height of $I^{\mathbf{t}}$ equals the height of the corresponding maximal ideal (7) of $Z(\mathcal{Q})$, which is l by Lemma 3.3. Therefore, (10) yields $\operatorname{GKdim}(U_q(\mathfrak{g}^+)/I^{\mathbf{t}}) = m - l$ for $\mathbf{t} \in (\mathbb{K}^*)^l$, a result which holds more generally for all $\mathbf{t} \in \mathbb{K}^l$, as seen in [18, 5.2.3].

Corollary 3.6. *Let $\mathbf{t} \in (\mathbb{K}^*)^l$. The factor algebra $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$ is a simple Noetherian domain with center $\mathbb{K}$ and GK dimension $m - l$. In particular, the GK dimension of $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$ is always even but $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$ is not isomorphic to a Weyl algebra $A_k(\mathbb{K})$ for any $k \geq 1$.*

Proof. The statement about the center of $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$ is a consequence of the Nullstellensatz for $U_q(\mathfrak{g}^+)$ (see the proof of Theorem 3.5 for more details), as the center of $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$ embeds in $\operatorname{End}_{U_q(\mathfrak{g}^+)}(M) \simeq \mathbb{K}$, where M is a simple $U_q(\mathfrak{g}^+)$-module with annihilator $I^{\mathbf{t}}$. The last statement follows since Δ_1 is invertible in $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$, as $\Delta_1 \Delta_n - t_1 \in I^{\mathbf{t}}$ and $t_1 \neq 0$, whereas the only invertible elements of $A_k(\mathbb{K})$ are the nonzero scalars. □

Remark 3.7. When $q = 1$ the central elements z_i (except for z_l if n is odd) specialize to the square of the generators of the center of $U(\mathfrak{g}^+)$, as determined in [9, Thè. 1]. Therefore, neither the Weyl-Hayashi algebra $U_q(\mathfrak{sl}_3^+)/(z_1 - 1)$ nor, more generally, the algebras $U_q(\mathfrak{g}^+)/I^{\mathbf{t}}$, $\mathbf{t} \in (\mathbb{K}^*)^l$, specialize to Weyl algebras at $q = 1$, as the resulting algebras are not even prime. The situation is different in the B_2 case, as in [17, Sec. 3] Launois constructs simple quotients of $U_q(\mathfrak{so}_5^+)$, some of which specialize to the first Weyl algebra when $q = 1$. It would be of interest to study the algebras of Corollary 3.6 in more detail and to compare them with other *quantum algebras* available in the literature.

4 The primitive ideals of $U_q(\mathfrak{sl}_4^+)$

In this section we study the primitive spectrum of $U_q(\mathfrak{sl}_4^+)$ in full detail. This was done for $U_q(\mathfrak{sl}_3^+)$ in [20], and for $U_q(\mathfrak{so}_5^+)$ in [17]. We begin by determining the set $\mathcal{H}$-Spec $U_q(\mathfrak{sl}_4^+)$ of all $\mathcal{H}$-invariant (completely) prime ideals of $U_q(\mathfrak{sl}_4^+)$. It is a finite set of cardinality $4! = 24$ whose elements parametrize the $\mathcal{H}$-orbits of primitive ideals of $U_q(\mathfrak{sl}_4^+)$, by the stratification theory of Goodearl and Letzter [12] and work of Gorelik [14]. Then, we explicitly describe each $\mathcal{H}$-stratum in Prim $U_q(\mathfrak{sl}_4^+)$, compute the height of all primitive ideals of $U_q(\mathfrak{sl}_4^+)$, and give an example of a simple $U_q(\mathfrak{sl}_4^+)$-module with annihilator P, for each $P \in \operatorname{Prim} U_q(\mathfrak{sl}_4^+)$. An interesting outcome of this analysis is that all primitive factor algebras of $U_q(\mathfrak{sl}_4^+)$ have even GK dimension, as occurs in the classical case.

4.1 Structure of $U_q(\mathfrak{sl}_4^+)$

A PBW basis of $U_q(\mathfrak{sl}_4^+)$, as seen in 2.2, is given by

$$X_1 = e_3, \qquad X_2 = e_2e_3 - q^{-1}e_3e_2, \qquad X_3 = e_2,$$

$$X_4 = e_1X_2 - q^{-1}X_2e_1, \qquad X_5 = e_1e_2 - q^{-1}e_2e_1, \qquad X_6 = e_1,$$

and we can take

$$\Delta_1 = X_4, \qquad \Delta_2 = X_2X_5 - q^{-1}X_3X_4,$$

$$\Delta_3 = q^{-2}\Big((q-q^{-1})^2X_1X_3X_6 - (q-q^{-1})X_1X_5 - (q-q^{-1})X_2X_6 + X_4\Big),$$

$$z_1 = \Delta_1\Delta_3, \qquad z_2 = \Delta_2,$$

so that $Z_q(\mathfrak{sl}_4^+) = \mathbb{K}[z_1, z_2]$.

Consider the *diagram automorphism* η of $U_q(\mathfrak{sl}_4^+)$, where $\eta(e_i) = e_{4-i}$ for $i = 1, 2, 3$. The element Δ_3 above was defined so that $\eta(\Delta_1) = \Delta_3$.

4.2 $\mathcal{H}$-Spec $U_q(\mathfrak{sl}_4^+)$

In [18, 5.3.3] we used the ideas of Goodearl and Letzter to obtain Proposition 4.1 below; specifically, we used [12, Lem. 3.2] and the proofs of [12, Lem. 3.3, Prop. 3.4]. As it is known that $|\mathcal{H}\text{-Spec}\, U_q(\mathfrak{sl}_4^+)| = 4!$, another approach to proving this result would be to show that the ideals listed therein are distinct prime ideals of $U_q(\mathfrak{sl}_4^+)$, as they are clearly $\mathcal{H}$-invariant. Notice that the automorphism η defined in 4.1 acts on $\mathcal{H}$-Spec $U_q(\mathfrak{sl}_4^+)$ and each η-orbit has size either one or two, since $\eta^2 = 1$.

Proposition 4.1. *The space $\mathcal{H}$-Spec $U_q(\mathfrak{sl}_4^+)$ consists of the 24 ideals in the following list, where ideals that are in the same η-orbit have been grouped together:*

1. (0);

2. (Δ_2);

3. (Δ_1), (Δ_3);

4. (Δ_1, Δ_3);

5. $(e_1e_2 - qe_2e_1)$, $(e_2e_3 - q^{-1}e_3e_2)$;

6. $(e_1e_2 - q^{-1}e_2e_1)$, $(e_2e_3 - qe_3e_2)$;

7. (e_1), (e_3);

8. $(e_1e_2 - qe_2e_1, e_2e_3 - qe_3e_2)$, $(e_1e_2 - q^{-1}e_2e_1, e_2e_3 - q^{-1}e_3e_2)$;

9. $(e_1e_2 - q^{-1}e_2e_1, e_2e_3 - qe_3e_2)$;

10. $(e_1e_2 - qe_2e_1, e_2e_3 - q^{-1}e_3e_2)$;

11. $(e_2e_3 - qe_3e_2, e_1)$, $(e_1e_2 - q^{-1}e_2e_1, e_3)$;

12. $(e_2e_3 - q^{-1}e_3e_2, e_1)$, $(e_1e_2 - qe_2e_1, e_3)$;

13. (e_1, e_2), (e_2, e_3);

14. (e_2);

15. (e_1, e_3);

16. (e_1, e_2, e_3).

4.3 Prim $U_q(\mathfrak{sl}_4^+)$

Finally, we can determine all primitive ideals of $U_q(\mathfrak{sl}_4^+)$ and compute their heights by studying each of the spaces $\mathrm{Prim}_J\, U_q(\mathfrak{sl}_4^+)$, for all possible choices of $J \in \mathcal{H}\text{-}\mathrm{Spec}\, U_q(\mathfrak{sl}_4^+)$. We denote the ideals given by Proposition 4.1 by J_i, $J_{i,a}$ or $J_{i,b}$ according to their position in that list, so that $J_{13,a} = (e_1, e_2)$, $J_{13,b} = (e_2, e_3)$ and $J_{14} = (e_2)$, for example. If M is a $U_q(\mathfrak{sl}_4^+)$-module, we denote its annihilator by $\mathrm{ann}\, M$.

