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Some asymptotic properties of SEIRS models with nonlinear incidence and random delays

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Abstract. This paper presents the dynamics of mosquitoes and humans with general nonlinear incidence rate and multiple distributed delays for the disease. The model is a SEIRS system of delay differential equations. The normalized dimensionless version is derived; analytical techniques are applied to find conditions for deterministic extinction and permanence of disease. The BRN R_0^* and ESPR $E(\mathrm{e}^{-(\mu_v T_1+\mu T_2)})$ are computed. Conditions for deterministic extinction and permanence are expressed in terms of R_0^* and $E(\mathrm{e}^{-(\mu_v T_1+\mu T_2)})$ and applied to a *P. vivax* malaria scenario. Numerical results are given.

Keywords: endemic equilibrium, basic reproduction number, permanence in the mean, Lyapunov functionals techniques, extinction rate.

1 Introduction

Malaria has exhibited an increasing alarming high mortality rate between 2015 and 2016. In fact, the latest WHO *World Malaria Report 2017* [14] estimates a total of 216 million cases of malaria from 91 countries in 2016, which constitutes a 5 million increase in the total malaria cases from the malaria statistics obtained previously in 2015. Moreover, the total death count was 445000, and sub-Saharan Africa accounts for 90% of the total estimated malaria cases. This rising trend in the malaria data signals a need for more learning about the disease, improvement of the existing control strategies and equipment, and also a need for more advanced resources etc. to fight and eradicate, or ameliorate the burdens of malaria.

Malaria and other mosquito-borne diseases such as dengue fever, yellow fever, zika fever, lymphatic filariasis, etc. exhibit some unique biological features. For instance, the incubation of the disease requires two hosts – the mosquito vector and human hosts, which may be either directly involved in a full life cycle of the infectious agent consisting of two separate and independent segments of sub-life cycles, which are completed

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separately inside the two hosts, or directly involved in two separate and independent half-life cycles of the infectious agent in the hosts. Therefore, there is a total latent time lapse of disease incubation, which extends over the two segments of delay incubation times, namely: (i) the incubation period of the infectious agent (or the half-life cycle) inside the vector, and (ii) the incubation period of the infectious agent (or the other half-life cycle) inside the human being (cf. [13, 14]). In fact, the malaria plasmodium undergoes the first developmental half-life cycle called the *sporogonic cycle* inside the female *Anopheles* mosquito lasting approximately 10–18 days, following a successful infected blood meal from a human through a mosquito bite. Moreover, the mosquito becomes infectious. The parasite completes the second developmental half-life cycle called the *exo-erythrocytic cycle* lasting about 7–30 days inside the exposed human being [13, 14], whenever the parasite is transferred to human being in the process of the infectious mosquito foraging for another blood meal.

The exposure and successful recovery from a malaria parasite, for example, *falciparum vivae* induces natural immunity against the disease, which can protect against subsequent severe outbreaks of the disease. Moreover, the effectiveness and duration of the naturally acquired immunity against malaria is determined by several factors such as the species and the frequency of exposure to the parasites (cf. [7, 14]).

Compartmental mathematical epidemic dynamic models have been used to investigate the dynamics of several different types of vector-borne diseases (cf. [1]). In general, these models are classified as SIS, SIR, SIRS, SEIRS, SEIR, etc. [3, 6, 7, 10] epidemic dynamic models depending on the compartments of the disease classes directly involved in the general disease dynamics. Many compartmental mathematical models with delays have been studied [6, 9].

Some important investigations in the study of population dynamic models expressed as systems of differential equations are the permanence, extinction of disease in the population, and also stability of the equilibria over sufficiently long time. Several papers in the literature [4, 10, 12] have addressed these topics. The extinction of disease seeks to find conditions that are sufficient for the disease related classes in the population, such as the exposed and infectious classes, to become extinct over sufficiently long time. The permanence of disease also answers the question about whether a significant number of people in the disease related classes will remain over sufficiently long time. Disease eradication or persistence of disease in the steady state population seeks to find conditions sufficient for the equilibria to be stable asymptotically.

The primary objectives of this paper include to investigate (i) the extinction, and (ii) the permanence of disease in a family of SEIRS epidemic models. In other words, we find conditions that are sufficient for a disease such as malaria, to become extinct from the population over time, and also conditions that cause the disease to be permanent in the population over time.

The rest of this paper is presented as follows: in Section 2, the mosquito–human models are derived. In Section 3, some model validation and preliminary results are presented. In Section 6, the results for the permanence of the disease are presented. Moreover, simulation results for the permanence of the disease in the population are presented in Section 7. In Section 4, the results for the extinction of the disease are

presented. Moreover, the numerical simulation results for the extinction of disease are presented in Section 7.

2 Derivation of the mosquito-host dynamics

The following assumptions are made to derive the epidemic model. Ideas from [5] will be used to derive the model for the mosquito–human dynamics.

(A) The delays T_1 and T_2 are the random incubation periods of the disease (plasmodium or dengue fever virus etc.) in the vector T_1 and in the human host T_2, respectively. T_3 is the random natural immunity period. f_{T_1} ($t_0 \leqslant T_1 \leqslant h_1$, $h_1 > 0$), f_{T_2} ($t_0 \leqslant T_2 \leqslant h_2,\ h_2 > 0$), and f_{T_3} ($t_0 \leqslant T_3 < \infty$) are the densities of T_i, $i = 1, 2, 3$ (cf. [7]).

(B) The vector (e.g., mosquito) population consists of two classes, namely: the susceptible vectors V_s and the infectious vectors V_i. Moreover, the total vector V_0 is constant at any time, i.e., $V_s(t) + V_i(t) = V_0 > 0$ for all $t \geqslant t_0$. Therefore, the birth and death rates of the vectors are equal and denoted $\hat{\mu}_v$. The susceptible vectors V_s are infected by infectious humans $\hat{I}$, and after the delay T_1, the exposed vectors become infectious V_i. There is homogenous mixing between the vector-host populations. It is assumed that the turnover of the vector population is very high, and the total number of vectors V_0 at any time t is very large, and as a result, $\hat{\mu}_v$ is sufficiently large number. In addition, it is assumed that the total vectors V_0 is exceedingly larger than the total humans present at any time t, denoted $\hat{N}(t)$, $t \geqslant t_0$. That is, $V_0 \gg \hat{N}(t)$, $t \geqslant t_0$.

(C) The humans consists of susceptible ($\hat{S}$), exposed ($\hat{E}$), infectious ($\hat{I}$), and removed ($\hat{R}$) classes. The susceptibles are infected by the infectious vectors V_i and become exposed (E). After the delay time T_2, the exposed individuals become infectious $\hat{I}$. The infectious class recovers from the disease with temporary or sufficiently long natural immunity and become ($\hat{R}$). Therefore, the total population at time t is $\hat{N}(t) = \hat{S}(t) + \hat{E}(t) + \hat{I}(t) + \hat{R}(t)$ for all $t \geqslant t_0$.

Furthermore, it is assumed that interaction between infectious vectors V_i and susceptible humans $\hat{S}$ exhibits nonlinear behavior due to overcrowding of vectors (see (B)), leading to change of behavior that limits the disease transmission rate. The nonlinear character for the incidence rate is characterized by the nonlinear function G. G satisfies the conditions of Assumption 1.

Assumption 1.

(A1) $G(0) = 0$;
(A2) $G(I)$ is strictly monotonic on $[0, \infty)$;
(A3) $G \in C^2([0, \infty), [0, \infty))$, and $G''(I) < 0$;
(A4) $\lim_{I \to \infty} G(I) = C,\ 0 \leqslant C < \infty$;
(A5) For all $I > 0,\ G(I) \leqslant I$;
(A6) For all $x, y \geqslant 0$,

$$\left(\frac{G(x)}{x} - \frac{G(y)}{y}\right)\big(G(x) - G(y)\big) \leqslant 0. \tag{1}$$

These assumptions clearly form an extension of the assumptions in [3, 7, 8]. Some examples of incidence functions include $G(x) = x/(1+\theta x)$, $\theta > 0$, etc.

