Nonlinear Analysis: Modelling and Control, Vilnius, IMI, 2000, No 5

Blending Cylinders and Cones using Canal Surfaces

M.Kazakevičiūtė, R.Krasauskas
Faculty of Mathematics and Informatics, Vilnius University
Naugarduko 24, 2600 Vilnius, Lithuania
margarita.kazakeviciute@telecom.lt, rimvydas.krasauaskas@maf.vu.lt

Received: 29.07.2000
Accepted: 11.10.2000

Abstract

There is reviewed the construction of a rational blending surface between cylinders and cones in some interlocation cases. This surface is constructed as a patch of rolling ball envelope, i.e. as a patch of tangent canal surface of rational-variable radius. This construction defines rational parameterization of a blending surface. The constructed surface is Laguerre invariant.

1 Introduction

The continuous blending problem of two surfaces is frequently being considered in geometric design (CAGD). Even in such simple blending cases of spheres, circular cylinders and cones, i.e. natural quadrics, the blending surface construction may be quite complex. In this paper the construction of rational blending surface for two natural quadrics in general position will be reviewed. It will be described more detailed in the case, when two cylinders or cylinder and cone intersect by two closed curves.

The blending surface is constructed as an envelope of rational variable rolling ball, i.e. as a canal surface with a rational radius. A tangent ball rolls along the intersection curve of two quadrics, determining continuous blending surface. The radius of rolling ball may be constant or changes according to some law. The center of ball follows some spatial curve. Rational blending surface is constructed according to common natural quadrics blending method [1] and applying the cyclographic model of Laguerre geometry [2], [3].

2 Short Review of cyclographic Model of Laguerre Geometry

Laguerre geometry analyses oriented spheres and oriented contacts between them. Several geometric design problems may be solved in a simple way using Laguerre geometry. The main element of Laguerre geometry is an oriented sphere in Euclidean space $\mathbf{R}^3$. The orientation of a sphere is determined by a signed radius. There may be determined a correspondence between the oriented sphere with center $(m_1; m_2; m_3) \in \mathbf{R}^3$, radius r and the point $(m_1; m_2; m_3; r) \in \mathbf{R}^4$. Two spheres are the oriented contact if they are tangent and their signed radii coincide in the point of contact. This kind of relationship is called a cyclographic model.

The Euclidean metric on $\mathbf{R}^3$ has a natural pseudo-euclidean extension to the whole $\mathbf{R}^4$, which is called a PE-metric and is defined as follows:

$$\|u\|_{PE} = \sqrt{\langle u;u \rangle_{PE}} \text{ , if } \langle u;u \rangle_{PE} \geq 0,$$

here $\langle u;v \rangle_{PE} = u_1v_1 + u_2v_2 + u_3v_3 - u_4v_4$ is a PE-scalar product of vectors $\boldsymbol{u}=(u_1;u_2;u_3;u_4)$ and $\boldsymbol{v}=(v_1;v_2;v_3;v_4)$.

PE-distance between two points $C=(c,r_c)$, $D=(d,r_d) \in \mathbf{R}^4$ equals to a tangential distance between corresponding spheres (if one sphere does not contain the other inside):

$$\|C-D\|_{PE} = \sqrt{\langle C;D \rangle_{PE}} = \sqrt{\langle c;d \rangle - r_c r_d}.$$

If $\|C-D\|_{PE} = 0$ then spheres are in oriented contact.

According to this definition all the lines in space $\mathbf{R}^4$ may be distributed into three cases (denote $\boldsymbol{l}$ as a direction vector of a line):

if $<l;l>_{PE} > 0$, then the line is hyperbolic;

if $<l;l>_{PE} = 0$, then the line is parabolic;

if $<l;l>_{PE} < 0$, then the line is elliptic.

The Laguerre transformation is defined as a special kind of affine transformation as follows:

$$x \mapsto \begin{pmatrix} 1 & 0 \\ t & \lambda\ A \end{pmatrix} x$$

here A is a pseudo-orthogonal 4×4 matrix, i.e. $A^T E_{PE} A = E_{PE}$, $E_{PE} = diag(1, 1, 1, -1)$; vector $\boldsymbol{t} \in \mathbf{R}^2$ describes the translation of affine mapping; $\lambda \neq 0$ is a coefficient of similarity.