Recall from [1] that the center of $U_q(\mathfrak{sl}_3^+)$ is the polynomial algebra in the quantum Casimir element

$$\Omega = (\dot{e}_1\dot{e}_2 - q^{-1}\dot{e}_2\dot{e}_1)(\dot{e}_1\dot{e}_2 - q\dot{e}_2\dot{e}_1) \in U_q(\mathfrak{sl}_3^+), \tag{11}$$

where $\dot{e}_1$ and $\dot{e}_2$ denote the Chevalley generators of $U_q(\mathfrak{sl}_3^+)$. For $i = 1, 2$ let S_i be the subalgebra of $U_q(\mathfrak{sl}_4^+)$ generated by e_i and e_{i+1}. Then $S_i \simeq U_q(\mathfrak{sl}_3^+)$ and therefore the center of S_i is the polynomial algebra $\mathbb{K}[\Omega_i]$ in the variable

$$\Omega_i = (e_ie_{i+1} - q^{-1}e_{i+1}e_i)(e_ie_{i+1} - qe_{i+1}e_i). \tag{12}$$

4.3.1 Annihilators of the finite-dimensional simple $U_q(\mathfrak{sl}_4^+)$-modules: the $\mathcal{H}$-strata of (e_2), (e_1, e_2), (e_2, e_3), (e_1, e_3) and (e_1, e_2, e_3)

As a result of all prime ideals of $U_q(\mathfrak{sl}_4^+)$ being completely prime, and given the assumption that $\mathbb{K}$ is algebraically closed, it can be shown that the simple finite-dimensional $U_q(\mathfrak{sl}_4^+)$-modules are one-dimensional. If $V = \mathbb{K}v_0$ is such a one-dimensional module, then there exist scalars $\alpha_i \in \mathbb{K}$ with either $\alpha_2 = 0$, or $\alpha_2 \neq 0$ and $\alpha_1 = 0 = \alpha_3$, satisfying $e_i.v_0 = \alpha_i v_0$, for $i = 1, 2, 3$ (the conditions on the scalars α_i follow directly from the quantum Serre relations (2)). Consequently, the primitive ideals that occur as annihilators of finite-dimensional simple $U_q(\mathfrak{sl}_4^+)$-modules are the maximal ideals of the form $(e_1-\alpha_1, e_2-\alpha_2, e_3-\alpha_3)$, with the α_i as above; these belong to one of the following $\mathcal{H}$-strata: (e_1, e_2, e_3), (e_1, e_2), (e_2, e_3), (e_1, e_3), (e_2).

Proposition 4.2. (a) $\mathrm{Prim}_{(e_1,e_2,e_3)}\, U_q(\mathfrak{sl}_4^+) = \{(e_1, e_2, e_3)\}$;

(b) $\mathrm{Prim}_{(e_1,e_2)}\, U_q(\mathfrak{sl}_4^+) = \{(e_1, e_2, e_3 - \alpha) \mid \alpha \in \mathbb{K}^*\}$;

(c) $\mathrm{Prim}_{(e_2,e_3)}\, U_q(\mathfrak{sl}_4^+) = \{(e_1 - \alpha, e_2, e_3) \mid \alpha \in \mathbb{K}^*\}$;

(d) $\mathrm{Prim}_{(e_1,e_3)}\, U_q(\mathfrak{sl}_4^+) = \{(e_1, e_2 - \alpha, e_3) \mid \alpha \in \mathbb{K}^*\}$;

(e) $\mathrm{Prim}_{(e_2)}\, U_q(\mathfrak{sl}_4^+) = \{(e_1 - \alpha, e_2, e_3 - \beta) \mid \alpha, \beta \in \mathbb{K}^*\}$.

The primitive ideals described above have height 6*.*

Proof. Parts (a)–(e) follow easily from the (commutative) Nullstellensatz, as $U_q(\mathfrak{sl}_4^+)/(e_2) \simeq \mathbb{K}[x, y]$ and $U_q(\mathfrak{sl}_4^+)/(e_1, e_3) \simeq \mathbb{K}[x]$. The last statement follows from Tauvel's height formula (10) since $\mathrm{GKdim}(U_q(\mathfrak{sl}_4^+)) = 6$ and $U_q(\mathfrak{sl}_4^+)/P \simeq \mathbb{K}$ for any of the primitive ideals listed in (a)–(e). □

4.3.2 The $\mathcal{H}$-stratum of (0)

Proposition 4.3. *Let z_1 and z_2 be the generators of the center of $U_q(\mathfrak{sl}_4^+)$, as defined in 4.1. Then,*

$$\mathrm{Prim}_{(0)}\, U_q(\mathfrak{sl}_4^+) = \{(z_1 - \alpha, z_2 - \beta) \mid \alpha, \beta \in \mathbb{K}^*\} \tag{13}$$

and these primitive ideals have height 2*.*

Proof. The equality (13) was proved in Proposition 3.4, and the height of the ideal $(z_1 - \alpha, z_2 - \beta)$, for $\alpha, \beta \in \mathbb{K}^*$, was computed in 3.3. □

In [19, 6.3] we defined $U_q(\mathfrak{sl}_4^+)$-modules $M_{(\alpha,\beta)}$, for $(\alpha, \beta) \in \mathbb{K}^2$, which were shown to be pairwise non-isomorphic, and simple under the assumption that $\alpha \neq 0$. It was also shown that $(z_1 - \alpha^2, z_2 - \beta) \subseteq \mathrm{ann}\, M_{(\alpha,\beta)}$. In particular, if $\alpha, \beta \in \mathbb{K}^*$ and α' is a square root of α in $\mathbb{K}$, then

$$\mathrm{ann}\, M_{(\alpha',\beta)} = (z_1 - \alpha, z_2 - \beta), \tag{14}$$

since the ideal on the right-hand side of (14) is maximal.

4.3.3 The $\mathcal{H}$-stratum of (Δ_2)

Let $Q \in \mathrm{Prim}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$. Since $U_q(\mathfrak{sl}_4^+)$ satisfies the Nullstellensatz over $\mathbb{K}$ (see the proof of Theorem 3.5), there exist scalars $\alpha, \beta \in \mathbb{K}$ such that $z_1 - \alpha, z_2 - \beta \in Q$; hence $\beta = 0$ as, by hypothesis, $z_2 = \Delta_2 \in Q$. If $\alpha = 0$ then $z_1 = \Delta_1\Delta_3 \in Q$, which implies that $\Delta_1 \in (\Delta_2)$ or $\Delta_3 \in (\Delta_2)$ because z_1 is a weight element and Q is completely prime. This is a contradiction, as $(\Delta_i, \Delta_2) \neq (\Delta_2)$ for $i = 1, 3$, and hence $z_1 - \alpha \in Q$ for some $\alpha \in \mathbb{K}^*$. Indeed, let $U_q(\mathfrak{sl}_4^+)$ act on the vector space $A = \mathbb{K}[x, y]$ of all polynomials in the variables x and y, by the formulas $(a, b \geq 0)$:

$$\begin{aligned} e_1.x^a y^b &= [a]x^{a-1}y^b, \\ e_2.x^a y^b &= x^{a+1}y^{b+1}, \\ e_3.x^a y^b &= [b]x^a y^{b-1}, \end{aligned} \tag{15}$$

where $e_1.y^b = 0 = e_3 x^a$.

Lemma 4.4. *The formulas (15) above endow A with the structure of a simple $U_q(\mathfrak{sl}_4^+)$-module with* $\mathrm{ann}\, A \in \mathrm{Prim}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$. *We also have* $z_1 - q^{-2} \in \mathrm{ann}\, A$.

Proof. It is easy to check that the formulas in (15) extend to give an action of $U_q(\mathfrak{sl}_4^+)$ on A, and that A thus becomes a simple $U_q(\mathfrak{sl}_4^+)$-module. Also, straightforward computations yield:

$$\begin{aligned} \Delta_1.x^a y^b &= -q^{a-b-1}x^a y^b, \\ \Delta_2.x^a y^b &= 0, \\ \Delta_3.x^a y^b &= -q^{b-a-1}x^a y^b, \qquad \text{for all } a, b \geq 0. \end{aligned}$$

Therefore, $z_1 - q^{-2}, z_2 \in \mathrm{ann}\, A$. Let $P = \mathrm{ann}\, A$. We use Proposition 4.1 to see that $P \in \mathrm{Prim}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$: noting that

$$P \cap \{e_1, e_2, e_3, e_1e_2 - q^{\pm 1}e_2e_1, e_2e_3 - q^{\pm 1}e_3e_2, \Delta_1, \Delta_3\} = \emptyset$$

and $\Delta_2 \in P$ (recall that $P = \mathrm{ann}\, A$ and it is thus easy to check if a given element is in P), it must be that $(P : \mathcal{H}) = (\Delta_2)$. □

Our next goal is to find generators for the primitive ideal $\mathrm{ann}\, A$ and its conjugates under the action of the torus $\mathcal{H} = (\mathbb{K}^*)^3$, as this group acts transitively on $\mathrm{Prim}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$. We begin by computing the center of $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$. For $u \in U_q(\mathfrak{sl}_4^+)$, $\bar{u}$ denotes its canonical image in $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$.

