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Design of multidimensional Recursive Systems through Padé Type Rational Approximation

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Abstract

The results obtained in classical 1-D rational approximation are extended in this paper to rational approximation of M-D functions. A full analog of classical Montessus de Ballore theorem for the convergence of the rows of Padé's tables is obtained. It is shown that the appropriate theorem for uniform convergence in $\mathbb{C}^2$ will really take place only in the case of choosing the necessary determinative (interpolation) sets $\mathbf{I}_j(\mathbf{n}, \mathbf{m}), j = 1, 2$. This theorem allows us to handle the problem of deriving the transfer function of a M-D digital system, that is described by its state-space representation[1]. The results of computer modelling by using MATLAB software are presented. Both the convergence theorem and results of modelling show that from theoretical and practical points of view the proposed approach is promising.

Keywords: multi-dimensional rational approximants, interpolation set, convergence, meromorphic function, state-space representation, transfer function

1 Introduction

Recently there is a great deal of interest expressed to the analysis and synthesis of a broad class of problems in multi-dimensional (M-D) systems, which may be characterized by rational functions (or matrices) in several complex variables.

Rational approximation theory has found many interpretations and different applications in signal processing and systems theory. They include the design of digital filters from a prescribed impulse response sequence [1], [2], model reduction of control systems [3], network synthesis and minimal partial realization problem [4], [2].

During the last decades a group of mathematicians have been researching the problem of approximating M-D functions ([5], [6], [7], [8], [9]). The method of Padé approximation for single-variable functions has been extended to the approximation of two (or more) variable functions. A lot of different applications of rational approximation and a huge bibliography is mentioned in [10].

In the rational modelling of M-D systems, the system to be modelled may be characterized by:

- state-space description;
- its impulse response;
- its autocorrelation data.

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The method outlined in this paper is developed for a two-variable system. The system is described by the state-variable model. The generalization to multiple dimensions (more than 2) is straightforward.

2 2-D Padé-type approximants in $\mathbb{C}^2$

1. Let $\mathbf{H}(z_1, z_2)$

$$\mathbf{H}(z_1, z_2) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} h_{i,j} z_1^i z_2^j, \tag{1}$$

be a **holomorphic** function near the origin.

For any integer sets $\mathbf{n} = (n_1, n_2)$ and $\mathbf{m} = (m_1, m_2)$, i.e. for any $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^2$, let

$$\mathcal{R}(\mathbf{n}, \mathbf{m}) = \left\{ \mathbf{r} = \frac{\mathbf{p}}{\mathbf{q}},\ \mathbf{p}(z_1, z_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{i,j} z_1^i z_2^j, \right.$$
$$\left. \mathbf{q}(z_1, z_2) = \sum_{k=0}^{m_1} \sum_{l=0}^{m_2} q_{k,l} z_1^k z_2^l,\ \mathbf{q}(0,0) = 1 \right\},$$

be the class of rational functions, i.e. the ratio of 2-D polynomials whose degrees don't exceed $\mathbf{n} = (n_1, n_2)$ and $\mathbf{m} = (m_1, m_2)$ correspondingly for each variable. In short it may be written that $\deg(\mathbf{p}) \leq \mathbf{n}, \deg(\mathbf{q}) \leq \mathbf{m}$.

Every rational function $\mathbf{r} \in \mathcal{R}(\mathbf{n}, \mathbf{m})$ may be identified with its power series (1) that converges in some neighborhood of the origin. It should be mentioned that $\mathbf{r} = \dfrac{\mathbf{p}}{\mathbf{q}} \in \mathcal{R}$ depends on

$$\tau_{\mathbf{n},\mathbf{m}} = (n_1 + 1)(n_2 + 1) + (m_1 + 1)(m_2 + 1) - 1$$

parameters (the coefficients of p and q).

2. The set of integer points $\mathbf{I}(\mathbf{n}, \mathbf{m}) \subset \mathbb{Z}_+^2$ for fixed $\mathbf{n} = (n_1, n_2)$ and $\mathbf{m} = (m_1, m_2)$ is called the **determinative (interpolation) set**, if it has the next properties:

1. $dim\mathbf{I}(\mathbf{n},\mathbf{m}) = \tau_{\mathbf{n},\mathbf{m}}$

2. $(n_1 + m_1, 0), (0, n_2 + m_2) \in \mathbf{I}(\mathbf{n},\mathbf{m})$. This property guarantees that in the case when $z_1 = 0$ (or $z_2 = 0$) one would have the **classical 1-D rational approximation** of Padé type.

3. $\mathbf{n} = (n_1, n_2) \in \mathbf{I}(\mathbf{n},\mathbf{m})$;

4. if $(k_1, k_2) \in \mathbf{I}(\mathbf{n},\mathbf{m})$ then $[0,\mathbf{k}] \subset \mathbf{I}(\mathbf{n},\mathbf{m})$, where $[0,\mathbf{k}] = \left\{(s_1, s_2) \in \mathbb{Z}_+^2 : 0 \le s_j \le k_j, j = 1, 2\right\}$ - the **rectangle** rule.

5. $(n_1 + m_1, m_2) \in \mathbf{I}_1(\mathbf{n},\mathbf{m})$ and $(m_1, n_2 + m_2) \in \mathbf{I}_2(\mathbf{n},\mathbf{m})$ (see below).

Two and only two possible variants of these sets satisfying requirements are:

$$
\begin{aligned}
1.\ \mathbf{I}_1(\mathbf{n},\mathbf{m}) = \{(i,j) : &[0 \le i \le n_1, 0 \le j \le n_2] \cup \\
&[n_1 + 1 \le i \le n_1 + m_1, 0 \le j \le m_2] \cup \\
&[0 = i, n_2 + 1 \le j \le n_2 + m_2]\},
\end{aligned}
\tag{2}
$$

(see Appendix A).

$$
\begin{aligned}
2.\ \mathbf{I}_2(\mathbf{n},\mathbf{m}) = \{(i,j) : &[0 \le i \le n_1, 0 \le j \le n_2] \cup \\
&[0 \le i \le m_1, n_2 + 1 \le j \le n_2 + m_2] \cup \\
&[n_1 + 1 \le i \le n_1 + m_1, 0 = j]\},
\end{aligned}
\tag{3}
$$

(see Appendix B).

