

On Joint Distribution of General Dirichlet Series *

J. Genys¹, A. Laurinčikas²

¹Department of Mathematics, Šiauliai University
Vytauto 84, LT-5400 Šiauliai, Lithuania
mat.kat@fm.su.lt

²Department of Probability Theory and Number Theory, Vilnius University
Naugarduko 24, LT-2600 Vilnius, Lithuania
antanas.laurincikas@maf.vu.lt

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Abstract. In the paper a joint limit theorem in the sense of the weak convergence in the space of meromorphic functions for general Dirichlet series is proved under weaker conditions as in [1].

Keywords: distribution, general Dirichlet series, probability measure, weak convergence.

1 Introduction

Let $s = \sigma + it$ denote a complex variable, and let for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_{mj}s}, \quad j = 1, \dots, n,$$

be a collection of general Dirichlet series. Here a_{mj} are complex numbers, and $\{\lambda_{mj}\}$ is an increasing sequence of positive numbers, $\lim_{m \rightarrow \infty} \lambda_{mj} = +\infty$, $j = 1, \dots, n$. In [1] a joint limit theorem for the functions $f_1(s), \dots, f_n(s)$ has been considered. To state it we need some notation and assumptions. We assume that the functions $f_1(s), \dots, f_n(s)$ are meromorphically continuable to the half-planes $\sigma > \sigma_{11}, \sigma_{11} < \sigma_{a1}, \dots, \sigma > \sigma_{1n}, \sigma_{1n} < \sigma_{an}$, respectively,

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and all poles in the regions are included in a compact set. We also suppose that, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(s) = B|t|^{\delta_j}, \quad |t| \geq t_0, \quad \delta_j > 0, \quad (1)$$

and

$$\int_{-T}^T |f_j(\sigma + it)|^2 dt = BT, \quad T \rightarrow \infty, \quad (2)$$

$j = 1, \dots, n$, are satisfied, where B denotes a quantity bounded by a constant. Moreover, we assume that

$$\lambda_{mj} \geq c_j(\log m)^{\theta_j} \quad (3)$$

with some positive constants c_j and θ_j , $j = 1, \dots, n$.

Denote by $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ the unit circle on the complex plane \mathbb{C} , and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \geq 1$, be the infinite-dimensional torus. With product topology and pointwise multiplication the torus Ω becomes a compact topological Abelian group. Therefore, on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S , the probability Haar measure m_H exists, and this leads to a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(m)$ the projection of $\omega \in \Omega$ to the coordinate space γ_m .

Let G be a region on \mathbb{C} . Denote by $H(G)$ the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $D_j = \{s \in \mathbb{C} : \sigma > \sigma_{1j}\}$, and put

$$H_n = H_n(D_1, \dots, D_n) = H(D_1) \times \dots \times H(D_n).$$

Now on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ we define an H_n -valued random element $F(s_1, \dots, s_n; \omega)$ by the formula

$$F(s_1, \dots, s_n; \omega) = (f_1(s_1, \omega), \dots, f_n(s_n, \omega)),$$

where

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj} \omega(m) e^{-\lambda_{mj} s_j}, \quad s \in D_j, \quad j = 1, \dots, n.$$

Now we define the space of meromorphic functions. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere with the spherical metric given by the formulas

$$\begin{aligned} d(s_1, s_2) &= \frac{2|s_1 - s_2|}{\sqrt{1 + |s_1|^2} \sqrt{1 + |s_2|^2}}, \\ d(s, \infty) &= \frac{2}{\sqrt{1 + |s|^2}}, \quad d(\infty, \infty) = 0, \\ s_1, s_2, s &\in \mathbb{C}. \end{aligned}$$

Let $M(G)$ stand for the space of meromorphic functions $g: G \rightarrow (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $g_n(s) \in M(G)$ converges to a function $g(s) \in M(G)$, if

$$d(g_n(s), g(s)) \rightarrow 0$$

as $n \rightarrow \infty$ uniformly on compact subsets of G . We put

$$M_n = M_n(D_1, \dots, D_n) = M(D_1) \times \dots \times M(D_n),$$

and let, for $T > 0$,

$$\nu_T(\dots) = \frac{1}{T} \text{meas}\{\tau \in [0, T]: \dots\},$$

where the dots denote some condition satisfied by τ . Then in [1] the following statement was given.