Lemma 4.5. *The center of $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ is the polynomial algebra in the variable $\bar{z}_1$.*

Proof. Let $N = \mathbb{K}[\Delta_1, \Delta_2, \Delta_3]$ be the polynomial subalgebra of $U_q(\mathfrak{sl}_4^+)$ generated by the q-central elements Δ_i, $i = 1, 2, 3$. By our separation of variables result [19, Prop. 1], $U_q(\mathfrak{sl}_4^+)$ is free over N, with basis (over N):

$$\begin{aligned} &\{X_1^a X_2^b X_3^c \mid (a,b,c) \in \mathbb{N}^3\} \cup \{X_1^a X_3^b X_5^c \mid (a,b,c) \in \mathbb{N}^3\} \\ &\quad \cup \{X_1^a X_2^b X_6^c \mid (a,b,c) \in \mathbb{N}^3\} \cup \{X_1^a X_5^b X_6^c \mid (a,b,c) \in \mathbb{N}^3\} \\ &\qquad \cup \{X_2^a X_3^b X_6^c \mid (a,b,c) \in \mathbb{N}^3\} \cup \{X_3^a X_5^b X_6^c \mid (a,b,c) \in \mathbb{N}^3\} \end{aligned} \tag{16}$$

(see [19, p. 597]). It follows easily that $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ is free over $N/\Delta_2 N = \mathbb{K}[\bar{\Delta}_1, \bar{\Delta}_3]$, with a basis obtained from (16) by replacing X_i with $\bar{X}_i$.

It is clear that $\bar{z}_1$ is central and that it generates a polynomial algebra (in fact, a subalgebra of $\mathbb{K}[\bar{\Delta}_1, \bar{\Delta}_3]$), so we need to prove that any central element of $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ is a polynomial in $\bar{z}_1$.

Let $\bar{\theta} \in U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ be central. In particular, $\bar{\theta}\bar{\Delta}_1 = \bar{\Delta}_1\bar{\theta}$. Since

$$\Delta_1 \left(X_1^a X_2^b X_3^c X_5^d X_6^e\right) = q^{a+b-d-e} \left(X_1^a X_2^b X_3^c X_5^d X_6^e\right) \Delta_1,$$

$\bar{\theta}$ can be written (uniquely) as

$$\begin{aligned} \sum_{a>0, b\geq 0} \bar{X}_1^a \bar{X}_2^b \bar{X}_6^{a+b}\, T_{a,b}^{(1)} + \sum_{a,b>0} \bar{X}_1^{a+b} \bar{X}_5^a \bar{X}_6^b\, T_{a,b}^{(2)} \\ + \sum_{a,b\geq 0} \bar{X}_2^a \bar{X}_3^b \bar{X}_6^a\, T_{a,b}^{(3)} + \sum_{a>0, b\geq 0} \bar{X}_1^a \bar{X}_3^b \bar{X}_5^a\, T_{a,b}^{(4)} \end{aligned} \tag{17}$$

for some polynomials $T_{a,b}^{(i)} \in \mathbb{K}[\bar{\Delta}_1, \bar{\Delta}_3]$. As Δ_2 is a weight element, the algebra $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ inherits a Q^+-grading from $U_q(\mathfrak{sl}_4^+)$ and the center of $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ is a graded subalgebra. Therefore it can be assumed that $\bar{\theta}$ is a weight element. By decomposing (17) into weight components, we are thus assuming that either

$$\bar{\theta} = \sum_{0\leq b\leq t} \bar{X}_1^a \bar{X}_2^b \bar{X}_6^{a+b}\, p_{t-b}^{(1)} + \sum_{0<b\leq t} \bar{X}_1^{a+b} \bar{X}_5^b \bar{X}_6^a\, p_{t-b}^{(2)} \tag{18}$$

for some fixed $t \geq 0$ and $a > 0$, or otherwise

$$\bar{\theta} = \sum_{0\leq a\leq t} \bar{X}_2^a \bar{X}_3^b \bar{X}_6^a\, p_{t-a}^{(3)} + \sum_{0<a\leq t} \bar{X}_1^a \bar{X}_3^b \bar{X}_5^a\, p_{t-a}^{(4)} \tag{19}$$

for fixed $t \geq 0$ and $b \geq 0$. In (18) and (19), $p_k^{(i)}$ is a polynomial in $\mathbb{K}[\bar{\Delta}_1, \bar{\Delta}_3]$ which is a linear combination of monomials in $\bar{\Delta}_1$ and $\bar{\Delta}_3$ all of which have total degree k.

We first consider the case that $\bar{\theta}$ is given by (18). Using the relations

$$\begin{aligned} &\bar{X}_1\bar{\Delta}_1 = q^{-1}\bar{\Delta}_1\bar{X}_1, \\ &\bar{X}_1\bar{\Delta}_3 = q\bar{\Delta}_3\bar{X}_1 \\ &\bar{X}_1\bar{X}_6 = \bar{X}_6\bar{X}_1 \\ &\bar{X}_1\bar{X}_2 = q^{-1}\bar{X}_2\bar{X}_1 \\ &\bar{X}_5^k\bar{X}_1 = q^{-k}\bar{X}_1\bar{X}_5^k + q^{-(k-1)}[k]\bar{X}_5^{k-1}\bar{\Delta}_1 \\ &\bar{X}_6\bar{\Delta}_1 = q\bar{\Delta}_1\bar{X}_6 \end{aligned}$$

and comparing the expressions of both sides of the equality $\bar{X}_1\bar{\theta} = \bar{\theta}\bar{X}_1$ in the given basis of $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ over $\mathbb{K}[\bar{\Delta}_1, \bar{\Delta}_3]$, we arrive at the following equations ($b \geq 1$):

$$p_t^{(1)}(\bar{\Delta}_1, \bar{\Delta}_3) = p_t^{(1)}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3) + q^{-a}\Delta_1 p_{t-1}^{(2)}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3) \tag{20}$$

$$p_{t-b}^{(1)}(\bar{\Delta}_1, \bar{\Delta}_3) = q^b p_{t-b}^{(1)}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3) \tag{21}$$

$$\begin{aligned} p_{t-b}^{(2)}(\bar{\Delta}_1, \bar{\Delta}_3) = q^{-b} p_{t-b}^{(2)}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3) \\ + q^{-(a+b)}[b+1]\Delta_1 p_{t-(b+1)}^{(2)}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3) \end{aligned} \tag{22}$$

Similarly, by considering the equality $\bar{X}_6\bar{\theta} = \bar{\theta}\bar{X}_6$, we obtain:

$$p^{(2)}_{t-b}(q^{-1}\bar{\Delta}_1, q\bar{\Delta}_3) = q^b p^{(2)}_{t-b}(\bar{\Delta}_1, \bar{\Delta}_3) \tag{23}$$

$$p^{(1)}_{t-b}(q^{-1}\bar{\Delta}_1, q\bar{\Delta}_3) = q^{-b} p^{(1)}_{t-b}(\bar{\Delta}_1, \bar{\Delta}_3) + q^{-(a+b+1)}[b+1]\Delta_1 p^{(1)}_{t-(b+1)}(\bar{\Delta}_1, \bar{\Delta}_3) \tag{24}$$

where equation (23) holds for $b \geq 1$ whereas equation (24) holds for $b \geq 0$. Assume there is $b \geq 1$ such that $p^{(2)}_{t-b} \neq 0$ and let $\tilde{b}$ be maximum with this property. By (22),

$$p^{(2)}_{t-\tilde{b}}(\bar{\Delta}_1, \bar{\Delta}_3) = q^{-\tilde{b}} p^{(2)}_{t-\tilde{b}}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3), \tag{25}$$

which combined with (23) yields also

$$p^{(2)}_{t-\tilde{b}}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3) = p^{(2)}_{t-\tilde{b}}(q^{-1}\bar{\Delta}_1, q\bar{\Delta}_3). \tag{26}$$

From this last equation it follows that $p^{(2)}_{t-\tilde{b}}(\bar{\Delta}_1, \bar{\Delta}_3)$ is a polynomial in $\bar{\Delta}_1\bar{\Delta}_3$, as q is not a root of unity, and from (25) we conclude that $\tilde{b} = 0$, a contradiction. Thus $p^{(2)}_{t-b} = 0$ for all $b \geq 1$. Now assume there is $b \geq 1$ such that $p^{(1)}_{t-b} \neq 0$ and let $\tilde{b}$ be maximum with this property. By (24),

$$p^{(1)}_{t-\tilde{b}}(q^{-1}\bar{\Delta}_1, q\bar{\Delta}_3) = q^{-\tilde{b}} p^{(1)}_{t-\tilde{b}}(\bar{\Delta}_1, \bar{\Delta}_3)$$

and by (21),

$$p^{(1)}_{t-\tilde{b}}(\bar{\Delta}_1, \bar{\Delta}_3) = q^{\tilde{b}} p^{(1)}_{t-\tilde{b}}(q\bar{\Delta}_1, q^{-1}\bar{\Delta}_3).$$