3 Principles of Canal Blending Surface Construction

According to the Laguerre geometry, shortly described above, the ways of two quadrics interlocation may be distributed into some groups. Any interlocation of two quadrics can be ascribed to one of the groups directly or after applying appropriate Laguerre transformation.

Any two cylinders or cones correspond to two hyperbolic lines L_1 and L_2 in $\mathbf{R}^4$. There will be analyzed the case when L_1 and L_2 are skew lines, since otherwise if L_1 and L_2 intersect Dupine cyclides may be used for blending [4]. Thus two hyperbolic lines will be considered instead of two quadrics.

According to the signatures of affine span of $L_1 \cup L_2$ there may be pointed out such interlocation cases:

(++)+ - case of two skew cylinders, which do not intersect, or intersect by one closed curve;

(++)– - case of two cylinders, which intersect by two closed curves;

(+–)+ - case of cylinder and cone, which intersect by two closed curves;

here the signs in brackets correspond to two-dimensional subspace, generated by directional vectors of L_1 and $L_{2,}$, with response to PE-metrics.

In this paper there will be considered the construction of blending surfaces in (++)– and (+–)+ cases, also the rational parameterization of the surfaces will be presented [1]. In addition the construction of blending surface for various quadric types will be reviewed.

Each cylinder can be transformed into cone applying PE-rotation and vice versa. Also two (++)– interlocated cylinders (cylinder and cone) with skew axes can be transformed into two cylinders with intersecting axes applying PE-translation and vice versa. Consequently combining these Laguerre transformations on natural quadrics and corresponding blending surface with respect to the simplest case of two cylinders with intersecting axes we may obtain blending surfaces for various natural quadric types in more complicated blending cases.

3.1 Construction of Blending Surface when Quadrics are in (++)–type Interlocation

Construction of blending surface when quadrics are in (++)– type interlocation will be described more detailed in cases of blending two cylinders with intersecting or skew axes and in case of cylinder and cone.

3.1.1 Blending of two Cylinders with Intersecting Axes

Consider two perpendicular cylinders Q_1 of radius r_1 and Q_2 of radius r_2 with intersecting axes. They form (++)– type interlocation. Assume, that the axis of Q_1 coincides with coordinate axis x_1, and axis of Q_2 with coordinate axis x_2:

$$Q_1=(x_0, x_1, 0, 0, r_1)^T \quad , Q_2=(x_0, 0, x_2, 0, r_2)^T.$$

These two cylinders in $\mathbf{R}^3$ correspond to two skew lines in $\mathbf{R}^4$, which lay on parallel hyper planes $x_4 = r_1$ and $x_4 = r_2$.

On each of the lines there is a point, the square PE-distance between which is negative. Here we see, that such an interlocation of cylinders determines two directional vectors of hyperbolic lines in $\mathbf{R}^4$ with signatures (++) and one vector with signature –.

Determine all the spheres, which are tangent to both cylinders. Further to cyclographic model of Laguerre geometry, the set of all tangent spheres in three dimensional space $\mathbf{R}^3$ correspond to the surface of centers T^4 in four dimensional space $\mathbf{R}^4$, which is described by equations as follows:

$$T^4\!: \begin{cases} {x_2}^2 + {x_3}^2 = (x_4 - x_0 r_1)^2 \\ {x_1}^2 + {x_3}^2 = (x_4 - x_0 r_2)^2 \end{cases} .$$

Consequently the corresponding parameterization of T^4 is deduced:

$$T^4(s,t) = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/4(s^2u + t^2u - t^2s^2 - u^2) \\ 1/2(t^2u - s^2u) \\ 1/2(u^2 - t^2s^2) \\ tsu \\ s^2u + t^2u - 1/2t^2s^2 - 1/2u^2 \end{pmatrix} . \qquad (1)$$

This surface of centers (Fig. 1 shows the projection of T^4 into space $\mathbf{R}^3$) consists of 4 corner parts, the points of which correspond to spheres, which are exterior tangent to both cylinders, and one part in the middle, the points of which correspond to spheres, which are interior tangent to both cylinders. In addition this surface has 4 singular points and 4 lines. The coordinate lines of T^4 are of degree 2.

The rational curve on the surface T^4 may be defined by ascribing some rational parameterization to variables t and s.

