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Lagrange problem for fractional ordinary elliptic system via Dubovitskii–Milyutin method

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Abstract. In the paper, we investigate a weak maximum principle for Lagrange problem described by a fractional ordinary elliptic system with Dirichlet boundary conditions. The Dubovitskii–Milyutin approach is used to find the necessary conditions. The fractional Laplacian is understood in the sense of Stone–von Neumann operator calculus.

Keywords: fractional Laplacian, Dirichlet boundary conditions, optimal control, maximum principle, Dubovitskii–Milyutin theorem.

1 Introduction

In the paper, we consider ordinary elliptic nonlinear control system of fractional order

$$(-\Delta)^{\beta} x(t) = f\bigl(t, x(t), u(t)\bigr), \quad t \in (0, \pi) \text{ a.e.}, \tag{1}$$

with a cost functional

$$F_0(x, u) = \int_0^{\pi} f_0\bigl(t, x(t), u(t)\bigr)\,\mathrm{d}t \tag{2}$$

where $\beta > 1/4$, $f_0 : (0,\pi) \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}$, $f : (0,\pi) \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m$, $(-\Delta)^{\beta}$ is the β-power of the Dirichlet–Laplace operator $-\Delta : H_0^1 \cap H^2 \to L^2$ (cf. Section 2.3) and $H_0^1 = H_0^1((0,\pi), \mathbb{R}^m)$, $H^2 = H^2((0,\pi), \mathbb{R}^m)$ are Sobolev spaces. $L^2 = L^2((0,\pi), \mathbb{R}^m)$ is the space of square integrable functions. On the controls u, we assume that they belong to the space $L^{\infty} = L^{\infty}((0,\pi), \mathbb{R}^r)$ of essentially bounded functions and satisfy a constraint $u(t) \in M \subset \mathbb{R}^r$.

In the last years, one can observe the growing interest in the subject of fractional Laplacians. It follows from the numerous applications of them, for example, in mathematical finance (infinitesimal generators of Lévy processes), elastostatics (Signorini obstacle

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problem), hydrodynamics (fractional Navier–Stokes equation and model of the flow in porous media); see [2–4] and references therein. System (1) can be viewed as a generalization of the Poisson equation. To our best knowledge, it was investigated for the first time in paper [2], where optimal control for problem (1)–(2) was examined in the context of the existence of solutions. More precisely, in the case of $\beta \in (0,1)$, $m = 1$, with the interval $(0,\pi)$ replaced by a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ and with exterior homogenous Dirichlet boundary conditions, the results concerning the continuous dependence of the sets of solutions on controls (with respect to the strong and weak convergences of controls) as well as theorem on the existence of a solution to problem (1)-(2) have been obtained. The variational "$\min$" structure of (2) is used in [2]. Similar results are presented in papers [4] and [3] for the control systems possessing the "minimax" and "mountain pass" structures, respectively. For some results concerning the existence, uniqueness, stability and sensitivity of a solution to problem (1), we refer to [2–4, 14].

Our aim is to derive the necessary conditions for problem (1)–(2). To the best of our knowledge, this issue was not investigated up to now. In our paper, we derive such conditions using the Dubovitskii–Milyutin approach. Specially, we are interested in fractional case, but we do not exclude from our study the classical case of $\beta = 1$. In optimal control theory, three main approaches for necessary conditions are known: two general (abstract) approaches and one of a direct character. The first general approach is based on the smooth-convex extremum principle by Ioffe and Tikchomirov (cf. [16]), and the second one – on the Dubovitskii–Milyutin theorem (cf. [11]). The third approach is based on the method of variation of controls. Such a "variational" approach in the fractional case seems an open problem. Its difficulty lies in global character of $(-\Delta)^{\beta}x$ – in the case of the systems containing the classical derivative, the local (pointwise) character of this derivative is essentially used in such an approach. We use Dubovitskii–Milyutin method because it allows us to avoid convexity-type assumption on f and f_0 (an assumption of such a type is necessary in the case of the smooth-convex extremum principle). The only assumptions on f, f_0 and their gradients (excluding assumptions of Proposition 7 on the surjectivity of differential $F'(x_*, u_*)$) ensure differentiability of the mappings used in Dubovitskii–Milyutin approach and are quite standard from the point of view of differential calculus in function spaces.

Our result is analogous to the corresponding Pontryagin maximum principle for classical ordinary systems – necessary conditions take the form of an adjoint system and a minimum condition. Since in this condition the gradients $(f)_u$ and $(f_0)_u$ appear instead of f and f_0, therefore we name our principle the weak maximum principle. It is worth of point out that if f, f_0 are smooth and convex in u, then minimum condition with $(f)_u$, $(f_0)_u$ and minumum condition with f, f_0 are equivalent.

The idea of Dubovitskii–Milyutin method has been presented in papers [9] and [10] but without proofs of the main results. A systematic exposition of this approach and proofs are contained in the book [11]. Some generalizations of the method have been derived by the second author in [26] and by Ledzewicz in [19–22]. In monograph [18] of 2015, applications of the Dubovitskii–Milyutin method and of its generalization in set-valued optimization are presented (cf. also [17]).

Existence of optimal solutions and stability results concerning the case of $\beta = 1$ can be found in [5, 6, 23]. Necessary first-order optimality conditions for $\beta = 1$ can be deduced in some cases from the results obtained in [12] and [13]. More precisely, in [12], the *scalar* control system

$$\frac{\mathrm{d}}{\mathrm{d}t} a\big(t, x(t), x'(t)\big) = b\big(t, x(t), x'(t), u(t)\big), \quad t \in (0, \pi) \text{ a.e.},$$
$$x(0) = x(\pi) = 0$$

with the cost functional

$$J(x, u) = \int_0^\pi f_0\big(t, x(t), x'(t), u(t)\big)\,\mathrm{d}t \tag{3}$$

is investigated under additional equality and inequality constraints. Using the direct method (McShane variations of controls), the authors derived maximum principle of Pontryagin type (it is important that a does not depend on u). In [13], the control system (not necessary scalar)

$$\frac{\mathrm{d}}{\mathrm{d}t} D_{x'} F\big(t, x(t), x'(t), u(t)\big) = D_x F\big(t, x(t), x'(t), u(t)\big), \quad t \in (0, \pi) \text{ a.e.},$$
$$x(0) = x(\pi) = 0$$

with cost functional (3) is studied via Ioffe–Tikchomirov approach. The Pontryagin-type maximum principle has been obtained under a convexity assumption on $(D_x F, D_{x'} F, f_0)$.

Paper consists of two main parts. In the first part, we give some basics from the area of Dubovitskii–Milyutin method and Stone–von Neumann operator calculus with an application to one-dimensional Dirichlet–Laplace operator. In the second part, we derive necessary optimality conditions in form of a minimum condition and an adjoint system.

