Nonlinear Analysis: Modelling and Control, Vol. 25, No. 2, 206–224
https://doi.org/10.15388/namc.2020.25.16513
ISSN: 1392-5113
eISSN: 2335-8963

Finite-time passivity for neutral-type neural networks with time-varying delays – via auxiliary function-based integral inequalities*

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Received: October 1, 2018 / **Revised:** May 11, 2019 / **Published online:** March 2, 2020

Abstract. In this paper, we investigated the problem of the finite-time boundedness and finite-time passivity for neural networks with time-varying delays. A triple, quadrable and five integral terms with the delay information are introduced in the new Lyapunov–Krasovskii functional (LKF). Based on the auxiliary integral inequality, Writinger integral inequality and Jensen's inequality, several sufficient conditions are derived. Finally, numerical examples are provided to verify the effectiveness of the proposed criterion. There results are compared with the existing results.

Keywords: auxilary integral inequality, finite-time boundedness, finite-time passivity, Lyapunov–Krasovskii functional, neural networks, time-varying delays.

1 Introduction

Recently, neural networks have received much attention of their extensive applications in signal processing, solving optimization problems, pattern recognition, pattern classification, image processing, model identification and other engineering fields. The stability problem of neural networks with time-varying delays has been deeply investigated in [8–10, 12, 25, 37]. Time-delay phenomena are inevitable in studying real systems. The existence of time delay makes the system dynamic performance worse or even leads to system instability. Therefore, the stability and control problem of time-delay system have attracted a lot of scholars attention, and some nice results have been obtained on linear and nonlinear time-delay neural networks during the past few decades. Moreover, the delay-dependent stability conditions are generally less conservative than delay-

*This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah (grant No. D-263-130-1441). The authors gratefully acknowledge DSR technical and financial support.
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independent conditions, in particularly when the size of the delay is small. Therefore, considerable attention has been focused on the derivation of delay-dependent stability results, and many effective approaches have been provided to reduce the conservatism of stability results for further improving the quality of delay-dependent stability criteria. Also, due to the finite speed of information processing in the implementation of the network, time delay occurs in many neural networks; see [22,29,30]. It is well known that time delay often causes undesirable dynamic behaviors such as oscillation and instability of the networks. Thus, delay-dependent stability and stabilization problem for neural networks with time delay have been paid more attention than delay-independent ones because the information on the size of time delays is utilized in delay-dependent criteria, which lead to reduce the conservatism of stability and stabilization criteria, (see [24, 31] and references therein).

In recent years, as a powerful tool, passivity has played an important role in the network control, process control, group coordination, analysis, design of linear and nonlinear systems, energy management and so on. Thus, passivity theory has become a most focused topic in the recent years. The problems of passivity and passification have been an active area of research over the last ten years due to its importance in engineering applications such as in safety-critical and high-integrity systems [19]. The essence of the passivity theory is that the passive properties of a system can keep the system internal stability. This class of systems was introduced by Krasovskii and Lidskii in 1961 [11]. Since then, the passivity of neural networks with delays have been studied in [13, 34, 36]. Based on the Lyapunov–Krasovskii method and utilization of zero equalities, the passivity properties for delayed neural networks were studied in [13]. However, in most existing works on the passivity, it is assumed that the input and output variables are only dependent on the time. But, in reality, the input and output variables are not only dependent on the time, but also intensively dependent on space variable in many circumstances. Thus, it is important and interesting to study the passivity of systems in which the input and output variables are varied with the time and space variables. Several sufficient conditions on passivity were derived for various neural networks such as time-invariant, time-varying, uncertain and stochastic network models.

At present, many researches have been extensively developed the area of finite-time stability for neural networks, see, for instance, [4–7,18,27,28]. Also, finite-time boundedness and stability results are investigated in [1,33,35]. Actually, it may be more reasonable that passivity is achieved over a finite-time interval in many practical applications. Thus, it is of great significance to study finite-time dynamical behaviors, which has become a very hot research topic recently [2,23]. Regrettably, few researchers devoted themselves to investigating the finite-time passivity of finite-time passive filtering for neutral system is studied in [2]. The finite-time passive control for nonlinear system with time delays was reported in [23]. In [33], the results on finite-time stability and H_∞ control of switched neutral systems using the average dwell time approach are proposed.

On the other hand, in recent years, many researchers have extensively investigated neutral-type neural networks and systems with time-varying delays, see, for instance, [14, 26]. There are many new methods expressed in less conservative results for neutral-type systems. Delay-dependent neutral-type neural networks with mixed random-time varying delays have been investigated by employing LMI technique in [22]. In [27], the authors have studied finite-time neutral delay uncertain neural networks. Passivity analysis for neural networks of neutral type has been studied in [32]. The passivity analysis for memristor-based stochastic BAM neural networks of neutral type was presented in [26]. To the best of the authors' knowledge up to now, the finite-time passivity of neural networks with neutral-type time-varying delays has not been completely studied in the literature, which motivates our research in this paper.

With the above motivation, in this article, the issue of finite-time boundedness and finite-time passivity criteria of neutral-type neural networks with time-varying delay based on the auxiliary function-based integral inequality technique is explored. As result, in this note, there still exists some less conservatism for neural networks with interval time-varying delay to be further improved. To achieve this, at the end, several numerical examples are addressed to show the effectiveness of the developed stability criteria. The highlights and major contributions of this paper are reflected in the subsequent key points:

(i) In this paper, we considered the system with time-varying delays, additionally the effect of neutral delay has also been taken into account to showing feasibility on a problem.
(ii) Some simplest LMI-based criterion has been launched with the help of integral inequality technique together with the auxiliary function-based integral inequality combined with Writinger integral inequality, Jensen's inequality.
(iii) Then we derived finite-time boundedness, finite-stability and finite time passivity conditions in the theorems.
(iv) Several examples have been investigated to verify the correctness of the main theorem and the corollaries.

The outline of the paper is structured as follows. In Section 2, the system models and some necessary mathematical preliminaries are declared. In Section 3, we present the main results for the neural network model in which neutral delay is taken into account. Simulation examples are given in Section 4, and conclusions follow in Section 5.

Notations. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, and $\mathbb{R}^{m\times n}$ is the set of all $m \times n$ real matrices. The superscript "T" denotes matrix transposition, and $A \geqslant B$ (respectively, $A < B$), where A and B are symmetric matrices (respectively, positive definite). $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$. If Q is a square matrix, $\lambda_{\max}(Q)$ (respectively, $\lambda_{\min}(Q)$) means the largest (respectively, smallest) eigenvalue of Q. The asterisk "$*$" in a symmetric matrix is used to denote term, which is induced by symmetry; $\operatorname{diag}\{\cdot\}$ stands for the diagonal matrix.

2 Problem formulation and preliminaries

Consider the neutral-type neural networks with time-varying delays as follows:

$$\dot{x}(t) - \mathbf{D}\dot{x}(t-d) = -\mathbf{A}x(t) + \mathbf{B}f\bigl(x(t)\bigr) + \mathbf{C}f\bigl(x\bigl(t-h(t)\bigr)\bigr) + \mathbf{E}w(t), \tag{1}$$

$$y(t) = \mathbf{G}f\bigl(x(t)\bigr) + \mathbf{H}w(t), \tag{2}$$

$$x(\theta) = \phi(\theta), \quad \theta \in [-H_d, 0]. \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the neural state vector, $v(t)$ is the exogenous disturbance input vector belongs to $L_2[0,\infty)$, and $y(t)$ is the output vector of the neural networks, $f(x(t))$ is the neuron activation function, $\mathbf{A} = \operatorname{diag}\{a_1, a_2, \dots, a_n\} > 0$ is a diagonal matrix, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ are connection weight matrices. $\phi(\theta)$ denotes the continuous vector-valued initial function. $h(t)$ denotes the time-varying delay, and d is neutral delay. We define the interval $t_{k+1} - t_k = h_k + \varDelta h_k \leqslant h + \varDelta_h(t)$. Here $|\varDelta_h(t)| < \rho < h$, where ρ is very small scalar. The intervals can be written as $t_k = t - (t - t_k) = t - h(t)$, $k = 0, 1, 2, \dots$, where $0 \leqslant h(t) \leqslant h \leqslant h + \rho$, $\dot{h}(t) \leqslant \mu$, $H_d = \max\{h, d\}$.