Theorem 1. *For $j = 1, \dots, n$, suppose that the sets $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$ are linearly independent over the field of rational numbers, and that for $f_j(s)$ the conditions (1)–(3) are satisfied. Then the probability measure*

$$\nu_T\left(\left(f_1(s + i\tau), \dots, f_n(s + i\tau)\right) \in A\right), \quad A \in \mathcal{B}(M_n),$$

converges weakly to the distribution of the random element $F(s_1, \dots, s_n; \omega)$ as $T \rightarrow \infty$.

However, the proof of Theorem 1 has a gap. For its validity the hypothesis on the linear independence of the sets $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$, $j = 1, \dots, n$, must be replaced by that on the linear independence of the set $\{\log 2\} \cup \bigcup_{j=1}^n \bigcup_{m=1}^{\infty} \{\lambda_{mj}\}$. However, the main shortcoming of Theorem 1 is the presence of the number $\log 2$ in its hypotheses. This is not natural and not convenient.

The aim of this note is to consider a collection of general Dirichlet series with the same exponents and to remove the number $\log 2$ from the hypothesis on the linear independence.

Let, for $\sigma > \sigma_{aj}$,

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_m s}, \quad j = 1, \dots, n,$$

and let in the definition of $F(s_1, \dots, s_n; \omega)$

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj} \omega(m) e^{-\lambda_m s_j}, \quad s \in D_j, \quad j = 1, \dots, n.$$

Theorem 2. *Suppose that the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers, and that for $f_j(s)$, $j = 1, \dots, n$, the conditions (1)–(3) are satisfied. Then the assertion of Theorem 1 is valid.*

The proof of the theorem is similar to that of Theorem 1 but simpler and shorter than in [1].

2 A limit theorem in H_{2n}

We begin with a limit theorem in the space

$$H_{2n} = H_{2n}(D_1, \dots, D_n) = H^2(D_1) \times \dots \times H^2(D_n).$$

Denote the poles of the function $f_j(s)$ in the region $\sigma > \sigma_{1j}$ by $s_{1j}, \dots, s_{r_j j}$, $j = 1, \dots, n$, and define

$$f_{1j}(s) = \prod_{l=1}^{r_j} (1 - e^{\lambda_1(s_{lj} - s)}).$$

Then, clearly, $f_{1j}(s_{lj}) = 0$ for $l = 1, \dots, r_j$, $j = 1, \dots, n$. This shows that the function

$$f_{2j}(s) = f_{1j}(s)f_j(s)$$

is regular on D_j , $j = 1, \dots, n$. We write

$$f_{1j}(s_j, \omega) = \prod_{l=1}^{r_j} (1 - e^{\lambda_1(s_{lj}-s)}\omega(1))$$

and

$$f_{2j}(s_j, \omega) = \sum_{l=0}^{r_j} \sum_{m=1}^{\infty} a_{m,l}^{(j)} \omega^l(1) \omega(m) e^{-(\lambda_m + l\lambda_1)s},$$

$$s_j \in D_j, \quad j = 1, \dots, n,$$

where the coefficients $a_{m,l}^{(j)}$ are defined by

$$f_{2j}(s) = \sum_{l=0}^{r_j} \sum_{m=1}^{\infty} a_{m,l}^{(j)} e^{-(\lambda_m + l\lambda_1)s}, \quad \sigma > \sigma_{a_j}, \quad j = 1, \dots, n.$$

Moreover, we set

$$Q_j(s, \omega) = (f_{1j}(s, \omega), f_{2j}(s, \omega)), \quad s_j \in D_j, \quad j = 1, \dots, n,$$

and define a probability measure

$$Q_{T,j}(A) = \nu_T((f_{1j}(s + i\tau), f_{2j}(s + i\tau)) \in A),$$

$$A \in \mathcal{B}(H^2(D_j)), \quad j = 1, \dots, n.$$

Lemma 1. *For $j = 1, \dots, n$, the probability measure $Q_{T,j}$ converges weakly to the distribution of the random element Q_j as $T \rightarrow \infty$.*

The lemma is Lemma 10 from [2].