For example, define curve $SS(v)$ by substituting parameterization:

$t = \dfrac{v}{\sqrt{1+v^2}}$, $s = \dfrac{1}{\sqrt{1+v^2}}$ and $u = 2$ to *(1)*.

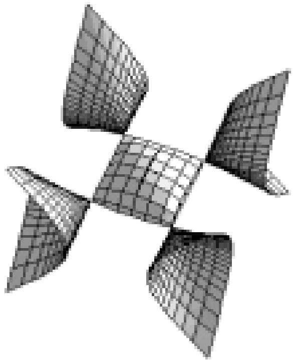

Fig 1. Surface of centers, (++)– case.

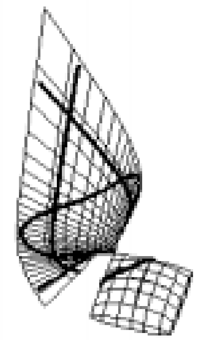

Fig. 2. Curves of degree 2 and 4 on surface of centers, (++)– case.

Corresponding curve of degree 4 (Fig.2) is represented as follows:

$$SS(v) = \left(x_0(v); x_1(v); x_2(v); x_3(v); x_4(v)\right)^T = \left(-\frac{1}{2} - \frac{v^2}{4(1+v^2)^2};\ \frac{v^2-1}{1+v^2};\ 2 - \frac{v^2}{2(1+v^2)^2};\ \frac{2v}{1+v^2};\ \frac{-v^2}{2(1+v^2)^2}\right)^T$$

Degree 4 is the lowest possible degree for the blending proper closed curve on the surface T^4, since closed curve must be of even degree and must wrap around the axis of one quadric. There are closed curves of degree 2 on T^4, though they wrap around the axes of both quadrics.

This curve *SS(v)* corresponds to a set of spheres *S(v)* in $\mathbf{R}^3$. The envelope surface of this set of spheres is a canal surface (Fig. 3), which is C^1 continuously tangent to both cylinders. The canal surface touches each cylinder along closed tangency curve. That curves bound the canal surface patch, which is the resulting blending canal surface of the cylinders.

Following task is to find the arcs of circles on each sphere, which are tangent to cylinders. First we determined the closed tangency curves on each cylinder. There is a circle on each sphere from the set *S(v)*, which is tangent to both cylinders. This circle passes through the points P_1 and P_2, in which sphere *S(v)* touches correspondingly Q_1 and Q_2:

$$P_1(v) = \left(\frac{x_1(v)}{x_0(v)},\ (1-p_1)\frac{x_2(v)}{x_0(v)},\ (1-p_1)\frac{x_3(v)}{x_0(v)}\right),\quad p_1 = \frac{x_4(v)}{x_4(v) + \operatorname{sgn}(x_0(v))\ x_0(v)\ r_1}.$$

$$P_2(v) = \left((1-p_2)\frac{x_1(v)}{x_0(v)},\ \frac{x_2(v)}{x_0(v)},\ (1-p_2)\frac{x_3(v)}{x_0(v)}\right),\quad p_2 = \frac{x_4(v)}{x_4(v) + \operatorname{sgn}(x_0(v))\ x_0(v)\ r_2}.$$

These points form corresponding tangency curves on each cylinder (Fig. 4).

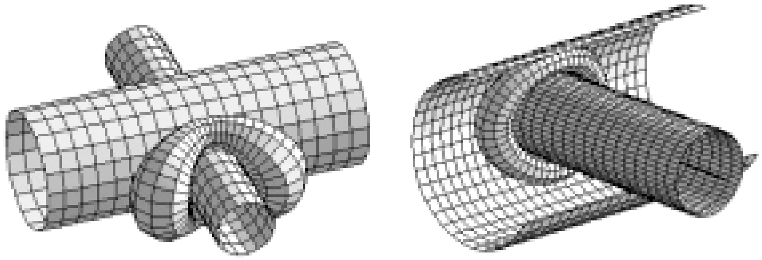

Fig. 3. Tangent canal surface, (++)– case.