3. The generalized Padé approximant (Chisholm [6], Vavilov [7]) of $\mathbf{H}(z_1, z_2)$ (see (1)) for given $\mathbf{n} = (n_1, n_2)$ and $\mathbf{m} = (m_1, m_2)$ is defined as the *rational function* $\mathbf{f}_{\mathbf{n},\mathbf{m}} \in \mathcal{R}(\mathbf{n},\mathbf{m})$ for which

$$
\mathbf{T}_{\mathbf{i},\mathbf{j}}(\mathbf{H} - \mathbf{f}_{\mathbf{n},\mathbf{m}}) = \mathbf{0}, \quad \text{for all} \quad (i,j) \in \mathbf{I}(\mathbf{n},\mathbf{m}), \tag{4}
$$

where $\mathbf{T}_{\mathbf{i},\mathbf{j}}(\phi)$ are the Taylor's coefficients of the power series for the function ϕ. The rational function $\mathbf{f}_{\mathbf{n},\mathbf{m}}$ is called the **2-D Padé-type approximant** of $\mathbf{H}(z_1, z_2)$ (see (1)) which corresponds to the determinative set $(i,j) \in \mathbf{I}(\mathbf{n},\mathbf{m})$.

As in the case of 1-D functions, the existence and uniqueness of Padé-type approximants (in the sense of the given above definition) for $\mathbb{C}^2$ require special type of analysis. It should be mentioned that Padé-type approximants do not always exist in the sense of the given definition. If for given $\mathbf{n}, \mathbf{m}, \mathbf{I}(\mathbf{n}, \mathbf{m})$ the Padé-type approximant of $\mathbf{H}(z_1, z_2)$ (see (1)) exists and $\mathbf{f}_{\mathbf{n},\mathbf{m}} = \dfrac{\mathbf{p}_{\mathbf{n},\mathbf{m}}}{\mathbf{q}_{\mathbf{n},\mathbf{m}}}$, where $\mathbf{p}_{\mathbf{n},\mathbf{m}}, \mathbf{q}_{\mathbf{n},\mathbf{m}}$ are mutually prime and $\mathbf{q}_{\mathbf{n},\mathbf{m}}(0,0) \neq 0$, then from (4) it follows that the (linear) interpolation relationship is true:

$$\mathbf{T}_{\mathbf{i},\mathbf{j}}(\mathbf{qH} - \mathbf{p}) = \mathbf{0}, \quad \text{for all } \ (i,j) \in \mathbf{I}(\mathbf{n}, \mathbf{m}), \tag{5}$$

with $\mathbf{p} = \mathbf{p}_{\mathbf{n},\mathbf{m}}$ and $\mathbf{q} = \mathbf{q}_{\mathbf{n},\mathbf{m}}$. The opposite statement is also true: if for given $\mathbf{n}, \mathbf{m}, \mathbf{I}(\mathbf{n}, \mathbf{m})$ there exist the polynomials $\mathbf{p}, \mathbf{q}$ that satisfy (5) and such that

$$\deg \mathbf{p} \leq \mathbf{n}, \deg \mathbf{q} \leq \mathbf{m}, \mathbf{q}(0,0) \neq 0,$$

then the Padé type approximant of (1) (that corresponds to the given determinative set $\mathbf{I}(\mathbf{n}, \mathbf{m})$) exists and $\mathbf{f}_{\mathbf{n},\mathbf{m}} = \dfrac{\mathbf{p}}{\mathbf{q}}$.

For the power series (1) and for any determinative set $\mathbf{I}(\mathbf{n}, \mathbf{m}), \mathbf{n}, \mathbf{m} \in \mathbb{Z}_+^2$ there always exist polynomials $\mathbf{p}, \mathbf{q}$ satisfying (5) and such that

$$\deg \mathbf{p} \leq \mathbf{n}, \deg \mathbf{q} \leq \mathbf{m}, \mathbf{q}(0,0) \neq 0. \tag{6}$$

In order to find the coefficients of $\mathbf{p}$ and $\mathbf{q}$ it is enough to find firstly the polynomial $\mathbf{q}$ whose coefficients satisfy the system of linear equations

$$\mathbf{T}_{\mathbf{i},\mathbf{j}}(\mathbf{qH}) = \mathbf{0}, \quad \text{for all } \ (i,j) \in \mathbf{I}(\mathbf{n}, \mathbf{m}) \setminus [0; \mathbf{n}]. \tag{7}$$

and after that to substitute into

$$\mathbf{p}(\mathbf{z}) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \mathbf{T}_{\mathbf{i},\mathbf{j}}(\mathbf{qH}) z_1^i z_2^j. \tag{8}$$

The system (7) is a homogeneous system of linear equations for the unknown coefficients of the polynomial $\mathbf{q}$. The matrix of the system is determined by the coefficients $h_{i,j}$ of the power series (1). The number of the unknowns is one plus the number of the equations. That is why the system (7) has always a nontrivial solution.

Fact 1 *If for the given power series (1) and for given* $\mathbf{n}, \mathbf{m}, \mathbf{I}(\mathbf{n}, \mathbf{m})$ *any pair of polynomials* $\mathbf{p}, \mathbf{q}$ *(satisfying (6, 7, 8)) has the property that* $\mathbf{q}(0,0) \neq 0$*, then there exists the unique Padé type approximant for (1) for the given determinative set* $\mathbf{I}(\mathbf{n}, \mathbf{m})$.