Now let

$$Q = Q(s_1, \dots, s_n; \omega) = (Q_1(s_1, \omega), \dots, Q_n(s_n, \omega)),$$

$$s_j \in D_j, \quad j = 1, \dots, n,$$

and

$$Q_T(A) = \nu_T((f_{11}(s_1 + i\tau), f_{21}(s_1 + i\tau), \dots, f_{1n}(s_n + i\tau), f_{2n}(s_n + i\tau)) \in A), \quad A \in \mathcal{B}(H^{2n}).$$

Lemma 2. *The probability measure Q_T converges weakly to the distribution of the random element Q as $T \rightarrow \infty$.*

Proof. First we prove that the family of probability measures $\{Q_T\}$ is relatively compact, i.e. every sequence of $\{Q_T\}$ contains a weakly convergent subsequence. By Lemma 1, for every $j = 1, \dots, n$, the probability measure

$$\nu_T((f_{1j}(s + i\tau), f_{2j}(s + i\tau)) \in A), \quad A \in \mathcal{B}(H^2(D_j)),$$

converges weakly to the distribution of the random element $Q_j(s, \omega)$ as $T \rightarrow \infty$. Therefore, the family of probability measures $\{Q_{T,j}\}$ is relatively compact, $j = 1, \dots, n$. Since $H^2(D_j)$ is a complete separable space, by the Prokhorov theorem [3], hence we have that the family $\{Q_{T,j}\}$ is tight, i.e. for an arbitrary $\varepsilon > 0$ there exists a compact subset $K_j \subset H^2(D_j)$ such that

$$Q_{T,j}(H^2(D_j) \setminus K_j) < \frac{\varepsilon}{n}, \quad j = 1, \dots, n, \quad (4)$$

for all $T > 0$. Let a random variable η_T be defined on a probability space $(\widehat{\Omega}, \mathcal{F}, \mathbb{P})$ and have the distribution

$$\mathbb{P}(\eta_T \in A) = \frac{\text{meas}\{A \cap [0, T]\}}{T}, \quad A \in \mathcal{B}(R).$$

Consider the $H^2(D_j)$ -valued random element $f_{T,j}(s)$ defined by

$$f_{T,j}(s) = ((f_{1j}(s + i\eta_T), f_{2j}(s + i\eta_T))), \quad j = 1, \dots, n.$$

Taking into account (4), we have

$$\mathbb{P}(f_{T,j}(s) \in H^2(D_j) \setminus K_j) < \frac{\varepsilon}{n}, \quad j = 1, \dots, n. \quad (5)$$

Now let

$$f_T(s_1, \dots, s_n) = (f_{T,1}(s_1), \dots, f_{T,n}(s_n)), \quad s_j \in D_j, \quad j = 1, \dots, n,$$

and let $K = K_1 \times \dots \times K_n$. Then K is a compact subset of the space H_{2n} .

Moreover, (5) yields

$$\begin{aligned} Q_T(H_{2n} \setminus K) &= \mathbb{P}(f_T(s_1, \dots, s_n) \in H_{2n} \setminus K) \\ &= \mathbb{P}\left(\bigcup_{j=1}^n (f_{T,j}(s) \in H_2(D_j) \setminus K_j)\right) \\ &\leq \sum_{j=1}^n \mathbb{P}(f_{T,j}(s) \in H_2(D_j) \setminus K_j) < \varepsilon \end{aligned}$$

for all $T > 0$. This means that the family of probability measures $\{Q_T\}$ is tight. Hence, by the Prokhorov theorem, it is relatively compact.

Now we take arbitrary points $s_1^{(j)}, \dots, s_k^{(j)}$ in the region D_j , and put

$$\sigma_1^{(j)} = \min_{1 \leq l \leq k} \operatorname{Re}(s_l^{(j)}), \quad j = 1, \dots, n.$$

Clearly,

$$\sigma_2^{(j)} \stackrel{\text{def}}{=} \sigma_{1j} - \sigma_1^{(j)} < 0, \quad j = 1, \dots, n.$$

Define a region D by

$$D = \{s \in \mathbb{C}: \sigma > \max_{1 \leq j \leq n} \sigma_2^{(j)}\}.$$

Let $u_{jl}, j = 1, \dots, n, l = 1, \dots, k$, be arbitrary complex numbers. Define a function $h: H_{2n} \rightarrow H(D)$ by the formula

$$h(g_{11}, g_{21}, \dots, g_{1n}, g_{2n}; s) = \sum_{r=1}^2 \sum_{j=1}^n \sum_{l=1}^k u_{jl} g_{rj}(s_s^{(j)} + s),$$

where $s \in D$, and $g_{rj} \in H(D_j), r = 1, 2, j = 1, \dots, n$. Moreover, let

$$\varphi_h(s) = h(f_{11}(s_1), f_{21}(s_1), \dots, f_{1n}(s_n), f_{2n}(s_n); s).$$