2 Preliminaries

In this section, we give some basics concerning Dubovitskii–Milyutin method, Stone–von Neumann operator calculus and one-dimensional Dirichlet–Laplace operator of fractional order.

2.1 Dubovitskii–Milyutin method

For the results of this section, we refer to [11]. Let X be a linear topological space with the dual space (the space of linear continuous functionals on X) denoted by X'. If K is a cone in X with vertex at the point 0 (by a cone we mean a set K such that $tK = K$ for any $t > 0$), then the conjugate cone K^* we define by

$$K^* = \{f \in X';\ f(x) \geqslant 0 \text{ for any } x \in K\}.$$

Of course, the conjugate cone is the convex cone with vertex at 0.

Let $F : X \to \mathbb{R}$ be a functional. We say that a vector $h \in X$ is a direction of decrease of functional F at a point x_0 if there exist a neighborhood U of h, $\varepsilon_0 > 0$ and $\alpha < 0$ such that

$$F(x_0 + \varepsilon\overline{h}) \leqslant F(x_0) + \varepsilon\alpha$$

for $\varepsilon \in (0, \varepsilon_0)$, $\overline{h} \in U$.

We say that a vector $h \in X$ is a feasible direction for the set $Q \subset X$ at a point $x_0 \in \overline{Q}$ ($\overline{Q}$ denotes the closure of the set Q) if there exist a neighborhood U of h and $\varepsilon_0 > 0$ such that

$$x_0 + \varepsilon\overline{h} \in Q$$

for $\varepsilon \in (0, \varepsilon_0)$, $\overline{h} \in U$.

We say that a vector $h \in X$ is a tangent direction to a set Q at a point $x_0 \in Q$ if there exist $\varepsilon_0 > 0$ and mapping $r : (0, \varepsilon_0) \to X$ such that

$$x_0 + \varepsilon h + r(\varepsilon) \in Q$$

for $\varepsilon \in (0, \varepsilon_0)$ and $r(\varepsilon)/\varepsilon \to 0$, $\varepsilon \to 0^+$ (i.e., for any neighborhood U of 0, there exists $\varepsilon_1 > 0$ such that $r(\varepsilon)/\varepsilon \in U$ for $\varepsilon \in (0, \varepsilon_1)$; when X is a Banach space, we write in such a case $r(\varepsilon) = o(\varepsilon)$).

One proves that the set of directions of decrease of functional F at a point x_0 and the set of feasible directions for the set Q at a point x_0 are open cones; the set of tangent directions to a set Q at a point x_0 is a cone.

Now, assume X is locally convex and consider the problem

$$F(x) \to \min, \qquad x \in Q := \bigcap_{i=1}^{n+1} Q_i, \tag{4}$$

where $F : X \to \mathbb{R}$, $Q_i \subset X$, $i = 1, \dots, n+1$. We say that $x_0 \in Q$ is a local minimum point of F if there exists a neighborhood V of x_0 such that

$$F(x_0) = \min_{x \in Q \cap V} F(x).$$

The main role in rest part of the paper is played by the following theorem (cf. [11, Thm. 6.1 and Remark 3 following the theorem]).

Theorem 1. *Let x_0 be a local minimum point for problem* (4)*, and the cone K_0 of directions of decrease of functional F at x_0 is nonempty and convex, cones K_i, $i = 1, \dots, n$, of feasible directions for the sets Q_i at x_0 are nonempty and convex, and the cone K_{n+1} of tangent directions for the set Q_{n+1} at x_0 is nonempty and convex. Then there exist functionals $f_i \in K_i^*$, $i = 0, 1, \dots, n+1$, not all identically zero, such that*

$$f_0 + f_1 + \dots + f_n + f_{n+1} = 0,$$

and if $\bigcap_{i=1}^{n+1} K_i \neq \emptyset$, then $f_0 \neq 0$.

Now, we shall give characterizations of the defined cones in some typical situations. These results can be found in [11, Thms. 7.5, 8.2, 9.1, resp.].

Proposition 1. *If X is a Banach space, $F : X \to \mathbb{R}$ – a functional differentiable at x_0 in Frechet sense, then the cone K_d of directions of decrease of functional F at x_0 has the form*

$$K_d = \{h \in X;\ F'(x_0)h < 0\}.$$

Proposition 2. *If Q is a convex set in linear topological space E, then the cone K_f of feasible directions for the set Q at a point $x_0 \in \overline{Q}$ is convex and has the form*

$$K_f = \{\rho(\operatorname{Int} Q - x_0);\ \rho > 0\}.$$

Theorem 2. *If X, Y are Banach spaces, $P : X \to Y$ – operator of class C^1, $P(x_0) = 0$ and $\Im P'(x_0) = Y$, then the cone K_t of tangent directions for the set $\{x \in X;\ P(x) = 0\}$ at x_0 is a subspace of the form*

$$K_t = \{h \in X;\ P'(x_0)h = 0\}.$$

We also have (cf. [11, Thms. 10.2, 10.5, resp.])

Proposition 3. *Let E be a linear topological space, $f \in E'$ and*

$$K = \{x \in E;\ f(x) > 0\}.$$

Then

$$K^* = \begin{cases} E' & \text{if } f = 0, \\ \{\lambda f;\ \lambda \geqslant 0\} & \text{if } f \neq 0. \end{cases}$$

Proposition 4. *Let Q be a nonempty convex closed set in linear topological space E and $x_0 \in Q$. If $\operatorname{Int} Q \neq \emptyset$, then*

$$K_f^* = \{g \in E';\ g(x) \geqslant g(x_0) \text{ for } x \in Q\}$$

where K_f is the cone of feasible directions for Q at x_0.

2.2 Basics of Stone–von Neumann operator calculus

Basics of Stone–von Neumann operator calculus presented in this section comes from [1, 24, 25][1] (cf. also [15] for a more comprehensive coverage of the topic). We give them here for the convenience of the reader.

Let H be a real Hilbert space with a scalar product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{R}$, and $\varPi(H)$ – the set of all projections of H on closed linear subspaces of H. By the spectral measure in $\mathbb{R}$ we mean a set function $E : \mathcal{B} \to \varPi(H)$, where $\mathcal{B}$ is the σ-algebra of Borel subsets

[1] In [1, 24, 25] the results are presented in the case of complex Hilbert space but their proofs can be moved to the case of real Hilbert space.

of $\mathbb{R}$, that satisfies the following conditions:

- for any $x \in H$, the function
$$\mathcal{B} \ni P \longmapsto E(P)x \in H \tag{5}$$
is a vector measure;
- $E(\mathbb{R}) = I$;
- $E(P \cap Q) = E(P) \circ E(Q)$ for $P, Q \in \mathcal{B}$.

By a support of a spectral measure E we mean the complement of the sum of all open subsets of $\mathbb{R}$ with zero spectral measure.