4. Let $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}_+^2$ be fixed. Let the class

$$\mathbf{M}_\mathbf{m} = \mathbf{M}_\mathbf{m}(\mathbb{C}^2) = \left\{ f : f(z_1, z_2) = \frac{\mathbf{F}(z_1, z_2)}{\mathbf{Q}_\mathbf{m}(z_1, z_2)} \right\} \tag{9}$$

be defined as a class of functions with the properties:

- $\mathbf{F}(z_1, z_2)$ is an entire function;
- $\deg \mathbf{Q}_\mathbf{m} = \mathbf{m}$, i.e. $\deg \mathbf{Q}_\mathbf{m}(z_1, 0) = m_1, \deg \mathbf{Q}_\mathbf{m}(0, z_2) = m_2$;
- $\mathbf{Q}_\mathbf{m}(0,0) = 1$;
- The functions $\mathbf{F}(z_1, 0), \mathbf{F}(0, z_2)$ and the polynomials $\mathbf{Q}_\mathbf{m}(z_1, 0)$, $\mathbf{Q}_\mathbf{m}(0, z_2)$ are not equal to zero simultaneously.

The most important theorem is the next:

Theorem 1 ([7]) *Let* $\mathbf{H}(\mathbf{z}) \in \mathbf{M}_\mathbf{m}$ *be given by the power series (1),* $\mathbf{m} = (m_1, m_2), \mathbf{m} \in \mathbb{Z}_+^2$ *be fixed and* $\mathbf{n} = (n_1, n_2), \mathbf{n} \in \mathbb{Z}_+^2$*. Then:*

1. *For all* $\acute{n} = \min(n_1, n_2)$ *that are enough big, there exists* **unique** *Padé type approximant* $\mathbf{f}^j_{\mathbf{n},\mathbf{m}} = \dfrac{\mathbf{p}^j_\mathbf{n}}{\mathbf{q}^j_\mathbf{m}}$ *for each of the determinative sets* $\mathbf{I}_j(\mathbf{n}, \mathbf{m}), j = 1, 2$*;*
2. *The sequence* $\mathbf{f}^j_{\mathbf{n},\mathbf{m}}$ *for* $\acute{n} = \min(n_1, n_2) \to \infty$ *converges uniformly to the function* $\mathbf{H}(z_1, z_2)$ *inside the compact subsets of* $\mathcal{G} = \mathbb{C}^2 \backslash \{\mathbf{Q}_\mathbf{m} = 0\}$*. For any compact* $\mathbf{E} \subset \mathbb{C}^2$ *the following relationships are true:*

$$\lim_{n' \to \infty} \left\| \mathbf{Q}_\mathbf{m} - \mathbf{q}^j_\mathbf{n} \right\|_\mathbf{E}^{1/n'} = 0, \tag{10}$$

$$\lim_{n' \to \infty} \left\| \mathbf{H} - \mathbf{f}^j_{\mathbf{n},\mathbf{m}} \right\|_\mathbf{E}^{1/n'} = 0, \tag{11}$$

where $j = 1, 2$ *and*

$$\| * \|_\mathbf{E} = \sup_{\mathbf{z} \in \mathbf{E}} | * |.$$

The choice of the determinative set $\mathbf{I}(\mathbf{n},\mathbf{m})$ may be made in different ways. But this theorem may be stated and proved in its *full measure* **only** for two determinative sets - $\mathbf{I}_1(\mathbf{n},\mathbf{m})$ and $\mathbf{I}_2(\mathbf{n},\mathbf{m})$.

5. As it was mentioned above, the denominator $\mathbf{q}(z_1, z_2)$ should "follow" the singularities of the meromorphic function $\mathbf{H}(z_1, z_2)$. Ideally the highest degree (degrees) of $\mathbf{q}(z_1, z_2)$ is (are) known or may be estimated. After evaluating $\mathbf{m}$, the degree of the numerator $\mathbf{n}$ may increase in order to approximate $\mathbf{H}(z_1, z_2)$.

If $\mathbf{m}$ is unknown then some experimental steps should be performed in order to choose the necessary degree.

One of the most important problems in M-D rational approximation is the choice of the determinative (interpolation) set $\mathbf{I}(\mathbf{n},\mathbf{m})$. In different publications the improper choice of this set has given rise to incorrect conclusions about nonexistence or non-uniqueness of 2-D Padé approximants [1], [4], [11], [3].

The new results obtained recently by Vavilov [7] allow one to make the right choice of the set $\mathbf{I}(\mathbf{n},\mathbf{m})$, namely $\mathbf{I}_1(\mathbf{n},\mathbf{m})$ and $\mathbf{I}_2(\mathbf{n},\mathbf{m})$.

6. One more reason that gave rise to these results was the desire to obtain some **recursive algorithms** for finding the M-D Padé type approximant.

There should be mentioned that the new choice of the determinative set given above leads to recursive algorithms for this so-called Canterbury interpolation (see [7]), which are quite fast and efficient.

But the task under consideration does not really need to handle very big degrees for both numerator and denominator (no more than 8-10). Unlike the one-dimensional case, the M-D case for recursive computations requires relatively a large number of computations for low degrees $\mathbf{n},\mathbf{m}$. That is why the recursive algorithms (developed in [7]) are not presented here.

On another hand, though the choice of the determinative set given in [1] gives a recursive algorithm, the approximant itself leads not only to much worse approximation error (see below) but also does not provide any type of convergence (see above).