(D) There is constant birthrate of humans $\hat{B}$ in the population, and all births are susceptibles. The natural deathrate of humans in the population is $\hat{\mu}$, and individuals die additionally due to disease related causes at the rate $\hat{d}$. From a biological point of view, $1/\hat{\mu}_v \ll 1/\hat{\mu}$. Thus, assuming exponential lifetime for all individuals (both vector and host) in the population, then the survival probabilities over intervals of length $T_1 = s \in [t_0, h_1]$ and $T_2 = s \in [t_0, h_2]$ satisfy

$$\mathrm{e}^{-\hat{\mu}_v T_1} \ll \mathrm{e}^{-\hat{\mu} T_1} \quad \text{and} \quad \mathrm{e}^{-\hat{\mu}_v T_1 - \hat{\mu} T_2} \ll \mathrm{e}^{-\hat{\mu}(T_1+T_2)}. \tag{2}$$

Applying similar ideas in [5], the vector dynamics from (A)–(D) follows the system

$$\mathrm{d}V_s(t) = \big[-\Lambda \mathrm{e}^{-\hat{\mu}_v T_1} \hat{I}(t-T_1) V_s(t-T_1) - \hat{\mu}_v V_s(t) + \hat{\mu}_v \big(V_s(t) + V_i(t)\big)\big]\,\mathrm{d}t, \tag{3}$$

$$\mathrm{d}V_i(t) = \big[\Lambda \mathrm{e}^{-\hat{\mu}_v T_1} \hat{I}(t-T_1) V_s(t-T_1) - \hat{\mu}_v V_i(t)\big]\,\mathrm{d}t, \tag{4}$$

$$V_0 = V_s(t) + V_i(t) \quad \forall t \geqslant t_0,\ t_0 \geqslant 0, \tag{5}$$

where Λ is the effective disease transmission rate from infectious humans to susceptible vectors. Observe, the incidence rate $\Lambda \mathrm{e}^{-\hat{\mu}_v T_1} \hat{I}(t-T_1) V_s(t-T_1)$ represents new infectious vectors occurring at time t, which became exposed at time $t-T_1$, and survive natural death over the incubation period T_1 with survival probability rate $\mathrm{e}^{-\hat{\mu}_v T_1}$, and are infectious at time t.

For the host population dynamics, at time t, it follows from (C) that when susceptible humans $\hat{S}$ and infectious vectors V_i interact with $\hat{\beta}$ effective contacts per vector, per unit time, then $\hat{\beta}\hat{S}(t)V_i(t)$ is the incidence rate of the disease into humans. Also, due to overcrowding effects of the vectors, it follows from (C) that the incidence rate becomes

$$\hat{\beta}\hat{S}(t) G\big(V_i(t)\big), \tag{6}$$

where G is the nonlinear incidence function satisfying the conditions in Assumption 1.

It follows easily (cf. [7]) from assumptions (A)–(D) and (6) that for T_j, $j = 1, 2, 3$, fixed in the population, the dynamics of malaria in the human population is given by the system

$$\mathrm{d}\hat{S}(t) = \big[\hat{B} - \hat{\beta}\hat{S}(t) G\big(V_i(t)\big) - \hat{\mu}\hat{S}(t) + \hat{\alpha}\hat{I}(t-T_3)\mathrm{e}^{-\hat{\mu}T_3}\big]\,\mathrm{d}t, \tag{7}$$

$$\mathrm{d}\hat{E}(t) = \big[\hat{\beta}\hat{S}(t) G\big(V_i(t)\big) - \hat{\mu}\hat{E}(t) - \hat{\beta}\hat{S}(t-T_2)\mathrm{e}^{-\hat{\mu}T_2} G\big(V_i(t-T_2)\big)\big]\,\mathrm{d}t, \tag{8}$$

$$\mathrm{d}\hat{I}(t) = \big[\hat{\beta}\hat{S}(t-T_2)\mathrm{e}^{-\hat{\mu}T_2} G\big(V_i(t-T_2)\big) - (\hat{\mu} + \hat{d} + \hat{\alpha})\hat{I}(t)\big]\,\mathrm{d}t, \tag{9}$$

$$\mathrm{d}\hat{R}(t) = \big[\hat{\alpha}\hat{I}(t) - \hat{\mu}\hat{R}(t) - \hat{\alpha}\hat{I}(t-T_3)\mathrm{e}^{-\hat{\mu}T_3}\big]\,\mathrm{d}t. \tag{10}$$

Furthermore, the function G satisfies Assumption 1, and the initial conditions are given as

$$\begin{aligned}
&\big(\hat{S}(t), \hat{E}(t), \hat{I}(t), \hat{R}(t)\big) = \big(\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t)\big), \quad t \in (-T_{\max}, t_0],\\
&\varphi_k \in \mathcal{C}\big((-T_{\max}, t_0], \mathbb{R}_+\big) \quad \forall k = 1, 2, 3, 4,\\
&\varphi_k(t_0) > 0 \quad \forall k = 1, 2, 3, 4, \quad \text{and} \quad \max_{\substack{t_0 \leqslant T_1 \leqslant h_1\\ t_0 \leqslant T_2 \leqslant h_2,\, T_3 \geqslant t_0}} (T_1 + T_2,\, T_3) = T_{\max},
\end{aligned} \tag{11}$$

where $\mathcal{C}((-T_{\max}, t_0], \mathbb{R}_+)$ is the space of continuous functions with the supremum norm

$$\|\varphi\|_\infty = \sup_{t \leqslant t_0} \bigl|\varphi(t)\bigr|.$$

Applying similar approximation technique in Wanduku [8] (see pp. 3800–3806, Appendix A, replace (G) with Assumption 1), the vector-host dynamics in (3)–(5) and (7)–(11) are combined to give the malaria model in [7], which omits the dynamics of the vector population under assumptions (A)–(D)[1]. That is, we obtain dimensionless variables for the humans

$$\begin{aligned} &S(t) = \frac{\hat{S}(t)}{(\frac{\hat{B}}{\hat{\mu}})}, \quad I(t) = \frac{\hat{I}(t)}{(\frac{\hat{B}}{\hat{\mu}})}, \quad E(t) = \frac{\hat{E}(t)}{(\frac{\hat{B}}{\hat{\mu}})}, \\ &R(t) = \frac{\hat{R}(t)}{(\frac{\hat{B}}{\hat{\mu}})}, \quad \text{and} \quad N(t) = \frac{\hat{N}(t)}{(\frac{\hat{B}}{\hat{\mu}})}, \end{aligned} \tag{12}$$

where $0 < \hat{N}(t) \leqslant \hat{B}/\hat{\mu}$ for all $t \geqslant t_0$, whenever $\hat{N}(t_0) \leqslant \hat{B}/\hat{\mu}$, and from (12) we see that

$$0 < S(t) + E(t) + I(t) + R(t) = N(t) \leqslant 1 \quad \forall t \geqslant t_0.$$

On a "slow" time scale (defined in [8]) $\eta = (\hat{B}/\hat{\mu})\Lambda t$, we obtain the approximated dimensionless human dynamics

$$\begin{aligned} \mathrm{d}S(\eta) = {} & \bigl[B - \beta S(\eta)\hat{G}\bigl(I(\eta - T_{1\eta})\bigr)\mathrm{e}^{-\mu_v T_{1\eta}} - \mu S(\eta) \\ & + \alpha I(\eta - T_{3\eta})\mathrm{e}^{-\mu T_{3\eta}}\bigr]\,\mathrm{d}\eta, \end{aligned} \tag{13}$$

$$\begin{aligned} \mathrm{d}E(\eta) = {} & \bigl[\beta S(\eta)\hat{G}\bigl(I(\eta - T_{1\eta})\bigr)\mathrm{e}^{-\mu_v T_{1\eta}} - \mu E(\eta) \\ & - \beta S(\eta - T_{2\eta})\hat{G}(I(\eta - T_{1\eta} - T_{2\eta}))\mathrm{e}^{-\mu_v T_{1\eta} - \mu T_{2\eta}}\bigr]\,\mathrm{d}\eta, \end{aligned} \tag{14}$$

$$\begin{aligned} \mathrm{d}I(\eta) = {} & \bigl[\beta S(\eta - T_{2\eta})\hat{G}\bigl(I(\eta - T_{1\eta} - T_{2\eta})\bigr)\mathrm{e}^{-\mu_v T_{1\eta} - \mu T_{2\eta}} - \mu I(\eta) \\ & - (\mu + d + \alpha)I(\eta)\bigr]\,\mathrm{d}\eta, \end{aligned} \tag{15}$$

$$\mathrm{d}R(\eta) = \bigl[\alpha I(\eta) - \mu R(\eta) - \alpha I(\eta - T_{3\eta})\mathrm{e}^{-\mu T_{3\eta}}\bigr]\,\mathrm{d}\eta, \tag{16}$$

where

$$\begin{aligned} &B = \frac{\hat{B}}{(\frac{\hat{B}}{\hat{\mu}})^2\Lambda}, \qquad \beta = \frac{\hat{\beta}V_0}{\hat{\mu}_v}, \qquad \mu = \frac{\hat{\mu}}{(\frac{\hat{B}}{\hat{\mu}})\Lambda}, \qquad \alpha = \frac{\hat{\alpha}}{(\frac{\hat{B}}{\hat{\mu}})\Lambda}, \\ &\mu_v = \frac{\hat{\mu_v}}{(\frac{\hat{B}}{\hat{\mu}})\Lambda}, \qquad d = \frac{\hat{d}}{(\frac{\hat{B}}{\hat{\mu}})\Lambda}, \qquad T_{j\eta} = \left(\frac{\hat{B}}{\hat{\mu}}\right)\Lambda T_j \quad \forall j = 1, 2, 3. \end{aligned} \tag{17}$$

System (13)–(16) describes the dynamics of malaria on a slower time scale η (see [8]). Furthermore, the analysis of the model (13)–(16) is considered only on the η timescale.