If $b : \mathbb{R} \to \mathbb{R}$ is a bounded Borel measurable function, defined E-a.e., then the integral $\int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda)$ is defined by

$$\left(\int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda) \right) x = \int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda) x$$

for any $x \in H$, where the integral $\int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda)x$ (with respect to the vector measure (5)) is defined in a standard way, with the aid of the sequence of simple functions converging $E(\mathrm{d}\lambda)x$ – a.e. to b (see [1]).

If $b : \mathbb{R} \to \mathbb{R}$ is an unbounded Borel measurable function, defined E-a.e., then, for any $x \in H$ such that

$$\int_{-\infty}^{\infty} \left|b(\lambda)\right|^2 \left\|E(\mathrm{d}\lambda)x\right\|^2 < \infty \tag{6}$$

(the above integral is taken with respect to the nonnegative measure $\mathcal{B} \ni P \mapsto \|E(P)x\|^2 \in \mathbb{R}_0^+$), there exists the limit

$$\lim \int_{-\infty}^{\infty} b_n(\lambda)\, E(\mathrm{d}\lambda)x$$

of integrals, where

$$b_n : \mathbb{R} \ni \lambda \longmapsto \begin{cases} b(\lambda) & \text{as } |b(\lambda)| \leqslant n, \\ 0 & \text{as } |b(\lambda)| > n \end{cases}$$

for $n \in \mathbb{N}$. Let us denote the set of all points x with property (6) by D. One proves that D is a dense linear subspace of H, and by $\int_{-\infty}^{\infty} b(\lambda) E(\mathrm{d}\lambda)$ one denotes the operator

$$\int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda) : D \subset H \to H$$

given by

$$\left(\int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda) \right) x = \lim \int_{-\infty}^{\infty} b_n(\lambda)\, E(\mathrm{d}\lambda)x.$$

Of course, $D = H$ and

$$\lim \int_{-\infty}^{\infty} b_n(\lambda)\, E(\mathrm{d}\lambda)x = \int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda)x$$

when $b : \mathbb{R} \to \mathbb{R}$ is a bounded Borel measurable function, defined E-a.e. One can prove that

$$\left\| \int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda)x \right\|^2 = \int_{-\infty}^{\infty} \left|b(\lambda)\right|^2 \left\|E(\mathrm{d}\lambda)x\right\|^2$$

for $x \in D$. The main property of the integral is its self-adjointness, i.e.,

$$\left(\int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda) \right)^* = \int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda).$$

If $\sigma \in \mathcal{B}$, then by the integral

$$\int_{\sigma} b(\lambda) E(\mathrm{d}\lambda)$$

we mean the integral

$$\int_{-\infty}^{\infty} \chi_\sigma(\lambda) b(\lambda)\, E(\mathrm{d}\lambda),$$

where χ_σ is the characteristic function of the set σ.

Next theorem plays the fundamental role in the spectral theory of self-adjoint operators (below, $\sigma(A)$ is the spectrum of an operator A).

Theorem 3 [Integral representation of self-adjoint operator]. *If $A : D(A) \subset H \to H$ is self-adjoint and the resolvent set $\rho(A)$ is nonempty, then there exists a unique spectral measure E with the closed support $\Lambda = \sigma(A)$ such that*

$$A = \int_{-\infty}^{\infty} \lambda\, E(\mathrm{d}\lambda) = \int_{\sigma(A)} \lambda\, E(\mathrm{d}\lambda).$$

The basic notion in the Stone–von Neumann operator calculus is a function of a self-adjoint operator. Namely, if $A : D(A) \subset H \to H$ is self-adjoint and E is the spectral measure determined according to the above theorem, then, for any Borel measurable function $b : \mathbb{R} \to \mathbb{R}$, one defines the operator

$$b(A) = \int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda) = \int_{\sigma(A)} b(\lambda)\, E(\mathrm{d}\lambda).$$

It is known that the spectrum $\sigma(b(A))$ of $b(A)$ is given by

$$\sigma\bigl(b(A)\bigr) = \overline{b\bigl(\sigma(A)\bigr)},$$

provided that b is continuous (it is sufficient to assume continuity of b on $\sigma(A)$).

Remark 1. To make sense to the integral $\int_{-\infty}^{\infty} b(\lambda)\, E(\mathrm{d}\lambda)$ in the case of Borel measurable function $b : B \to \mathbb{R}$, where B is a Borel set containing the support of the measure E, it is sufficient to extend b on $\mathbb{R}$ to a whichever Borel measurable function (putting, for example, $b(\lambda) = 0$ for $\lambda \notin B$).

2.3 Fractional one-dimensional Dirichlet–Laplace operator

For the results of this section, we refer to [14]. By the one-dimensional Dirichlet–Laplace operator on the interval $(0, \pi)$ we mean the operator

$$-\Delta : H_0^1 \cap H^2 \subset L^2 \to L^2$$

given by

$$-\Delta x(t) = -x''(t).$$

It is known that $-\Delta$ is self-adjoint,

$$\sigma(-\Delta) = \sigma_p(-\Delta) = \bigl\{j^2;\ j \in \mathbb{N}\bigr\}$$

($\sigma_p(-\Delta)$ is the pointwise spectrum of $(-\Delta)$) and the eigenspace $N(j^2)$ corresponding to the eigenvalue $\lambda_j = j^2$ is the set $\{c \sin jt;\ c \in \mathbb{R}^m\}$. The system of functions

$$c_{j,i} = \Bigl(0, \dots, 0, \underbrace{\sqrt{\frac{2}{\pi}} \sin jt}_{i\text{th}}, 0, \dots, 0\Bigr), \quad j = 1, 2, \dots,\ i = 1, \dots, m,$$

is the Hilbertian basis (complete orthonormal system) in L^2.

By fractional Dirichlet–Laplace operator of order $\beta > 0$ we mean the operator

$$(-\Delta)^\beta : D\bigl((-\Delta)^\beta\bigr) \subset L^2 \to L^2$$

where

$$D\bigl((-\Delta)^\beta\bigr) = \Biggl\{x(t) \in L^2;\ \int\limits_{\sigma(-\Delta)} \bigl|\lambda^\beta\bigr|^2 t \bigl\|\, E(\mathrm{d}\lambda) x\bigr\|^2 = \sum_{j=1}^{\infty} \bigl(\bigl(j^2\bigr)^\beta\bigr)^2 \bigl|a_j\bigr|^2 < \infty,$$
$$\text{where } x(t) = \Biggl(\ \int\limits_{\sigma(-\Delta)} 1\, E(\mathrm{d}\lambda) x\Biggr)(t) = \sum_{j=1}^{\infty} a_j \sqrt{\frac{2}{\pi}} \sin jt\Biggr\}.$$

Here E is the spectral measure given by Theorem 3 for the operator $(-\Delta)$, $a_j\sqrt{2/\pi}\sin jt$ is the projection of x on the m-dimensional eigenspace $N(j^2)$ of the operator $(-\Delta)$, and

$$(-\Delta)^\beta x(t) = \biggl(\int\limits_{\sigma(-\Delta)} \lambda^\beta\, E(\mathrm{d}\lambda)x\biggr)(t) = \sum_{j=1}^{\infty} \bigl(j^2\bigr)^\beta a_j\sqrt{\frac{2}{\pi}}\sin jt$$

for $x(t) = \sum_{j=1}^{\infty} a_j\sqrt{2/\pi}\sin jt \in D((-\Delta)^\beta)$ (from the Carleson theorem it follows that the equality $x(t) = \sum_{j=1}^{\infty} a_j\sqrt{2/\pi}\sin jt$ meant in L^2 implies the pointwise one a.e. on $(0,\pi)$ (cf. [7, Thm. 5.17])).