We will prove that

$$\varphi_h(s + i\eta T) \xrightarrow[T \rightarrow \infty]{\mathcal{D}} h(Q; s),$$

where $\xrightarrow[T \rightarrow \infty]{\mathcal{D}}$ means the convergence in distribution. Clearly, for all $j=1, \dots, n$,

$$f_{1j}(s) = \sum_{m=0}^{r_j} b_{mj} e^{-\lambda_1 m s}$$

is a Dirichlet polynomial. In the region of absolute convergence $\sigma > \sigma_{aj}$ we have that

$$f_{2j}(s) = \sum_{l=0}^{r_j} \sum_{m=1}^{\infty} a_{m,l}^{(j)} e^{-(\lambda_m + l\lambda_1)s}, \quad \sigma > \sigma_{aj}, \quad j = 1, \dots, n.$$

We put $r = \max_{1 \leq j \leq n} r_j$,

$$\widehat{b}_{mj} = \begin{cases} b_{mj}, & m \leq r_j, \\ 0, & m > r_j, \end{cases}$$

$$v_m = \sum_{l=1}^k u_{j,l} \widehat{b}_{mj} e^{-\lambda_1 m s_l^{(j)}},$$

$$\widehat{a}_{m,\theta}^{(j)} = \begin{cases} a_{m,\theta}^{(j)}, & \theta \leq r_j, \\ 0, & \theta > r_j, \end{cases}$$

and

$$b_{m,\theta,l}^{(j)} = \widehat{a}_{m,\theta}^{(j)} e^{-(\lambda_m + \theta \lambda_1) s_l^{(j)}}.$$

Now suppose that

$$\sigma > \max_{1 \leq j \leq n} (\sigma_2^{(j)} + (\sigma_{aj} - \sigma_{1j})).$$

From the definition of the function h we find

$$\begin{aligned} \varphi_h(s) &= \sum_{j=1}^n \sum_{l=1}^k u_{jl} \sum_{m=0}^{r_j} b_{mj} e^{-\lambda_1 m (s_l^{(j)} + s)} \\ &\quad + \sum_{j=1}^n \sum_{l=1}^k u_{jl} \sum_{\theta=0}^{r_j} \sum_{m=1}^{\infty} a_{m,\theta}^{(j)} e^{-(\lambda_m + \theta \lambda_1) (s_l^{(j)} + s)} \\ &= \sum_{m=0}^r v_m e^{-\lambda_1 m s} + \sum_{j=1}^n \sum_{l=1}^k u_{jl} \sum_{\theta=0}^r \sum_{m=1}^{\infty} b_{m,\theta,l}^{(j)} e^{-(\lambda_m + \theta \lambda_1) s} \\ &\stackrel{def}{=} D_r(s) + \widehat{D}_r(s), \end{aligned}$$

where

$$D_r(s) = \sum_{m=0}^r v_m e^{-\lambda_1 m s}$$

is a Dirichlet polynomial, and $\widehat{D}_r(s)$ is a linear combination of Dirichlet series satisfying conditions (1)–(3). Clearly, the function $\widehat{D}_r(s)$ is regular on D .

Since the set $\{\lambda_m\}$ is linearly independent over the field of rational numbers it can be obtained by using standard arguments, see, for example, [4], Chapter 5, that the probability measure

$$\nu_T((D_r(s + i\tau) + \widehat{D}_r(s + i\tau)) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges weakly to the distribution of the $H(D)$ -valued random element

$$\begin{aligned} \varphi_h(s, \omega) &\stackrel{\text{def}}{=} \sum_{m=0}^r v_m \omega^m(1) e^{-\lambda_1 m s} \\ &+ \sum_{j=1}^n \sum_{l=1}^k u_{jl} \sum_{\theta=0}^r \sum_{m=1}^{\infty} b_{m, \theta, l}^{(j)} \omega^\theta(1) \omega(m) e^{-(\lambda_m + \theta \lambda_1) s}. \end{aligned} \quad (6)$$

Thus, we have proved that the probability measure

$$\nu_T(\varphi_h(s + i\tau) \in A), \quad A \in \mathcal{B}(H(D)),$$

converges to the distribution of the random element (6) as $T \rightarrow \infty$. However, by the definition of h

$$\begin{aligned} \varphi_h(s, \omega) &= \sum_{j=1}^n \sum_{l=1}^k u_{jl} \sum_{m=0}^{r_j} b_{mj} \omega^m(1) e^{-\lambda_1 m (s_l^{(j)} + s)} \\ &+ \sum_{j=1}^n \sum_{l=1}^k u_{jl} \sum_{\theta=0}^{r_j} \sum_{m=1}^{\infty} a_{m, l}^{(j)} \omega^\theta(1) \omega(m) e^{-(\lambda_m + \theta \lambda_1) (s_l^{(j)} + s)} \\ &= \sum_{r=1}^2 \sum_{j=1}^n \sum_{l=1}^k u_{jl} f_{rj} f(s_j^{(j)} + s, \omega) = h(Q; s), \end{aligned}$$

and therefore

$$\varphi_h(s + i\eta T) \xrightarrow[T \rightarrow \infty]{\mathcal{D}} h(Q; s). \quad (7)$$

