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Stabilization of Continuous-Time Adaptive Control Systems with Possible Input Saturation through a Controllable Modified Estimation Model*

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Abstract. This paper presents an indirect adaptive control scheme for linear continuous-time systems. The estimated plant model is controllable and then the adaptive scheme is free from singularities. Such singularities are avoided through a modification of the estimated plant parameter vector so that its associated Sylvester matrix is guaranteed to be nonsingular. That property is achieved by ensuring that the absolute value of its determinant does not lie below a positive threshold. An alternative modification scheme based on the achievement of a modified diagonally dominant Sylvester matrix of the parameter estimates is also proposed. This diagonal dominance is achieved through estimates modification as a way to guarantee the controllability of the modified estimated model when a controllability measure of the estimation model without modification fails. In both schemes, the use of a hysteresis switching function for the modification of the estimates is not required to ensure the controllability of the modified estimated model. Both schemes ensure that chattering due to switches associated with the modification is not present. The results are extended to the first-order case when the input is subject to saturation being modeled as a sigmoid function. In this case, a hysteresis-type switching law is used to implement the estimates modification.

Keywords: adaptive control, parameter modification, singular Sylvester matrix.

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1 Introduction

The adaptive stabilization and control of linear continuous and discrete systems has been successfully developed in the two last decades, [1]–[4]. Usually, the plant is assumed to be inversely stable and its relative degree and its high-frequency gain sign are assumed to be known together with an absolute upper-bound for that gain in the discrete case. The assumption on the knowledge of the order can be relaxed by assuming a nominal known order and considering the exceeding modes as unmodeled dynamics, [5]–[9]. The assumption on the knowledge of the high frequency gain has been removed in [4] and [9] and the assumption of the plant being inversely stable has been successfully removed in the discrete case and, more recently, in the continuous one [10, 11], [5]–[7]. The problem is solved by using either excitation of the plant signals or a modification of the least-squares estimation by either using excitation of the plant signals or exploiting the properties of the standard least-squares covariance matrix, [11], [5, 8], [12]. In a set of papers, the assumption of the plant being inversely stable has been removed by using either excitation of the plant signals or estimates modification by using hysteresis switching functions which generate the controllability of the estimated plant model while exploiting the properties of the covariance matrix, [11, 7, 8] and references therein focused on a deterministic approach. An alternative modification strategy was the use of a random search-type algorithm to avoid the degeneracy of the Sylvester matrix, [6]. In [5], a recursive coordinate modification method was given which ensue convergence in a stochastic sense. This paper presents an adaptive stabilization algorithm for continuous-time systems which can have unstable zeros . *The adaptive scheme uses a parameter modification scheme which neither involves hysteresis switching nor takes advantage of the properties of the covariance matrix while guarantees that the absolute value of the determinant of the Sylvester matrix associated with the parameter estimates is bounded from below by a positive threshold. An alternative modification procedure which is based upon the achievement of a diagonally dominant Sylvester matrix of the modified estimates is also proposed.* This modification is an alternative method in the case when a sufficiency test on maintenance of controllability of the unmodified estimated model fails. Such a test consists of guaranteeing through the manipulation of matrix norms that the maximum absolute eigenvalue of the Sylvester matrix of such a model is bounded above by a finite real constant while the minimum one is bounded from below by a positive real constant. The boundedness and convergence of all the estimates and controller parameters is guaranteed in both the ideal perfectly modeled case and when the wide class of unmodelled dynamics and bounded disturbances considered in [7]–[15] are present. The plant input and output are bounded and converge to zero in the ideal perfectly modeled case while they are bounded in the above mentioned non ideal situation. Section 2 is devoted to the synthesis of the adaptive stabilizer in the perfectly modelled case for unknown continuous-time plants. The basic estimation scheme, used prior to the modification procedure, is of least-squares type. The two above mentioned estimation modification procedures are also given. Section 3 presents the convergence and stability properties of the proposed scheme. Some robustness issues against the presence of unmodeled dynamics and bounded disturbances are also pointed out the mechanism used to guarantee robustness is the variation of the basic estimation scheme by adding a relative dead zone so that the estimation and covariance matrix adaptation are frozen when the adaptation error is small compared to an absolute overbounding function of the contribution to the uncertainties to the output. The modification procedures that ensure controllability of the estimated model are kept as in the ideal case. The scheme's modifications to operate in the case of presence of unmodelled dynamics and/or bounded disturbances are also given. A numerical example is given in Section 4 and, finally, conclusions end the paper. The mathematical proofs of the results are developed in Appendix.