One can show that

$$\sigma\bigl((-\Delta)^\beta\bigr) = \sigma_p\bigl((-\Delta)^\beta\bigr) = \bigl\{\bigl(j^2\bigr)^\beta;\ j\in\mathbb{N}\bigr\},$$

and the corresponding eigenspaces for $(-\Delta)$ and $(-\Delta)^\beta$ are the same.

The function $(-\Delta)^\beta x$ will be called the *Dirichlet–Laplacian of order β of x*. We have the following lemmas.

Lemma 1. *$D((-\Delta)^\beta)$ is complete with the scalar products*

$$\langle x,y\rangle_\beta = \langle x,y\rangle_{L^2} + \bigl\langle(-\Delta)^\beta x, (-\Delta)^\beta y\bigr\rangle_{L^2}$$

and

$$\langle x,y\rangle_{\sim\beta} = \bigl\langle(-\Delta)^\beta x, (-\Delta)^\beta y\bigr\rangle_{L^2},$$

and norm generated by these products are equivalent.

Lemma 2. *If $\beta > 1/4$, then*

$$D\bigl((-\Delta)^\beta\bigr) \subset L^\infty,$$

and this embedding is continuous, more precisely,

$$\|x\|_{L^\infty} \leqslant \sqrt{\frac{2}{\pi}\zeta(4\beta)}\|x\|_{\sim\beta} \tag{7}$$

for $x \in D((-\Delta)^\beta)$, where $\zeta(4\beta)$ is the value of the Riemann zeta function $\zeta(\gamma) = \sum_{j=1}^{\infty}(1/j^\gamma)$ at $\gamma = 4\beta$.

3 Maximum principle

Let us consider the following problem:

$$F_0(x,u) \to \min, \quad F(x,u) = 0, \quad u\in U, \tag{8}$$

where

$$F_0 : D\bigl((-\Delta)^\beta\bigr)\times L^\infty \ni (x,u) \longmapsto \int\limits_0^\pi f_0\bigl(t,x(t),u(t)\bigr)\,\mathrm{d}t \in \mathbb{R},$$

$$F : D\bigl((-\Delta)^\beta\bigr)\times L^\infty \ni (x,u) \longmapsto (-\Delta)^\beta x(t) - f\bigl(t,x(t),u(t)\bigr) \in L^2$$

with $\beta > 1/4$, f_0, $f : (0,\pi)\times\mathbb{R}^m\times\mathbb{R}^r \to \mathbb{R}^m$, $U = L^\infty((0,\pi),M)$, where $M\subset\mathbb{R}^r$ is closed convex set such that $\operatorname{Int} M \neq \emptyset$.

3.1 Assumptions

In paper [14], the following proposition has been derived.

Proposition 5. *If $\beta > 1/4$, function f is measurable in $t \in (0,\pi)$, continuously differentiable in $(x,u) \in \mathbb{R}^m \times \mathbb{R}^r$ and*

$$\bigl|f(t,x,u)\bigr|,\ \bigl|f_x(t,x,u)\bigr|,\ \bigl|f_u(t,x,u)\bigr| \leqslant a(t)\gamma\bigl(|x|\bigr) + b(t)\delta\bigl(|u|\bigr) \tag{9}$$

for $(t,x,u) \in (0,\pi) \times \mathbb{R}^m \times \mathbb{R}^r$, where $a, b \in L^2$ and $\gamma, \delta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous functions, then F is of class C^1, and the differential $F'(x,u) : D((-\Delta)^\beta) \times L^\infty \to L^2$ of F at (x,u) is given by

$$F'(x,u)(h,v) = (-\Delta)^\beta h(t) - f_x\bigl(t,x(t),u(t)\bigr)h(t) - f_u\bigl(t,x(t),u(t)\bigr)v(t)$$

for $(h,v) \in D((-\Delta)^\beta) \times L^\infty$.

In a similar way, we obtain a differentiability property of F_0.

Proposition 6. *If $\beta > 1/4$, function f_0 is measurable in $t \in (0,\pi)$, continuously differentiable in $(x,u) \in \mathbb{R}^m \times \mathbb{R}^r$ and*

$$\bigl|f_0(t,x,u)\bigr|,\ \bigl|(f_0)_x(t,x,u)\bigr|,\ \bigl|(f_0)_u(t,x,u)\bigr| \leqslant d(t)c\bigl(|x|,|u|\bigr) \tag{10}$$

for $(t,x,u) \in (0,\pi) \times \mathbb{R}^m \times \mathbb{R}^r$, where $d \in L^1 = L^1((0,\pi),\mathbb{R}_0^+)$, $c : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous function, then F_0 is differentiable on $D((-\Delta)^\beta) \times L^\infty$, and

$$F_0'(x,u)(h,v) = \int_0^\pi \bigl((f_0)_x\bigl(t,x(t),u(t)\bigr)h(t) + (f_0)_u\bigl(t,x(t),u(t)\bigr)v(t)\bigr)\,\mathrm{d}t$$

for $(h,v) \in D((-\Delta)^\beta) \times L^\infty$.