Now we are ready to prove Lemma 2. We have seen that the family of probability measures $\{Q_T\}$ is relatively compact. Hence we can find a sequence $T_1 \rightarrow \infty$ such that the measure Q_{T_1} converge weakly to some probability measure Q_0 on $(H_{2n}, \mathcal{B}(H_{2n}))$ as $T_1 \rightarrow \infty$. This shows that there exists an H_{2n} -valued random element

$$\widehat{f} = \widehat{f}(s_1, \dots, s_n) = (\widehat{f}_{11}(s_1), \widehat{f}_{21}(s_1), \dots, \widehat{f}_{1n}(s_n), \widehat{f}_{2n}(s_n))$$

with distribution Q_0 defined, say, on a certain probability space $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$. Write the random element $f_T(s_1, \dots, s_n)$ in the form

$$f_T = f_T(s_1, \dots, s_n) = (f_{T,11}(s_1), f_{T,21}(s_1), \dots, f_{T,1n}(s_n), f_{T,2n}(s_n)),$$

where

$$\begin{aligned} f_{T,1j}(s_j) &= f_{1j}(s_j + i\eta_T), \\ f_{T,2j}(s_j) &= f_{2j}(s_j + i\eta_T), \end{aligned}$$

and $s_j \in D_j, j = 1, \dots, n$. By the choice of T_1 we have that $f_{T_1} \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} \hat{f}$, and therefore

$$h(f_{T_1}) \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} h(\hat{f}; s).$$

Hence, in view of the definition of $\varphi_h(s)$,

$$\varphi_h(s + i\eta_{T_1}) \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} h(\hat{f}; s). \quad (8)$$

On the other hand, (7) shows that

$$\varphi_h(s + i\eta_{T_1}) \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} h(Q; s).$$

This and (8) yield

$$h(Q; s) \stackrel{\mathcal{D}}{=} h(\hat{f}; s). \quad (9)$$

Let the function $u: H(D) \rightarrow \mathbb{C}$ be given by the formula

$$u(f) = f(0), \quad f \in H(D). \quad (10)$$

The topology of the space $H(D)$ shows that the function u is measurable, and therefore by (9)

$$u(h(Q; s)) \stackrel{\mathcal{D}}{=} u(h(\hat{f}; s)),$$

and

$$h(Q; 0) \stackrel{\mathcal{D}}{=} h(\hat{f}; 0)$$

by (10). From this it follows that

$$\sum_{r=1}^2 \sum_{j=1}^n \sum_{l=1}^k u_{jl} f_{rj}(s_l^{(j)}, \omega) \stackrel{\mathcal{D}}{=} \sum_{r=1}^2 \sum_{j=1}^n \sum_{l=1}^k u_{jl} \widehat{f}_{rj}(s_l^{(j)}) \quad (11)$$

for arbitrary complex numbers u_{jl} . It is well known [3] that all hyperplanes in \mathbb{R}^{4nk} generates a determining class, and therefore they generate a determining class in \mathbb{C}^{2nk} . Consequently, in view of (11) the \mathbb{C}^{2nk} -valued random elements $f_{rj}(s_l^{(j)}, \omega)$ and $\widehat{f}_{rj}(s_l^{(j)})$, $r = 1, 2$, $j = 1, \dots, n$, $l = 1, \dots, k$, have the same distributions.