2 Adaptive stabilizer for a continuous-time plant

In the sequel, the time-argument is suppressed unless confusion can arise and the constant parameters are denoted by a superscript "$*$". Consider the following continuous-time controllable system

$$A^*(D)y(t) = B^*(D)u(t), \quad D^i y(0) = y_0^{(i)} \quad (i = 0, 1, \ldots, n-1) \qquad (1)$$

where $D^i \equiv \dfrac{d^i}{dt^i}$ $(i = 0, 1, \ldots, n-1)$ is the i-th time-derivative operator, $A^*(D) = D^n + \sum_{i=1}^{n} a_i^* D^{n-i}$ and $B^*(D) = \sum_{i=0}^{m} b_i^* D^{m-i}$ with $n \geq m$. Since (1) is controllable then its associated $(n+m)$ Sylvester resultant matrix

$$S(\theta_0^*) = \begin{bmatrix} 1 & 0 & \cdots & 0 & b_0^* & 0 & \cdots & \cdots & 0 \\ a_1^* & 1 & \ddots & \vdots & b_1^* & b_0^* & & & \vdots \\ \vdots & a_1^* & \ddots & 0 & \vdots & b_1^* & \ddots & & \vdots \\ \vdots & \vdots & & 1 & b_m^* & \vdots & \ddots & \ddots & 0 \\ a_n^* & \vdots & & a_1^* & 0 & b_m^* & & \ddots & b_0^* \\ 0 & a_n^* & & \vdots & \vdots & & \ddots & & b_1^* \\ \vdots & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ \underbrace{0 \quad \cdots \quad 0 \quad a_n^*}_{m} & & & & \underbrace{0 \quad \cdots \quad \cdots \quad 0 \quad b_m^*}_{n} \end{bmatrix}$$

is nonsingular. Define the filtered signals:

$$E^*(D)u_f = u, \quad E^*(D)y_f = y, \quad E^*(D) = D^n + \sum_{i=1}^{n-1} e_i^* D^{n-i} \tag{2}$$

with $E^*(D)$ being a strictly Hurwitz polynomial. The filtered control law for a known plant (1) is generated as

$$S^*(D)u_f = -R^*(D)y_f \tag{3}$$

where $S^*(D) = D^n + \sum_{i=1}^{n} s_i^* D^{n-i}$, $R^*(D) = D^n + \sum_{i=0}^{m-1} r_i^* D^{m-i-1}$, satisfy the diophantine equation:

$$A^*(D)S^*(D) + B^*(D)R^*(D) = C^*(D)$$

where $C^*(D) = D^n + \sum_{i=1}^{n^*-1} c_i^* D^{n^*-i}$ of prefixed degree fulfilling the constraint $n^* \leq n + \deg\big(S^*(D)\big) \leq 2n$ is a strictly Hurwitz polynomial (i.e., with roots in $ReD < 0$) which defines the suited closed-loop dynamics. $S^*(D)$ and $R^*(D)$ are the unique solution to the above diophantine equations since $A^*(D)$ and $B^*(D)$ are coprime because of the controllability of (1) and the constraints