Proof. Indeed, it is sufficient to observe that the mapping

$$\begin{aligned}&(F_0)'\bigl(x(\cdot),u(\cdot)\bigr) : D((-\Delta)^\beta) \times L^\infty \ni (h,v)\\ &\qquad \longmapsto \int_0^\pi (f_0)_x\bigl(t,x(t),u(t)\bigr)h(t) + (f_0)_u\bigl(t,x(t),u(t)\bigr)v(t)\,\mathrm{d}t \in \mathbb{R}\end{aligned}$$

is the Gâteaux differential of F_0 at any fixed point $(x(\cdot),u(\cdot)) \in D((-\Delta)^\beta) \times L^\infty$ and the mapping

$$D\bigl((-\Delta)^\beta\bigr) \times L^\infty \ni \bigl(x(\cdot),u(\cdot)\bigr) \longmapsto (F_0)'\bigl(x(\cdot),u(\cdot)\bigr) \in \mathcal{L}\bigl(D\bigl((-\Delta)^\beta\bigr) \times L^\infty, \mathbb{R}\bigr)$$

is continuous. Linearity of $(F_0)'(x(\cdot),u(\cdot))$ is obvious. Its continuity follows from the estimation (cf. (7))

$$\bigl|h(t)\bigr| \leqslant \sqrt{\frac{2}{\pi}\zeta(4\beta)}\|h\|_{\sim\beta}, \quad t \in (0,\pi) \text{ a.e.},$$

for $h(\cdot) \in D((-\Delta)^\beta)$. So, to prove that $(F_0)'(x(\cdot), u(\cdot))$ is the Gâteaux differential of F_0, it is sufficient to check that $(F_0)'(x(\cdot), u(\cdot))(h(\cdot), v(\cdot))$ is the derivative at the point $\lambda = 0$ of the real function of one (real) variable

$$\psi : (-1, 1) \ni \lambda \longmapsto F_0\bigl(x(\cdot) + \lambda h(\cdot), u(\cdot) + \lambda v(\cdot)\bigr) = \int_0^\pi g(\lambda, t)\, \mathrm{d}t \in \mathbb{R},$$

where

$$g : (-1, 1) \times [0, 1] \ni (\lambda, t) \longmapsto f_0\bigl(t, x(t) + \lambda h(t),\, u(t) + \lambda v(\cdot)\bigr) \in \mathbb{R}.$$

In turn, the differentiability of ψ at $\lambda = 0$ means the differentiability of an integral with respect to the parameter and follows from the classical theorem on such a differentiability. Now, let $(x_n, u_n) \to (x_0, u_n)$ as $n \to \infty$ in $D((-\Delta)^\beta) \times L^\infty$. Then

$$\begin{aligned}
&\bigl|\bigl((F_0)'(x_n,\, u_n) - (F_0)'(x_0, u_0)\bigr)(h, v)\bigr| \\
&\quad\leqslant \int_0^\pi \bigl|(f_0)_x\bigl(t, x_n(t),\, u_n(t)\bigr) - (f_0)_x\bigl(t, x_0(t), u_0(t)\bigr)\bigr|\bigl|h(t)\bigr|\, \mathrm{d}t \\
&\qquad + \int_0^\pi \bigl|(f_0)_u\bigl(t, x_n(t), u_n(t)\bigr) - (f_0)_u\bigl(t, x_0(t), u_0(t)\bigr)\bigr|\bigl|v(t)\bigr|\, \mathrm{d}t \\
&\quad\leqslant \sqrt{\frac{2}{\pi}\zeta(4\beta)} \int_0^\pi \bigl|(f_0)_x\bigl(t, x_n(t),\, u(t)\bigr) - (f_0)_x\bigl(t, x_0(t), u_0(t)\bigr)\bigr|\, \mathrm{d}t \|h\|_{\sim\beta} \\
&\qquad + \int_0^\pi \bigl|(f_0)_u\bigl(t, x_n(t),\, u_n(t)\bigr) - (f_0)_u\bigl(t, x_0(t), u_0(t)\bigr)\bigr|\, \mathrm{d}t \|v\|_{L^\infty}
\end{aligned}$$

for $(h, v) \in D((-\Delta)^\beta) \times L^\infty$. Thus,

$$\begin{aligned}
&\bigl\|(F_0)'(x_n, u_n) - (F_0)'(x_0, u_0)\bigr\|_{\mathcal{L}(AC \times L^\infty, \mathbb{R})} \\
&\quad\leqslant \sqrt{\frac{2}{\pi}\zeta(4\beta)} \int_0^\pi \bigl|(f_0)_x\bigl(t, x_n(t),\, u_n(t)\bigr) - (f_0)_x\bigl(t, x_0(t), u_0(t)\bigr)\bigr|\, \mathrm{d}t \\
&\qquad + \int_0^\pi \bigl|(f_0)_u(t, x_n(t),\, u_n(t)\bigr) - (f_0)_u\bigl(t, x_0(t), u_0(t)\bigr)\bigr|\, \mathrm{d}t.
\end{aligned}$$

From the fact that the convergence in $D((-\Delta)^\beta)$ implies the almost uniform one and from the growth conditions (10) it follows that right-hand side of the above inequality tends to 0 as $n \to \infty$. □

We have the following proposition.

Proposition 7. *Let the assumptions of Proposition 5 be satisfied, and a pair $(x,u) \in D((-\Delta)^\beta) \times L^\infty$ be fixed. If one of the following conditions is satisfied*

(a) $\beta > 1/2$ *and* $\|\Lambda\|_{L^1_{m\times m}} < \pi/(2\zeta(2\beta))$;
(b) $\beta > 1/2$ *and* $\Lambda(t) \leqslant 0$, *i.e., matrix* $\Lambda(t)$ *is nonpositive, for* $t \in (0,\pi)$ *a.e.*;
(c) $\beta > 1/4$ *and* $\Lambda \in L^\infty_{m\times m}$ *and* $\|\Lambda\|_\infty < 1$,

where $\Lambda(t) := f_x(t,x(t),u(t))$, $L^\infty_{m\times m} = L^\infty((0,\pi),\mathbb{R}^{m\times m})$[2], *then the differential* $F'(x,u) : D((-\Delta)^\beta) \times L^\infty \to L^2$ *is "onto".*

Proof. To show that, for any function $g \in L^2$, there exists a pair $(h,v) \in D((-\Delta)^\beta) \times L^\infty$ such that

$$(-\Delta)^\beta h(t) - f_x\big(t,x(t),u(t)\big)h(t) - f_u\big(t,x(t),u(t)\big)v(t) = g(t),$$

it is sufficient to consider the pairs $(h,0) \in D((-\Delta)^\beta) \times L^\infty$ and use Proposition 5.2 from [14] and the remark following that proposition. □

3.2 Conjugate cones

Let $(x_*,u_*) \in D((-\Delta)^\beta) \times L^\infty$ be fixed. From Propositions 1 and 6 it follows that the cone K_d of directions of decrease of functional F_0 at the point (x_*,u_*) has the form

$$K_d = \Bigg\{ (h,v) \in AC \times L^\infty; \; \int_0^\pi (f_0)_x\big(t,x_*(t),u_*(t)\big)h(t) + (f_0)_u\big(t,x_*(t),u_*(t)\big)v(t)\,\mathrm{d}t < 0 \Bigg\}.$$

This set is convex and nonempty, provided that $(F_0)'(x_*,u_*) \neq 0$ (vanishing of the differential is equivalent to the equality $K_d = \emptyset$). In such a case, in view of Proposition 3, the conjugate cone is the following:

$$K_d^* = \big\{ \lambda_0\big(-(F_0)'\big(x_*(\cdot),u_*(\cdot)\big)\big);\ \lambda_0 \geqslant 0 \big\}.$$

Using Theorem 2, we assert that if one of the conditions of Proposition 7 is fulfilled for $\Lambda(t) = f_x(t,x_*(t),u_*(t))$, then the cone K_t of tangent directions for the set

$$\big\{ (x,u) \in D\big((-\Delta)^\beta\big) \times L^\infty;\ F(x,u) = (-\Delta)^\beta x(t) - f\big(t,x(t),u(t)\big) = 0 \big\}$$

at the point (x_*,u_*) has the form

$$K_t = \ker F'(x_*,u_*).$$

[2] The norm in the space $L^\infty_{m\times m}$ of $\mathbb{R}^{m\times m}$-valued functions is defined in the same way as in the space L^∞ of $\mathbb{R}^m$-valued functions.