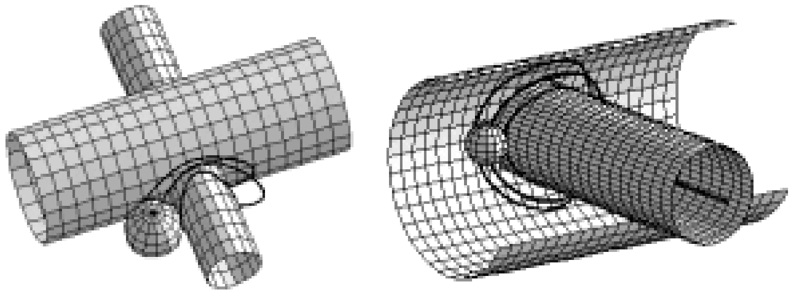

Fig. 4. Center curves and corresponding tangency curves on cylinders, (++)– case.

Next we determine tangent circles on each sphere. Sphere $S(v)$ corresponds to a point $(x_1(v)/x_0(v); x_2(v)/x_0(v); x_3(v)/x_0(v); x_4(v)/x_0(v))$ on a curve of centers $SS(v)$ in $\mathbf{R}^4$. The tangent line of the curve $SS(v)$ in that point in space $\mathbf{R}^4$ is represented as a cone in $\mathbf{R}^3$, which is tangent to sphere $S(v)$. That cone touches sphere $S(v)$ along the circle. This circle coincides with the tangent circle on the sphere we are looking for, i.e. the circle, which passes through the points P_1 and P_2 on Q_1 and Q_2 and has a center $C(v)$.

Finally denote $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ as vectors, starting in center $C(v)$, and ending in corresponding points P_1 and P_2, and φ as a half of angle between $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$. Then the arc of tangent circle, trimmed by the points P_1 and P_2, can be expressed in form of Bezier curve in $\mathbf{RP}^3$ as follows (denote $w_0 = cos^2(\varphi)$):

$$DD(v,t) = (1-t)^2\begin{pmatrix} 1 \\ e_{11} \\ e_{12} \\ e_{13} \end{pmatrix} + 2t(1-t)\begin{pmatrix} w_0 \\ (e_{11}+e_{21})/2 \\ (e_{12}+e_{22})/2 \\ (e_{13}+e_{23})/2 \end{pmatrix} + t^2\begin{pmatrix} w_0 \\ w_0 e_{21} \\ w_0 e_{22} \\ w_0 e_{23} \end{pmatrix}.$$

Thus the tangent arc, trimmed by the points P_1 and P_2, can be expressed in form of rational Bezier curve in $\mathbf{R}^3$ as follows:

$$D(v,t)=\left(\frac{DD_1(v,t)}{DD_0(v,t)}+x_C(v)\quad \frac{DD_2(v,t)}{DD_0(v,t)}+y_C(v)\quad \frac{DD_3(v,t)}{DD_0(v,t)}+z_C(v)\right).$$

The set of all tangent arcs *D(v,t)* form rational blending surface of Q_1 and Q_2 (Fig. 5).

There are the main points of rational canal blending surface construction in the case of two perpendicular cylinders with intersecting axes.

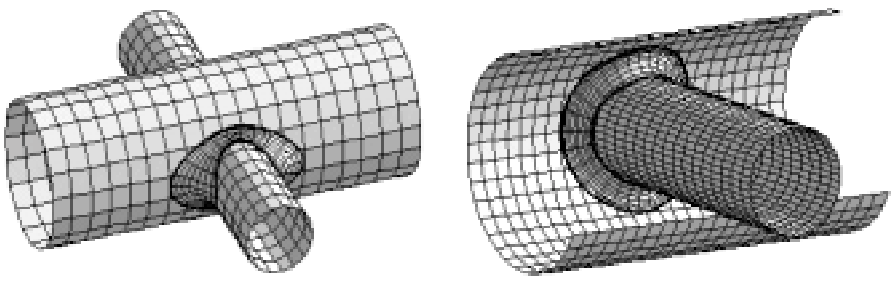

Fig. 5. Rational blending canal surface in (++)– case, when cylinders are perpendicular, and their axes intersect

The same blending surface construction holds for the general case of the two cylinders, when their axes intersect with some angle α (Fig.6).