$\deg\big(S^*(D)\big) \le \deg\big(E^*(D)\big) \le n$ and $\deg\big(R^*(D)\big) < \deg\big(A^*(D)\big)$. (In particular, if $E^*(D)$ satisfies $\deg\big(E^*(D)\big) \le n-1$ then its appropriate coefficients in (2) are zeroed). Equation (3) is equivalent to its unfiltered version:

$$u = \big(E^*(D) - S^*(D)\big)u_f - R^*(D)y_f. \tag{4}$$

The control objective in the adaptive case for unknown plant is to update the controller parameters s_i and r_j $(i = 1, 2, \ldots, n,\ j = 0, 1, \ldots, m)$ in an adaptive way so that the plant (1), subject to the control law (4) when replacing the parameters by their estimates, is asymptotically stable in the large in the absence of disturbances. Under bounded noise and a standard class of unmodelled dynamics, the scheme is guaranteed to be globally stable. Simple direct calculus with (1), (2) yields for filtered signals:

$$D^n y_f = \theta^{*T}\varphi \tag{5}$$

$$\begin{aligned}
\text{with}\quad \theta^* &= [\theta_0^{*T} \vdots \varepsilon_0^{*T}]^T \\
&= [\theta_1^*, \theta_2^*, \ldots, \theta_{n+m+1}^* \vdots \theta_{n+m+2}^*, \theta_{n+m+3}^*, \ldots, \theta_{2n+m+1}^*]^T \\
&= [b_0^*, b_1^*, \ldots, b_m^* a_1^*, a_2^*, \ldots, a_n^* \vdots \varepsilon_{01}^*, \varepsilon_{02}^*, \ldots, \varepsilon_{0n}^*]^T, && \text{(6a)}\\
\varphi(t) &= \left[\varphi_0^T(t), i_\varphi^T(t)\right]^T = [D^m u_f, D^{m-1}u_f, \ldots, u_f, && \text{(6b)}\\
&\quad - D^{n-1}y_f, -D^{n-2}y_f, \ldots, y_f, i_1, i_2, \ldots, i_n]^T && \text{(6c)}
\end{aligned}$$

where $g(t) = \varepsilon_0^T(t)i(t)$ is an exponentially decaying term that depends on initial conditions and each $i_j(t)$ is known and it has the form $t^\ell e^{\lambda_k^* t}$ for $\ell = 0, 1, \ldots, m_k - 1$ with m_k being the multiplicity of the root λ_k^* of $C^*(D)$. There are m_k terms $i_{(\cdot)}(t)$ of such a form for each λ_k^*. The parameter vector θ^* is estimated by using an standard least-squares algorithms of covariance matrix $P(t)$ and estimated vector $\theta(t) = \big(\theta_0^T(t), \varepsilon_0^T(t)\big)^T$ with $\varepsilon_0(t)$ being the estimation of the initial conditions of ε_0^*. The estimation algorithm consists of an estimation algorithm and a rule to modify such estimates as follows:

2.1 Parameter estimation

$$e = D^n y_f - \theta^T\varphi \quad \text{(prediction error)}, \tag{7a}$$

$$\dot{\theta} = P\varphi e, \tag{7b}$$
$$\dot{P} = -P\varphi\varphi^T P; \quad P(0) = P^T(0) > 0. \tag{7c}$$

The basic modification of the estimated plant model is performed when necessary to maintain the controllability of the estimated model in the sense that $\left|\mathrm{Det}\big(S(\bar{\theta}_0)\big)\right| \geq \rho > 0$ even if $\left|\mathrm{Det}\big(S(\theta_0)\big)\right| < \rho$ for some positive real constant ρ while the Sylvester matrices of the "a priori" and modified estimates have the same structures as $S(\theta_0^*)$ and their values are obtained by replacing θ_0^* with θ_0 and $\bar{\theta}_0$, respectively. The modification scheme to calculate $\bar{\theta}$ from θ is implemented according to the following scheme:

2.2 Basic modification of the estimation

The plant parameter estimates through the algorithm (7) are then modified as follows. First, define the strictly positive piecewise constant real time-function $h(\cdot)$ and nonnegative time-functions δ_α and α as follows for positive real constants ρ and $\rho' \geq \rho$:

$$h(0)=\rho,\ h(t^+)=\begin{cases} \rho \ \text{ if } h(t)=\rho \ \text{ and } \left|\mathrm{Det}\big(S(\theta_0)\big)\right| \neq \rho \ \text{ for } t=t^-, \\ \rho \ \text{ if } h(t)=\rho' \text{ and } \left|\mathrm{Det}\big(S(\theta_0)\big)\right| = \rho' \text{ for } t=t^-, \\ \rho' \text{ if } h(t)=\rho' \text{ and } \left|\mathrm{Det}\big(S(\theta_0)\big)\right| \neq \rho' \text{ for } t=t^-, \\ \rho' \text{ if } h(t)=\rho \ \text{ and } \left|\mathrm{Det}\big(S(\theta_0)\big)\right| = \rho \ \text{ for } t=t^-, \end{cases} \tag{8a}$$

$$\delta_\alpha=\begin{cases} \dfrac{3h-\mathrm{Det}\big(S(\theta_0)\big)}{\bar{C}}=\dfrac{3h-\left|\mathrm{Det}\big(S(\theta_0)\big)\right|\mathrm{Sign}(\bar{C})\big(\mathrm{Det}\big(S(\theta_0)\big)\big)}{\bar{C}} \\ \quad \text{if} \quad \left|\mathrm{Det}\big(S(\theta_0)\big)\right| < h, \\ 0 \quad \text{if} \quad \left|\mathrm{Det}\big(S(\theta_0)\big)\right| \geq h, \end{cases} \tag{8b}$$

$$\alpha = \begin{cases} \delta_\alpha \bar{C} & \text{if} \quad \delta_\alpha \bar{C} \geq 1, \\ (\delta_\alpha \bar{C})^{\frac{1}{n+m}} & \text{if} \quad \delta_\alpha \bar{C} < 1 \end{cases} \tag{8c}$$

for some small prefixed positive real constant ρ of upper-bound specified later and

$$\bar{C} = \Big\{ C(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{n+m+1})\colon \big|C(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{n+m+1})\big| = \max_{\sigma_i \in \{0,-1,1\}} \big|C(\bar{\sigma}_1, \ldots, \bar{\sigma}_{n+m+1})\big| \Big\}, \tag{8d}$$

$$C(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{n+m+1}) = \sum_{k=1}^{n+m} \sum_{i_1, i_2, \ldots, i_k = 1}^{n+m+1} \frac{1}{k!} \mathrm{Trace}\big(S_{\theta_{i_1}}(\theta_0) \widetilde{S}_{\theta_{i_1} \ldots \theta_{i_k}}(\theta_0)\big) \prod_{j=i_1}^{i_k} [\sigma_j], \tag{8e}$$

$$(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{n+m+1}) = \big\{ \mathrm{Arg}(\sigma_1, \sigma_2, \ldots, \sigma_{n+m+1})\colon \bar{C} = C(\sigma_1, \sigma_2, \ldots, \sigma_{n+m+1}) \text{ and } \sigma_i \in \{0, -1, 1\},\ i = 1, 2, \ldots, n+m+1 \big\} \tag{8f}$$

where $\widetilde{S}(\theta_0)$ is the matrix of cofactors of $S(\theta_0)$, with subscripts denoting partial first or higher-order derivatives with respect to the respective arguments, and the first-order derivatives with respect to the parameter estimates are:

$$S_{a_i}(\theta_0) = \frac{dS}{da_i}\Big|_{\theta_0} = \begin{bmatrix} 0_{i \times (n+m)} \\ \cdots\ \cdots\ \cdots \\ I_m \quad 0_{m \times n} \\ \cdots\ \cdots\ \cdots \\ 0_{(n-i) \times (n+m)} \end{bmatrix} \begin{matrix} \leftarrow (i+1)\text{-th row} \\ (i = 1, \ldots, n) \end{matrix}$$

$$S_{b_j}(\theta_0) = \frac{dS}{db_j}\Big|_{\theta_0} = \begin{bmatrix} 0_{j \times (n+m)} \\ \cdots\cdots\cdots \\ I_m \quad 0_{m \times n} \\ \cdots\cdots\cdots \\ 0_{(n-i) \times (n+m)} \end{bmatrix} \begin{matrix} \leftarrow (j+1)\text{-th row} \\ (j = 0, 1, \ldots, m) \end{matrix} \tag{8g}$$