[1]This nontrivial process is omitted to conserve space (see [8]).

To reduce heavy notation, substitute t for η, and the delays T_j substitute $T_{j\eta}$ for all $j = 1, 2, 3$. Moreover, since the delays are distributed with densities f_{T_j} for all $j = 1, 2, 3$, it follows from (A)–(D), (13)–(16), and (11) that the average SEIRS dynamics is given as follows:

$$\mathrm{d}S(t) = \Bigg[B - \beta S(t) \int_{t_0}^{h_1} f_{T_1}(s)\mathrm{e}^{-\mu_v s} G\big(I(t-s)\big)\,\mathrm{d}s - \mu S(t) + \alpha \int_{t_0}^{\infty} f_{T_3}(r) I(t-r)\mathrm{e}^{-\mu r}\mathrm{d}r\Bigg]\,\mathrm{d}t, \tag{18}$$

$$\mathrm{d}E(t) = \Bigg[\beta S(t) \int_{t_0}^{h_1} f_{T_1}(s)\mathrm{e}^{-\mu_v s} G\big(I(t-s)\big)\,\mathrm{d}s - \mu E(t) - \beta \int_{t_0}^{h_2} f_{T_2}(u) S(t-u) \int_{t_0}^{h_1} f_{T_1}(s)\mathrm{e}^{-\mu_v s - \mu u} G\big(I(t-s-u)\big)\,\mathrm{d}s\,\mathrm{d}u\Bigg]\,\mathrm{d}t, \tag{19}$$

$$\mathrm{d}I(t) = \Bigg[\beta \int_{t_0}^{h_2} f_{T_2}(u) S(t-u) \int_{t_0}^{h_1} f_{T_1}(s)\mathrm{e}^{-\mu_v s - \mu u} G\big(I(t-s-u)\big)\,\mathrm{d}s\,\mathrm{d}u - (\mu + d + \alpha) I(t)\Bigg]\,\mathrm{d}t, \tag{20}$$

$$\mathrm{d}R(t) = \Bigg[\alpha I(t) - \mu R(t) - \alpha \int_{t_0}^{\infty} f_{T_3}(r) I(t-r)\mathrm{e}^{-\mu s}\,\mathrm{d}r\Bigg]\,\mathrm{d}t, \tag{21}$$

where the initial conditions are as follows. Let $h = h_1 + h_2$ and define

$$\begin{aligned} &\big(S(t), E(t), I(t), R(t)\big) = \big(\varphi_1(t), \varphi_2(t), \varphi_3(t), \varphi_4(t)\big), \quad t \in (-\infty, t_0], \\ &\varphi_k \in UC_g \subset \mathcal{C}\big((-\infty, t_0], \mathbb{R}_+\big), \quad \varphi_k(t_0) > 0 \quad \forall k = 1, 2, 3, 4, \end{aligned} \tag{22}$$

where UC_g is some fading memory sub-space of the Banach space $\mathcal{C}((-\infty, t_0], \mathbb{R}_+)$ endowed with the norm

$$\|\varphi\|_g = \sup_{t \leqslant t_0} \frac{|\varphi(t)|}{g(t)}, \tag{23}$$

and g is some continuous function with the following properties:

(P1) $g((-\infty, t_0]) \subseteq [1, \infty)$is nonincreasing, and $g(t_0) = 1$;
(P2) $\lim_{u \to t_0^-} g(t+u)/g(t) = 1$ uniformly on $[t_0, \infty)$; $\lim_{t \to -\infty} g(t) = \infty$.

An example of such a function is $g(t) = \mathrm{e}^{-at}$, $a > 0$ (cf. [2]). Note that for any g satisfying (P1)–(P2), the Banach space $\mathcal{C}((-\infty, t_0], \mathbb{R}_+)$ is continuously embedded in UC_g, which allows structural properties for $\mathcal{C}((-\infty, t_0], \mathbb{R}_+)$ with the uniform norm to hold in UC_g with $\|\cdot\|_g$ norm. Moreover, for $\varphi \in UC_g$, there exists g if and only if $\|\varphi\|_g < \infty$ and $|\varphi(t)|/g(t)$ is uniformly continuous on $(-\infty, t_0]$. Also, the function G in (18)–(21) satisfies the conditions of Assumption 1.

Observe the equations for E and R decouple from (18)–(21). Therefore, the results in this paper are exhibited for the decoupled system (18) and (20) for S and I.

Denote

$$\begin{aligned} &Y(t) = \big(S(t), E(t), I(t), R(t)\big)^{\mathrm{T}}, \quad X(t) = \big(S(t), E(t), I(t)\big)^{\mathrm{T}}, \\ &N(t) = S(t) + E(t) + I(t) + R(t). \end{aligned} \tag{24}$$

Whilst permanence or extinction has been investigated in some delay type systems (cf. [4,10,12]), the permanence and extinction in the sense of [12] in systems with multiple random delays is underdeveloped in the literature. Furthermore, as far as we know, no other paper has addressed extinction and persistence of malaria in a mosquito–human population dynamics involving delay differential equations in the line of thinking of [4, 12]. We recall the following definition from [11, 12].

Definition 1.

(i) A population $x(t)$ is called strongly permanent if $\liminf_{t\to+\infty} x(t) > 0$;
(ii) $x(t)$ is said to go extinct if $\lim_{t\to+\infty} x(t) = 0$;
(iii) $x(t)$ is said to be weakly permanent in the mean if $\limsup_{t\to+\infty} \int_0^t x(s)\,\mathrm{d}s/t > 0$;
(iv) $x(t)$ is said to be strongly permanent in the mean if $\liminf_{t\to+\infty} \int_0^t x(s)\,\mathrm{d}s/t > 0$;
(v) $x(t)$ is said to be stable in the mean if $\lim_{t\to\infty} \int_{t_0}^t x(s)\,\mathrm{d}s/t = c > 0$.

3 Model validation results

The consistency results for system (18)–(21) are given. Observe from (17) that expression B/μ simplifies to 1, i.e., $B/\mu \equiv 1$.

Theorem 1. *For the given initial conditions* (22)–(23)*, system* (18)–(21) *has a unique positive solution* $Y(t) \in \mathbb{R}^4_+$*. Moreover,*

$$\limsup_{t\to\infty} N(t) \leqslant S_0^* = \frac{B}{\mu} \equiv 1. \tag{25}$$

Furthermore, there is a positive self invariant space for the system denoted

$$D(\infty) = \bar{B}^{(-\infty,\infty)}_{\mathbb{R}^4_+,}\left(0, \frac{B}{\mu} \equiv 1\right),$$

where $D(\infty)$ *is the closed unit ball in* $\mathbb{R}^4_+$ *centered at the origin with radius* $B/\mu \equiv 1$ *containing all positive solutions defined over* $(-\infty, \infty)$.

Proof. The proof of this result is standard and easy to follow applying the notations (24) to system (18)–(21). □

Theorem 2. *The feasible region for the unique positive solutions* $Y(t)$*,* $t \geqslant t_0$*, of system* (18)–(21) *in the phase plane that lie in the self-invariant unit ball*

$$D(\infty) = \bar{B}^{(-\infty,\infty)}_{\mathbb{R}^4_+,}\left(0, \frac{B}{\mu} \equiv 1\right) = \bar{B}^{(-\infty,\infty)}_{\mathbb{R}^4_+,}(0, 1)$$

or the system, also lie in a much smaller space $D^{\mathrm{expl}}(\infty) \subset D(\infty)$*, where*

$$D^{\mathrm{expl}}(\infty) = \left\{ Y(t) \in \mathbb{R}_+^4 \colon \frac{B}{\mu + d} \leqslant N(t) = S(t) + E(t) + I(t) + R(t) \leqslant \frac{B}{\mu} \right. \\ \left. \forall t \in (-\infty, \infty) \right\}. \tag{26}$$

Moreover, the space $D^{\mathrm{expl}}(\infty)$ *is also self-invariant with respect to system* (18)–(21).