We conclude, as before, that $\tilde{b} = 0$. Hence, $p^{(1)}_{t-b} = 0$ for all $b \geq 1$ and, for $b = 0$, (24) yields

$$p^{(1)}_t(q^{-1}\bar{\Delta}_1, q\bar{\Delta}_3) = p^{(1)}_t(\bar{\Delta}_1, \bar{\Delta}_3),$$

which means that $p^{(1)}_t$ is a polynomial in $\bar{z}_1 = \bar{\Delta}_1\bar{\Delta}_3$. In particular,

$$\bar{\theta} = \bar{X}_1^a \bar{X}_6^a \, p^{(1)}_t$$

with $p^{(1)}_t$ central. This implies that

$$0 = \bar{\theta}\bar{X}_2 - \bar{X}_2\bar{\theta} = (q^{-a} - q^a)\bar{X}_1^a \bar{X}_2 \bar{X}_6^a \, p^{(1)}_t + [a]q^{-(a-1)}\bar{X}_1^a \bar{X}_6^{a-1} \Delta_1 p^{(1)}_t.$$

Since we are assuming that $a > 0$, it must be that $p^{(1)}_t = 0 = \bar{\theta}$. This concludes the case where $\bar{\theta}$ is of the form (18). In case $\bar{\theta}$ is given by (19) we proceed in a similar fashion, first using the equality $\bar{X}_3\bar{\theta} = \bar{\theta}\bar{X}_3$, along with the relation $\bar{X}_2\bar{X}_5 = q^{-1}\bar{X}_3\bar{\Delta}_1$ to deduce that $\bar{\theta} = \bar{X}_3^b \, p^{(3)}_t$, for some $b \geq 0$. Then, we use the equation $\bar{\theta}\bar{X}_1 = \bar{X}_1\bar{\theta}$ to infer that either $b = 0$ and $p^{(3)}_t$ is a polynomial in $\bar{z}_1$ or $p^{(3)}_t = 0 = \bar{\theta}$. We leave these details to the reader. □

The following proposition determines the $\mathcal{H}$-stratum of (Δ_2). The proof is essentially that of [17, Prop. 2.3], which we include for completeness.

Proposition 4.6. *The $\mathcal{H}$-stratum of (Δ_2) in $\operatorname{Prim} U_q(\mathfrak{sl}_4^+)$ is*

$$\operatorname{Prim}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+) = \{(\Delta_2, z_1 - \alpha) \mid \alpha \in \mathbb{K}^*\},$$

and these primitive ideals have height 2.

Proof. Let $Q \in \operatorname{Spec}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$ and assume $z_1 \in Q$. As z_1 is a weight element, it follows that $z_1 \in (Q : \mathcal{H}) = (\Delta_2)$, which is a contradiction (as seen at the beginning of 4.3.3). Hence $\Delta_2 \in Q$ and $z_1 \notin Q$. Conversely, suppose Q is a prime ideal of $U_q(\mathfrak{sl}_4^+)$ such that $\Delta_2 \in Q$ and $z_1 \notin Q$. If we set $J = (Q : \mathcal{H})$ then $\Delta_2 \in J$ and $\Delta_i \notin J$ for $i = 1, 3$. By Proposition 4.1, the only possibility is $J = (\Delta_2)$. We have thus seen that

$$\operatorname{Spec}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+) = \{Q \in \operatorname{Spec} U_q(\mathfrak{sl}_4^+) \mid \Delta_2 \in Q \text{ and } z_1 \notin Q\}. \tag{27}$$

Let T be the localization of $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ at the multiplicatively closed set generated by the central element $\bar{z}_1$. Since both Δ_2 and z_1 are $\mathcal{H}$-eigenvectors, the action of $\mathcal{H}$ on $U_q(\mathfrak{sl}_4^+)$ induces a rational action of $\mathcal{H}$ on T by automorphisms (see [4, Ex. II.3.A]). By (27) and standard results of localization theory, the localization map $U_q(\mathfrak{sl}_4^+) \to U_q(\mathfrak{sl}_4^+)/(\Delta_2) \to T$ induces a homeomorphism of $\operatorname{Spec}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$ onto $\operatorname{Spec} T$. Furthermore, as this map is $\mathcal{H}$-equivariant and (Δ_2) is the only $\mathcal{H}$-invariant ideal in $\operatorname{Spec}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$, it results that (0) is the only $\mathcal{H}$-invariant prime ideal of T. Now, every proper $\mathcal{H}$-invariant ideal of T is contained in a prime ideal of T which is $\mathcal{H}$-invariant (one can take, for example, a minimal prime over it and show that this prime ideal is necessarily $\mathcal{H}$-invariant); hence, T is $\mathcal{H}$-simple, i.e., (0) is the only proper $\mathcal{H}$-invariant ideal of T.

By the last paragraph, we can apply [4, Cor. II.3.9] to conclude that contraction and extension provide mutually inverse homeomorphisms between $\operatorname{Spec} T$ and $\operatorname{Spec} Z(T)$, where $Z(T)$ is the center of T. From Lemma 4.5 we know that the center of $U_q(\mathfrak{sl}_4^+)/(\Delta_2)$ is the polynomial algebra $\mathbb{K}[\bar{z}_1]$, and so it is easily deduced that $Z(T) = \mathbb{K}[\bar{z}_1^{\pm 1}]$. As we are assuming that $\mathbb{K}$ is algebraically closed, we have $\operatorname{Spec} T = \{(0), (\bar{z}_1 - \alpha)T \mid \alpha \in \mathbb{K}^*\}$.

Claim: For $\alpha \in \mathbb{K}^*$, $(\bar{z}_1 - \alpha)T \cap U_q(\mathfrak{sl}_4^+)/(\Delta_2) = (\bar{z}_1 - \alpha)U_q(\mathfrak{sl}_4^+)/(\Delta_2)$.
Proof of claim: The claim can be proved in the usual way (see [17, Prop. 2.3]).

It follows from the claim that $\operatorname{Spec}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+) = \{(\Delta_2), (\Delta_2, z_1 - \alpha) \mid \alpha \in \mathbb{K}^*\}$ and $\operatorname{Prim}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+) = \{(\Delta_2, z_1 - \alpha) \mid \alpha \in \mathbb{K}^*\}$, as the latter space consists of the ideals which are maximal in $\operatorname{Spec}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$.

The chain of prime ideals

$$(0) \subsetneq (\Delta_2) \subsetneq (\Delta_2, z_1 - \alpha)$$

gives $\operatorname{height}(\Delta_2, z_1 - \alpha) \geq 2$, for $\alpha \neq 0$. The reverse inequality can be obtained as in [17, Prop. 2.3], or by using the Generalized principal ideal theorem (see for example [21, 4.1.13]). $\square$

Let $P = \operatorname{ann} A$, where A is the simple $U_q(\mathfrak{sl}_4^+)$-module defined in (15). As a result of the previous proposition and Lemma 4.4, we have that $P = (\Delta_2, z_1 - q^{-2})$. The task of finding a simple $U_q(\mathfrak{sl}_4^+)$-module with given annihilator $Q \in \operatorname{Prim}_{(\Delta_2)} U_q(\mathfrak{sl}_4^+)$ is now trivial: if $Q = \sigma_{\bar{\lambda}}(P)$, then Q is the annihilator of the simple module obtained by twisting A by the automorphism $\sigma_{\bar{\lambda}}^{-1} = \sigma_{\bar{\lambda}^{-1}}$.

4.3.4 The $\mathcal{H}$-strata of (Δ_1) and (Δ_3)

Consider the $U_q(\mathfrak{sl}_4^+)$-module $P_{(\alpha,\beta,\gamma)} = \mathbb{K}[x, y^{\pm 1}]$, defined in [19, 6.3] in terms of the parameters $\alpha, \beta, \gamma \in \mathbb{K}$. Taking $\alpha = 0, \beta = 1$ and $\gamma = q^{-1}$, we obtain the following formulas for the action of $U_q(\mathfrak{sl}_4^+)$ on $P_{(0,1,q^{-1})}$:

$$e_1.x^a y^b = \begin{cases} q^{-(a+b)}(x^a y^b + [a]x^{a-1}y^{b-1}) & \text{if } b \geq 1 \\ q^{-(a+b)}x^a y^b + [a]x^{a-1}y^{b-1} & \text{if } b \leq 0, \end{cases}$$

$$e_2.x^a y^b = \begin{cases} q^b x^{a+1} y^b & \text{if } b \geq 0 \\ x^{a+1}y^b & \text{if } b \leq 0, \end{cases}$$

$$e_3.x^a y^b = \begin{cases} -q^{a-b}[a]x^{a-1}y^{b+1} & \text{if } b \geq 0 \\ -[a]x^{a-1}y^{b+1} & \text{if } b \leq -1. \end{cases}$$

Lemma 4.7. *The module $P_{(0,1,q^{-1})}$ is simple and $P \in \operatorname{Prim}_{(\Delta_1)} U_q(\mathfrak{sl}_4^+)$, where $P = \operatorname{ann} P_{(0,1,q^{-1})}$.*