3 Design of recursive systems

1. System description The N-dimensional discrete system is a dynamic system described by the equations in N independent variables for $N \geq 2$ ([12]). In the particular case $N = 2$ one obtains the **generalised 2-D model (G2-DM)**:

$$\mathbf{E}\mathbf{x_{i+1,j+1}} =$$

$$= \mathbf{A_0}\mathbf{x_{i,j}} + \mathbf{A_1}\mathbf{x_{i+1,j}} + \mathbf{A_2}\mathbf{x_{i,j+1}} + \mathbf{B_0}\mathbf{u_{i,j}} + \mathbf{B_1}\mathbf{u_{i+1,j}} + \mathbf{B_2}\mathbf{u_{i,j+1}}$$

or the **generalised 2-D Roesser model (G2-DRM)**:

$$\mathbf{E}\begin{bmatrix} \mathbf{x^h_{i+1,j}} \\ \mathbf{x^v_{i,j+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{A_{11}}\mathbf{A_{12}} \\ \mathbf{A_{21}}\mathbf{A_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{x^h_{i,j}} \\ \mathbf{x^v_{i,j}} \end{bmatrix} + \begin{bmatrix} \mathbf{B_{10}} \\ \mathbf{B_{20}} \end{bmatrix} \mathbf{u_{i,j}},$$

$$\mathbf{y_{i,j}} = \begin{bmatrix} \mathbf{C_{10}}\mathbf{C_{20}} \end{bmatrix} \begin{bmatrix} \mathbf{x^h_{i,j}} \\ \mathbf{x^v_{i,j}} \end{bmatrix} + \mathbf{D}\mathbf{u_{i,j}}.$$

where $\mathbf{x_{i,j}} = \begin{bmatrix} \mathbf{x^h_{i,j}} \\ \mathbf{x^v_{i,j}} \end{bmatrix}$, $\mathbf{x_{i,j}} \in \mathbf{R^2}$ is the local semistate vector at point (i,j), $\mathbf{x^h_{i,j}} \in \mathbf{R}$, $\mathbf{x^v_{i,j}} \in \mathbf{R}$ are the horizontal and vertical local semistate vectors, $\mathbf{u_{i,j}} \in \mathbf{R^2}$ is the input, $\mathbf{y_{i,j}} \in \mathbf{R^2}$ is the output, and the matrices $\mathbf{A_{l,m}}, \mathbf{B_{l,m}}, \mathbf{C_{l,m}}, \mathbf{E}$ and $\mathbf{D}$ are real matrices of appropriate dimensions.

If $\mathbf{E}$ is nonsingular the model is regular. The system state and output equations are given by

$$\begin{aligned} \begin{bmatrix} \mathbf{x^h_{i+1,j}} \\ \mathbf{x^v_{i,j+1}} \end{bmatrix} &= \mathbf{A}\mathbf{x_{i,j}} + \mathbf{B}\mathbf{u_{i,j}}, \\ \mathbf{y_{i,j}} &= \mathbf{C}\mathbf{x_{i,j}}. \end{aligned}$$

In this case the 2-D z-transform may give that the transfer matrix of the system is

$$\mathbf{H}(z_1, z_2) = \mathbf{C}(\begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} - \mathbf{A})^{-1}\mathbf{B}. \tag{12}$$

The goal is to develop a method for computing $\mathbf{H}(z_1, z_2)$ by using the 2-D rational approximation.

From (12) it is clear that the **meromorphic** function $\mathbf{H}(z_1, z_2)$ will have singularities that should be approximated by a rational function, whose denominator plays the most important role in this approximation process.

The approach developed in section [2] may be applied for solving the problem.

2. Computing the transfer function The transfer function $\mathbf{H}(z_1, z_2)$ may be expanded into an infinite power series. The entries of $\mathbf{H}(z_1, z_2)$ are computed from the infinite series using the procedure of the previous section.

Two cases must be considered.

Case 1 .
The matrix $\mathbf{A}$ *is invertible.*

If the matrix $\mathbf{A}$ is invertible then one can use that

$$(\mathbf{W} - \mathbf{A})^{-1} = -\sum_{i=0}^{\infty} \mathbf{A}^{-1}(\mathbf{W}\mathbf{A}^{-1})^i \tag{13}$$

with $(\mathbf{W}\mathbf{A}^{-1})^0 = \mathbf{I}$. Equations (12) and (13) give a simple method to calculate the reduced transfer matrix. Each entry $\mathbf{h}_{ij}(z_1, z_2)$ is expanded into a power series in z_1 and z_2.

Case 2 .
The matrix $\mathbf{A}$ *is non-invertible.*

The matrix $(\mathbf{A} - \xi\mathbf{I})$ is an invertible matrix for ξ which is not an eigenvalue of $\mathbf{A}$. The technique just described can be applied with $\mathbf{A}$ replaced by $\mathbf{D} = \mathbf{A} - \xi\mathbf{I}$, or, equivalently, with z_1 and z_2 replaced in $\mathbf{W}$ by $z_1 + \xi$ and $z_2 + \xi$, respectively. The above technique may be applied and after reversing the change of variables one gets the desired transfer matrix $\mathbf{H}(z_1, z_2)$.

3. Examples. *Example 1.* Consider the double power series from [1], which is to be approximated by a rational function in (14) where the degrees in each variable in the numerator and denominator polynomials are each set to 1:

$$\begin{aligned}\mathbf{H}(z_1, z_2) = 1 + z_1 + z_2 + 2z_1^2 + 3z_2^2 - z_1 z_2 + \\ z_1^2 z_2 - z_1^2 z_2^2 + z_1 z_2^3 - z_1^3 z_2^2 + 2z_1^2 z_2^3 + \ldots\end{aligned}$$

$$\hat{\mathbf{H}}(z_1, z_2) = \frac{P_{00} + P_{10} z_1 + P_{01} z_2 + P_{11} z_1 z_2}{1 + Q_{10} z_1 + Q_{01} z_2 + Q_{11} z_1 z_2}. \tag{14}$$

An inaccurate choice of the determinative set $\mathbf{I}(\mathbf{n}, \mathbf{m})$ leads to the erroneous statement that the Padé approximation is nonunique. That choice was based on the determinative set $\mathbf{I}(\mathbf{n}, \mathbf{m}) = \{(0,0), (1,0), (0,1), (1,1), (2,2), (2,3), (2,3)\}$.