Proof. Suppose $Y(t) \in D(\infty)$, then it follows from (18)–(21) and (24) that the total population $N(t) = S(t) + E(t) + I(t) + R(t)$ satisfies the following inequality:

$$\big[B - (\mu + d)N(t)\big]\,\mathrm{d}t \leqslant \mathrm{d}N(t) \leqslant \big[B - (\mu)N(t)\big]\,\mathrm{d}t. \tag{27}$$

It is easy to see from (27) that

$$\frac{B}{\mu + d} \leqslant \liminf_{t \to \infty} N(t) \leqslant \limsup_{t \to \infty} N(t) \leqslant \frac{B}{\mu},$$

and (26) follows immediately. □

Remark 1. Theorem 2 signifies that every solution for (18)–(21) that starts in the unit ball $D(\infty)$ in the phase plane, oscillates continuously inside $D(\infty)$. Moreover, if the solution oscillates and enters the space

$$D^{\mathrm{expl}}(\infty) = \bar{B}^{(-\infty,\infty)}_{\mathbb{R}^4_+,}\left(0, \frac{B}{\mu} \equiv 1\right) \cap (\bar{B}^{(-\infty,\infty)}_{\mathbb{R}^4_+,}\left(0, \frac{B}{\mu + d}\right)^c,$$

the solution stays in $D^{\mathrm{expl}}(\infty)$ for all time.

Biologically, observe that B/μ and $B/(\mu+d)$ represent the total births that occur over the average lifespans $1/\mu$ and $1/(\mu + d)$ of a human being in a malaria-free population and in a malaria- epidemic population, respectively. Thus, Theorem 2 signifies that when the population grows into $N(t) \in [B/(\mu+d), B/\mu]$, it stays within that range for all time.

Also, it is easy to see that system (18)–(21) has a DFE $E_0 = (S_0^*, 0, 0) = (B/\mu \equiv 1, 0, 0) = (1, 0, 0)$. The basic reproduction number (BRN) for the disease when the delays in the system T_1, T_2, and T_3 are constant, is given by

$$\hat{R}_0^* = \frac{\beta}{\mu + d + \alpha}. \tag{28}$$

Furthermore, when $\hat{R}_0^* < 1$, then $E_0 = X_0^* = (S_0^*, 0, 0) = (1, 0, 0)$ is asymptotically stable, and the disease can be eradicated from the population. Also, when the delays in the system T_i, $i = 1, 2, 3$, are distributed, the BRN is proportional to

$$R_0 \propto \frac{\beta}{(\mu + d + \alpha)} + \frac{\alpha}{(\mu + d + \alpha)}.$$

And disease is eradicated from the system, whenever $R_0 \leqslant 1$.

The following result can be made about the nonzero steady state of the dimensionless system (18)–(21) when Assumption 1 is satisfied.

Theorem 3. *Let the conditions of Assumption* 1 *be satisfied. Suppose* $R_0 > 1$ (*or* $\hat{R}_0^* > 1$) *and the expected survival probability rate of the plasmodium satisfies*

$$E\big(\mathrm{e}^{-\mu_v T_1 - \mu T_2}\big) \geqslant \frac{R_0}{(R_0 - \frac{\alpha}{\mu+d+\alpha})G'(0)}.$$

Then there exists a nonzero endemic equilibrium $E_1 = (S_1^*, E_1^*, I_1^*)$ *for the dimensionless system* (18)–(21)*, where*

$$E\big(\mathrm{e}^{-\mu_v T_1 - \mu T_2}\big) = \int_{t_0}^{h_2}\int_{t_0}^{h_1} \mathrm{e}^{-\mu_v s - \mu u} f_{T_2}(u) f_{T_1}(s)\,\mathrm{d}s\,\mathrm{d}u.$$

Proof. The dimensionless endemic equilibrium $E_1 = (S_1^*, I_1^*)$ of the decoupled (18)–(21) is a solution to the following system:

$$B - \beta E\big(\mathrm{e}^{-\mu_v T_1}\big) SG(I) - \mu S + \alpha E\big(\mathrm{e}^{-\mu T_3}\big) I = 0, \tag{29}$$

$$\beta E\big(\mathrm{e}^{-\mu_v T_1 - \mu T_2}\big) SG(I) - (\mu + d + \alpha) I = 0. \tag{30}$$

Solving for S from (30) and substituting the result into (29) give the following equation:

$$H(I) = 0, \tag{31}$$

where

$$H(I) = B - \frac{1}{E(\mathrm{e}^{-\mu_v T_1 + \mu T_2})} I \bigg[\frac{(\mu + d + \alpha)\mu}{\beta G(I)} + (\mu + d) E\big(\mathrm{e}^{-\mu_v T_1}\big) + \alpha E\big(\mathrm{e}^{-\mu_v T_1}\big)\big(1 - E\big(\mathrm{e}^{-\mu T_2}\big) E\big(\mathrm{e}^{-\mu T_3}\big)\big)\bigg]. \tag{32}$$

Note that $0 < E(\mathrm{e}^{-\mu T_i}) \leqslant 1$, $i = 1, 2, 3$, and $\lim_{I\to\infty} G(I) = C < \infty$, hence for sufficiently large positive value of I, $H(I) < 0$. Furthermore, the derivative of $H(I)$ is given by

$$H'(I) = -\frac{(\mu + d + \alpha)\mu}{\beta E(\mathrm{e}^{-\mu_v T_1 - \mu T_2})} \frac{(G(I) - IG'(I))}{G^2(I)} - \frac{1}{E(\mathrm{e}^{-\mu_v T_1 - \mu T_2})} \times \big((\mu + d) E\big(\mathrm{e}^{-\mu_v T_1}\big) + \alpha E\big(\mathrm{e}^{-\mu_v T_1}\big)\big(1 - E\big(\mathrm{e}^{-\mu T_2}\big) E\big(\mathrm{e}^{-\mu T_3}\big)\big)\big).$$

Assume without loss of generality that $G'(I) > 0$. It follows from the other properties of G in Assumption 1, that is, $G(0) = 0$, $G''(I) < 0$, that $(G(I) - IG'(I)) > 0$, and this further implies that $H'(I) < 0$ for all $I > 0$. That is, $H(I)$ is a decreasing function over all $I > 0$. Therefore, a positive root of equation (31) requires that $H(0) > 0$. Observe from (32) and the dimensionless expressions in (17)

$$\begin{aligned} H(0) &= B\bigg(1 - \frac{(\mu + d + \alpha)}{\beta G'(0) E(\mathrm{e}^{-\mu_v T_1 - \mu T_2})}\bigg) \\ &= B\bigg(1 - \frac{1}{(R_0 - \frac{\alpha}{\mu+d+\alpha}) G'(0) E(\mathrm{e}^{-\mu_v T_1 - \mu T_2})}\bigg) \geqslant B\bigg(1 - \frac{1}{R_0}\bigg). \end{aligned}$$

For $R_0 > 1$, it is easy to see that $H(0) > 0$. □

The extinction of disease will be investigated in the neighborhood of the zero steady state E_0, and the permanence of disease will be investigated in the neighborhood of E_1.

4 Extinction of disease

In this section, the extinction of malaria from system (18)–(21) is investigated.

Lemma 1. *Let the assumptions of Theorem* 2 *hold, and define the following Lyapunov functional in* $D^{\mathrm{expl}}(\infty)$:

$$\tilde{V}(t) = V(t) + \beta\Bigg[\int_{t_0}^{h_2}\int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s)\mathrm{e}^{-(\mu_v s+\mu u)} \int_{t-u}^{t} S(\theta)\frac{G(I(\theta-s))}{I(t)}\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}u$$
$$+\int_{t_0}^{h_2}\int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s)\mathrm{e}^{-(\mu_v s+\mu u)} \int_{t-s}^{t} S(t)\frac{G(I(\theta))}{I(t)}\,d\theta\,\mathrm{d}s\,\mathrm{d}u\Bigg], \tag{33}$$

where $V(t) = \log I(t)$. *It follows that*

$$\limsup_{t\to\infty}\frac{1}{t}\log\bigl(I(t)\bigr) \leqslant \beta\frac{B}{\mu}E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr) - (\mu+d+\alpha). \tag{34}$$

Proof. The differential operator $\dot{V}$ applied to the Lyapunov functional $\tilde{V}(t)$ with respect to system (18) leads to the following:

$$\dot{\tilde{V}}(t) = \beta\int_{t_0}^{h_2} f_{T_2}(u)\int_{t_0}^{h_1} f_{T_1}(s)\mathrm{e}^{-(\mu_v s+\mu u)} S(t)\frac{G(I(t))}{I(t)}\,\mathrm{d}s\,\mathrm{d}u - (\mu+d+\alpha). \tag{35}$$

Since $S(t), I(t) \in D^{\mathrm{expl}}(\infty)$ and G satisfies the conditions of Assumption 1, it follows easily from (35) that

$$\dot{\tilde{V}}(t) \leqslant \beta\frac{B}{\mu}E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr) - (\mu+d+\alpha). \tag{36}$$

Now, integrating both sides of (36) over the interval $[t_0, t]$, it follows from (36) and (33) that

$$\begin{aligned}\log I(t) &\leqslant \tilde{V}(t)\\ &\leqslant \tilde{V}(t_0) + \Bigg[\beta\frac{B}{\mu}E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr) - (\mu+d+\alpha)\Bigg](t-t_0).\end{aligned} \tag{37}$$

Diving both sides of (37) by t and taking the limit supremum as $t\to\infty$, it is easy to see that (37) reduces to

$$\limsup_{t\to\infty}\frac{1}{t}\log I(t) \leqslant \Bigg[\beta\frac{B}{\mu}E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr) - (\mu+d+\alpha)\Bigg]. \tag{38}$$

And the result (34) follows immediately from (38). □

The extinction conditions for the infectious population over time are expressed in terms: (i) the BRN R_0^* in (28), and (ii) the expected survival probability rate (ESPR) of the parasites $E(\mathrm{e}^{-(\mu_v T_1+\mu T_2)})$, also defined in [7, Thm. 5.1].