Proof. First, we show that $P_{(0,1,q^{-1})}$ is simple. Let $(0) \neq W \subseteq P_{(0,1,q^{-1})}$ be a submodule and take $p = p(x,y) \in W \setminus \{0\}$ such that the x-degree of p is as small as possible, say $a \geq 0$. If $a > 0$, then $e_3.p$ is a nonzero element of W with smaller x-degree, which is a contradiction. Thus $a = 0$ and $p = p(y) \in \mathbb{K}[y^{\pm 1}]$. Notice that $e_3e_2.y^k \in \mathbb{K}^*.y^{k+1}$ for all $k \in \mathbb{Z}$ and hence, acting by a high enough power of e_3e_2, we can assume that $p \in \mathbb{K}[y]$ and choose such a nonzero element of W with minimum y-degree, say $b \geq 0$. Assume $b \geq 1$ and write

$$p = c_0 + \cdots + c_{b-1}y^{b-1} + y^b$$

with all $c_i \in \mathbb{K}$. Since $e_1.y^k = q^{-k}y^k$ for all $k \in \mathbb{Z}$, we have

$$c_{b-1}(q^{-b} - q^{-(b-1)})y^{b-1} + \cdots + c_0(q^{-b} - 1) = q^{-b}p - e_1.p \in W. \tag{28}$$

Given the minimality of b, the element in (28) must be equal to 0; hence $c_r = 0$ for all $0 \leq r < b$ and $p = y^b$. Computing still, we obtain

$$y^{b-1} = (qe_1e_2 - q^{-b}e_2).y^b \in W,$$

which contradicts the minimality of b. Therefore $b = 0$ and $1 \in W$. By construction, the modules $P_{(\alpha,\beta,\gamma)}$ are generated by $1 \in \mathbb{K}[x, y^{\pm 1}]$ and hence $W = P_{(0,1,q^{-1})}$, proving the simplicity of $P_{(0,1,q^{-1})}$.

Let $P = \operatorname{ann} P_{(0,1,q^{-1})}$. We have just seen that P is a primitive ideal of $U_q(\mathfrak{sl}_4^+)$; thus $P \in \operatorname{Prim}_J U_q(\mathfrak{sl}_4^+)$, for some $J \in \mathcal{H}\text{-Spec}\, U_q(\mathfrak{sl}_4^+)$. It is easy to check that none of the following weight elements is in P: e_1, e_2, e_3, $e_1e_2 - q^{\pm 1}e_2e_1$, $e_2e_3 - q^{\pm 1}e_3e_2$ (neither, except for e_3, annihilate $1 \in P_{(0,1,q^{-1})}$, and e_3 does not annihilate x, for example). Therefore, by Proposition 4.1, J must be one of the following ideals (0), (Δ_1), (Δ_2), (Δ_3) or (Δ_1, Δ_3). Again by [19, 6.3] (or by direct computation) we see that $\Delta_1 \in P$ and $\Delta_2 - q^{-1} \in P$, and we can also check that $\Delta_3 \notin P$ (for example, $\Delta_3.1 = -(q - q^{-1})y$). Thus $J = (\Delta_1)$. $\square$

Let $Q_{(0,1,q^{-1})}$ be the $U_q(\mathfrak{sl}_4^+)$-module obtained by twisting $P_{(0,1,q^{-1})}$ by the diagram automorphism η of 4.1. Recall that if $P_{(0,1,q^{-1})}$ is given by the representation ρ, then $Q_{(0,1,q^{-1})}$ is defined by the representation $\rho \circ \eta$. The following lemma is an easy consequence of Lemma 4.7.

Lemma 4.8. *The module $Q_{(0,1,q^{-1})}$ is simple and $Q \in \operatorname{Prim}_{(\Delta_3)} U_q(\mathfrak{sl}_4^+)$, where $Q = \operatorname{ann} Q_{(0,1,q^{-1})}$.*

Proof. The simplicity of $Q_{(0,1,q^{-1})}$ follows from Lemma 4.7 and the fact that η is onto. In particular, Q is a primitive ideal and $Q = \ker \rho \circ \eta = \eta(P)$, as $\eta^2 = 1$. Furthermore,

$$\sigma_{(\lambda_1,\lambda_2,\lambda_3)} \circ \eta = \eta \circ \sigma_{(\lambda_3,\lambda_2,\lambda_1)}, \qquad \text{for all } (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{H}, \tag{29}$$

and thus $(Q : \mathcal{H}) = \eta((P : \mathcal{H})) = (\Delta_3)$, since $\eta(\Delta_1) = \Delta_3$. Therefore, $Q \in \operatorname{Prim}_{(\Delta_3)} U_q(\mathfrak{sl}_4^+)$. □

We can finally characterize the $\mathcal{H}$-strata of (Δ_1) and (Δ_3) in $\operatorname{Prim} U_q(\mathfrak{sl}_4^+)$:

Proposition 4.9. (a) $\operatorname{Prim}_{(\Delta_1)} U_q(\mathfrak{sl}_4^+) = \{(\Delta_1, \Delta_2 - \alpha) \mid \alpha \in \mathbb{K}^*\}$*;*

(b) $\operatorname{Prim}_{(\Delta_3)} U_q(\mathfrak{sl}_4^+) = \{(\Delta_3, \Delta_2 - \alpha) \mid \alpha \in \mathbb{K}^*\}$.

The primitive ideals described above have height 2*.*

Proof. For the proof of part (a) we can use arguments similar to those of the proofs of Lemma 4.5 and Proposition 4.6.

Part (b) follows from part (a) by applying the diagram automorphism η. □

4.3.5 The $\mathcal{H}$-stratum of (Δ_1, Δ_3)

Let $B = \mathbb{K}[t]$ be the vector space of all polynomials in the variable t. It is easy to see that the formulas

$$\begin{aligned} e_1.t^k &= [k]t^{k-1}, \\ e_2.t^k &= t^{k+1}, \\ e_3.t^k &= [k]t^{k-1}, \end{aligned}$$

define an action of $U_q(\mathfrak{sl}_4^+)$ on B. Let $P = \operatorname{ann} B$.

Lemma 4.10. *With the action described above, B becomes a simple $U_q(\mathfrak{sl}_4^+)$-module and $P \in \operatorname{Prim}_{(\Delta_1,\Delta_3)} U_q(\mathfrak{sl}_4^+)$.*

Proof. It is routine to check that B is simple; therefore the ideal P is primitive and $P \in \operatorname{Prim}_J U_q(\mathfrak{sl}_4^+)$, for some $J \in \mathcal{H}\text{-}\operatorname{Spec} U_q(\mathfrak{sl}_4^+)$. As none of the weight elements e_1, e_2, e_3, $e_1e_2 - q^{\pm 1}e_2e_1$, $e_2e_3 - q^{\pm 1}e_3e_2$, Δ_2 annihilates B, yet Δ_1 and Δ_3 do annihilate it, it must be that $J = (\Delta_1, \Delta_3)$, as desired. □

Recall the quantum Casimir element $\Omega \in U_q(\mathfrak{sl}_3^+)$ and the elements Ω_i, $i = 1, 2$, of $U_q(\mathfrak{sl}_4^+)$ defined in 4.3. Let $P' = (\Delta_1, \Delta_3, e_1 - e_3, \Omega_1 - 1)$. By computing, we check that

$$e_1 - e_3, \Omega_1 - 1 \in P = \operatorname{ann} B,$$

and consequently $P' \subseteq P$.

Lemma 4.11. *Let P' be as before. Then,*

$$U_q(\mathfrak{sl}_3^+)/(\Omega - 1) \simeq U_q(\mathfrak{sl}_4^+)/P', \tag{30}$$

and $P = P'$.

Proof. The inclusion $U_q(\mathfrak{sl}_3^+) \to U_q(\mathfrak{sl}_4^+)$, $\dot{e}_i \mapsto e_i$, $i = 1, 2$, induces an algebra homomorphism $\phi : U_q(\mathfrak{sl}_3^+) \to U_q(\mathfrak{sl}_4^+)/P'$, which is onto because $e_1 - e_3 \in P'$. Moreover, since $\phi(\Omega - 1) = (\Omega_1 - 1) + P' = P'$, it follows that $\Omega - 1 \in \ker \phi$. By Theorem 3.5, $(\Omega - 1)$ is a maximal ideal of $U_q(\mathfrak{sl}_3^+)$; thus $\ker \phi = (\Omega - 1)$ and ϕ induces the desired isomorphism (30). In particular, $U_q(\mathfrak{sl}_4^+)/P'$ is a simple algebra and hence P' is a maximal ideal of $U_q(\mathfrak{sl}_4^+)$. This shows that $P' = P$, as we had already observed that $P' \subseteq P$. $\square$

Proposition 4.12. *The $\mathcal{H}$-stratum of* (Δ_1, Δ_3) *in* $\operatorname{Prim} U_q(\mathfrak{sl}_4^+)$ *is*

$$\operatorname{Prim}_{(\Delta_1,\Delta_3)} U_q(\mathfrak{sl}_4^+) = \{(\Delta_1, \Delta_3, e_1 - \alpha e_3, \Omega_1 - \beta) \mid \alpha, \beta \in \mathbb{K}^*\},$$

and these primitive ideals have height 4.