Assumption 1. For a given positive parameter δ, the external disturbance input $w(t)$ is time varying and satisfies

$$\int_0^{\mathrm{T}} w^{\mathrm{T}}(t) w(t)\,\mathrm{d}t \leqslant \delta, \quad \delta > 0.$$

Assumption 2. The activation functions satisfy the following condition for any $s = 1, 2, \dots, n$, there exist constants ϱ_i^-, ϱ_i^+ such that

$$\varrho_i^- \leqslant \frac{f_s(x_1) - f_s(x_2)}{x_1 - x_2} \leqslant \varrho_i^+ \quad \forall x_1, x_2 \in \mathbb{R},\ x_1 \neq x_2.$$

For presentation convenience, we denote

$$\begin{aligned}
\varDelta_1 &= \operatorname{diag}\{\varrho_1^- \varrho_1^+, \varrho_2^- \varrho_2^+, \dots, \varrho_n^- \varrho_n^+\},\\
\varDelta_2 &= \operatorname{diag}\left\{\frac{\varrho_1^- + \varrho_1^+}{2}, \frac{\varrho_2^- + \varrho_2^+}{2}, \dots, \frac{\varrho_n^- + \varrho_n^+}{2}\right\}.
\end{aligned}$$

Definition 1 [Finite-time boundedness]. (See [1].) For a given time constant $c_1 > 0$, $\delta > 0$, T and symmetric matrix $R > 0$, the neural networks (1)–(3) is said to be finite-time bounded with respect to (c_1, c_2, T, R, δ) if there exist constants $c_2 > c_1 > 0$ such that

$$x^{\mathrm{T}}(t_0) R x(t_0) \leqslant c_1 \implies x^{\mathrm{T}}(t) R x(t) \leqslant c_2 \quad \forall t_0 \in [-H_d, 0],\ t \in [0, T].$$

Definition 2 [Finite-time stability]. (See [33].) For a given time constant $c_1 > 0$, T and symmetric matrix $R > 0$, the neural networks (1)–(3) with $w(t) = 0$ is said to be finite-time stable with respect to (c_1, c_2, T, R) if there exist constants $c_2 > c_1 > 0$ such that

$$x^{\mathrm{T}}(t_0) R x(t_0) \leqslant c_1 \implies x^{\mathrm{T}}(t) R x(t) \leqslant c_2 \quad \forall t_0 \in [-H_d, 0],\ t \in [0, T].$$

Definition 3 [Finite-time passivity]. (See [23].) System (1)–(3) is said to be a finite-time passive with respect to $(c_1, c_2, R, \gamma, \delta)$, where $0 < c_1 < c_2$, γ is a prescribed dissipation performance level $\gamma > 0$ and $R > 0$ if the following conditions holds:

(a) System (1)–(3) is finite-time bounded for all external disturbances $w(t)$;
(b) Under zero initial condition, the following relation hold for a given positive scalar $\gamma > 0$:

$$\int_0^{\mathrm{T}} w^{\mathrm{T}}(t)y(t)\,\mathrm{d}t \geqslant \gamma \int_0^{\mathrm{T}} w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t.$$

Lemma 1. (*See* [20].) *For a positive definite matrix M, a differentiable function $x(u)$, $u \in (\alpha, \beta)$, and a polynomial auxiliary function $p_i(u) = (u-\alpha)^i$, the following inequality holds for $0 \leqslant n \leqslant 3$:*

$$\int_\alpha^\beta \dot{x}^{\mathrm{T}}(\theta)M\dot{x}(\theta)\,\mathrm{d}\theta \geqslant \sum_{k=0}^{n} \frac{2k+1}{\beta-\alpha} \Pi_k^{\mathrm{T}} M \Pi_k,$$

where

$$\Pi_0 = x(\beta) - x(\alpha), \qquad \Pi_1 = x(\beta) + x(\alpha) - \frac{2}{\beta-\alpha}\int_\alpha^\beta x(\theta)\,\mathrm{d}\theta,$$

$$\Pi_2 = x(\beta) - x(\alpha) + \frac{6}{\beta-\alpha}\int_\alpha^\beta x(\theta)\,\mathrm{d}\theta - \frac{12}{(\beta-\alpha)^2}\int_\alpha^\beta\int_\zeta^\beta x(\theta)\,\mathrm{d}\theta\,\mathrm{d}\zeta,$$

$$\Pi_3 = x(\beta) + x(\alpha) - \frac{12}{\beta-\alpha}\int_\alpha^\beta x(\theta)\,\mathrm{d}\theta + \frac{60}{(\beta-\alpha)^2}\int_\alpha^\beta\int_\zeta^\beta x(\theta)\,\mathrm{d}\theta\,\mathrm{d}\zeta$$
$$- \frac{120}{(\beta-\alpha)^3}\int_\alpha^\beta\int_\varsigma^\beta\int_\zeta^\beta x(\theta)\,\mathrm{d}\theta\,\mathrm{d}\zeta\,\mathrm{d}\varsigma.$$

Lemma 2. (*See* [21].) *For any constant matrix $M > 0$, the following inequality holds for all continuously differentiable function x in $[\alpha, \beta] \to \mathbb{R}^n$:*

$$(\beta-\alpha)\int_\alpha^\beta x^{\mathrm{T}}(s)Mx(s)\,\mathrm{d}s \geqslant \left[\int_\alpha^\beta x(s)\,\mathrm{d}s\right]^{\mathrm{T}} M \left[\int_\alpha^\beta x(s)\,\mathrm{d}s\right] + 3\Pi^{\mathrm{T}}M\Pi,$$

where $\Pi = \int_\alpha^\beta x(s)\,\mathrm{d}s - (2/(\beta-\alpha))\int_\alpha^\beta\int_\alpha^s x(u)\,\mathrm{d}u\,\mathrm{d}s$.