It clear that $K_t \neq \emptyset$. Of course, K_t is a subspace (consequently, convex cone). So,

$$K_t^* = \big(\ker F'(x_*, u_*)\big)^\perp,$$

where $(\ker F'(x_*, u_*))^\perp$ is the set of linear continuous functionals on $D((-\Delta)^\beta) \times L^\infty$ vanishing on the subspace $\ker F'(x_*, u_*)$; the set $(\ker F'(x_*, u_*))^\perp$ is called the anulator of $\ker F'(x_*, u_*)$. From the lemma on the anulator[3] it follows that

$$\big(\ker F'(x_*, u_*)\big)^\perp = \Im\big(\big(F'(x_*, u_*)\big)^*\big).$$

Definition of the adjoint operator $(F'(x_*, u_*))^*$ to the operator $F'(x_*, u_*) : D((-\Delta)^\beta) \times L^\infty \to L^2$ gives

$$\begin{aligned} K_t^* &= \Im\big(\big(F'(x_*, u_*)\big)^*\big) \\ &= \Bigg\{ g \in \big(D((-\Delta)^\beta) \times L^\infty\big)'; \text{ there exists } \lambda \in L^2 \text{ such that} \\ &\qquad g(h, v) = \int_0^\pi \lambda(t)\big((-\Delta)^\beta h(t) - f_x\big(t, x(t), u(t)\big)h(t) - f_u\big(t, x(t), u(t)\big)v(t)\big)\,\mathrm{d}t \\ &\qquad \text{for any } (h, v) \in D\big((-\Delta)^\beta\big) \times L^\infty \Bigg\}. \end{aligned}$$

Finally, writing the constraint $u \in U$ in the form

$$(x, u) \in Q := \big\{(x, u) \in D\big((-\Delta)^\beta\big) \times L^\infty;\ u \in U\big\} = D\big((-\Delta)^\beta\big) \times U$$

and using the fact that $\operatorname{Int} Q \neq \emptyset$ (because $\operatorname{Int} U \neq \emptyset$), we assert (cf. Proposition 2) that the cone K_f of feasible directions for the set Q at the point $(x_*, u_*) \in Q$ is the following:

$$\begin{aligned} K_f &= \big\{\rho\big(\operatorname{Int}\big(D\big((-\Delta)^\beta\big) \times U\big) - (x_*, u_*)\big);\ \rho > 0\big\} \\ &= \big\{\rho\big(\big(D\big((-\Delta)^\beta\big) \times \operatorname{Int} U\big) - (x_*, u_*)\big);\ \rho > 0\big\}. \end{aligned}$$

Consequently, it is nonempty and convex. From Proposition 4 (closedness of M implies closedness of Q) it follows that conjugate cone K_f^* has the form

$$\begin{aligned} K_f^* = \big\{& g_1 = (\mu, \xi) \in \big(D\big((-\Delta)^\beta\big) \times L^\infty\big)'; \\ & \mu(x) + \xi(u) = g_1(x, u) \geqslant g_1(x_*, u_*) = \mu(x_*) + \xi(u_*) \text{ for } (x, u) \in Q\big\} \\ = \big\{& (0, \xi) \in \big(D\big((-\Delta)^\beta\big) \times L^\infty\big)';\ \xi(u) \geqslant \xi(u_*) \text{ for } u \in U\big\}. \end{aligned}$$

[3] If X, Y are Banach spaces and $\Lambda : X \to Y$ is linear bounded operator such that $\Im\Lambda = Y$, then $(\ker \Lambda)^\perp = \Im\Lambda^*$, where $\Lambda^* : Y' \to X'$ is adjoint operator to Λ (see [16]).

3.3 Thesis

So, if $(x_*, u_*) \in D((-\Delta)^\beta) \times L^\infty$ is a local minimum point, then there exist functionals $g_0 \in K_d^*$, $g_1 \in K_f^*$, $g_2 \in K_t^*$, not all identically zero, such that

$$g_0 + g_1 + g_2 = 0.$$

In other words, there exist $\lambda_0 \leqslant 0$, $\xi \in (L^\infty)'$, $\lambda \in L^2$, not all zero, such that

$$\begin{aligned}
&\lambda_0 \int_0^\pi (f_0)_x\big(t, x_*(t), u_*(t)\big)h(t) + (f_0)_u\big(t, x_*(t), u_*(t)\big)v(t)\,\mathrm{d}t + \xi(v) \\
&\quad + \int_0^\pi \lambda(t)\big((-\Delta)^\beta h(t) - f_x\big(t, x_*(t), u_*(t)\big)h(t) \\
&\qquad - f_u\big(t, x_*(t), u_*(t)\big)v(t)\big)\,\mathrm{d}t = 0
\end{aligned} \tag{11}$$

for $(h, v) \in D((-\Delta)^\beta) \times L^\infty$ and

$$\xi(u) \geqslant \xi(u_*) \tag{12}$$

for $u \in U$.

Taking in (11) points $(h, v) = (h, 0) \in D((-\Delta)^\beta) \times L^\infty$, we obtain

$$\begin{aligned}
&\lambda_0 \int_0^\pi (f_0)_x\big(t, x_*(t), u_*(t)\big)h(t) \\
&\quad + \int_0^\pi \lambda(t)\big((-\Delta)^\beta h(t) - f_x(t, x_*(t), u_*(t))h(t)\big)\,\mathrm{d}t = 0
\end{aligned} \tag{13}$$

for $h \in D((-\Delta)^\beta)$. On the other hand, taking in (11) points $(h, v) = (0, v) \in D((-\Delta)^\beta) \times L^\infty$, we obtain

$$\begin{aligned}
&\lambda_0 \int_0^\pi (f_0)_u\big(t, x_*(t), u_*(t)\big)v(t)\,\mathrm{d}t \\
&\quad + \xi(v) - \int_0^\pi \lambda(t) f_u\big(t, x_*(t), u_*(t)\big)v(t)\,\mathrm{d}t = 0
\end{aligned} \tag{14}$$

for $v \in L^\infty$. Conditions (12) and (14) imply the following inequality for any $u \in U$:

$$\begin{aligned}
&-\lambda_0 \int_0^\pi (f_0)_u\big(t, x_*(t), u_*(t)\big)u(t)\,\mathrm{d}t + \int_0^\pi \lambda(t) f_u\big(t, x_*(t), u_*(t)\big)u(t)\,\mathrm{d}t \\
&\quad \geqslant -\lambda_0 \int_0^\pi (f_0)_u\big(t, x_*(t), u_*(t)\big)u_*(t)\,\mathrm{d}t + \int_0^\pi \lambda(t) f_u\big(t, x_*(t), u_*(t)\big)u_*(t)\,\mathrm{d}t
\end{aligned}$$