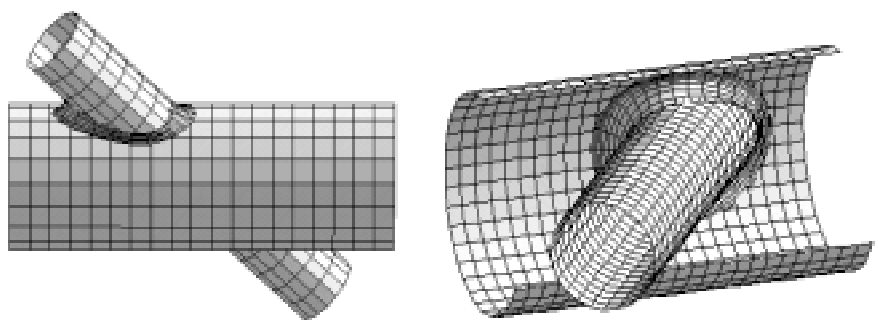

Fig. 6. Rational blending canal surface in (++)– case, when their axes intersect

3.1.2 Blending of two Cylinders with Skew Axes

Consider two perpendicular cylinders Q_1 of radius r_1 and Q_2 of radius r_2 with skew axes. They form (++)– type interlocation. Assume, that the axis of Q_1 is parallel to coordinate axis x_1 and is vertically above it in distance q, and axis of Q_2 is parallel to coordinate axis x_2 and is vertically below it in distance q.

$$Q_1=(x_0, x_1, 0, q, r_1)^T \quad , Q_2=(x_0, 0, x_2, -q, r_2)^T.$$

Applying Laguerre transformation, i.e. PE-rotation, to Q_1 and Q_2 one may obtain two perpendicular cylinders with intersecting axes:

$$\overline{\overline{Q}}_1 = \widetilde{A} Q_1 = (x_0,\ x_1,\ 0,\ q\, ch\beta + r_1 sh\beta,\ q\, sh\beta + r_1 ch\beta)^T,$$

$$\overline{\overline{Q}}_2 = \widetilde{A} Q_2 = (x_0,\ 0,\ x_2,\ -q\, ch\beta + r_2 sh\beta,\ -q\, sh\beta + r_2 ch\beta)^T,$$

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & ch\beta & sh\beta \\ 0 & 0 & 0 & sh\beta & ch\beta \end{pmatrix},$$

here $ch\ \beta$ and $sh\ \beta$ are hyperbolic cosine and sine, matrix $\widetilde{A}$ represents PE-rotation, rotation angle β has to satisfy equality:

$$th\beta = \frac{2q}{r_2 - r_1}.$$

The construction of the rational blending surface *D(v,t)* for two cylinders with intersecting axes is described in paragraph 2.1.1. The original blending surface of two cylinders with skew axes is obtained applying inverse PE-transformation to *D(v,t)*:

$$\overline{\overline{D}}(v,t) = \widetilde{A}^{-1} D(v,t).$$

This inverse transformation maps rational blending surface *D(v,t)* to the rational blending surface of two cylinders with skew axes (Fig. 7).

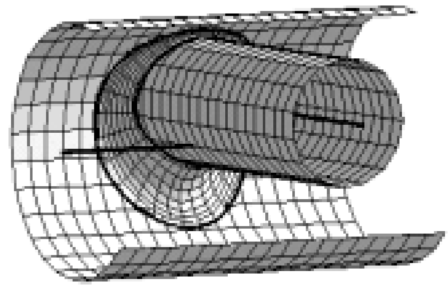

Fig. 7. Rational blending surface of two cylinders with skew axes, (++)–case.

3.1.3 Blending of Cylinder and Cone with Intersecting Axes

Consider perpendicular cylinder Q_1 of radius r_1 and cone Q_2 with intersecting axes. They form (++)– type interlocation. Assume, that the axis of Q_1 coincides with coordinate axis x_1, and axis of Q_2 with coordinate axis x_2.

$$Q_1=(x_0, x_1, 0, 0, r_1)^T \quad , Q_2=(x_0, 0, (x_2 - a), 0, (x_2 - a) \sin\varphi)^T,$$

here a is some positive or negative constant, φ equals to half of cone angle.