Let K_j be an arbitrary compact subset of D_j , and let $v_{rj}(s) \in H(D_j)$, $r = 1, 2$. Now we suppose that the set $\{s_l^{(j)}, 1 \leq l < \infty\}$ is dense in K_j , $j = 1, \dots, n$. Consider the sets of functions

$$G = \{(g_{11}, g_{21}, \dots, g_{1n}, g_{2n}) \in H_{2n} : \sup_{s \in K_j} |g_{rj}(s) - v_{rj}(s)| \leq \varepsilon, \\ j = 1, \dots, n, r = 1, 2\} \quad \text{and}$$

$$G_k = \{(g_{11}, g_{21}, \dots, g_{1n}, g_{2n}) \in H_{2n} : |g_{rj}(s_l^{(j)}) - v_{rj}(s_l^{(j)})| \leq \varepsilon, \\ j = 1, \dots, n, l = 1, \dots, k, r = 1, 2\}.$$

Since the distributions of the random elements $f_{rj}(s_l^{(j)}, \omega)$ and $\widehat{f}_{rj}(s_l^{(j)})$, $r = 1, 2$, $j = 1, \dots, n$, $l = 1, \dots, k$, coincide we have

$$m_H(\omega \in \Omega : Q(s_1, \dots, s_n, \omega) \in G_k) = \mathbb{P}_0(\widehat{f}(s_1, \dots, s_n) \in G_k). \quad (12)$$

Clearly, $G_1 \supset G_2 \supset \dots$. This and the denseness of $\{s_l^{(j)}, 1 \leq l < \infty\}$ show that $G_k \rightarrow G$ as $k \rightarrow \infty$. Hence and from (12) we deduce that

$$m_H(\omega \in \Omega : Q(s_1, \dots, s_n, \omega) \in G) = \mathbb{P}_0(\widehat{f}(s_1, \dots, s_n) \in G). \quad (13)$$

Since H_{2n} is a separable space, finite intersections of spheres in it form a determining class [3]. Therefore, (13) yields

$$Q \stackrel{\mathcal{D}}{=} \widehat{f}.$$

Since $f_{T_1} \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} \widehat{f}$, hence it follows that

$$f_{T_1} \xrightarrow[T_1 \rightarrow \infty]{\mathcal{D}} Q.$$

In other words, the probability measure Q_{T_1} converges weakly to the distribution \widehat{Q} of the random element Q as $T_1 \rightarrow \infty$. Since the family of probability measures $\{Q_T\}$ is relatively compact, the measure \widehat{Q} is independent on the choice of the sequence Q_{T_1} . Since Q_T converges weakly to \widehat{Q} as $T \rightarrow \infty$ if and only if every subsequence $\{Q_{T_1}\}$ of $\{Q_T\}$ contains another subsequence $\{Q_{T_2}\}$ weakly convergent to \widehat{Q} as $T_2 \rightarrow \infty$, hence we obtain the lemma. \square

3 Proof of Theorem 2

Define a metric on the spaces H_{2n} and M_n as the maximum of the metrics on the coordinate spaces. Let the function $u: H_{2n} \rightarrow M_n$ be given by the formula

$$u(g_{11}, g_{21}, \dots, g_{1n}, g_{2n}) = \left(\frac{g_{21}}{g_{11}}, \dots, \frac{g_{2n}}{g_{1n}} \right),$$

where $g_{1j}, g_{2j} \in H_j, j = 1, \dots, n$. In virtue of the equality

$$d(g_1, g_2) = d\left(\frac{1}{g_1}, \frac{1}{g_2}\right)$$

for the spherical metric we have that u is a continuous function. Therefore, by Lemma 2 the probability measure

$$\begin{aligned} & \nu_T((f_1(s_1 + i\tau), \dots, f_n(s_n + i\tau)) \in A) \\ &= \nu_T\left(\left(\frac{f_{21}(s_1 + i\tau)}{f_{11}(s_1 + i\tau)}, \dots, \frac{f_{2n}(s_n + i\tau)}{f_{1n}(s_n + i\tau)}\right) \in A\right), \quad A \in \mathcal{B}(M_n), \end{aligned}$$

converges weakly to the distribution of the random element

$$\left(\frac{f_{21}(s_1, \omega)}{f_{11}(s_1, \omega)}, \dots, \frac{f_{2n}(s_n, \omega)}{f_{1n}(s_n, \omega)}\right),$$

where

$$\begin{aligned} \frac{f_{2j}(s_j, \omega)}{f_{11}(s_j, \omega)} &= \frac{\sum_{l=0}^{r_j} \sum_{m=1}^{\infty} a_{m,l}^{(j)} \omega^l(1) \omega(m) e^{-(\lambda_m + l\lambda_1)s_j}}{\prod_{l=1}^{r_j} (1 - \omega(1) e^{\lambda_1(s_{lj} - s_j)})} \\ &= \sum_{m=1}^{\infty} a_{mj} \omega(1) \omega(m) e^{-\lambda_m s_j} = f_j(s_j, \omega), \\ & \quad s_j \in D_j, \quad j = 1, \dots, n. \end{aligned}$$

\square

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