Proof. We have seen that $P = (\Delta_1, \Delta_3, e_1 - e_3, \Omega_1 - 1) \in \operatorname{Prim}_{(\Delta_1,\Delta_3)} U_q(\mathfrak{sl}_4^+)$. Since $\mathbb{K}$ is algebraically closed, the space $\operatorname{Prim}_{(\Delta_1,\Delta_3)} U_q(\mathfrak{sl}_4^+)$ consists of a single $\mathcal{H}$-orbit and it is easy to check that

$$\mathcal{H}.P = \{(\Delta_1, \Delta_3, e_1 - \alpha e_3, \Omega_1 - \beta) \mid \alpha, \beta \in \mathbb{K}^*\}.$$

The statement about the height of the primitive ideals in the $\mathcal{H}$-orbit of P follows from Tauvel's height formula and Lemma 4.11, as $\operatorname{GKdim}(U_q(\mathfrak{sl}_3^+)/(\Omega - 1)) = 2$ by Corollary 3.6. $\square$

4.3.6 The $\mathcal{H}$-strata of $(e_1e_2 - q^{\pm 1}e_2e_1)$ and $(e_2e_3 - q^{\pm 1}e_3e_2)$

We consider only the $\mathcal{H}$-stratum of $J_{5,b} = (e_2e_3 - q^{-1}e_3e_2)$ in detail, the cases of the $\mathcal{H}$-strata of $J_{5,a}$, $J_{6,a}$ and $J_{6,b}$ being similar. The factor algebra $U_q(\mathfrak{sl}_4^+)/J_{5,b}$ is isomorphic to $R := S_1[Y; \nu]$, where ν is the algebra automorphism of S_1 given by $\nu(e_1) = e_1$ and $\nu(e_2) = qe_2$. Indeed, there is a surjective algebra homomorphism $\phi : U_q(\mathfrak{sl}_4^+) \to R$ such that $\phi(e_i) = e_i$, $i = 1, 2$, and $\phi(e_3) = Y$. As $e_2e_3 - q^{-1}e_3e_2 \in \ker \phi$, ϕ induces a surjective map, which we still denote by ϕ, $U_q(\mathfrak{sl}_4^+)/J_{5,b} \to R$. The natural map $S_1 \to U_q(\mathfrak{sl}_4^+) \to U_q(\mathfrak{sl}_4^+)/J_{5,b}$ can be extended to an algebra homomorphism $\psi : R \to U_q(\mathfrak{sl}_4^+)/J_{5,b}$ such that $\psi(Y) = e_3 + J_{5,b}$, by the universal property of Ore extensions. The maps ϕ and ψ just defined are inverses of each other; in particular, $\phi : U_q(\mathfrak{sl}_4^+)/J_{5,b} \to R$ is an algebra isomorphism.

The algebra R is Q^+-graded so that ϕ becomes an isomorphism of graded algebras. Hence, the spaces $\operatorname{Spec}_{J_{5,b}} U_q(\mathfrak{sl}_4^+)$ and $\operatorname{Spec}_{(0)} R$ can be identified via ϕ.

Lemma 4.13. *The ideal $J_{5,b}$ of $U_q(\mathfrak{sl}_4^+)$ is primitive.*

Proof. It suffices to show that $\operatorname{Spec}_{(0)} R = \{(0)\}$, as $\operatorname{Prim}_{J_{5,b}} U_q(\mathfrak{sl}_4^+)$ consists of the maximal elements of $\operatorname{Spec}_{J_{5,b}} U_q(\mathfrak{sl}_4^+)$, by the stratification theory of Goodearl and Letzter [12].

Let $P \in \operatorname{Spec}_{(0)} R$ and assume $P \neq (0)$. As P is a completely prime ideal of R (because R is a homomorphic image of $U_q(\mathfrak{sl}_4^+)$ and the latter algebra has the property that all of its prime ideals are completely prime), it follows that $P' = P \cap S_1$ is a (completely) prime ideal of $S_1 \simeq U_q(\mathfrak{sl}_3^+)$. Since P contains no nonzero weight elements, the same is true of P' and by [20, Thm. 2.4], either $\Omega_1 - \alpha \in P'$ for some $\alpha \in \mathbb{K}^*$ or $P' = (0)$.

Assume $\Omega_1 - \alpha \in P'$, where $\alpha \in \mathbb{K}^*$. In R, we have

$$Y\Omega_1 = q^2\Omega_1 Y, \tag{31}$$

and thus

$$(q^2-1)\alpha Y = Y(\Omega_1 - \alpha) - q^2(\Omega_1 - \alpha)Y \in P. \tag{32}$$

This is a contradiction, as Y is a nonzero weight element of R and (32) implies that $Y \in P$. Therefore $P' = (0)$.

Let $f \in P \setminus (0)$, say

$$f = u_0 + \cdots + u_k Y^k,$$

with $k \geq 0$, $u_0, \ldots, u_k \in S_1$ and $u_k \neq 0$, and assume such an element was chosen with minimum k. Then, by (31) and the fact that Ω_1 is central in S_1,

$$\left[(1-q^{2k})u_0 + \cdots + (1-q^2)u_{k-1}Y^{k-1}\right]\Omega_1 = f\Omega_1 - q^{2k}\Omega_1 f \in P. \tag{33}$$

Since P is completely prime and $\Omega_1 \notin P$, the minimality of k implies that $u_r = 0$ for all $r < k$. Hence $f = u_k Y^k$ and again it must be that $u_k \in P$, as $Y \notin P$. Thus $u_k \in P \cap S_1 = (0)$, a contradiction. The contradiction resulted from our assumption that $P \neq (0)$, so $\mathrm{Spec}_{(0)} R = \{(0)\}$ and $\mathrm{Spec}_{J_{5,b}} U_q(\mathfrak{sl}_4^+) = \{J_{5,b}\}$. $\square$

Proposition 4.14. (a) $\mathrm{Prim}_{J_{5,a}} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - qe_2e_1)\}$;

(b) $\mathrm{Prim}_{J_{5,b}} U_q(\mathfrak{sl}_4^+) = \{(e_2e_3 - q^{-1}e_3e_2)\}$;

(c) $\mathrm{Prim}_{J_{6,a}} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - q^{-1}e_2e_1)\}$;

(d) $\mathrm{Prim}_{J_{6,b}} U_q(\mathfrak{sl}_4^+) = \{(e_2e_3 - qe_3e_2)\}$.

The primitive ideals described above have height 2.

Proof. Part (b) of the proposition is Lemma 4.13 and the others are similar (for example, for (a) we need only use the automorphism η). The height of these primitive ideals can be computed using Tauvel's height formula. For example, $\mathrm{height}(J_{5,b}) = 2$ because $\mathrm{GKdim}(U_q(\mathfrak{sl}_4^+)/J_{5,b}) = \mathrm{GKdim}(R) = 4$. $\square$

To finish this paragraph, we give an example of a simple $U_q(\mathfrak{sl}_4^+)$-module C with annihilator $J_{5,b}$. Simple $U_q(\mathfrak{sl}_4^+)$-modules with annihilators $J_{5,a}$, $J_{6,a}$ and $J_{6,b}$ can be readily obtained by twisting C by η and/or replacing q by q^{-1} in the formulas below. Let $C = \mathbb{K}[x, y^{\pm 1}]$ be the vector space with basis $\{x^a y^b \mid a \geq 0, b \in \mathbb{Z}\}$, and define an action of $U_q(\mathfrak{sl}_4^+)$ on C by

$$\begin{array}{rcll} e_1.x^ay^b &=& [a]x^{a-1}y^b, & \\ e_2.x^ay^b &=& q^{-b}x^{a+1}y^{b-1}, & \\ e_3.x^ay^b &=& x^ay^{b+1}, & a \geq 0, b \in \mathbb{Z}. \end{array} \tag{34}$$

(We leave it as an exercise to show that (34) does give a well-defined action of $U_q(\mathfrak{sl}_4^+)$ on C.)

Lemma 4.15. *The* $U_q(\mathfrak{sl}_4^+)$*-module* C *defined above is simple and* $\operatorname{ann} C = J_{5,b}$.