Lemma 3 [Jensen's inequality]. (*See* [3].) *For any constant matrix $M \in \mathbb{R}^{m\times m}$, $M = M^{\mathrm{T}} > 0$, take scalars $a < b$ and vector function $x(s) : [a, b] \in \mathbb{R}^n$ such that the following integration is well defined, then*

$$(b-a)\int_a^b x^{\mathrm{T}}(s)Mx(s)\,\mathrm{d}s \geqslant \left[\int_a^b x(s)\,\mathrm{d}s\right]^{\mathrm{T}} M \left[\int_a^b x(s)\,\mathrm{d}s\right],$$

$$(b-a)^2\left[\int_a^b\int_\theta^b x^{\mathrm{T}}(s)Mx(s)\,\mathrm{d}s\,\mathrm{d}\theta\right] \geqslant 2\left[\int_a^b\int_\theta^b x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right]^{\mathrm{T}} M\left[\int_a^b\int_\theta^b x(s)\,\mathrm{d}s\,\mathrm{d}\theta\right],$$

$$(b-a)^3\left[\int_a^b\int_\theta^b\int_\lambda^b x(s)^{\mathrm{T}}Mx(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\lambda\right]$$
$$\geqslant 6\left[\int_a^b\int_\theta^b\int_\lambda^b x(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\lambda\right]^{\mathrm{T}} M\left[\int_a^b\int_\theta^b\int_\lambda^b x(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\lambda\right].$$

3 Main results

3.1 Finite-time boundedness

In this section, we investigate finite-time boundedness for the following delayed neural networks (1)–(3):

$$\dot{x}(t) = -\mathbf{A}x(t) + \mathbf{B}f\big(x(t)\big) + \mathbf{C}f\big(x\big(t-h(t)\big)\big) + \mathbf{D}\dot{x}(t-d) + \mathbf{E}w(t), \tag{4}$$
$$x(\theta) = \phi(\theta)\,\mathrm{d}\theta \in [-H_d, 0], \tag{5}$$

where $\phi(\theta)$ is a continuous vector-valued initial function, and we define the following vectors:

$$\begin{aligned}\xi^{\mathrm{T}}(t) = \mathrm{col}\Bigg[&x^{\mathrm{T}}(t),\ x^{\mathrm{T}}\big(t-h(t)\big),\ x^{\mathrm{T}}(t-h),\ f^{\mathrm{T}}\big(x(t)\big),\ f^{\mathrm{T}}\big(x\big(t-h(t)\big)\big),\ \dot{x}(t)\\
&\dot{x}(t-d),\ \int_{t-h(t)}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s,\ \int_{t-h}^{t-h(t)} x^{\mathrm{T}}(s)\,\mathrm{d}s,\ \int_{-h(t)}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta\\
&\int_{-h}^{-h(t)}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta,\ \int_{-h(t)}^{0}\int_{\nu}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\nu,\\
&\int_{-h}^{-h(t)}\int_{\nu}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\nu,\ w^{\mathrm{T}}(t)\Bigg],\end{aligned}$$
$$e_k = \mathrm{col}\big[0_{(k-1)n\times n},\ I_n,\ 0_{(k-1)n\times n}\big],\quad k = (1, 2, \dots, 14).$$

Theorem 1. *For given scalars h, μ, d, δ, α, β, c_1, c_2 and T, the neural networks* (4)–(5) *is finite-time bounded if there exist positive symmetric matrices P, Q_i ($i = 1, 2, \dots, 10$), any diagonal matrices U, S and matrices N_1, N_2 with appropriate dimensions such that* *the following LMIs holds*:

$$\hat{\Theta} = \Upsilon_\kappa - e_1 \alpha P e_1^{\mathrm{T}} - e_{14} \alpha I e_{14}^{\mathrm{T}} < 0, \tag{6}$$

where

$$\Upsilon_\kappa = \Upsilon_1 + \Upsilon_2 + \cdots + \Upsilon_{11},$$

$$\lambda_1 c_2 \mathrm{e}^{-\alpha T} > \Lambda c_1 + \delta\bigl(1 - \mathrm{e}^{\alpha T}\bigr), \tag{7}$$

where

$$\begin{aligned}
\Upsilon_1 &= -2e_1 P\mathbf{A}e_1^{\mathrm{T}} + 2e_1 P\mathbf{B}e_4^{\mathrm{T}} + 2e_1 P\mathbf{C}e_5^{\mathrm{T}} + 2e_1 P\mathbf{D}e_7^{\mathrm{T}} + 2\alpha e_1 P\mathbf{E}e_{14}^{\mathrm{T}},\\
\Upsilon_2 &= e_1 Q_1 e_1^{\mathrm{T}} - e_3 Q_1 e_3^{\mathrm{T}} + e_1 Q_2 e_1^{\mathrm{T}} - (1-\mu) e_2 Q_2 e_2^{\mathrm{T}},\\
\Upsilon_3 &= e_6 Q_3 e_6^{\mathrm{T}} - e_7 Q_3 e_7^{\mathrm{T}} + e_4 Q_4 e_4^{\mathrm{T}} - e_5 Q_4 e_5^{\mathrm{T}},\\
\Upsilon_4 &= h e_1 Q_5 e_1^{\mathrm{T}} + \frac{1}{h} e_9 Q_5 e_9^{\mathrm{T}} - \frac{3}{h}\left[e_9^{\mathrm{T}} - \frac{2}{h} e_{11}^{\mathrm{T}}\right]^{\mathrm{T}} Q_5 \left[e_9^{\mathrm{T}} - \frac{2}{h} e_{11}^{\mathrm{T}}\right]\\
&\quad - \frac{1}{h+\rho} e_8 Q_5 e_8^{\mathrm{T}} - \frac{3}{h+\rho}\left[e_8^{\mathrm{T}} - \frac{2}{h+\rho} e_{10}^{\mathrm{T}}\right]^{\mathrm{T}} Q_5 \left[e_8^{\mathrm{T}} - \frac{2}{h+\rho} e_{10}^{\mathrm{T}}\right],\\
\Upsilon_5 &= h e_6 Q_6 e_6^{\mathrm{T}} - \frac{1}{h}\bigl[e_1^{\mathrm{T}} - e_3^{\mathrm{T}}\bigr]^{\mathrm{T}} Q_6 \bigl[e_1^{\mathrm{T}} - e_3^{\mathrm{T}}\bigr] - \frac{3}{h}\left[e_1^{\mathrm{T}} + e_3^{\mathrm{T}} - \frac{2}{h} e_8^{\mathrm{T}} - \frac{2}{h} e_9^{\mathrm{T}}\right]^{\mathrm{T}} Q_6\\
&\quad \times \left[e_1^{\mathrm{T}} + e_3^{\mathrm{T}} - \frac{2}{h} e_8^{\mathrm{T}} - \frac{2}{h} e_9^{\mathrm{T}}\right] - \frac{5}{h}\left[e_1^{\mathrm{T}} - e_3^{\mathrm{T}} + \frac{6}{h} e_8^{\mathrm{T}} + \frac{6}{h} e_9^{\mathrm{T}} - \frac{12}{h^2} e_{10}^{\mathrm{T}} - \frac{12}{h^2} e_{11}^{\mathrm{T}}\right]^{\mathrm{T}}\\
&\quad \times Q_6 \left[e_1^{\mathrm{T}} - e_3^{\mathrm{T}} + \frac{6}{h} e_8^{\mathrm{T}} + \frac{6}{h} e_9^{\mathrm{T}} - \frac{12}{h^2} e_{10}^{\mathrm{T}} - \frac{12}{h^2} e_{11}^{\mathrm{T}}\right]\\
&\quad - \frac{7}{h}\left[e_1^{\mathrm{T}} + e_3^{\mathrm{T}} - \frac{12}{h} e_8^{\mathrm{T}} - \frac{12}{h} e_9^{\mathrm{T}} + \frac{60}{h^2} e_{10}^{\mathrm{T}} + \frac{60}{h^2} e_{11}^{\mathrm{T}} - \frac{120}{h^3} e_{12}^{\mathrm{T}} - \frac{120}{h^3} e_{13}^{\mathrm{T}}\right]^{\mathrm{T}}\\
&\quad \times Q_6 \left[e_1^{\mathrm{T}} + e_3^{\mathrm{T}} - \frac{12}{h} e_8^{\mathrm{T}} - \frac{12}{h} e_9^{\mathrm{T}} + \frac{60}{h^2} e_{10}^{\mathrm{T}} + \frac{60}{h^2} e_{11}^{\mathrm{T}} - \frac{120}{h^3} e_{12}^{\mathrm{T}} - \frac{120}{h^3} e_{13}^{\mathrm{T}}\right],\\
\Upsilon_6 &= \frac{h^2}{2} e_1 Q_7 e_1^{\mathrm{T}} - \frac{2}{h^2}\bigl[e_{10}^{\mathrm{T}} + e_{11}^{\mathrm{T}}\bigr]^{\mathrm{T}} Q_7 \bigl[e_{10}^{\mathrm{T}} + e_{11}^{\mathrm{T}}\bigr],\\
\Upsilon_7 &= \frac{h^2}{2} e_6 Q_8 e_6^{\mathrm{T}} - 2\left[e_1^{\mathrm{T}} - \frac{1}{h} e_8^{\mathrm{T}} - \frac{1}{h} e_9^{\mathrm{T}}\right]^{\mathrm{T}} Q_8 \left[e_1^{\mathrm{T}} - \frac{1}{h} e_8^{\mathrm{T}} - \frac{1}{h} e_9^{\mathrm{T}}\right],\\
\Upsilon_8 &= \frac{h^3}{6} e_6 Q_9 e_6^{\mathrm{T}} - 6h\left[\frac{1}{2} e_1^{\mathrm{T}} - \frac{1}{h^2} e_{10}^{\mathrm{T}} - \frac{1}{h^2} e_{11}^{\mathrm{T}}\right]^{\mathrm{T}} Q_9 \left[\frac{1}{2} e_1^{\mathrm{T}} - \frac{1}{h^2} e_{10}^{\mathrm{T}} - \frac{1}{h^2} e_{11}^{\mathrm{T}}\right],\\
\Upsilon_9 &= \frac{h^4}{24} e_6 Q_{10} e_6^{\mathrm{T}} - 24h^2\left[\frac{1}{6} e_1^{\mathrm{T}} - \frac{1}{h^3} e_{12}^{\mathrm{T}} - \frac{1}{h^3} e_{13}^{\mathrm{T}}\right]^{\mathrm{T}} Q_{10} \left[\frac{1}{6} e_1^{\mathrm{T}} - \frac{1}{h^3} e_{12}^{\mathrm{T}} - \frac{1}{h^3} e_{13}^{\mathrm{T}}\right],\\
\Upsilon_{10} &= 2[e_1 + e_6](N_1 + N_2)\bigl[-\mathbf{A}e_1^{\mathrm{T}} + \mathbf{B}e_4^{\mathrm{T}} + \mathbf{C}e_5^{\mathrm{T}} + \mathbf{D}e_7^{\mathrm{T}} + \mathbf{E}e_{14} - e_6\bigr],\\
\Upsilon_{11} &= -e_1 \Delta_1 U e_1^{\mathrm{T}} + e_1 \Delta_2 U e_4^{\mathrm{T}} - e_4 U e_4^{\mathrm{T}} - e_2 \Delta_1 S e_2^{\mathrm{T}} + e_2 \Delta_2 S e_5^{\mathrm{T}} - e_5 S e_5^{\mathrm{T}}.
\end{aligned}$$