Theorem 4. *Suppose Lemma* 1 *is satisfied, and let the BRN* R_0^* *be defined as in* (28). *In addition, let one of the following conditions hold*:

(i) $R_0^* \geqslant 1$ *and* $E(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}) < 1/R_0^*$, *or*
(ii) $R_0^* < 1$.

Then

$$\limsup_{t\to\infty} \frac{1}{t}\log\bigl(I(t)\bigr) < -\lambda, \tag{39}$$

where $\lambda > 0$ *is some positive constant. In other words,* $I(t)$ *converges to zero exponentially.*

Proof. Suppose Theorem 4(i) holds, then from (34),

$$\limsup_{t\to\infty} \frac{1}{t}\log\bigl(I(t)\bigr) < \beta\frac{B}{\mu}\left(E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr) - \frac{1}{R_0^*}\right) \equiv -\lambda,$$

where the positive constant $\lambda > 0$ is taken to be as follows:

$$\begin{aligned}\lambda &\equiv (\mu+d+\alpha) - \beta\frac{B}{\mu}E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr)\\ &\equiv \beta\frac{B}{\mu}\left(\frac{1}{R_0^*} - E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr)\right) > 0.\end{aligned} \tag{40}$$

Also, suppose Theorem 4(ii) holds, then from (34),

$$\begin{aligned}\limsup_{t\to\infty} \frac{1}{t}\log\bigl(I(t)\bigr) &\leqslant \beta\frac{B}{\mu}E\bigl(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}\bigr) - (\mu+d+\alpha)\\ &< \beta\frac{B}{\mu} - (\mu+d+\alpha)\\ &= -(1-R_0^*)(\mu+d+\alpha)\\ &\equiv -\lambda,\end{aligned}$$

where the positive constant $\lambda > 0$ is taken to be as follows:

$$\lambda \equiv (1-R_0^*)(\mu+d+\alpha) > 0. \qquad \square \tag{41}$$

Remark 2. Theorems 4 and 2 signify that all trajectories of $(S(t), I(t))$ of (18) and (20) that start in $D(\infty)$ and grow into $D^{\mathrm{expl}}(\infty) \subset D(\infty)$ remain in $D^{\mathrm{expl}}(\infty)$. Moreover, on the phase plane of $(S(t), I(t))$, the trajectory of $I(t)$, $t \geqslant t_0$, ultimately turn to zero exponentially, whenever either the ESPR $E(\mathrm{e}^{-(\mu_v T_1+\mu T_2)}) < 1/R_0^*$ for $R_0^* \geqslant 1$ or whenever the BRN $R_0^* < 1$. Furthermore, the Lyapunov exponent from (39) is estimated by the term λ defined in (40) and (41).

5 Persistence of susceptibility and stability of zero equilibrium

Theorem 4 characterizes the behavior of $I(t)$ coordinate of the solution $(S(t), I(t))$ of (18) and (20) in the phase plane. The question remains about how $S(t)$ behave asymptotically in the phase plane.

Using Definition 1(iii)–(v), the average behavior of $S(t)$ over sufficiently long time is given below. Also, stability conditions for the DFE $E_0 = (S_0^*, 0) = (1, 0)$ are given, whenever Theorem 4 holds.

Theorem 5. *Let conditions* (i)–(ii) *of Theorem* 4 *be satisfied. In* $D^{\mathrm{expl}}(\infty)$*, the trajectories of* $S(t)$ *of the decoupled system* (18) *and* (20) *satisfy*

$$\lim_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi = \frac{B}{\mu} \equiv 1. \tag{42}$$

That is, the susceptible state is strongly persistent in the mean over long time (*see Definition* 1(iii)–(iv)). *Moreover, it is stable in the mean, and the average value of* $S(t)$ *over time is* $S(t) = S_0^* = B/\mu$*, obtained when the system is in steady state.*

Proof. Suppose either of conditions (i)–(ii) in Theorem 4 hold, then it follows clearly from Theorem 4 that for every $\epsilon > 0$, there is a positive constant $K_1(\epsilon) \equiv K_1 > 0$ such that

$$I(t) < \epsilon, \quad \text{whenever } t > K_1. \tag{43}$$

It follows from (43) that

$$I(t-s) < \epsilon, \quad \text{whenever } t > K_1 + h_1, \tag{44}$$

for all $s \in [t_0, h_1]$.

In $D^{\mathrm{expl}}(\infty)$, define

$$V_1(t) = S(t) + \alpha \int_{t_0}^{\infty} f_{T_3}(r)\mathrm{e}^{\mu r} \int_{t-r}^{t} I(\theta)\,\mathrm{d}\theta\,\mathrm{d}r. \tag{45}$$

The differential operator $\dot{V}_1$ applied to the Lyapunov functional $V_1(t)$ in (45) leads to the following:

$$\dot{V}_1(t) = g(S, I) - \mu S(t), \tag{46}$$

where

$$\begin{aligned} g(S, I) = B - \beta S(t) &\int_{t_0}^{h_1} f_{T_1}(s)\mathrm{e}^{-\mu_v s} G\big(I(t-s)\big)\,\mathrm{d}s \\ &+ \alpha E\big(\mathrm{e}^{-\mu T_3}\big) I(t). \end{aligned} \tag{47}$$

Estimating the right-hand-side of (46) in $D^{\mathrm{expl}}(\infty)$ and integrating over $[t_0, t]$, it follows from (43)–(44) that

$$\begin{aligned} V_1(t) &\leqslant V_1(t_0) + B(t-t_0) + \int_{t_0}^{K_1} \alpha I(\xi)\,\mathrm{d}\xi + \int_{K_1}^{t} \alpha I(\xi)\,\mathrm{d}\xi - \mu \int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi, \\ &\leqslant V_1(t_0) + B(t-t_0) + \alpha \frac{B}{\mu}(K_1 - t_0) + \alpha(t-K_1)\epsilon - \mu \int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{48}$$

Thus, dividing both sides of (48) by t and taking the limit supremum as $t \to \infty$, it follows that

$$\limsup_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi \leqslant \frac{B}{\mu} + \frac{\alpha}{\mu}\epsilon. \tag{49}$$

On the other hand, estimating $g(S, I)$ in (47) from below and using the conditions of Assumption 1 and (44), it is easy to see that in $D^{\mathrm{expl}}(\infty)$,

$$\begin{aligned} g(S,I) &\geqslant B - \beta S(t) \int_{t_0}^{h_1} f_{T_1}(s) \mathrm{e}^{-\mu_v s} \big(I(t-s)\big)\,\mathrm{d}s \\ &\geqslant B - \beta \frac{B}{\mu} E\big(\mathrm{e}^{-\mu_v T_1}\big)\epsilon \geqslant B - \beta \frac{B}{\mu}\epsilon \end{aligned} \tag{50}$$

for all $t > K_1 + h_1$. Moreover, for $t \in [t_0, K_1 + h_1]$,

$$g(S,I) \geqslant B - \beta \left(\frac{B}{\mu}\right)^2. \tag{51}$$

Therefore, applying (50)–(51) into (46), then integrating both sides of (46) over $[t_0, t]$, and diving the result by t, it is easy to see from (46) that

$$\begin{aligned} \frac{1}{t} V_1(t) \geqslant{}& \frac{1}{t} V_1(t_0) + B\left(1 - \frac{t_0}{t}\right) - \frac{1}{t}\beta \left(\frac{B}{\mu}\right)^2 (K_1 + h_1 - t_0) \\ &- \beta \frac{B}{\mu}\epsilon \left[1 - \frac{K_1 + h_1}{t}\right] - \frac{1}{t}\mu \int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{52}$$

Observe that in $D^{\mathrm{expl}}(\infty)$, $\lim_{t\to\infty} V_1(t)/t = 0$, and $\lim_{t\to\infty} V_1(t_0)/t = 0$. Therefore, rearranging (52) and taking the limit infimum of both sides as $t \to \infty$, it is easy to see that

$$\liminf_{t\to\infty} \frac{1}{t} \int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi \geqslant \frac{B}{\mu} - \frac{1}{\mu}\beta \frac{B}{\mu}\epsilon. \tag{53}$$

It follows from (49) and (53) that

$$\frac{B}{\mu} - \frac{1}{\mu}\beta\frac{B}{\mu}\epsilon \leqslant \liminf_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi \leqslant \limsup_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi \leqslant \frac{B}{\mu} + \frac{\alpha}{\mu}\epsilon. \tag{54}$$

Hence, for ϵ arbitrarily small, the result in (42) follows immediately from (54). □

Remark 3. Theorem 5 signifies that the DFE E_0 is strongly persistent and stable in the mean by Definition 1(iii)–(v). That is, over sufficiently long time, on average the human population will be in the DFE E_0. Thus, the conditions in Theorem 4 are sufficient for malaria to be eradicated from the population when the population is in a steady state.