As it was mentioned above the correct choice is $\mathbf{I}_1(\mathbf{n}, \mathbf{m})$ or $\mathbf{I}_2(\mathbf{n}, \mathbf{m})$ in (2) or (3).

If $\mathbf{I}_1(\mathbf{n}, \mathbf{m})$ was chosen as a determinative set ($\mathbf{I}_1(\mathbf{n}, \mathbf{m}) = \{(0,0), (1,0), (0,1), (1,1), (2,0), (2,1), (0,3)\}$) (see Appendix C) then one obtains the next equations for the denominator's coefficients:

$$\begin{array}{ll} (2,0) & 0 = 2 + Q_{10} \\ (2,1) & 0 = 1 - Q_{10} + 2Q_{01} + Q_{11} \\ (0,3) & 0 = 3 + Q_{01}. \end{array}$$

Then the coefficients will be

$$\begin{aligned} Q_{10} &= -2 \\ Q_{01} &= -3 \\ Q_{11} &= 3. \end{aligned}$$

The numerator's coefficients may be found after substituting in (15) for the rest of the points from the determinative set $\mathbf{I}_1(\mathbf{n}, \mathbf{m})$:

$$
\begin{array}{ll}
(0,0) & P_{00} = 1 \\
(1,0) & P_{10} = 1 + Q_{10} \\
(0,1) & P_{01} = 1 + Q_{01} \\
(1,1) & P_{11} = -1 + Q_{10} + Q_{01} + Q_{11}.
\end{array} \tag{15}
$$

The final result for the 2-D Padé approximant will be

$$\hat{\mathbf{H}}(z_1, z_2) = \frac{1 - z_1 - 2z_2 - 3z_1z_2}{1 - 2z_1 - 3z_2 + 3z_1z_2}.$$

Example 2. The problem under consideration is to find the Padé type 2-D approximation for the power series $\mathbf{H}(z_1, z_2)$ obtained from the function

$$G(z_1, z_2) = \frac{(1 - z_1)^4 \cdot (1 - z_2)^5}{(1 - 0.5z_1 - 0.5z_2)}.$$

There was developed a special tool (by MATLAB) for M-D Padé type approximation. The results of modelling are presented as the plots of the error function $|\mathbf{H}(z_1, z_2) - \hat{\mathbf{H}}(z_1, z_2)|$, where $(\mathbf{z_1}, \mathbf{z_2}) \in \mathcal{A}$, $\mathcal{A} = \{z_1 = \rho e^{i\phi_1}, z_2 = \rho e^{i\phi_2}\}, \phi_1, \phi_2 = 0, ..., 2\pi, \rho = 0.99$ and in table 1 (the average approximation accuracy) for the following criteria:

- maximum of the discrepancy

$$\mathbf{Error1} = \max_{(\mathbf{z_1}, \mathbf{z_2}) \in \mathcal{A}} (\mathbf{H}(z_1, z_2) - \hat{\mathbf{H}}(z_1, z_2))$$

- mean square error

$$\mathbf{Error2} = \mathbf{MSE}(\mathbf{H}(z_1, z_2) - \hat{\mathbf{H}}(z_1, z_2)).$$

The determinative set $\mathbf{I}^*(\mathbf{n}, \mathbf{m})$ was chosen in accordance with the assumption(s) made in [1] and in other publications mentioned above.

From the results in table 1 it is obvious that:

- for any degrees of the numerator and denominator $\mathbf{n}$ and $\mathbf{m}$,
- for any approximation error criteria,

the proposed method of choosing the determinative set $\mathbf{I}(\mathbf{n}, \mathbf{m})$ always gives a much better result.

The corresponding plots of approximation error are given below (figures 1-6).

Table 1: **Modelling results**

Degrees		Maximum of the discrepancy		Mean square error	
n	m	for $I_1(n,m)$	for $I^*(n,m)$	for $I_1(n,m)$	for $I^*(n,m)$
5, 5	1, 1	$1.0 \cdot 10^{-4}$	0.78	$3.9 \cdot 10^{-2}$	3500
5, 5	5, 5	$1.5 \cdot 10^{-4}$	1004	0.29	$3.2 \cdot 10^{5}$
10, 10	1, 1	$1.5 \cdot 10^{-4}$	0.01	0.04	40.7
10, 10	4, 4	$1.7 \cdot 10^{-2}$	4.1	25	297

4 Summary

A new theorem concerning the extension of the results obtained in classical 1-D rational approximation for rational approximation of M-D functions was presented. There was obtained a full analog of the classical Montessus de Ballore theorem for the convergence of the rows of Padé's tables.

A method for computing the transfer function of M-D systems by their state-space representation was proposed. The algorithm is conceptually simple and computationally easy. The method gives a novel approach to identification and approximation of M-D discrete linear systems.

It was mentioned the possibility to perform recursive computations. The developed software tool allowed comparison of the approximation error for different choices of the determinative sets. The best accuracy is achieved for the determinative sets $\mathbf{I}_{1,2}(\mathbf{n},\mathbf{m})$, proposed in this paper, for which the analog of the classical Montessus de Ballore theorem was proved.

References

[1] N. Bose and S. Basu. Two-dimensional matrix Pade approximants: existence, nonuniqueness, and recursive computation. *IEEE Trans. Autom. Contr.*, 25(1):509–514, February 1980.

[2] S. Sakata. Partial realization of 2-D discrete linear system and 2-D Padé approximation and reduction of 2-D rational transfer function. *Proc. IEEE*, 78(4):604–613, April 1990.