Consider

$$\begin{aligned}
&\lambda_1 = \lambda_{\min}(\bar{P}), && \lambda_2 = \lambda_{\max}(\bar{P}), && \lambda_3 = \lambda_{\max}(\bar{Q}_1), && \lambda_4 = \lambda_{\max}(\bar{Q}_2),\\
&\lambda_5 = \lambda_{\max}(\bar{Q}_3), && \lambda_6 = \lambda_{\max}(\bar{Q}_4), && \lambda_7 = \lambda_{\max}(\bar{Q}_5), && \lambda_8 = \lambda_{\max}(\bar{Q}_6),\\
&\lambda_9 = \lambda_{\max}(\bar{Q}_7), && \lambda_{10} = \lambda_{\max}(\bar{Q}_8), && \lambda_{11} = \lambda_{\max}(\bar{Q}_9), && \lambda_{12} = \lambda_{\max}(\bar{Q}_{10}).
\end{aligned}$$

Proof. Consider the following Lyapunov–Krasovskii functional:

$$V(x_t, t) = \sum_{k=1}^{9} V_k\bigl(x(t)\bigr),$$

where

$$V_1(x_t, t) = x^{\mathrm{T}}(t) P x(t), \qquad V_2(x_t, t) = \int_{t-h}^{t} x^{\mathrm{T}}(s) Q_1 x(s)\,\mathrm{d}s + \int_{t-h(t)}^{t} x^{\mathrm{T}}(s) Q_2 x(s)\,\mathrm{d}s,$$

$$V_3(x_t, t) = \int_{t-d}^{t} \dot{x}^{\mathrm{T}}(s) Q_3 \dot{x}(s)\,\mathrm{d}s + \int_{t-h(t)}^{t} f^{\mathrm{T}}\bigl(x(s)\bigr) Q_4 f\bigl(x(s)\bigr)\,\mathrm{d}s,$$

$$V_4(x_t, t) = \int_{-h}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s) Q_5 x(s)\,\mathrm{d}s\,\mathrm{d}\theta, \qquad V_5(x_t, t) = \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Q_6 \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta,$$

$$V_6(x_t, t) = \int_{-h}^{0}\int_{\nu}^{0}\int_{t+\theta}^{t} x^{\mathrm{T}}(s) Q_7 x(s) ds\,\mathrm{d}\theta\,\mathrm{d}\nu, \qquad V_7(x_t, t) = \int_{-h}^{0}\int_{\nu}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Q_8 \dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\nu,$$

$$V_8(x_t, t) = \int_{-h}^{0}\int_{\varrho}^{0}\int_{\nu}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Q_9 \dot{x}(s) ds\,\mathrm{d}\theta\,\mathrm{d}\nu\,\mathrm{d}\varrho,$$

$$V_9(x_t, t) = \int_{-h}^{0}\int_{\omega}^{0}\int_{\varrho}^{0}\int_{\nu}^{0}\int_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s) Q_{10} \dot{x}(s) ds\,\mathrm{d}\theta\,\mathrm{d}\nu\,\mathrm{d}\varrho\,\mathrm{d}\omega.$$

Then we calculating the time derivative of $V(x(t))$:

$$\dot{V}_1(x_t, t) = 2x^{\mathrm{T}}(t) P_i \dot{x}(t) = \xi^{\mathrm{T}}(t) \Upsilon_1 \xi(t), \tag{8}$$

$$\begin{aligned}
\dot{V}_2(x_t, t) &= x^{\mathrm{T}}(t) Q_1 x(t) - x^{\mathrm{T}}(t-h) Q_1 x(t-h)\\
&\quad + x^{\mathrm{T}}(t) Q_2 x(t) - (1-\mu) x^{\mathrm{T}}(t-h(t)) Q_2 x\bigl(t-h(t)\bigr),\\
&= \xi^{\mathrm{T}}(t) \Upsilon_2 \xi(t),
\end{aligned} \tag{9}$$

$$\begin{aligned}
\dot{V}_3(x_t, t) &= \dot{x}^{\mathrm{T}}(t) Q_3 \dot{x}(t) - \dot{x}^{\mathrm{T}}(t-d) Q_3 \dot{x}(t-d)\\
&\quad + f^{\mathrm{T}}\bigl(x(t)\bigr) Q_4 f\bigl(x(t)\bigr) - f^{\mathrm{T}}\bigl(x\bigl(t-h(t)\bigr)\bigr) Q_4 f\bigl(x\bigl(t-h(t)\bigr)\bigr),\\
&= \xi^{\mathrm{T}}(t) \Upsilon_3 \xi(t),
\end{aligned} \tag{10}$$

$$\dot{V}_4(x_t,t) = hx^{\mathrm{T}}(t)Q_5x(t) - \int\limits_{t-h}^{t} x^{\mathrm{T}}(s)Q_5x(s)\,\mathrm{d}s, \tag{11}$$