Let us recall the following lemma proved in [13]: if $M \subset \mathbb{R}^m$, $\varphi = \varphi(t,u) : [0,1] \times M \to \mathbb{R}$ is measurable in t, continuous in u and

$$-\infty < \int\limits_0^1 \varphi\bigl(t, \widehat{u}(t)\bigr)\,\mathrm{d}t \leqslant \int\limits_0^1 \varphi\bigl(t, u(t)\bigr)\,\mathrm{d}t < \infty$$

for a fixed function $\widehat{u}(\cdot) \in L^2([0,1], M)$ and all functions $u(\cdot) \in L^2([0,1], M)$, then

$$\varphi\bigl(t, \widehat{u}(t)\bigr) \leqslant \varphi(t,u)$$

for $t \in [0,1]$ a.e. and all $u \in M$. Proof of this lemma can be moved without changes to the case of $L^\infty([0,1], M)$.

So, from L^∞-version of the above lemma it follows that, for $t \in [0,1]$ a.e.,

$$\begin{aligned}&\bigl(-\lambda_0 (f_0)_u\bigl(t, x_*(t), u_*(t)\bigr) + \lambda(t) f_u\bigl(t, x_*(t), u_*(t)\bigr)\bigr)u \\ &\quad \geqslant \bigl(-\lambda_0 (f_0)_u\bigl(t, x_*(t), u_*(t)\bigr) + \lambda(t) f_u\bigl(t, x_*(t), u_*(t)\bigr)\bigr)u_*(t)\end{aligned}$$

for any $u \in M$.

Now, let us denote

$$V(t) = \lambda_0 (f_0)_x\bigl(t, x_*(t), u_*(t)\bigr) - \lambda(t) f_x\bigl(t, x_*(t), u_*(t)\bigr), \quad t \in [0,1] \text{ a.e.}$$

Equality (13) takes the form

$$\int\limits_0^\pi V(t)h(t) + \int\limits_0^\pi \lambda(t)(-\Delta)^\beta h(t)\,\mathrm{d}t = 0$$

for $h \in D((-\Delta)^\beta)$. Using the fact that $(-\Delta)^\beta$ is self-adjoint from the above condition, we obtain that $\lambda \in D((-\Delta)^\beta)$ and

$$(-\Delta)^\beta \lambda = \bigl[f_x\bigl(t, x_*(t), u_*(t)\bigr)\bigr]^\top \lambda(t) - \lambda_0 \bigl(f_0\bigr)_x\bigl(t, x_*(t), u_*(t)\bigr), \quad t \in [0,1] \text{ a.e.}$$

At the end, let us observe that one can assume that $\lambda_0 = -1$. Indeed, let us consider a control $u_m(\cdot) \equiv m$, where $m \in \operatorname{Int} M$, and let $h_{u_m - u_*} \in D((-\Delta)^\beta)$ be the solution of the system

$$(-\Delta)^\beta h(t) - f_x\bigl(t, x_*(t), u_*(t)\bigr)h(t) - f_u\bigl(t, x_*(t), u_*(t)\bigr)\bigl(u_m(t) - u_*(t)\bigr) = 0$$

(existence of such a solution follows from [14, Prop. 5.2]). Then $(x_* + h_{u_m - u_*}, u_m) \in D((-\Delta)^\beta) \times \operatorname{Int} U = \operatorname{Int}(D((-\Delta)^\beta) \times U)$, and, for $\rho = 1$, we have

$$\begin{aligned}K_f &\ni \rho\bigl((x_* + h_{u_m - u_*}, u_m) - (x_*, u_*)\bigr) \\ &= (x_* + h_{u_m - u_*} - x_*, u_m - u_*)(h_{u_m - u_*}, u_m - u_*).\end{aligned}$$

Since $(h_{u_m - u_*}, u_m - u_*) \in \ker F'(x_*, u_*) = K_t$, therefore the intersection $K_t \cap K_f$ is nonempty, and one can assume that $\lambda_0 = -1$.

3.4 Final result

To formulate the main result of the paper, it will be convenient to introduce the classical Hamiltonian function:

$$H : [0,\pi] \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}:$$
$$H(t,x,\lambda,u) = \lambda^\top f(t,x,u) + f_0(t,x,u).$$

Thus, we have proved

Theorem 4 [Weak maximum principle]. *Let $M \subset \mathbb{R}^r$ be a closed convex set with nonempty interior. If $(x_*, u_*) \in D((-\Delta)^\beta) \times L^\infty$ is a local minimum point for problem* (8)*, conditions* (9)*,* (10) *are satisfied, one of the conditions of Proposition* 7 *is fulfilled for $\Lambda(t) = f_x(t, x_*(t), u_*(t))$ and $(f_0)_x(\cdot, x_*(\cdot), u_*(\cdot))$, $(f_0)_u(\cdot, x_*(\cdot), u_*(\cdot))$ are not all identically zero, then there exists a function $\lambda \in D((-\Delta)^\beta)$ such that*

$$(-\Delta)^\beta \lambda(t) = H_x\big(t, x_*(t), \lambda(t), u_*(t)\big), \quad t \in [0,\pi] \text{ a.e.,}$$

and

$$H_u\big(t, x_*(t), \lambda(t), u_*(t)\big)\big(u - u_*(t)\big) \geqslant 0, \quad t \in [0,\pi] \text{ a.e.,}$$

for any $u \in M$.

Remark 2. In fact, in the above theorem, the assumption that $(f_0)_x(\cdot, x_*(\cdot), u_*(\cdot))$, $(f_0)_u(\cdot, x_*(\cdot), u_*(\cdot))$ are not all identically zero can be dropped, but if it is not the case, then the unique solution of conjugate system is $\lambda = 0$, and, consequently, the minimum condition takes the form

$$0u \geqslant 0u_*(t), \quad t \in [0,\pi],$$

for $u \in M$. So, it gives no information on u_*.