[3] A. Sahani and S. Nagar. Design of digital controllers for multivariable systems via time-moments matching. *Computers & Electrical Engineering*, 24(4):335–347, 1998.

[4] N. Bose. *Multidimensional systems theory*, chapter *Multivariate rational approximants of the Padé-Type in systems theory* . D. Reidel Publ. Comp., 1985.

[5] G. Baker and P. Graves-Morris. *Padé approximants.* Cambridge University Press, 2nd edition, 1996.

[6] J. Chisholm. Rational approximants defined from double power series. *Math. Comp.*, 27:43–52, 1973.

[7] A. Gonchar and V. Vavilov. Rational approximations of functions of several complex variables. *Journ. of Approx. Theory*, to appear.

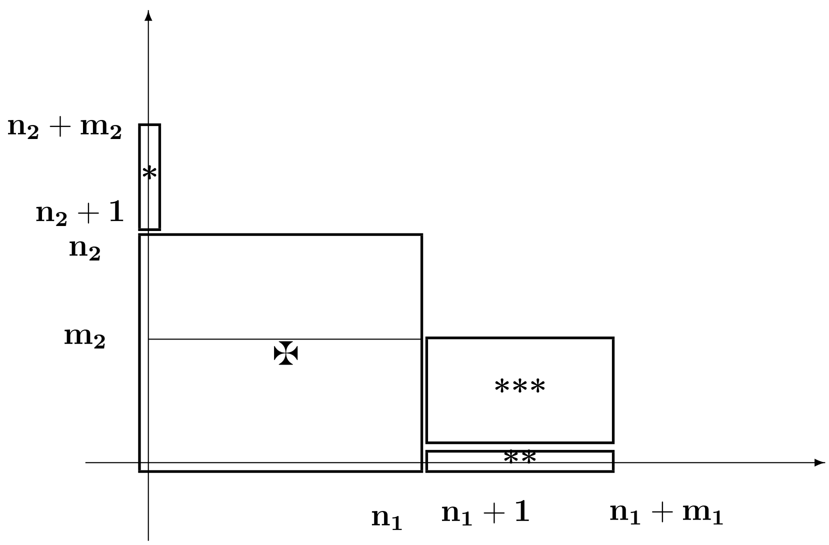
[8] A. Cuyt. A comparison of some multivariate Padé type approximants. *SIAM Journal on Mathematical Analysis*, 14:195–207, 1983.

[9] A. Cuyt, K. Driver, and D. Lubinsky. Kronecker type theorems, normality and continuity of the multivariate padé operator. *Numerische Mathematik*, 73:311–328, 1996.

[10] H. Werner and H. Bűnger, editors. *Padé approximation and its applications*, volume 1071 of *Lecture notes in Mathematics, Springer-Verlag.* Bad Honnef, 1983.

[11] B. Bandyopadhyay, A. Rao, and H. Singh. On Padé approximation of multivariable systems. *IEEE Trans. Circ. and Syst.*, 36(4):638–639, Apr 1989.

[12] T. Kaczorek. *Linear control systems: Synthesis of multivariable systems and multidimensional systems*, volume 2. John Wiley & Sons Inc., Industrial control, computers and communication series edition, 1993.



* - $\mathbf{m_2}$

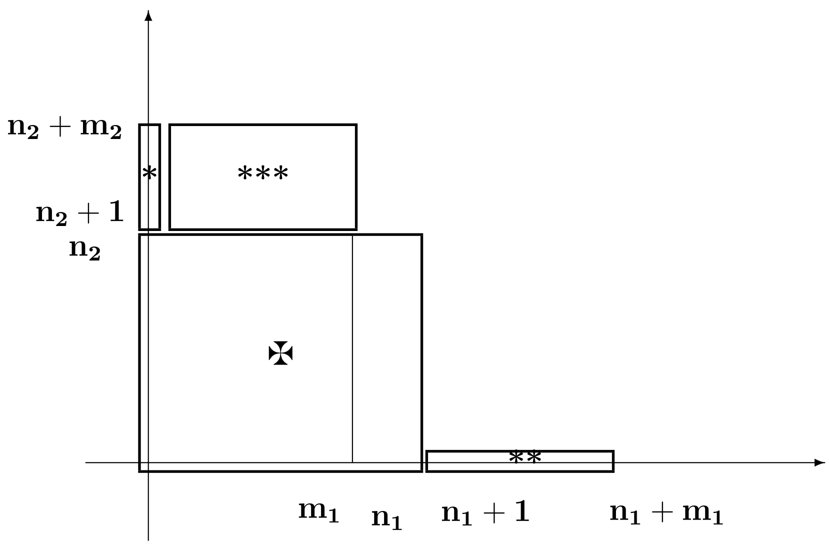
** - $\mathbf{m_1}$

*** - $\mathbf{m_1 \cdot m_2}$

✠ - $(\mathbf{n_1 + 1}) \cdot (\mathbf{n_2 + 1})$

Determinative set $\mathbf{I_1(n, m)}$

Appendix B



* - $\mathbf{m_2}$

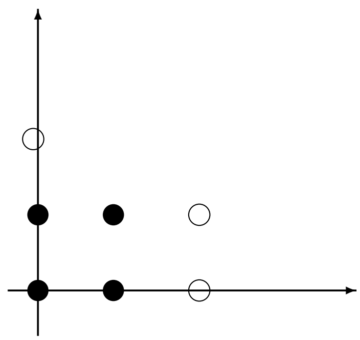
** - $\mathbf{m_1}$

*** - $\mathbf{m_1 \cdot m_2}$

✠ - $(\mathbf{n_1 + 1}) \cdot (\mathbf{n_2 + 1})$

Determinative set $\mathbf{I_2(n, m)}$

Appendix C



● $\{(0,0),(1,0),(0,1),(1,1)\} = [0,\mathbf{n}]$

○ $(2,0),(2,1),(0,2) = \mathbf{I}(\mathbf{n},\mathbf{m})\backslash\,[0,\mathbf{n}]$