Using Lemma 2 in (11), we can get

$$\dot{V}_4(x_t,t) \leqslant \xi^{\mathrm{T}}(t)\Upsilon_4\xi(t), \tag{12}$$

$$\dot{V}_5(x_t,t) = h\dot{x}^{\mathrm{T}}(t)Q_6\dot{x}(t) - \int\limits_{t-h}^{t} \dot{x}^{\mathrm{T}}(s)Q_6\dot{x}(s)\,\mathrm{d}s, \tag{13}$$

and applying Lemma 1 in $\dot{V}_5(x(t))$, we get

$$\dot{V}_5(x_t,t) \leqslant \xi^{\mathrm{T}}(t)\Upsilon_5\xi(t), \tag{14}$$

$$\dot{V}_6(x_t,t) = \frac{h^2}{2}x^{\mathrm{T}}(t)Q_7x(t) - \int\limits_{-h}^{0}\int\limits_{t+\theta}^{t} x^{\mathrm{T}}(s)Q_7x(s)\,\mathrm{d}s\,\mathrm{d}\theta. \tag{15}$$

By applying Lemma 3 we get

$$\dot{V}_6(x_t,t) \leqslant \xi^{\mathrm{T}}(t)\Upsilon_6\xi(t), \tag{16}$$

$$\dot{V}_7(x_t,t) = \frac{h^2}{2}\dot{x}^{\mathrm{T}}(t)Q_8\dot{x}(t) - \int\limits_{-h}^{0}\int\limits_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Q_8\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta. \tag{17}$$

By Lemma 3 we obtain

$$\dot{V}_7(x_t,t) \leqslant \xi^{\mathrm{T}}(t)\Upsilon_7\xi(t), \tag{18}$$

$$\dot{V}_8(x_t,t) = \frac{h^3}{6}\dot{x}^{\mathrm{T}}(t)Q_9\dot{x}(t) - \int\limits_{-h}^{0}\int\limits_{\nu}^{0}\int\limits_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Q_9\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\nu. \tag{19}$$

Also, by using the Lemma 3 we can get

$$\dot{V}_8(x_t,t) \leqslant \xi^{\mathrm{T}}(t)\Upsilon_8\xi(t), \tag{20}$$

$$\dot{V}_9(x_t,t) = \frac{h^4}{24}\dot{x}^{\mathrm{T}}(t)Q_{10}\dot{x}(t) - \int\limits_{-h}^{0}\int\limits_{\varrho}^{0}\int\limits_{\nu}^{0}\int\limits_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)Q_{10}\dot{x}(s)\,\mathrm{d}s\,\mathrm{d}\theta. \tag{21}$$

By Lemma 3 we get

$$\dot{V}_9(x_t,t) \leqslant \xi^{\mathrm{T}}(t)\Upsilon_9\xi(t). \tag{22}$$

Furthermore, the following equality holds for any real matrices N_1 and N_2 with compatible dimensions:

$$0 = 2\big[x^{\mathrm{T}}(t) + \dot{x}^{\mathrm{T}}(t)\big](N_1 + N_2)\big[-\mathbf{A}x(t) + \mathbf{B}f\big(x(t)\big) \\ + \mathbf{C}f\big(x\big(t-h(t)\big)\big) + \mathbf{D}\dot{x}(t-d) + \mathbf{E}w(t) - \dot{x}(t)\big], \tag{23}$$

$$0 = 2\big[e_1^{\mathrm{T}}(t) + e_6^{\mathrm{T}}(t)\big](N_1 + N_2)\big[-\mathbf{A}e_1 + \mathbf{B}e_4 + \mathbf{C}e_5 + \mathbf{D}e_7 + \mathbf{E}e_{14} - e_6\big], \tag{24}$$

$$0 = \xi^{\mathrm{T}}(t)\Upsilon_{10}\xi(t). \tag{25}$$

Based on Assumption 2, for $i = 1, 2, \dots, n$, we obtain

$$\big[f_i\big(x_i(t)\big) - \varrho_i^- x_i(t)\big]\big[f_i\big(x_i(t)\big) - \varrho_i^+ x_i(t)\big] \leqslant 0,$$
$$\big[f_i\big(x_i\big(t-\tau(t)\big)\big) - \varrho_i^- x_i\big(t-\tau(t)\big)\big]\big[f_i\big(x_i\big(t-\tau(t)\big)\big) - \varrho_i^+ x_i\big(t-\tau(t)\big)\big] \leqslant 0,$$

which is equivalent to

$$\begin{bmatrix} x_i(t) \\ f_i(x_i(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \varrho_i^- \varrho_i^+ m_i m_i^{\mathrm{T}} & -\frac{\varrho_i^- \varrho_i^+}{2} m_i m_i^{\mathrm{T}} \\ * & m_i m_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_i(t) \\ f_i(x_i(t)) \end{bmatrix} \leqslant 0, \quad i = 1, 2, \dots, n,$$

where m_i denotes the unit column vector having 1 on its ith row and zeros elsewhere. Let $U = \operatorname{diag}\{u_1, u_2, \dots, u_n\}$, $S = \operatorname{diag}\{s_1, s_2, \dots, s_n\}$.

$$\sum_{i=1}^{n} u_i \begin{bmatrix} x_i(t) \\ f_i(x_i(t)) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \varrho_i^- \varrho_i^+ m_i m_i^{\mathrm{T}} & -\frac{\varrho_i^- \varrho_i^+}{2} m_i m_i^{\mathrm{T}} \\ * & m_i m_i^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_i(t) \\ f_i(x_i(t)) \end{bmatrix} \leqslant 0$$

can be written as

$$\begin{bmatrix} x(t) \\ f\big(x(t)\big) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta_1 U & -\Delta_2 U \\ * & U \end{bmatrix} \begin{bmatrix} x(t) \\ f\big(x(t)\big) \end{bmatrix} \leqslant 0.$$

Similarly, we get

$$\begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Delta_1 S & -\Delta_2 S \\ * & S \end{bmatrix} \begin{bmatrix} x(t-h(t)) \\ f(x(t-h(t))) \end{bmatrix} \leqslant 0,$$

which implies that

$$\xi^{\mathrm{T}}(t)\Upsilon_{11}\xi(t) \leqslant 0. \tag{26}$$

Define the following function:

$$J_1 = \dot{V}(t) - \alpha V(t) - \alpha w^{\mathrm{T}}(t)w(t). \tag{27}$$

Combining (8)–(27) results

$$J_1 \leqslant \xi^{\mathrm{T}}(t)\hat{\Theta}\xi(t) < 0.$$

Then $\alpha > 0$, so the inequality $J_1 > 0$ holds, and we can get

$$\dot{V}(t) < \alpha V(t) + \alpha w^{\mathrm{T}}(t)w(t).$$

Multiplication by $\mathrm{e}^{-\alpha t}$ of both sides yields

$$\mathrm{e}^{-\alpha t}V\big(x(t)\big) < \alpha \mathrm{e}^{-\alpha t}w^{\mathrm{T}}(t)w(t). \tag{28}$$