$$\bar{\theta} = \theta + \bar{\delta}, \tag{9a}$$

$$\bar{\delta} = [\delta\theta_1, \delta\theta_2, \ldots, \delta\theta_{n+m+1}, 0, \overbrace{\ldots}^{n}, 0]^T = [\bar{\delta}_0^T, 0, 0^T]^T = [\delta b_0, \delta b_1, \ldots, \delta b_m, \delta a_1, \delta a_2, \ldots, \delta a_n, 0, \ldots, 0]^T, \tag{9b}$$

$$\bar{a}_i = a_i + \delta a_i = a_i + \alpha \bar{\sigma}_i, \quad \bar{b}_j = b_j + \delta b_j = b_j + \alpha \bar{\sigma}_{n+1+j}, \quad i = 1, 2, \ldots, n,\ j = 0, 1, \ldots, m. \tag{9c}$$

Note that $(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_{n+m+1})^T$ is a non necessarily unique vector, whose components take values in the set $\{1, 0, -1\}$ which maximizes the function $C(\sigma_1, \sigma_2, \ldots, \sigma_{n+m+1})$ for the of constraints $\sigma_i \in \{1, 0, -1\}$ for $i = 1, 2, \ldots,$ $n+m+1$. The idea behind the above modification method (8), (9) is basically the following. Two different thresholds are used to modify the parameter components. The use of two thresholds ρ and ρ' is only made for purposes of avoiding chattering by involving the mechanism of switching between them each time that a discontinuity in the modification is found. These thresholds are sufficiently small compared to the stability abscissa of the objective polynomial $C^*(D)$ in order to guarantee the closed-loop stability. Each absolute value of a parameter estimate is either modified with a maximum amount $\alpha(t)$ or such a parameter becomes unmodified (see (9)). The maximum value of depends on the thresholds ρ and ρ' (see (8a)–(8c)). The mechanism which ensures that the absolute value modified Sylvester determinant exceeds the corresponding threshold is to manipulate its Taylor expansion around its unmodified value by checking the maximum allowable absolute increase by increasing each of all the estimates in $\pm\alpha$ or leaving them unmodified. See (8d)–($8g$).

More in detail, assume that each i-th parameter component of θ_0 is modified by an additive increment so that the modification $\alpha\sigma_i$ scheme is $\bar{\theta}_0 = \theta_0 + \alpha(\sigma_1, \ldots, \sigma_{n+m+1})^T$. A well-known equation from Linear Algebra is

$$\frac{d}{d\theta_{0i}} \big(\mathrm{Det}\big(S(\theta_0)\big)\big)\big|_{\theta_0 = q_0} = \mathrm{Trace}\big(S_{\theta_{0i}}(q_0)\widetilde{S}(q_0)\big),$$

[13], from which higher-order derivatives with respect to the various parameter vector components. Thus, by using a series Taylor expansion of the analytic multivariable function of the modified estimates $\mathrm{Det}\big(S(\bar{\theta}_{01}, \ldots, \bar{\theta}_{0,n+m+1})\big)$ around $\mathrm{Det}\big(S(\theta_{01,\ldots,n+m+1})\big)$ (later denoted $\mathrm{Det}\big(S(\theta_0)\big)$ by for notation simplicity purposes) which is considered as a multivariable function of all the parameter components, the identity $\mathrm{Det}\big(S(\bar{\theta}_0)\big) = \big(S(\theta_0)\big) + C\alpha$, with the function C being calculated from (8e). The switches in $h(t)$ between ρ and ρ' given by (8a) have as objective of avoiding chattering so that the existence of solution is ensured for all time. Chattering could potentially arise if the Sylvester determinant would converge to a constant function h while, at the

same time, its time-derivative converges to zero with changing sign. This phenomenon is avoided in this approach by using the switching rule (8a) by taking advantage of the fact that the unmodified and modified parameter estimates converge asymptotically to finite limits. Thus if the Sylvester determinant converges to ρ (or ρ') after a large but finite time it cannot converge to ρ' (or ρ) while it remains in a certain small neighborhood centered at ρ' (or ρ). The avoidance of chattering guarantees the existence of solution. These features will be proved in the following section of the paper.