Applying Laguerre transformation, i.e. PE-rotation, to Q_1 and Q_2, one may obtain two perpendicular cylinders with intersecting axes, moreover the (++)– interlocation holds, i.e. cylinders intersect along two closed curves:

$$\overline{\overline{Q}}_1 = \widetilde{A}\, Q_1 = (x_0,\ x_1,\ r_1 sh\beta,\ 0,\ r_1 ch\beta)^T,$$

$$\overline{\overline{Q}}_2 = \widetilde{A}\, Q_2 = (x_0,\ 0,\ \widetilde{x}_2(ch\beta + \sin\varphi\ sh\beta),\ 0,\ \widetilde{x}_2(sh\beta + \sin\varphi\ ch\beta))^T,$$

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & ch\beta & 0 & sh\beta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & sh\beta & 0 & ch\beta \end{pmatrix}$$

here $\widetilde{x}_2 = x_2 - a$,

here $ch\,\beta$ and $sh\,\beta$ are hyperbolic cosine and sine, matrix $\widetilde{A}$ represents PE-rotation, rotation angle β has to satisfy equality: $th\,\beta = \sin\varphi$.

The construction of the rational blending surface $D(v,t)$ for two cylinders with intersecting axes is described in paragraph 2.1.1. The original blending surface of cylinder and cone with intersecting axes is obtained applying inverse PE-transformation to $D(v,t)$:

$$\overline{\overline{D}}(v,t) = \widetilde{A}^{-1} D(v,t).$$

This inverse transformation maps rational blending surface $D(v,t)$ to the rational blending surface of cylinder and cone with intersecting axes (Fig. 8).

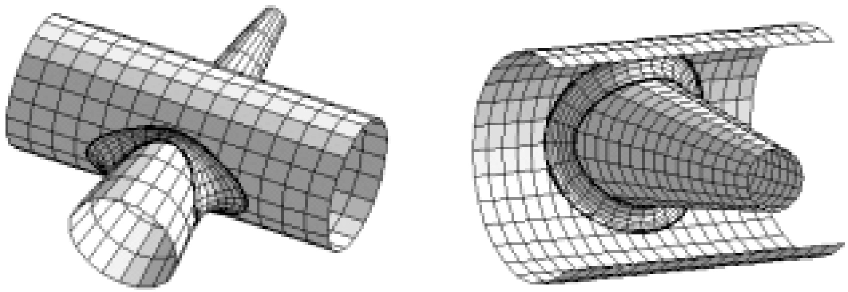

Fig. 8. Rational blending surface of cylinder and cone with intersecting axes, (++)– case.

3.2 Construction of Blending Surface when Quadrics are in (+–)+ type Interlocation

Construction of blending surface when quadrics are in (+–)+ type interlocation will be described more detailed in case of blending cylinders and cone with parallel axes.

3.2.1 Blending of Cylinder and Cone with Parallel Axes

Consider cylinder Q_1 of radius r_1 and cone Q_2 with parallel axes. They form (+–)+ type interlocation. Assume, that the axis of Q_1 is parallel to coordinate axis x_2 and is vertically above it in distance q, and axis of Q_2 is parallel to coordinate axis x_2 and is vertically below it in distance q.

Determine all the spheres, which are tangent to both quadrics. Further to cyclographic model of Laguerre geometry, the set of all tangent spheres in three-dimensional space $\mathbf{R}^3$ correspond to the surface of centers T^4 in four-dimensional space $\mathbf{R}^4$. It is described by equations as follows:

$$T^4\colon \begin{cases} (x_1 - x_0 q)^2 + x_3{}^2 = (x_4 - x_0 r_1)^2 \\ (x_1 + x_0 q)^2 + x_3{}^2 = (\sqrt{2}\ \ x_4 - x_2)^2 \end{cases}.$$

Consequently the corresponding parameterization of T^4 is deduced (Fig. 9):

$$T^4(s,t) = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1/8(s^2u - t^2u + t^2s^2 - u^2) \\ 1/4(-s^2u + t^2u + t^2s^2 - u^2) \\ \dfrac{5\sqrt{2}}{8}s^2u + \dfrac{3\sqrt{2}}{8}t^2u + \left(\dfrac{\sqrt{2}}{8} - \dfrac{1}{2}\right)t^2s^2 - \left(\dfrac{\sqrt{2}}{8} + \dfrac{1}{2}\right)u^2 \\ tsu \\ 1/8(5s^2u + 3t^2u + t^2s^2 - u^2) \end{pmatrix}. \quad (2)$$

The rational curve on the surface T^4 may be defined by ascribing some rational parameterization to variables *t* and *s*.

For example, define curve *SS(v)* by substituting parameterization:

$t = \frac{v}{\sqrt{1+v^2}}$, $s = \frac{1}{\sqrt{1+v^2}}$ and $u = 2$ to *(2)*.