Theorem 6. *Suppose any of the conditions in the hypothesis of Theorem* 4 *are satisfied. Also, suppose the conditions of Theorem* 5 *hold. It follows that in* $D^{\mathrm{expl}}(\infty)$*, the DFE* $E_0 = (S_0^*, 0) = (B/\mu, 0) = (1, 0)$ *is stable in the sense of Lyapunov.*

Proof. It is left to show that every trajectory that starts near E_0 remains near E_0 asymptotically. Indeed, if Theorem 4(i)–(ii) holds, then all trajectories of $I(t)$ converge asymptotically and exponentially to $I_0^* = 0$. It remains to show that if the trajectories of $S(t)$ from Theorem 5 (42), converge asymptotically in the mean to $S_0^* = B/\mu$, then they must remain asymptotically near $S_0^* = B/\mu$.

Indeed, if on the contrary, there exist a trajectory for $S(t)$ starting near $S_0^* = B/\mu \equiv 1$ that does not stay near $S_0^* = B/\mu$ asymptotically, that is, there exists some $\epsilon_0 > 0$ and $\delta(t_0, \epsilon_0) > 0$ such that $\|S(t_0) - S_0^*\| < \delta$, but $\|S(t) - S_0^*\| \geqslant \epsilon_0$ for all $t \geqslant t_0$, then clearly from (42), either

$$S_0^* = \lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi \geqslant S_0^* + \epsilon_0 \quad \text{or} \quad S_0^* = \lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} S(\xi)\,\mathrm{d}\xi \leqslant S_0^* - \epsilon_0. \tag{55}$$

Thus, ϵ_0 must be zero, otherwise (55) is a contradiction. Hence, $E_0 = (S_0^*, 0)$ is stable in the sense of Lyapunov. □

Remark 4. Theorems 5, 4, and 2 signify that not only is the DFE $E_0 = (S_0^*, 0) = (1, 0)$ of (18) and (20) stable and persistent in the mean, but it is also stable in the sense of Lyapunov. Thus, the conditions in Theorem 4 are strong disease eradication conditions.

6 Permanence of infectivity near nonzero equilibrium

As remarked in Theorem 3, when $R_0^* > 1$, system (18) and (20) has a nonzero equilibrium $E_1 = (S_1^*, I_1^*)$. In this section, conditions for $I(t)$ to be strongly persistent (Definition 1(i)) in the neighborhood of E_1 are given.

Lemma 2. *Suppose the conditions of Theorems* 2 *and* 3 *are satisfied, and let the nonlinear incidence function G satisfy the assumptions of Assumption* 1.

Then every positive solution $(S(t), I(t)) \in D(\infty)$ of the decoupled system (18) *and* (20) *with initial conditions* (22) *and* (23) *satisfies the following conditions:*

$$\begin{aligned} &\liminf_{t\to\infty} S(t) \geqslant v_1 \equiv \frac{B}{\mu + \beta G(S_0)} \\ &\liminf_{t\to\infty} I(t) \geqslant v_2 \equiv qI_1^* \mathrm{e}^{-(\mu+d+\alpha)(\rho+1)h}, \end{aligned} \tag{56}$$

where $h = h_1 + h_2$, and $\rho > 0$ is a suitable positive constant, $S_1^* < \min\{S_0,\, S^{\vartriangle}\}$ and $0 < q < \bar{q} < 1$, given that

$$\begin{aligned} \bar{q} &= \frac{B\beta E(\mathrm{e}^{-\mu T_1})G(I_1^*) - \mu\alpha E(\mathrm{e}^{-\mu T_3})I_1^*}{(B + \alpha E(\mathrm{e}^{-\mu T_3})I_1^*)\beta I_1^*}, \\ S^{\vartriangle} &= \frac{B}{k}\bigl(1 - \mathrm{e}^{-k\rho h}\bigr), \quad k = \mu + \beta G(qI_1^*). \end{aligned} \tag{57}$$

Proof. Recall (25) asserts that for $N(t) = S(t)+E(t)+I(t)+R(t)$, $\limsup_{t\to\infty} N(t) \leqslant S_0^* = B/\mu$. This implies that $\limsup_{t\to\infty} S(t) \leqslant S_0^* \equiv 1$. This further implies that for any arbitrarily small $\epsilon > 0$, there exists a sufficiently large $\varLambda > 0$ such that

$$I(t) \leqslant S_0^* + \varepsilon, \quad \text{whenever } t \geqslant \varLambda.$$

Without loss of generality, let $\varLambda_1 > 0$ be sufficiently large such that

$$t \geqslant \varLambda \geqslant \max_{(s,r)\in[t_0,h_1]\times[t_0,\infty)} (\varLambda_1 + s,\, \varLambda_1 + r).$$

It follows from Assumption 1, (2), and (18) that

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &\geqslant B - \beta S(t) \int_{t_0}^{h_1} f_{T_1}(s)\mathrm{e}^{-\mu s} G(S_0^* + \epsilon)\,\mathrm{d}s - \mu S(t) \\ &\geqslant B - \bigl[\mu + \beta G(S_0^* + \epsilon)\bigr] S(t). \end{aligned} \tag{58}$$

From (58) it follows that

$$S(t) \geqslant \frac{B}{k_1} - \frac{B}{k_1}\mathrm{e}^{-k_1(t-t_0)} + S(t_0)\mathrm{e}^{-k_1(t-t_0)}, \tag{59}$$

where $k_1 = \mu + \beta G(S_0^* + \epsilon)$.

It is easy to see from (59)

$$\liminf_{t\to\infty} S(t) \geqslant \frac{B}{\mu + \beta G(S_0 + \epsilon)}.$$

Since $\epsilon > 0$ is arbitrarily small, then the first part of (56) follows immediately.

In the following, it is shown that $\liminf_{t\to\infty} I(t) \geqslant v_2$. In order to establish this result, it is first proved that it is impossible that $I(t) \leqslant qI_1^*$ for sufficiently large $t \geqslant t_0$, where $q \in (0,1)$ is defined in the hypothesis. Suppose on the contrary there exists some sufficiently large $\Lambda_0 > t_0 > 0$ such that $I(t) \leqslant qI_1^*$ for all $t \geqslant \Lambda_0$. It follows from (18) that

$$S_1^* = \frac{B + \alpha E(\mathrm{e}^{-\mu T_3})I_1^*}{\mu + \beta E(\mathrm{e}^{-\mu T_1})G(I_1^*)} = \frac{B}{\mu + \frac{B\beta E(\mathrm{e}^{-\mu T_1})G(I_1^*) - \mu\alpha E(\mathrm{e}^{-\mu T_3})I_1^*}{B+\alpha E(\mathrm{e}^{-\mu T_3})I_1^*}}. \tag{60}$$

But it can be easily seen from (18) and (20) that

$$\begin{aligned} &B\beta E\big(\mathrm{e}^{-\mu T_1}\big)G(I_1^*) - \mu\alpha E(\mathrm{e}^{-\mu T_3})I_1^* \\ &\quad = \frac{\mu(\mu+d+\alpha)[S_0^* - \frac{\alpha E(\mathrm{e}^{-\mu T_3})E(\mathrm{e}^{-\mu T_2})}{(\mu+d+\alpha)}S_1^*]}{E(\mathrm{e}^{-\mu T_2})S_1^*} I_1^* \geqslant \frac{\mu(\mu+d+\alpha)(S_0^* - S_1^*)}{E(\mathrm{e}^{-\mu T_2})S_1^*} \\ &\quad > 0 \end{aligned}$$

since $S_0^* = B/\mu \geqslant S_1^*$. Therefore, from (60) it follows that

$$S_1^* < \frac{B}{\mu + \beta I_1^* q} \leqslant \frac{B}{\mu + \beta G(qI_1^*)}, \tag{61}$$

where $0 < q < \bar{q}$, and $\bar{q}$ is defined in (57).