Proof. Let $P = \operatorname{ann} C$. The simplicity of C is easy to check, as it is to verify that $J_{5,b} \subseteq P$. Thus, $P \in \operatorname{Prim}_J U_q(\mathfrak{sl}_4^+)$ for some $J_{5,b} \subseteq J \in \mathcal{H}\text{-Spec}\, U_q(\mathfrak{sl}_4^+)$. Since

$$\{e_1, e_2, e_3, e_1e_2 - q^{\pm 1}e_2e_1, e_2e_3 - qe_3e_2\} \cap P = \emptyset \tag{35}$$

and

$$e_2e_3 - q^{-1}e_3e_2 \notin (\Delta_1) \cup (\Delta_2) \cup (\Delta_3) \cup (\Delta_1, \Delta_3) \cup (0), \tag{36}$$

(note that (36) was verified when we studied each of the individual strata involved), it can only be that $J = J_{5,b}$, by Proposition 4.1, and therefore $P = J_{5,b}$, by Proposition 4.14. □

4.3.7 The $\mathcal{H}$-strata of (e_1) and (e_3)

Since $U_q(\mathfrak{sl}_4^+)/(e_1) \simeq U_q(\mathfrak{sl}_3^+)$, the spaces $\operatorname{Prim}_{(e_1)} U_q(\mathfrak{sl}_4^+)$ and $\operatorname{Prim}_{(0)} U_q(\mathfrak{sl}_3^+)$ can be naturally identified. As we have seen in 3.2, or otherwise by [20, Thm. 2.4],

$$\operatorname{Prim}_{(0)} U_q(\mathfrak{sl}_3^+) = \{(\Omega - \alpha) \mid \alpha \in \mathbb{K}^*\}, \tag{37}$$

where Ω is given by (11).

Proposition 4.16. *Let Ω_i, $i = 1, 2$, be defined as in (12). Then,*

(a) $\operatorname{Prim}_{(e_1)} U_q(\mathfrak{sl}_4^+) = \{(e_1, \Omega_2 - \alpha) \mid \alpha \in \mathbb{K}^*\}$;

(b) $\operatorname{Prim}_{(e_3)} U_q(\mathfrak{sl}_4^+) = \{(e_3, \Omega_1 - \alpha) \mid \alpha \in \mathbb{K}^*\}$.

The primitive ideals described above have height 4.

Proof. Part (a) follows from (37). Since

$$U_q(\mathfrak{sl}_4^+)/(e_1, \Omega_2 - \alpha) \simeq U_q(\mathfrak{sl}_3^+)/(\Omega - \alpha)$$

and $\operatorname{GKdim}(U_q(\mathfrak{sl}_3^+)/(\Omega - \alpha)) = 2$, by Corollary 3.6, the last statement follows from Tauvel's height formula. Part (b) is analogous and can be obtained from (a) via the automorphism η. □

An example of a simple $U_q(\mathfrak{sl}_3^+)$-module with annihilator $(\Omega - 1) \subseteq U_q(\mathfrak{sl}_3^+)$ is the vector space $D = \mathbb{K}[t]$, with action induced by

$$\begin{aligned} e_1.t^k &= [k]t^{k-1}, \\ e_2.t^k &= t^{k+1}, \qquad\qquad k \geq 0 \end{aligned}$$

(see [19, 6.2]). This action extends to $U_q(\mathfrak{sl}_4^+)$ by defining $e_3.t^k = 0$ for all $k \geq 0$, and hence we obtain a simple $U_q(\mathfrak{sl}_4^+)$-module with annihilator $(e_3, \Omega_1 - 1)$. Twisting this action by the automorphisms of the form $\sigma_{\bar{\lambda}}$, $\bar{\lambda} \in (\mathbb{K}^*)^3$, and η, we easily get examples of simple $U_q(\mathfrak{sl}_4^+)$-modules corresponding to each of the primitive ideals of Proposition 4.16.

4.3.8 The $\mathcal{H}$-strata of $(e_1e_2 - q^{\pm 1}e_2e_1, e_2e_3 - q^{\pm 1}e_3e_2)$

Let $\delta, \epsilon \in \{-1, 1\}$, and consider the quantum affine space $\mathbb{K}_{\delta,\epsilon}[x, y, z]$, generated by x, y, z with relations $xz = zx$, $xy = q^{\epsilon}yx$, $yz = q^{\delta}zy$. There are isomorphisms

$$U_q(\mathfrak{sl}_4^+)/J_{8,a} \simeq \mathbb{K}_{1,1}[x, y, z], \qquad U_q(\mathfrak{sl}_4^+)/J_{8,b} \simeq \mathbb{K}_{-1,-1}[x, y, z], \tag{38}$$

$$U_q(\mathfrak{sl}_4^+)/J_9 \simeq \mathbb{K}_{1,-1}[x, y, z], \qquad U_q(\mathfrak{sl}_4^+)/J_{10} \simeq \mathbb{K}_{-1,1}[x, y, z], \tag{39}$$

each sending e_1, e_2, e_3 to x, y, z, respectively.

Denote the localization of $\mathbb{K}_{\delta,\epsilon}[x, y, z]$ at the multiplicatively closed set generated by the normal elements x, y and z by $\mathbb{K}_{\delta,\epsilon}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. It follows from [12, Thm. 4.4] and Theorem 2.3 that the isomorphism $U_q(\mathfrak{sl}_4^+)/J_9 \simeq \mathbb{K}_{1,-1}[x, y, z]$ provides a homeomorphism between $\mathrm{Prim}_{J_9} U_q(\mathfrak{sl}_4^+)$ and the space of maximal ideals of $\mathbb{K}_{1,-1}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, and similarly for $J_{8,a}$, $J_{8,b}$ and J_{10}. Furthermore, by Theorem 2.3(b), contraction and extension induce mutually inverse homeomorphisms between the space of maximal ideals of $\mathbb{K}_{\delta,\epsilon}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ and the space of maximal ideals of $Z_{\delta,\epsilon}$, the center of $\mathbb{K}_{\delta,\epsilon}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

Lemma 4.17. *The center of $\mathbb{K}_{\delta,\epsilon}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ is a Laurent polynomial algebra in the variable $w_{\delta,\epsilon} = x^{\delta}z^{\epsilon}$:*

$$Z_{\delta,\epsilon} = \mathbb{K}[w_{\delta,\epsilon}^{\pm 1}].$$

Proof. Let $w \in \mathbb{K}_{\delta,\epsilon}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ be nonzero. Then $w \in Z_{\delta,\epsilon}$ exactly when $dw = wd$ for all $d \in \{x, y, z\}$. Write

$$w = \sum_{(a,b,c)\in\mathbb{Z}^3} \lambda(a, b, c)x^a y^b z^c,$$

where the $\lambda(a, b, c) \in \mathbb{K}$ are all but finitely many equal to 0. A simple computation shows that $xw = wx$ is equivalent to the equation

$$\sum_{(a,b,c)\in\mathbb{Z}^3} \lambda(a, b, c)x^{a+1}y^b z^c = \sum_{(a,b,c)\in\mathbb{Z}^3} \lambda(a, b, c)q^{-\epsilon b}x^{a+1}y^b z^c, \tag{40}$$

and thus, since q is not a root of unity, $b = 0$ whenever $\lambda(a, b, c) \neq 0$. So, we can write

$$w = \sum_{(a,c)\in\mathbb{Z}^2} \lambda(a, c)x^a z^c,$$

and it is clear that w now commutes with both x and z. Similarly, we can show that a necessary and sufficient condition for w to commute with y is that $\epsilon a = \delta c$ whenever $\lambda(a, c) \neq 0$. Hence, since $\delta^2 = 1$, we can express the central element w in the form

$$w = \sum_{a\in\mathbb{Z}} \lambda(a)x^a z^{\epsilon\delta a} = \sum_{a\in\mathbb{Z}} \lambda(a)(w_{\delta,\epsilon})^{\delta a} \in \mathbb{K}[w_{\delta,\epsilon}^{\pm 1}].$$

Note that the integer powers of $w_{\delta,\epsilon}$ are linearly independent, and therefore the subalgebra of $\mathbb{K}_{\delta,\epsilon}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ generated by $w_{\delta,\epsilon}$ and $w_{\delta,\epsilon}^{-1}$ is indeed a Laurent polynomial algebra. $\square$

It follows from the stratification theory of Goodearl and Letzter and the lemma above that the primitive ideals of $\mathbb{K}_{\delta,\epsilon}[x,y,z]$ with no nonzero homogeneous elements (relative to the grading induced by the isomorphisms of (38) and (39)) are those of the form

$$I^{\alpha}_{\delta,\epsilon} = \mathbb{K}_{\delta,\epsilon}[x,y,z] \cap \mathbb{K}_{\delta,\epsilon}[x^{\pm1}, y^{\pm1}, z^{\pm1}](w_{\delta,\epsilon} - \alpha), \tag{41}$$

for $\alpha \in \mathbb{K}^*$.