Integrating two sides of inequality (28) from 0 to t, $t \in [0, T]$, we have

$$\mathrm{e}^{-\alpha t}\big[V\big(x(t)\big) - V\big(x(0)\big)\big] < \alpha \int_0^t \mathrm{e}^{-\alpha s}w^{\mathrm{T}}(s)w(s)\,\mathrm{d}s,$$

which implies that

$$\begin{aligned} V\big(x(t)\big) &< \mathrm{e}^{\alpha t}V\big(x(0)\big) + \alpha \mathrm{e}^{\alpha t}\int_0^t \mathrm{e}^{-\alpha s}w^{\mathrm{T}}(s)w(s)\,\mathrm{d}s. \\ &< \mathrm{e}^{\alpha t}V\big(x(0)\big) + \delta \mathrm{e}^{\alpha T}\big(1 - \mathrm{e}^{\alpha T}\big). \end{aligned} \tag{29}$$

On the other hand, we define

$$\bar{P} = R^{-1/2}PR^{-1/2}, \qquad \bar{Q}_i = R^{-1/2}Q_iR^{-1/2}, \quad i = 1, 2, \dots 10.$$

From (29) and by the same method used in [27] it follows that

$$\begin{aligned} V\big(x(t)\big) &\leqslant \mathrm{e}^{\alpha T}\bigg(\lambda_2 + h\lambda_3 + h\lambda_4 + d\lambda_5 + h\lambda_6 + \frac{h^2}{2}\lambda_7 + \frac{h^2}{2}\lambda_8 \\ &\quad + \frac{h^3}{6}\lambda_9 + \frac{h^3}{6}\lambda_{10} + \frac{h^4}{24}\lambda_{11} + \frac{h^5}{120}\lambda_{12}\bigg)c_1 + \delta \mathrm{e}^{\alpha T}\big(1 - e^{\alpha T}\big), \\ &\leqslant e^{\alpha T}\big[\varLambda c_1 + \delta\big(1 - \mathrm{e}^{\alpha T}\big)\big], \end{aligned} \tag{30}$$

where

$$\begin{aligned} \varLambda = \lambda_2 + h\lambda_3 + h\lambda_4 + d\lambda_5 + h\lambda_6 \\ + \frac{h^2}{2}\lambda_7 + \frac{h^2}{2}\lambda_8 + \frac{h^3}{6}\lambda_9 + \frac{h^3}{6}\lambda_{10} + \frac{h^4}{24}\lambda_{11} + \frac{h^5}{120}\lambda_{12}. \end{aligned}$$

On the other hand,

$$V\big(x(t)\big) \geqslant \lambda_{\min}(\bar{P})x^{\mathrm{T}}(t)Rx(t) = \lambda_1 x^{\mathrm{T}}(t)Rx(t). \tag{31}$$

Combining (30) and (31), we obtain

$$x^{\mathrm{T}}(t)Rx(t) \leqslant \frac{\mathrm{e}^{\alpha T}[\varLambda c_1 + \delta(1 - e^{\alpha T})]}{\lambda_1}.$$

Condition (7) implies that for all $t \in [0, T]$, $x^{\mathrm{T}}(t)Rx(t) < c_2$. This completes the proof. Hence, by the Definition 1 the considered system (4)–(5) is finite-time bounded. □

3.2 Finite-time stability analysis

Remark 1. If the external disturbance $w(t) = 0$, system (4)–(5) becomes

$$\dot{x}(t) = -\mathbf{A}x(t) + \mathbf{B}f\bigl(x(t)\bigr) + \mathbf{C}f\bigl(x\bigl(t-h(t)\bigr)\bigr) + \mathbf{D}\dot{x}(t-d), \tag{32}$$

$$x(\theta) = \phi(\theta), \quad \theta \in [-H_d, 0]. \tag{33}$$

We define

$$\begin{aligned}
\xi^{\mathrm{T}}(t) = \mathrm{col}\Biggl[&x^{\mathrm{T}}(t),\ x^{\mathrm{T}}\bigl(t-h(t)\bigr),\ x^{\mathrm{T}}(t-h),\ f^{\mathrm{T}}\bigl(x(t)\bigr),\ f^{\mathrm{T}}\bigl(x\bigl(t-h(t)\bigr)\bigr),\ \dot{x}(t),\\
&\dot{x}(t-d),\ \int\limits_{t-h(t)}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s,\ \int\limits_{t-h}^{t-h(t)} x^{\mathrm{T}}(s)\,\mathrm{d}s,\ \int\limits_{-h(t)}^{0}\int\limits_{t+\theta}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta,\\
&\int\limits_{-h}^{-h(t)}\int\limits_{t+\theta}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta,\ \int\limits_{-h(t)}^{0}\int\limits_{\nu}^{0}\int\limits_{t+\theta}^{t} x^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\nu,\\
&\int\limits_{-h}^{-h(t)}\int\limits_{\nu}^{0}\int\limits_{t+\theta}^{t} \dot{x}^{\mathrm{T}}(s)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}\nu\Biggr],
\end{aligned}$$

$$e_k = \mathrm{col}\bigl[0_{(k-1)n\times n},\ I_n,\ 0_{(k-1)n\times n}\bigr], \quad k = (1, 2, \dots 13).$$

Corollary 1. *For given scalars h, μ, α, c_1, c_2 and T, the neural networks* (32)–(33) *is finite-time stable if there exist positive symmetric matrices P, Q_i $(i = 1, 2, \dots, 10)$, any diagonal matrices U, S and matrices N_1, N_2 with appropriate dimensions such that the following LMIs holds*:

$$\Upsilon_m - e_1\alpha P e_1^{\mathrm{T}} < 0,$$

where

$$\Upsilon_m = \Upsilon_{b1} + \Upsilon_2 + \Upsilon_3 + \Upsilon_4 + \Upsilon_5 + \Upsilon_6 + \Upsilon_7 + \Upsilon_8 + \Upsilon_9 + \Upsilon_{b10} + \Upsilon_{11},$$

$$\lambda_1 c_2 \mathrm{e}^{-\alpha T} > \Lambda c_1,$$

where

$$\Upsilon_{b1} = -2e_1 P\mathbf{A}e_1^{\mathrm{T}} + 2e_1 P\mathbf{B}e_4^{\mathrm{T}} + 2e_1 P\mathbf{C}e_5^{\mathrm{T}} + 2e_1 P\mathbf{D}e_7^{\mathrm{T}},$$

$$\Upsilon_{b10} = 2[e_1 + e_6](N_1 + N_2)\bigl[-\mathbf{A}e_1^{\mathrm{T}} + \mathbf{B}e_4^{\mathrm{T}} + \mathbf{C}e_5^{\mathrm{T}} + \mathbf{D}e_7^{\mathrm{T}} - e_6\bigr]$$

with $\Upsilon_2, \dots \Upsilon_9, \Upsilon_{11}$ as described in Theorem 1.