The above modification procedure basically operates as follows. Assume that θ_i is any estimate $a_{(\cdot)}$ or $b_{(\cdot)}$. If $\sigma = 0$ then such a parameter does not contribute to the maximum C (i.e., to $\bar{C}$). That means that if the parameter were accounted for in (8c) for eventual parameter modification with both signs, i.e., $\sigma = \pm 1$, then C would have less absolute value. If $\sigma = \pm 1$, then the parameter contributes to $\bar{C}$, i. e., if it is accounted for to calculate $\bar{C}$ which reaches a larger absolute value than for any other possibilities for accounting or not all the remaining parameter estimates. At the end of the modification procedure, all the estimates whose corresponding $\bar{\sigma}_{(\cdot)}$ is ± 1 become modified while those ones whose corresponding $\bar{\sigma}_{_{(\cdot)}}$ is zero remain unmodified. The use of two distinct values ρ and ρ' to deal with switches in the determinant test is just to avoid that the potential situation of the determinant converging to one of those values implies the non existence of solution in the closed-loop system. Therefore, an isolated discontinuity (the test for switching $h_{(\cdot)}$ from one value to the other in (8a)) ensures the existence of solution and the problem of convergence of the determinant of the Sylvester matrix of the unmodified estimates to one of those values is avoided since in finite, but large, time the determinant is close to its limit, since the estimates have a limit, as proved in Theorem 1 and the corresponding discontinuity of f ensures that ho new switches would arise. *It is proved in Appendix, as an intermediate step in the proof of the subsequent controllability result, that for all time because not all the derivatives in* (8e) *with respect to the estimates evaluated at the parameter vector estimated from the algorithm* (7) *are zero.* This feature makes possible that the Sylvester determinant of the modified estimates can always be modified with respect to its value prior to modification. It becomes obvious from the above modification philosophy that $|\bar{C}|$ can be replaced by any value of $|C|$ which be bounded from below by a positive constant. The main idea behind its proof is that the scalar function $\mathrm{Det}\big(S(\theta_0)\big)$ whose $n+m$ arguments are all the estimates $a_{(\cdot)}$ and $b_{(\cdot)}$ built through (8), (9) is not constant at any real interval. This property will follow from the fact that at least one of its first-order derivatives (i.e., the components of its gradient with respect to the estimated parameters)or of its successive higher order derivatives in the parameter space of estimates is nonzero. Therefore, the modification rule (8), (9) allows the modification of the estimates when necessary so that the constraint $\big|\mathrm{Det}\big(S(\bar{\theta}_0)\big)\big| \geq \rho$ is fulfilled. The following result relies on the controllability of the modified estimated model:

Proposition 1. *Assume that* $\rho < \dfrac{|\sigma|}{6(n+m)}$ *where* $(-\sigma)$ *is the convergence abscissa of* $C^*(D)$. *Thus, modified estimation scheme* (8), (9) *of the plant model estimated from* (7) *fulfils at all time* $\big|\mathrm{Det}\big(S(\bar{\theta}_0)\big)\big| \geq \rho > 0$ *so that such a model is controllable. Furthermore, there is no chattering caused by switches in the estimates modification rule* (8a)–(8c).