Corresponding curve of degree 4 (Fig. 10) is represented as follows:

$$SS(v) = \begin{pmatrix} \frac{1-v^2}{4(1+v^2)} + \frac{v^2}{8(1+v^2)^2} - \frac{1}{2} \\ \frac{v^2-1}{2(1+v^2)} + \frac{v^2}{4(1+v^2)^2} - 1 \\ \frac{\sqrt{2}(5+3v^2)}{4(1+v^2)} + \frac{(\sqrt{2}-4)v^2}{8(1+v^2)^2} - 2 - \frac{\sqrt{2}}{2} \\ \frac{5+3v^2}{4(1+v^2)} + \frac{v^2}{8(1+v^2)^2} - \frac{1}{2} \end{pmatrix}.$$

According to this center curve parameterization, rational blending surface is described in analogous way as in case of two perpendicular cylinders with intersecting axes (Fig. 11).

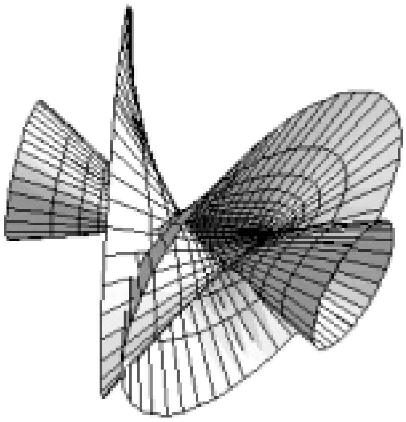

Fig. 9. Surface of centers (+–)+ case.

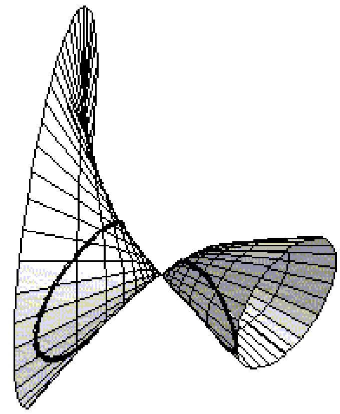

Fig. 10. Degree 4 curves on surface of centers
(+–)+ case.

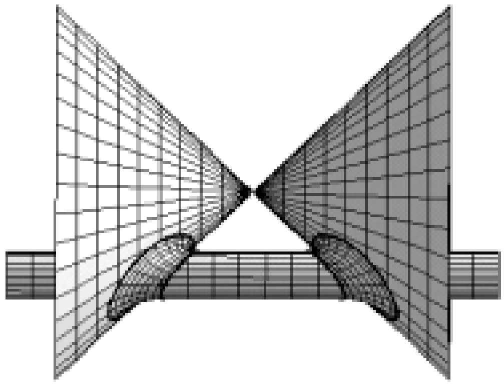

Fig.11. Rational blending surface of cylinder and cone with parallel axes, (+–)+ case.

4 Conclusions

The construction of rational C^1–blending between two natural quadrics is reviewed based on the cases of cylinders and cones blending, when quadrics intersect along two closed curves. The blending surface is constructed as a patch of rational radius rolling ball envelope, i.e. as a patch of a canal surface with rational radius, bounded by rational curves.

With the approach of Laguerre geometry rational canal surfaces are treated as cyclographic images of rational curves in R^4. This leads to a simple control structure for canal surfaces, i.e. applying Laguerre transformations to the blends in the several simplest positions of natural quadrics we may construct rational canal blends in more general cases.

Main characteristics of construction:

1 The constructed blending surface is rational, i.e. has precise analytical expression.

2 The construction of surface is stable with respect to Laguerre transformations, i.e. when applying Laguerre transformations (PE-rotations, PE-translations, including offset constructions), blending canal surface remains rational of the same degree.

3 The rational blending surface may be converted to the NURBS form, which is widely applied in various geometric design systems.

4 The problem of determining the curve, along which the center of rolling ball moves, is not resolved completely. In this paper the construction was made for example of the 4-degree center curve. From the modeling point of view the way how blending surface looks may be essentially improved by

rising a degree of center curve. Though this leads to high degree of blending surface, and consequently to higher difficulty of computation. The solution for this problem may be the construction of center curve using splines. By that means the quality of surface would rise and management of construction would become much easier.

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