Proof. The proof is similar to that of Theorem 1, so it is omitted here. □

3.3 Finite-time passivity analysis

In this section, we investigate the finite-time passivity analysis for the following delayed neutral-type neural networks:

$$\begin{aligned} \dot{x}(t) - \mathbf{D}\dot{x}(t-d) &= -\mathbf{A}x(t) + \mathbf{B}f\big(x(t)\big) + \mathbf{C}f\big(x\big(t-h(t)\big)\big) + \mathbf{E}w(t), \\ y(t) &= \mathbf{G}f\big(x(t)\big) + \mathbf{H}w(t), \qquad (34) \\ x(\theta) &= \phi(\theta), \quad \theta \in [-H_d, 0]. \qquad (35) \end{aligned}$$

Theorem 2. *For given scalars h, μ, δ, h, α, β, c_1, c_2 and T, the neural networks* (34)–(35) *is finite-time passive if there exist positive symmetric matrices P, Q_i ($i = 1, 2, \dots, 10$), any diagonal matrices U, S and matrices N_1, N_2 with appropriate dimensions such that the following LMIs holds*:

$$\Phi = \Upsilon_\kappa - e_1 \alpha P e_1^{\mathrm{T}} - e_4 G^{\mathrm{T}} e_{14} - e_{14}\big(\beta I - H^{\mathrm{T}} - H\big)e_{14}^{\mathrm{T}} < 0, \qquad (36)$$

where $\Upsilon_\kappa = \Upsilon_1 + \Upsilon_2 + \dots + \Upsilon_{11}$,

$$\lambda_1 c_2 \mathrm{e}^{-\alpha T} > \Lambda c_1 + \delta\big(1 - \mathrm{e}^{\alpha T}\big) \qquad (37)$$

with $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_{11}$ as described in Theorem 1 *with respect to* (c_1, c_2, R, δ, T).

Proof. By using LKF and the similar lines as that in the proof of Theorem 1,

$$J_2 = \dot{V}\big(x(t)\big) - \big[\alpha V\big(x(t)\big) - \beta w^{\mathrm{T}}(t)w(t) + 2w^{\mathrm{T}}(t)y(t)\big].$$

Since $J_2 < 0$,

$$\dot{V}\big(x(t)\big) - \big[\alpha V\big(x(t)\big) - \beta w^{\mathrm{T}}(t)w(t) + 2w^{\mathrm{T}}(t)y(t)\big] < 0,$$

we can obtain

$$\xi^{\mathrm{T}}(t)\Phi\xi(t) < 0.$$

Hence,

$$\dot{V}\big(x(t)\big) - \alpha V\big(x(t)\big) < 2w^{\mathrm{T}}(t)y(t) - \beta w^{\mathrm{T}}(t)w(t).$$

Multiplying the above inequality by $\mathrm{e}^{-\alpha T}$ and integrating between 0 and T, we have

$$\begin{aligned} \mathrm{e}^{\alpha T} V\big(x(t)\big) &< 2\int_0^{\mathrm{T}} e^{-\alpha t} w^{\mathrm{T}}(t)y(t)\,\mathrm{d}t - \beta\int_0^{\mathrm{T}} \mathrm{e}^{-\alpha t} w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t \\ &< 2\int_0^{\mathrm{T}} e^{-\alpha t} w^{\mathrm{T}}(t)y(t)\,\mathrm{d}t - \beta \mathrm{e}^{-\alpha T}\int_0^{\mathrm{T}} w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t, \end{aligned}$$

which implies that

$$V\big(x(t)\big) < 2\mathrm{e}^{\alpha T}\int_0^{\mathrm{T}} e^{-\alpha t} w^{\mathrm{T}}(t)y(t)\,\mathrm{d}t - \beta\int_0^{\mathrm{T}} w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t.$$

Therefore,

$$\gamma \int_0^{\mathrm{T}} w^{\mathrm{T}}(t)w(t)\,\mathrm{d}t < \int_0^{\mathrm{T}} w^{\mathrm{T}}(t)y(t)\,\mathrm{d}t.$$

where $\gamma = \beta e^{-\alpha T}/2$. This completes the proof. By Definition 3 the considered system (34)–(35) is finite-time passive. $\square$

Definition 4. System (34)–(35) is called passive if there exists a scalar $\gamma \geqslant 0$ such that

$$\gamma \int_0^{\mathrm{T}} w^{\mathrm{T}}(s)w(s)\,\mathrm{d}s \leqslant 2\int_0^{\mathrm{T}} y^{\mathrm{T}}(s)w(s)\,\mathrm{d}s$$

for all solution of (34)–(35) with $x(0) = 0$.

Remark 2. By Definition 4 we can obtain a passivity criterion of network (34)–(35) by the similar method of the proof of Theorem 2. The following LMI holds:

$$\Phi_1 = \Upsilon_\kappa - e_4 G^{\mathrm{T}} e_{14} - e_{14}\bigl(\beta I - H^{\mathrm{T}} - H\bigr)e_{14}^{\mathrm{T}} < 0,$$

where $\Upsilon_\kappa = \Upsilon_1 + \Upsilon_2 + \dots + \Upsilon_{11}$ with $\Upsilon_1, \Upsilon_2, \dots, \Upsilon_{11}$ described in Theorem 1.

Remark 3. From (1) we write the following neural networks without neutral delay, output vector and external disturbance (i.e., $D = 0$, $y(t) = 0$ and $w(t) = 0$):

$$\dot{x}(t) = -\mathbf{A}x(t) + \mathbf{B}f\bigl(x(t)\bigr) + \mathbf{C}f\bigl(x\bigl(t - h(t)\bigr)\bigr). \tag{38}$$

Corollary 2. *For given scalars h, μ the neural networks* (38) *is asymptotically stable if there exist positive symmetric matrices P, Q_i* ($i = 1, 2, 4 \dots, 10$)*, any diagonal matrices U, S and matrices N_1, N_2 with appropriate dimensions such that the following LMIs holds*:

$$\Upsilon_n < 0,$$

where

$$\begin{aligned}
\Upsilon_n &= \Upsilon_{\epsilon 1} + \Upsilon_2 + \Upsilon_{\epsilon 3} + \Upsilon_4 + \Upsilon_5 + \Upsilon_6 + \Upsilon_7 + \Upsilon_8 + \Upsilon_9 + \Upsilon_{\epsilon 10} + \Upsilon_{11},\\
\Upsilon_{\epsilon 1} &= -2e_1 P\mathbf{A}e_1^{\mathrm{T}} + 2e_1 P\mathbf{B}e_4^{\mathrm{T}} + 2e_1 P\mathbf{C}e_5^{\mathrm{T}}, \qquad \Upsilon_{\epsilon 3} = e_4 Q_4 e_4^{\mathrm{T}} - e_5 Q_4 e_5^{\mathrm{T}},\\
\Upsilon_{\epsilon 10} &= 2[e_1 + e_6](N_1 + N_2)\bigl[-\mathbf{A}e_1^{\mathrm{T}} + \mathbf{B}e_4^{\mathrm{T}} + \mathbf{C}e_5^{\mathrm{T}} - e_6\bigr],
\end{aligned}$$

Proof. Consider the same Lyapunov–Krasovskii functional from Theorem 1, except $\int_{t-d}^{t} \dot{x}^{\mathrm{T}}(s)Q_3\dot{x}(s)\,\mathrm{d}s$. The proof is similar to that of in Theorem 1. $\square$

4 Numerical examples

In this section, five examples are presented to illustrate the main theoretical results proposed. In the first example, we consider neutral-neural networks without output vectors. The delayed neutral-type neural networks with output vectors is provided in the second example. In the third to five examples, comparison results are given.

Example 1. Consider the neutral-type neural networks described in (4)–(5) with the following matrix parameters:

$$\mathbf{A} = \begin{bmatrix} 3.6 & 0 \\ 0 & 3.6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} -0.34 & 0 \\ -0.1 & -0.1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0.1 & 0.2 \\ -0.15 & -0.18 \end{bmatrix},$$
$$\mathbf{D} = \begin{bmatrix} -0.5 & 0 \\ 0.2 & 0.5 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0.41 & 0.5 \\ 0.69 & -0.31 \end{bmatrix}.$$

The values of delays c_1, T, δ are given as follows:

$$h = 1.3, \qquad d = 0.2, \qquad c_1 = 0.4, \qquad T = 6,$$
$$\alpha = 0.10, \qquad \delta = 0.005, \qquad \mu = 0.5, \qquad \delta = 0.10,$$

and $\Delta_1 = \operatorname{diag}\{0, 0\}$, $\Delta_2 = \operatorname{diag}\{1, 1\}$. Solving LMIs (6)–(7) by the toolbox of MATLAB, we can obtain $c_2 = 26.8320$, and hence, the considered neutral system is finite-time bounded.