For all vector values $(s,r) \in [t_0,h_1] \times [t_0,\infty)$, define

$$\Lambda_{0,\max} = \max_{(s,r)\in[t_0,h_1]\times[t_0,\infty)} (\Lambda_0 + s,\ \Lambda_0 + r).$$

It follows from Assumption 1 and (18) that for all $t \geqslant \Lambda_{0,\max}$,

$$S(t) \geqslant \frac{B}{k} - \frac{B}{k}\mathrm{e}^{-k(t-\Lambda_{0,\max})} + S(\Lambda_{0,\max})\mathrm{e}^{-k(t-\Lambda_{0,\max})}, \tag{62}$$

where k is defined in (57). For $t \geqslant \Lambda_{0,\max} + \rho h$, where $h = h_1 + h_2$, and $\rho > 0$ is sufficiently large, it follows from (62) that

$$S(t) \geqslant \frac{B}{k}\big[1 - \mathrm{e}^{-k(t-\Lambda_{0,\max})}\big] \geqslant \frac{B}{k}\big[1 - \mathrm{e}^{-k\rho h}\big] = S^{\vartriangle}. \tag{63}$$

Hence, from (61) and (63) it follows that for some suitable choice of $\rho > 0$ sufficiently large, then

$$S^{\vartriangle} > S_1^* \quad \forall t \geqslant \Lambda_{0,\max} + \rho h. \tag{64}$$

For $t \geqslant \Lambda_{0,\max} + \rho h$, define

$$\begin{aligned} V(t) = I(t) &+ \beta S_1^* \int_{t_0}^{h_2}\int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s)\mathrm{e}^{-\mu(s+u)} \int_{t-s}^{t} G\big(I(v-u)\big)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}u \\ &+ \beta S_1^* \int_{t_0}^{h_2}\int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s)\mathrm{e}^{-\mu(s+u)} \int_{t-u}^{t} G\big(I(v)\big)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}u. \end{aligned} \tag{65}$$

It is easy to see from system (18)–(20) and (65) that differentiating $V(t)$ with respect to system (18) and (20) leads to the following:

$$\dot{V}(t) = \beta \int_{t_0}^{h_2} \int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s) \mathrm{e}^{-\mu(s+u)} G(I(t-s-u)) \big[S(t-u) - S_1^*\big] \,\mathrm{d}s\,\mathrm{d}u + \left[\beta S_1^* E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) \frac{G(I(t))}{I(t)} - (\mu + d + \alpha)\right] I(t). \tag{66}$$

For all $t \geqslant \Lambda_{0,\max} + \rho h + h > \Lambda_{0,\max} + \rho h + h_2$, it follows from (1), (64), and (18)–(20) that

$$\begin{aligned}\dot{V}(t) &\geqslant \beta \int_{t_0}^{h_2} \int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s) \mathrm{e}^{-\mu(s+u)} G\big(I(t-s-u)\big) \big[S^{\vartriangle} - S_1^*\big] \,\mathrm{d}s\,\mathrm{d}u \\ &\quad + \left[\beta S_1^* E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) \frac{G(I_1^*)}{I_1^*} - (\mu + d + \alpha)\right] I(t) \\ &= \beta \int_{t_0}^{h_2} \int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s) \mathrm{e}^{-\mu(s+u)} G\big(I(t-s-u)\big) \big[S^{\vartriangle} - S_1^*\big] \,\mathrm{d}s\,\mathrm{d}u. \end{aligned} \tag{67}$$

Observe that the union of the subintervals $\bigcup_{(s,u)\in[t_0,h_1]\times[t_0,h_2]} [t_0 - (s+u),\, t_0] = [t_0 - h,\, t_0]$, where $h = h_1 + h_2$. Denote the following:

$$i_{\min} = \min_{\substack{\theta\in[t_0-h,\,t_0] \\ (s,u)\in[t_0,h_1]\times[t_0,h_2]}} I(\Lambda_{0,\max} + \rho h + h + s + u + \theta). \tag{68}$$

Note that (68) is equivalent to $i_{\min} = \min_{\theta\in[t_0-h,\,t_0]} I(\Lambda_{0,\max} + \rho h + h + h + \theta)$.

It is shown in the following that $I(t) \geqslant i_{\min}$ for all $t \geqslant \Lambda_{0,\max} + \rho h + h \geqslant \Lambda_{0,\max} + \rho h + u$ for all $u \in [t_0, h_2]$.

Suppose on the contrary there exists $\tau_1 \geqslant 0$ such that $I(t) \geqslant i_{\min}$ for all $t \in [\Lambda_{0,\max} + \rho h + h,\, \Lambda_{0,\max} + \rho h + h + h + \tau_1] \supset [\Lambda_{0,\max} + \rho h + h,\, \Lambda_{0,\max} + \rho h + h + s + u + \tau_1]$, for all $(s,u) \in [t_0, h_1] \times [t_0, h_2]$,

$$I(\Lambda_{0,\max} + \rho h + h + h + \tau_1) = i_{\min}, \quad \text{and} \quad \dot{I}(\Lambda_{0,\max} + \rho h + h + h + \tau_1) \leqslant 0. \tag{69}$$

For the value of $t = \Lambda_{0,\max} + \rho h + h + h + \tau_1$, it follows that $S(t-u) > S^{\vartriangle} > S_1^*$, and $t - s - u \in [\Lambda_{0,\max} + \rho h + h,\, \Lambda_{0,\max} + \rho h + h + h + \tau_1]$ for all $(s,u) \in [t_0, h_1] \times [t_0, h_2]$, and it can be further seen from (18)–(20), (64), and (1) that

$$\begin{aligned}\dot{I}(t) &\geqslant \beta E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) G(i_{\min}) S^{\vartriangle} - (\mu + d + \alpha) i_{\min} \\ &= \left[\beta E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) \frac{G(i_{\min})}{i_{\min}} S^{\vartriangle} - (\mu + d + \alpha)\right] i_{\min} \\ &> \left[\beta E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) \frac{G(I_1^*)}{I_1^*} S_1^* - (\mu + d + \alpha)\right] i_{\min} = 0. \end{aligned} \tag{70}$$

But (70) contradicts (69). Therefore, $I(t) \geqslant i_{\min}$ for all $t \geqslant \Lambda_{0,\max} + \rho h + h \geqslant \Lambda_{0,\max} + \rho h + u + s$, for all $(s,u) \in [t_0, h_1] \times [t_0, h_2]$.

It follows further from (66)–(68) and the Assumption 1 that for all $t \geqslant \Lambda_{0,\max} + \rho h + h + h \geqslant \Lambda_{0,\max} + \rho h + h + s + u$ and for all $(s,u) \in [t_0, h_1] \times [t_0, h_2]$,

$$\begin{aligned}\dot{V}(t) &\geqslant \beta \int_{t_0}^{h_2} \int_{t_0}^{h_1} f_{T_2}(u) f_{T_1}(s) \mathrm{e}^{-\mu(s+u)} G\big(I(t-s-u)\big)\big[S^{\vartriangle} - S_1^*\big]\,\mathrm{d}s\,\mathrm{d}u \\ &> \beta E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) G(i_{\min})\big(S^{\vartriangle} - S_1^*\big) > 0. \end{aligned} \tag{71}$$

From (71) it implies that $\limsup_{t\to\infty} V(t) = +\infty$.

On the contrary, it can be seen from (25) that $\limsup_{t\to\infty} N(t) \leqslant S_0^* = B/\mu$, which implies that $\limsup_{t\to\infty} I(t) \leqslant S_0^* = B/\mu$. This further implies that for every $\epsilon > 0$ infinitesimally small, there exists $\tau_2 > 0$ sufficiently large such that $I(t) \leqslant S_0^* + \varepsilon$ for all $t \geqslant \tau_2$. It follows from Assumption 1 that for all $v \in [t-s, t]$, $(s,u) \in [t_0, h_1] \times [t_0, h_2]$,

$$\begin{aligned} G\big(I(t-s-u)\big) &\leqslant G\big(I(v-u)\big) \leqslant G\big(I(t-u)\big) \leqslant G\big(I(t)\big) \\ &\leqslant G(S_0^* + \epsilon). \end{aligned} \tag{72}$$

From (72) it follows that

$$\limsup_{t\to\infty} G\big(I(t-s-u)\big) \leqslant \limsup_{t\to\infty} G\big(I(t)\big) \leqslant G(S_0^*). \tag{73}$$

It is easy to see from (65) and (73) that

$$\limsup_{t\to\infty} V(t) \leqslant S_0^* + \beta S_1^* G(S_0^*) E\big((T_1+T_2)\mathrm{e}^{-\mu(T_1+T_2)}\big) < \infty.$$

Therefore, it is impossible that $I(t) \leqslant qI_1^*$ for sufficiently large $t \geqslant t_0$, where $q \in (0,1)$.

Hence, the following are possible:

Case 1. $I(t) \geqslant qI_1^*$ for all t sufficiently large; and

Case 2. $I(t)$ oscillates about qI_1^* for sufficiently large t.