Example 1. It is easy to see that the optimal control problem

$$(-\Delta)^\beta x(t) = -\big(x(t)\big)^3 + u(t), \quad t \in (0,\pi) \text{ a.e.,} \tag{15}$$
$$\int_0^\pi \big(x(t)\big)^2 - \big(u(t)\big)^2 \, \mathrm{d}t \to \min$$

satisfies the assumptions of the above maximum principle with $\beta > 1/2$, $M = [-1,1]$,

$$f(t,x,u) = -x^3 + u, \qquad f_0(t,x,u) = x^2 - u^2$$

and

$$\gamma(z) = z^3 + 3z^2, \qquad \delta(w) = w + 1, \qquad a(t) \equiv b(t) \equiv 1,$$
$$c(z,w) = z^2 + 2z + w^2 + 2w, \qquad d(t) \equiv 1.$$

Clearly, $\Lambda(t) = -3(x_*(t))^2$ satisfies condition (b) of Proposition 7 for any $x_*(\cdot)$. So, if $(x_*(\cdot), u_*(\cdot))$ is a local minimum point for the problem under considerations, then there exists $\lambda(\cdot) \in D((-\Delta)^\beta)$ such that

$$(-\Delta)^\beta \lambda(t) = -3\bigl(x_*(t)\bigr)^2 \lambda(t) + 2x_*(t), \quad t \in (0, \pi) \text{ a.e.}, \tag{16}$$

and

$$\bigl(-2u_*(t) + \lambda(t)\bigr)u \geqslant \bigl(-2u_*(t) + \lambda(t)\bigr)u_*(t), \quad t \in (0, \pi) \text{ a.e.}, \tag{17}$$

for any $u \in [-1, 1]$.

Analysing the above conditions, we see that the pair $(x_*(\cdot), u_*(\cdot)) \equiv (0, 0)$ is suspected, i.e., it fulfiles (15), (16), (17) with $\lambda(t) \equiv 0$. Moreover, each pair of the form $(x_*(\cdot), u_*(\cdot)) \equiv (x_*(\cdot), 0)$ such that $x_*(\cdot)$ is not zero function cannot be locally optimal because in such a case (17) implies that $\lambda(t) = 0$ for $t \in (0, \pi)$ a.e., and we have contradiction with (16) and the fact that $x_*(\cdot)$ is different from zero function. Of course, any pair $(0, u_*)$ with $u_* \neq 0$ cannot be locally optimal because it does not satisfy the control system (15).

Example 2. Let us consider the optimal control problem

$$(-\Delta)^{1/2} x(t) = u(t), \quad t \in (0, \pi) \text{ a.e.}, \tag{18}$$

$$\int_0^\pi \sin t\, x(t)\, \mathrm{d}t \to \min \tag{19}$$

with $m = 1$, $r = 1$, $M = [-1, 1]$. It is easy to see that all assumptions of our weak maximum principle are satisfied. Thus, if $(x_*(\cdot), u_*(\cdot))$ is a local minimum point for the above problem, then

$$\lambda(t)u \geqslant \lambda(t)u_*(t), \quad u \in M, \tag{20}$$

for $t \in (0, \pi)$ a.e., where $\lambda(\cdot) \in D((-\Delta)^{1/2})$ is a unique solution of conjugate system

$$(-\Delta)^{1/2} \lambda(t) = \sin t, \quad t \in (0, \pi) \text{ a.e.}$$

Let $\lambda(t)$ be of the form $\lambda(t) = \sum_{j=1}^\infty b_j \sqrt{2/\pi} \sin jt$. One can write the above conjugate system as follows:

$$\sum_{j=1}^\infty \bigl(j^2\bigr)^{1/2} b_j \sqrt{\frac{2}{\pi}} \sin jt = \sum_{j=1}^\infty e_j \sqrt{\frac{2}{\pi}} \sin jt,$$

where $e_j = \int_0^\pi \sin t \sqrt{2/\pi} \sin jt$ for $j \in \mathbb{N}$. In consequence,

$$b_j = \frac{e_j}{j} = \begin{cases} \sqrt{\frac{\pi}{2}}; & j = 1, \\ 0; & j > 1, \end{cases} \quad \text{and} \quad \lambda(t) = \sin t, \quad t \in [0, \pi].$$

Thus, from (20) it follows that

$$u_*(t) = -1, \quad t \in (0,\pi) \text{ a.e.},$$

is a unique suspected control and

$$(-\Delta)^{1/2} x_*(t) = -1, \quad t \in (0,\pi) \text{ a.e.},$$

i.e.,

$$\sum_{j=1}^{\infty} \left(j^2\right)^{1/2} c_j \sqrt{\frac{2}{\pi}} \sin jt = \sum_{j=1}^{\infty} d_j \sqrt{\frac{2}{\pi}} \sin jt, \quad t \in (0,\pi) \text{ a.e.},$$

where

$$x_*(t) = \sum_{j=1}^{\infty} c_j \sqrt{\frac{2}{\pi}} \sin jt \quad \text{and} \quad -1 = \sum_{j=1}^{\infty} d_j \sqrt{\frac{2}{\pi}} \sin jt, \quad t \in (0,\pi) \text{ a.e.}$$

It is known (see [8]) that

$$d_j = \begin{cases} -\sqrt{\frac{8}{\pi}} \frac{1}{j}; & j \text{ is odd}, \\ 0; & j \text{ is even}. \end{cases}$$

So,

$$c_j = \frac{d_j}{j} = \begin{cases} -\sqrt{\frac{8}{\pi}} \frac{1}{j^2}; & j \text{ is odd}, \\ 0; & j \text{ is even}, \end{cases}$$

and

$$\begin{aligned} x_*(t) &= \sum_{j=1}^{\infty} c_j \sqrt{\frac{2}{\pi}} \sin jt \\ &= -\frac{4}{\pi} \left(\sin t + \frac{\sin 3t}{3^2} + \frac{\sin 5t}{5^2} + \cdots \right), \quad t \in (0,\pi) \text{ a.e.} \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^{\pi} \sin t \, x_*(t) \, \mathrm{d}t &= -\frac{4}{\pi} \int_0^{\pi} \sin t \left(\sin t + \frac{\sin 3t}{3^2} + \frac{\sin 5t}{5^2} + \cdots \right) \mathrm{d}t \\ &= -\frac{4}{\pi} \int_0^{\pi} \sin t \sin t \, \mathrm{d}t = -2. \end{aligned}$$

Summarizing, if a pair (x_*, u_*) is a local minimum point for problem (18)–(19) with $M = [-1,1]$, then, for $t \in (0,\pi)$ a.e.,

$$\begin{aligned} u_*(t) &= -1, \\ x_*(t) &= -\frac{4}{\pi} \left(\sin t + \frac{\sin 3t}{3^2} + \frac{\sin 5t}{5^2} + \cdots \right), \end{aligned}$$

and the minimal value of the cost functional (19) is equal to -2.

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