Example 2. Consider the neutral-type neural networks (34)–(35) with the following matrix parameters:

$$\mathbf{A} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1.1 & 0.2 \\ -0.1 & -1.1 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0.2 & 0 \\ 0.2 & -0.2 \end{bmatrix},$$
$$\mathbf{D} = \begin{bmatrix} -0.5 & 0.3 \\ 0.2 & 0.1 \end{bmatrix}, \qquad \mathbf{E} = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & -0.14 \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} 0.1 & 0.2 \\ -0.01 & 0.4 \end{bmatrix},$$
$$\mathbf{H} = \begin{bmatrix} 0.2 & -0.6 \\ 0.3 & 0.2 \end{bmatrix},$$

Let

$$h = 2.4, \qquad d = 1.2, \qquad c_1 = 0.5, \qquad c_2 = 6, \qquad T = 5,$$
$$\alpha = 0.10, \qquad \delta = 1, \qquad \mu = 0.9, \qquad \beta = 0.2,$$

and $\Delta_1 = \operatorname{diag}\{0, 0\}$, $\Delta_2 = \operatorname{diag}\{0.5, 0.9\}$. Solving LMI (36)–(37), we can obtain $\gamma = 0.0779$. Thus system (34)–(35) is finite-time passive.

Example 3. Consider the neural networks (34)–(35) with the following parameters:

$$\mathbf{A} = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix},$$
$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \Delta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{D} = \mathbf{E} = \mathbf{H} = 0,$$

and $f(x) = (|x + 1| - |x - 1|)/2$. In Table 1, the results of the upper bounds of time delay for guaranteeing passivity are compared with the previous results. The maximum allowable upper bounds of neural networks (34)–(35) when $0 \leqslant h(t) \leqslant 42.2438$ and $\mu \leqslant 0.5$ are given in Table 1.

Table 1. Maximum allowable bound h for different values μ in Example 3. (m is delay partitioning number.)

μ	0.5	0.7	0.9	Unknown
[34]	0.7230	0.6814	0.6791	0.6791
[36]	1.3752	1.3036	1.3027	1.3027
[13] ($m = 3$)	35.3121	9.0128	4.7568	4.7368
Remark 2	42.2438	14.9210	7.0010	5.9908

Example 4. Consider the neural networks with time-varying delays (38) with matrix parameters taken from [8–10,12,22,25,29,30,37]. In Table 2, the maximum allowable upper bounds of the time varying-delays h for different values μ are given. Also, we obtained less conservative results than the results in [8–10, 12, 22, 25, 29, 30, 37]. The response of the state trajectories for the delayed neural networks (38), which is asymptotically stable, is shown in Fig. 1.

Table 2. Maximum allowable bounds of h for different values μ in Example 4.

μ	[10]	[8]	[9]	[37]	[12]	[25]	[29]	[30]	[22]	Corollary 2
0.1	3.27	3.27	3.30	3.35	3.75	3.70	3.91	3.91	3.97	6.46
0.5	2.15	2.22	2.53	2.59	2.73	3.12	2.79	2.80	2.86	5.21
0.9	1.31	1.58	2.08	2.13	2.27	2.59	2.33	2.55	2.63	5.01

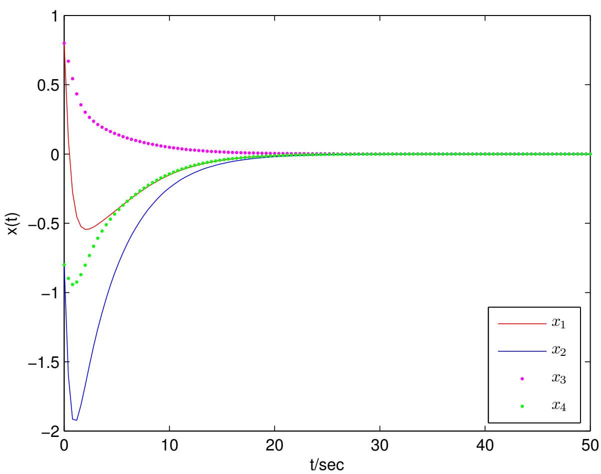

Figure 1. State trajectories of the system (38) with time varying delay $\mu = 0.5$ in Example 4.

Example 5. Consider the delayed neural networks (38) with following parameters:

$$\mathbf{A} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.7 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$

$$\varDelta_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \varDelta_2 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.8 \end{bmatrix}.$$

Table 3. Maximum allowable bounds of h for different values μ in Example 5.

Method	0.4	0.45	0.50	0.55	0.60	0.65
[24]	3.9972	3.2760	3.0594	2.9814	–	–
[25]	4.3814	3.6008	3.3377	3.2350	–	–
[31]	5.2420	4.4301	4.1055	3.9231	–	–
[22]	5.3165	4.5312	4.2581	4.0569	–	–
Corollary 2	6.4380	5.9025	5.2100	5.0301	4.8908	4.5268

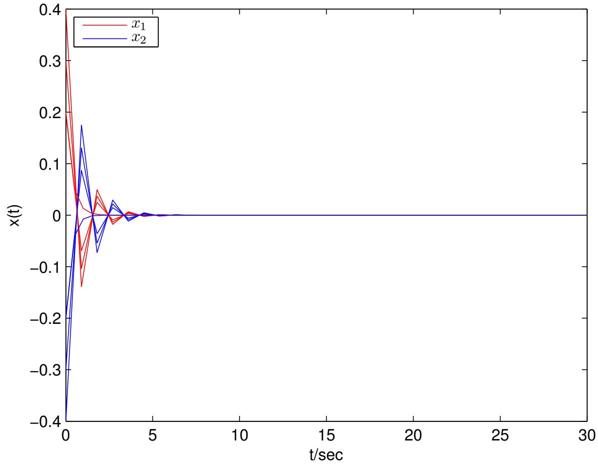

Figure 2. State trajectories of the system (38) with time varying delay $\mu = 0.50$ in Example 5.

In Table 3, for different values of μ we get the maximum allowable upper bounds of h. The results are compared with the results in [22, 24, 25, 31]. The response of the state trajectories for the delayed neural networks (38), which is asymptotically stable, is shown in Fig. 2.

5 Conclusion

In this article, we investigated the finite-time passivity of neutral-type neural networks with time-varying delays. By applying the Jensen-type integral inequality technique a delay-dependent criterion is developed to achieve the finite-time boundedness and finite-time stability for the neutral-type neural networks. Based on our proposed multiple integral forms of the Wirtinger-based integral inequality and the auxiliary function-based integral inequalities approach for high-order case, a novel delay-dependent condition is established to achieve the finite-time passivity neural networks. Numerical examples shows the effectiveness of the theoretical results and superiority to the existing results. Thus, the proposed technique can be extendable to spatial finite-time stabilization or synchronization: finite/fixed-time pinning synchronization of complex networks with stochastic disturbances [17]; discontinuous observers design for finite-time consensus of multi-agent systems with external disturbances [16]; nonsmooth finite-time synchronization of switched coupled neural networks [15]. This will occur in the near future.

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