Obviously, we need show only Case 2. Suppose t_1 and t_2 are sufficiently large values such that

$$I(t_1) = I(t_2) = qI_1^*, \quad \text{and} \quad I(t) < qI_1^* \quad \forall (t_1, t_2).$$

If for all $(s,u) \in [t_0, h_1] \times [t_0, h_2]$, $t_2 - t_1 \leqslant \rho h + h$, where $h = h_1 + h_2$, observe that $[t_1,\, t_1 + \rho h + s + u] \subseteq [t_1, t_1 + \rho h + h]$, and it is easy to see from (18) by integration that

$$I(t) \geqslant I(t_1)\mathrm{e}^{-(\mu+d+\alpha)(t-t_1)} \geqslant qI_1^* \mathrm{e}^{-(\mu+d+\alpha)(\rho+1)h} \equiv v_2.$$

If for all $(s,u) \in [t_0, h_1] \times [t_0, h_2]$, $t_2 - t_1 > \rho h + h \geqslant \rho h + s + u$, then it can be seen easily that $I(t) \geqslant v_2$ for all $t \in [t_1,\, t_1 + \rho h + s + u] \subseteq [t_1,\, t_1 + \rho h + h]$.

Now, for each $t \in (\rho h + h,\, t_2) \supseteq (\rho h + s + u,\, t_2)$ $\forall (s,u) \in [t_0, h_1] \times [t_0, h_2]$, one can also claim that $I(t) \geqslant v_2$. Indeed, as similarly shown above, suppose on the contrary for all $(s,u) \in [t_0, h_1] \times [t_0, h_2]$, there exists $T^* > 0$ such that $I(t) \geqslant v_2$ for all

$$t \in [t_1,\, t_1 + \rho h + h + T^*] \supseteq [t_1, t_1 + \rho h + s + u + T^*],$$
$$I(t_1 + \rho h + h + T^*) = v_2, \quad \text{but} \quad \dot{I}(t_1 + \rho h + h + T^*) \leqslant 0. \tag{74}$$

It follows from (18)–(20) and (1) that for the value of $t = t_1 + \rho h + h + T^*$,

$$\begin{aligned} I(t) &\geqslant \beta E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) G(v_2) S^{\vartriangle} - (\mu + d + \alpha) v_2 \\ &> \left[\beta E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) \frac{G(v_2)}{v_2} S_1^* - (\mu + d + \alpha)\right] v_2 \\ &\geqslant \left[\beta E\big(\mathrm{e}^{-\mu(T_1+T_2)}\big) \frac{G(I_1^*)}{I_1^*} S_1^* - (\mu + d + \alpha)\right] v_2 = 0. \end{aligned} \tag{75}$$

Observe that (75) contradicts (74). Therefore, $I(t) \geqslant v_2$ for $t \in [t_1, t_2]$. And since $[t_1, t_2]$ is arbitrary, it implies that $I(t) \geqslant v_2$ for all sufficiently large t. Therefore, (56) is satisfied. □

Theorem 7. *If the conditions of Lemma* 2 *are satisfied, then system* (18)–(20) *is strongly permanent for any total delay time* $h = h_1 + h_2$ *according to Definition* 1(i).

Remark 5.

(i) From Lemma 2, (56) observe that when $\beta = 0$, then $v_1 = B/\mu$. That is, when disease transmission stops, then asymptotically, the smallest S that remain are new births over the average lifespan $1/\mu$ of the population, i.e., the DFE $S_0^* = B/\mu \equiv 1$. Also, as $\beta \to \infty$, then the total susceptibles that remains $v_1 \to 0^+$. That is, as disease transmission rises, even the new births are either infected, or die from natural or disease related causes over time.

(ii) From (56) observe that $\mathrm{e}^{-(\mu+d+\alpha)(\rho+1)h}$ is the survival probability from natural death (μ), disease mortality (d), and from infectiousness (α), over the total life cycle of the parasite h. Thus, the smallest I that remains asymptotically $v_2 \equiv qI_1^*\mathrm{e}^{-(\mu+d+\alpha)(\rho+1)h}$ is a fraction $q \in (0,1)$ of the endemic equilibrium I_1^* that survives from death and disease over life cycle h.

7 Example: Application to *P. vivax* malaria

In this section, the extinction results are exhibited for the *P. vivax* malaria example in Wanduku [8]. This is accomplished by examining the trajectories of the decoupled system (18) and (20) relative to the zero and endemic equilibria. To conserve space, we recall the dimensionless parameters in [8, p. 3793, Table 1] given in Table 1, and the reader is referred to [8] for detailed description of the *P. vivax* malaria scenario.

The dimensional estimates for the parameters of the malaria model given in [8, p. 3792, (a)–(e)] are applied to (17) to find the dimensionless parameters for model (18)–(21) given in Table 1.

The Euler approximation scheme is used to generate trajectories for $S(t)$, $I(t)$ over time $[0, 1000]$ days in Fig. 1. We use $G(I) = a_1 I/(1+I)$, $a_1 = 0.05$ in [8]. Furthermore, the initial fractions of susceptible, exposed, infectious, and removed individuals in the

initial population size $\hat{N}(t_0) = 65000$ are used:

$$S(t) = \frac{10}{23} \approx \frac{28261}{65000}, \qquad E(t) = \frac{5}{23} \approx \frac{14131}{65000},$$
$$I(t) = \frac{6}{23} \approx \frac{16957}{65000}, \qquad R(t) = \frac{2}{23} \approx \frac{5653}{65000}$$

for all $t \in [-T, 0]$, $T = \max(T_1 + T_2, T_3) = 2.129167$. Recall Section 3, when $R_0^* > 1$, the endemic equilibrium $E_1 = (S_1^*, E_1^*, I_1^*, R_1^*)$ satisfies the following system:

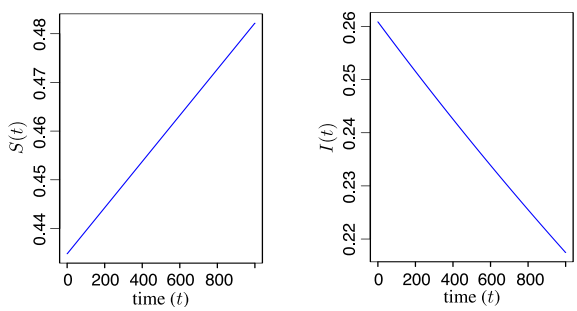
$$\begin{aligned}
&B - \beta S \mathrm{e}^{-\mu_v T_1} G(I) - \mu S + \alpha I \mathrm{e}^{-\mu T_3} = 0,\\
&\beta S \mathrm{e}^{-\mu_v T_1} G(I) - \mu E - \beta S \mathrm{e}^{-(\mu_v T_1 + \mu T_2)} G(I) = 0,\\
&\beta S \mathrm{e}^{-(\mu_v T_1 + \mu T_2)} G(I) - (\mu + d + \alpha) I = 0,\\
&\alpha I - \mu R - \alpha I \mathrm{e}^{-\mu T_3} = 0.
\end{aligned}$$

For the dimensionless parameter estimates in Table 1, the DFE is $E_0 = (S_0^*, 0, 0) = (1, 0, 0)$ and $E_1 = (S_1^*, E_1^*, I_1^*) = (0.002323845, 0.00068247, 0.04540019)$.

Example 1 [*Example for extinction of disease*]. For parameter estimates in Table 1, where $\beta = 0.02146383$, from (28) the BRN $\hat{R}_0^* = 0.2498732 < 1$. Therefore, E_0 is stable, and $E_1 = (S_1^*, E_1^*, I_1^*)$ fails to exist.

Table 1. A list of dimensionless values for the system parameters for Example 1.

Disease transmission rate	β	0.02146383
Constant Birth rate	B	$8.476678e{-}06$
Recovery rate	α	0.08571429
Disease death rate	d	0.0001761252
Natural death rate	μ, μ_v	$8.476678e{-}06$, 42.85714
Incubation delay in vector	T_1	0.105
Incubation delay in host	T_2	0.175
Immunity delay time	T_3	2.129167



(a) (b)

Figure 1. (a) and (b) show the trajectories of S and I, respectively, over time $t \in [0, 1000]$. The BRN in (28) is $R_0^* = 0.2498732 < 1$, and extinction rate of the disease in (41) is $\lambda = 0.06443506 > 0$.

Figure 1 verifies the results of Theorems 4 and 5. Indeed, since $R_0^* = 0.2498732 < 1$, Theorems 4(i) and 5 are satisfied, and from (41) the extinction rate $\lambda = 0.06443506 > 0$. That is, $\limsup_{t\to\infty} \log(I(t))/t \leqslant -\lambda = -0.06443506$. Figure 1(b) confirms that over time, since $\lambda = 0.06443506 > 0$, then $\lim_{t\to\infty} I(t) = 0$. Furthermore, the BRN $R_0^* = 0.2498732 < 1$, and Fig. 1(a) shows that $\lim_{t\to\infty} S(t) = 1$.

8 Conclusion

The vector-human population dynamic models are derived. The models have a general nonlinear incidence rate. The extinction and persistence of the vector-borne disease in the SEIRS epidemic models are studied. Numerical simulation results are given to confirm the results.

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