Lemma 4.18. *Let $\alpha \in \mathbb{K}^*$. Then,*

(a) $I^{\alpha}_{1,1} = \mathbb{K}_{1,1}[x,y,z](xz - \alpha)$;

(b) $I^{\alpha}_{-1,-1} = \mathbb{K}_{-1,-1}[x,y,z](xz - \alpha^{-1})$;

(c) $I^{\alpha}_{1,-1} = \mathbb{K}_{1,-1}[x,y,z](x - \alpha z)$;

(d) $I^{\alpha}_{-1,1} = \mathbb{K}_{-1,1}[x,y,z](z - \alpha x)$.

Proof. We prove only (c), the proofs of the other parts being similar.

First, observe that $x - \alpha z$ is a normal element of $\mathbb{K}_{1,-1}[x,y,z]$, so that $\mathbb{K}_{1,-1}[x,y,z](x-\alpha z)$ is indeed an ideal of $\mathbb{K}_{1,-1}[x,y,z]$; also, this ideal is prime as the factor algebra is a quantum plane. On the other hand, the prime ideal $\mathbb{K}_{1,-1}[x^{\pm1}, y^{\pm1}, z^{\pm1}](xz^{-1} - \alpha)$, $\alpha \neq 0$, of $\mathbb{K}_{1,-1}[x^{\pm1}, y^{\pm1}, z^{\pm1}]$ has height 1, by the Principal ideal theorem [21, 4.1.11], and hence so does the ideal $I^{\alpha}_{1,-1}$ of $\mathbb{K}_{1,-1}[x,y,z]$, by localization theory. Therefore, as

$$(0) \subsetneq \mathbb{K}_{1,-1}[x,y,z](x - \alpha z) \subseteq I^{\alpha}_{1,-1}, \tag{42}$$

equality must hold at the latter inclusion of (42). □

Proposition 4.19. (a) $\mathrm{Prim}_{J_{8,a}} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - qe_2e_1, e_1e_3 - \alpha) \mid \alpha \in \mathbb{K}^*\}$;

(b) $\mathrm{Prim}_{J_{8,b}} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - q^{-1}e_2e_1, e_1e_3 - \alpha) \mid \alpha \in \mathbb{K}^*\}$;

(c) $\mathrm{Prim}_{J_9} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - q^{-1}e_2e_1, e_1 - \alpha e_3) \mid \alpha \in \mathbb{K}^*\}$;

(d) $\mathrm{Prim}_{J_{10}} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - qe_2e_1, e_1 - \alpha e_3) \mid \alpha \in \mathbb{K}^*\}$.

The primitive ideals described above have height 4.

Proof. Parts (a)–(d) are a consequence of the isomorphisms of (38) and (39), Theorem 2.3, [12, Thm. 4.4] and Lemma 4.18, noting that for all $\alpha \in \mathbb{K}^*$

$$(e_1e_2 - qe_2e_1, e_2e_3 - qe_3e_2, e_1e_3 - \alpha) = (e_1e_2 - qe_2e_1, e_1e_3 - \alpha),$$

as two-sided ideals of $U_q(\mathfrak{sl}_4^+)$, and similarly for the other ideals involved in this proposition.

Let $P \in \mathrm{Prim}_{J_{8,a}} U_q(\mathfrak{sl}_4^+)$, say $e_1e_3 - \alpha \in P$, with $\alpha \neq 0$. There is an algebra isomorphism $\mathbb{K}_q[\rho^{\pm1}, \theta] \to U_q(\mathfrak{sl}_4^+)/P$ sending ρ, θ, ρ^{-1} to e_1, e_2, $\alpha^{-1}e_3$, respectively, where $\mathbb{K}_q[\rho^{\pm1}, \theta]$ is the algebra generated by ρ, ρ^{-1} and θ, subject to the relations

$$\rho\rho^{-1} = 1 = \rho^{-1}\rho, \qquad \rho\theta = q\theta\rho. \tag{43}$$

Since $\mathbb{K}_q[\rho^{\pm1}, \theta]$ has GK dimension 2, it follows from Tauvel's height formula that P has height $6 - 2 = 4$. The other cases are analogous. □

Consider the action of $U_q(\mathfrak{sl}_4^+)$ on the vector space $E = \mathbb{K}[t^{\pm 1}]$ of Laurent polynomials in the variable t, given by the formulas ($k \in \mathbb{Z}$, $\alpha \in \mathbb{K}^*$):

$$e_1.t^k = t^{k+1}, \qquad e_2.t^k = q^k t^{k-1}, \qquad e_3.t^k = \alpha t^{k-1}. \tag{44}$$

Then $E = E_\alpha$ becomes a simple $U_q(\mathfrak{sl}_4^+)$-module and it is easy to see that

$$\operatorname{ann} E_\alpha = (e_1e_2 - q^{-1}e_2e_1, e_1e_3 - \alpha).$$

To obtain simple $U_q(\mathfrak{sl}_4^+)$-modules with annihilators in $\operatorname{Prim}_{J_9} U_q(\mathfrak{sl}_4^+)$ we can consider the module E'_α, given by the following action of $U_q(\mathfrak{sl}_4^+)$ on $\mathbb{K}[t^{\pm 1}]$:

$$e_1.t^k = \alpha t^{k+1}, \qquad e_2.t^k = q^k t^{k-1}, \qquad e_3.t^k = t^{k+1}. \tag{45}$$

Finally, by twisting E_α by the automorphism η and changing q into q^{-1} in the formula for the action of e_2 on E'_α in (45), we get simple $U_q(\mathfrak{sl}_4^+)$-modules with annihilators in $\operatorname{Prim}_{J_{8,a}} U_q(\mathfrak{sl}_4^+)$ and $\operatorname{Prim}_{J_{10}} U_q(\mathfrak{sl}_4^+)$, respectively.

4.3.9 The $\mathcal{H}$-strata of $(e_2e_3 - q^{\pm 1}e_3e_2, e_1)$ and $(e_1e_2 - q^{\pm 1}e_2e_1, e_3)$

Let $\mathbb{K}_q[x, y]$ be the *quantum plane*, given by generators x and y, satisfying only the relation $yx = qxy$. It is well-known that $\mathbb{K}_q[x, y]$ is primitive; in fact, let $\mathbb{K}_q[x, y]$ act on $\overline{F} = \mathbb{K}[t^{\pm 1}]$ as follows:

$$x.t^k = t^{k+1}, \qquad y.t^k = q^k t^{k-1}, \qquad k \in \mathbb{Z}.$$

Then, $\overline{F}$ is a faithful representation of $\mathbb{K}_q[x, y]$ which is simple. Since there is an isomorphism

$$\mathbb{K}_q[x, y] \longrightarrow U_q(\mathfrak{sl}_4^+)/(e_2e_3 - qe_3e_2, e_1), \tag{46}$$

mapping x to $e_3 + (e_2e_3 - qe_3e_2, e_1)$ and y to $e_2 + (e_2e_3 - qe_3e_2, e_1)$, we obtain the simple $U_q(\mathfrak{sl}_4^+)$-module $F = \mathbb{K}[t^{\pm 1}]$ given by

$$e_1.t^k = 0, \qquad e_2.t^k = q^k t^{k-1}, \qquad e_3.t^k = t^{k+1}, \qquad k \in \mathbb{Z}. \tag{47}$$

Note that $\operatorname{ann} F = (e_2e_3 - qe_3e_2, e_1)$, by construction. Simple $U_q(\mathfrak{sl}_4^+)$-modules with annihilators $(e_2e_3 - q^{-1}e_3e_2, e_1)$ and $(e_1e_2 - q^{\pm 1}e_2e_1, e_3)$ can be obtained, as before, by interchanging q and q^{-1} in (47) above and by using the automorphism η.

Proposition 4.20. (a) $\operatorname{Prim}_{J_{11,a}} U_q(\mathfrak{sl}_4^+) = \{(e_2e_3 - qe_3e_2, e_1)\}$*;*

(b) $\operatorname{Prim}_{J_{11,b}} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - q^{-1}e_2e_1, e_3)\}$*;*

(c) $\operatorname{Prim}_{J_{12,a}} U_q(\mathfrak{sl}_4^+) = \{(e_2e_3 - q^{-1}e_3e_2, e_1)\}$*;*

(d) $\operatorname{Prim}_{J_{12,b}} U_q(\mathfrak{sl}_4^+) = \{(e_1e_2 - qe_2e_1, e_3)\}$*.*

The primitive ideals described above have height 4.

Proof. For (a), we have just seen that $(e_2e_3 - qe_3e_2, e_1) \in \operatorname{Prim}_{J_{11,a}} U_q(\mathfrak{sl}_4^+)$. Equality holds because $\operatorname{Prim}_{J_{11,a}} U_q(\mathfrak{sl}_4^+)$ consists of a single $\mathcal{H}$-orbit. Since $\operatorname{GKdim}(\mathbb{K}_q[x, y]) = 2$, the last statement of the proposition follows from the isomorphism of (46) and Tauvel's height formula. Parts (b)–(d) are analogous. □

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