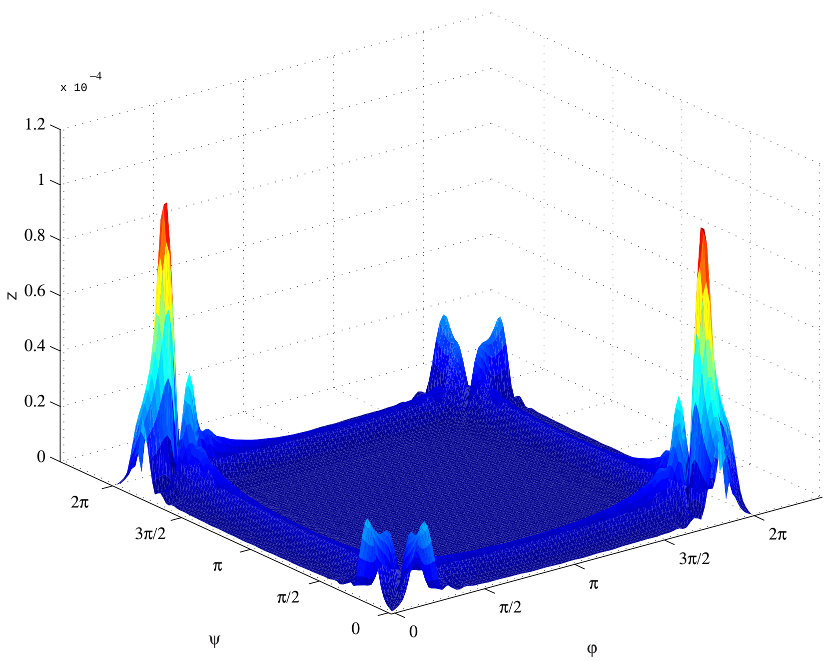


Figure 1: Approximation error for $\mathbf{I}_1(\mathbf{n}, \mathbf{m})$ and $\mathbf{n}, \mathbf{m} = ((10, 10), (1, 1))$.

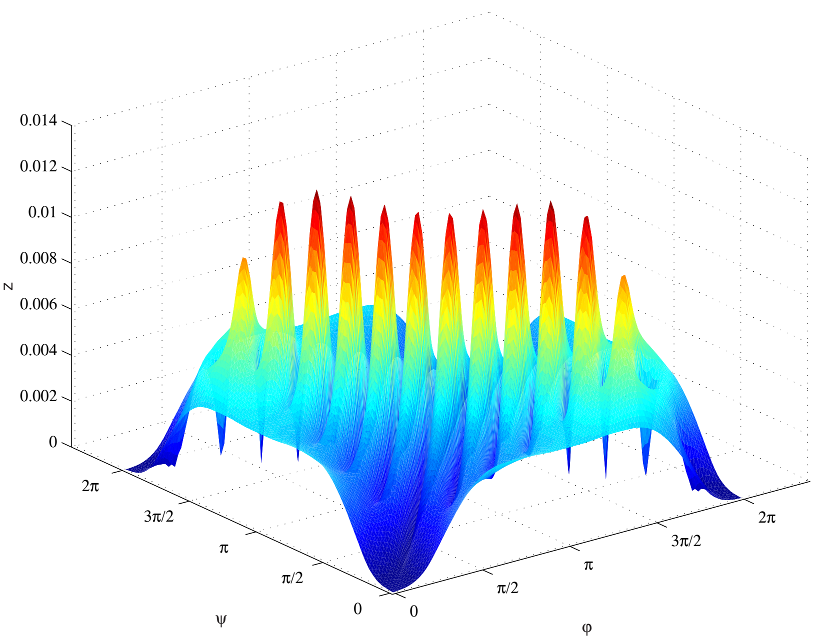


Figure 2: Approximation error for $\mathbf{I}^*(\mathbf{n}, \mathbf{m})$ and $\mathbf{n}, \mathbf{m} = ((10, 10), (1, 1))$.

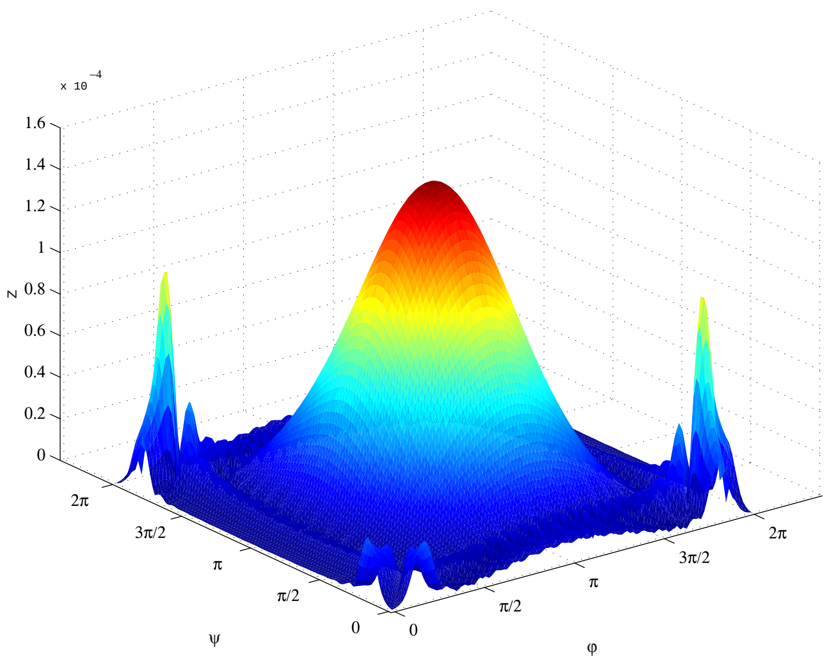


Figure 3: Approximation error for $\mathbf{I}_1(\mathbf{n}, \mathbf{m})$ and $\mathbf{n}, \mathbf{m} = ((5, 5), (5, 5))$.

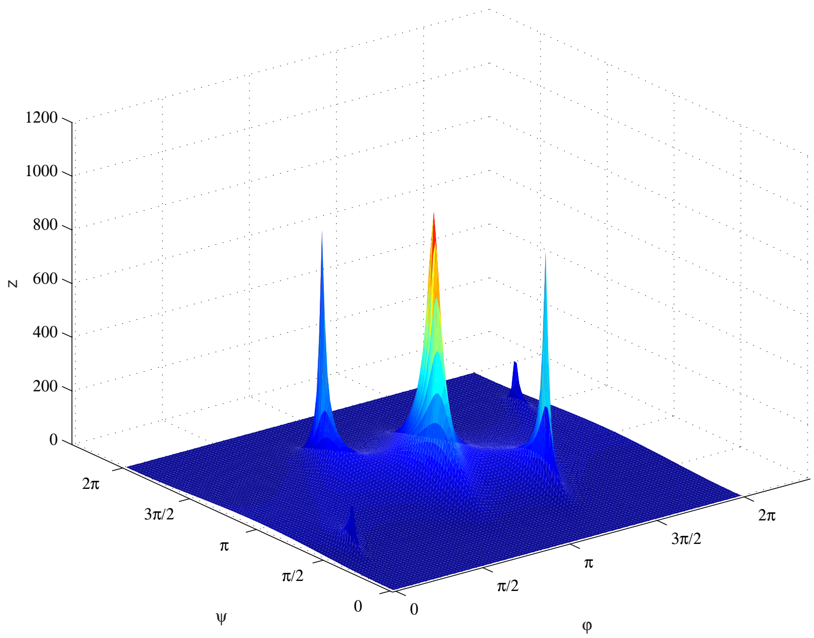


Figure 4: Approximation error for $\mathbf{I}^*(\mathbf{n}, \mathbf{m})$ and $\mathbf{n}, \mathbf{m} = ((5, 5), (5, 5))$.

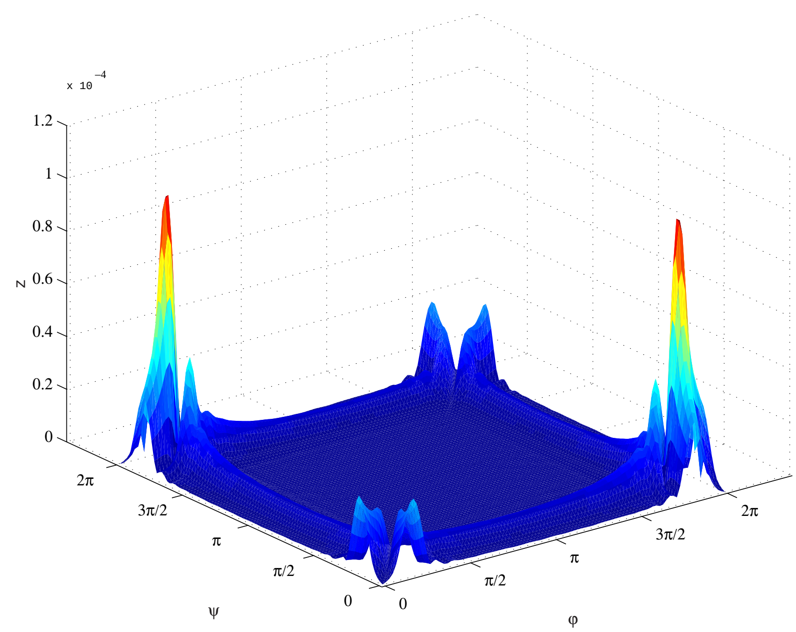


Figure 5: Approximation error for $\mathbf{I}_1(\mathbf{n}, \mathbf{m})$ and $\mathbf{n}, \mathbf{m} = ((5, 5), (1, 1))$.

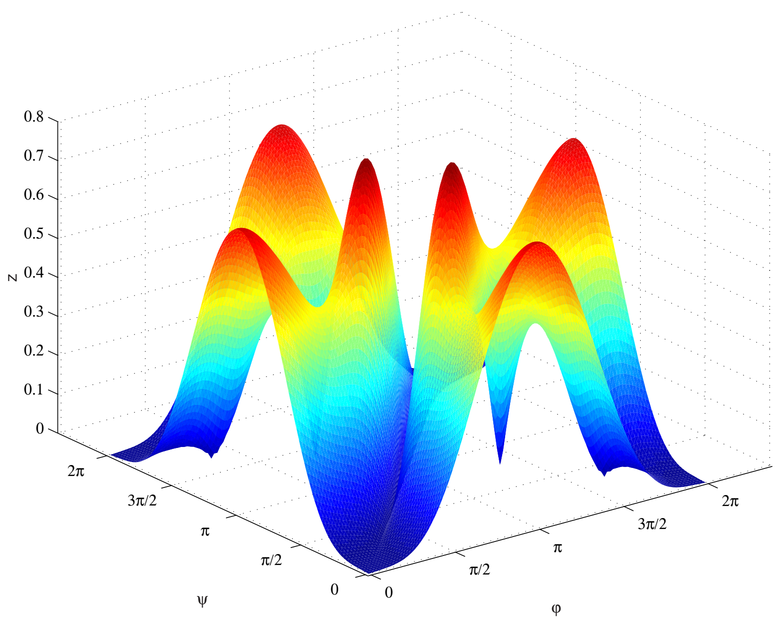


Figure 6: Approximation error for $\mathbf{I}^*(\mathbf{n}, \mathbf{m})$ and $\mathbf{n}, \mathbf{m} = ((5,5),(1,1))$.