2.3 Alternative modification of the estimation

A second variation of the above estimation modification rule of (8), (9) is given below by modifying the algorithm rules (8) and (9c). It is based on ensuring that the Sylvester matrix of the modified estimates is diagonally dominant in the case when that associated with estimates without modification is not guaranteed to be controllable under a sufficiency test. Such a test is based on the evaluation of matrix norms of $S(\theta_0)$ and it does not requires the computation of its eigenvalues. First, define small positive real constants $\varepsilon_{bi}, \varepsilon_{0i}$ and ε'_{0i} fulfilling $\varepsilon_{b2} \geq \varepsilon_{b1} + \bar{\varepsilon}_0,\ \varepsilon_{02} \geq \varepsilon_{01} + \bar{\varepsilon}_1,\ \varepsilon'_{02} \geq \varepsilon'_{01} + \bar{\varepsilon}'_1$ as well as an arbitrary large real constant $T > 0$ and an arbitrary large integer $N > 0$. Then, establish Condition 1 for controllability test purposes of the estimated model before modification at any time t as follows:

It is said that **Condition 1** holds at time t if

$$(n+m)^{\frac{1}{2}}\big\|S(\theta_0)\big\|_\infty \geq \frac{1}{\varepsilon'_0} \quad \text{and} \quad (n+m)^{-\frac{1}{2}}\big\|S(\theta_0)\big\|_1 \leq \frac{1}{\varepsilon_0} \qquad (10a)$$

with $\varepsilon_0(t) = \varepsilon_0(t-T) = \varepsilon_{0i},\ \varepsilon_0'(t) = \varepsilon_0'(t-T) = \varepsilon_{0i}'$ for some $i \in \{1,2\}$ if $N_t^- \leq N$, and $\varepsilon_0(t) = \varepsilon_{0j}$, $\varepsilon_0'(t) = \varepsilon_{0j}'$, for some $j \in \{1,2\}$ with $j \neq i$ if $N_t^- = N+1$ where N_t^- is the overall number of times where inequalities $(10a)$ are simultaneously violated with the same values for the constants on the finite but large time interval $[t-T,t)$, i.e. with either $(\varepsilon_{01}, \varepsilon_{01}')$ or $(\varepsilon_{02}, \varepsilon_{02}')$. After any switch in the values of both constants in $(10a)$, N_t is set to zero, i.e., if $N_t^- = N+1$ then $N_t = 0$.

Also, Condition 2 is now established for testing if b_m belongs to a small neighborhood around zero as follows :

It is said that **Condition 2** holds at time t if $|b_m| \geq \varepsilon_b$ with $\varepsilon_b(t) = \varepsilon_b(t-T) = \varepsilon_{bi}$ for some $i \in \{1,2\}$ if $N_t'^- \leq N$; and $\varepsilon_b(t) = \varepsilon_{bj}$ for some $j \in \{1,2\}$ with $j \neq i$ if $N_t'^- = N+1$ where $N_t'^-$ is a the overall set of consecutive violations of Condition 2 on the time interval $[t-T,t)$ which operates in the same way as for Condition 1

The parameter estimates are now modified as follows by using Conditions 1, 2. Modify (9c) as follows:

$$\delta a_i = \begin{cases} 0 & \text{if} \quad \text{Condition 1 holds,} \\ -\alpha a_i & \text{otherwise,} \end{cases}$$
$$\delta b_j = \begin{cases} 0 & \text{if} \quad \text{Condition 1 holds,} \\ -\alpha b_j & \text{otherwise,} \end{cases} \qquad (10b)$$
$$(i = 0, 1, \ldots, n,\ j = 0, 1, \ldots, m-1),$$

$$\delta b_m = \begin{cases} 0 & \text{if} \quad \text{Condition 1 holds,} \\ \beta b_m & \text{if} \quad \text{Condition 1 does not hold} \\ & \quad \text{and Condition 2 holds,} \\ \beta' & \text{if} \quad \text{Condition 1, 2 do not hold,} \end{cases} \qquad (10c)$$

$$\alpha = \frac{\sum_{i=1}^{n} |a_i| + \sum_{i=1}^{m} |b_i| + \rho_\alpha - 1}{\sum_{i=1}^{n} |a_i| + \sum_{i=1}^{m} |b_i|}, \qquad (11a)$$

$$\beta = \begin{cases} \frac{1}{b_m}\left\{(1-\alpha)\Big[\sum\limits_{i=1}^{n}|a_i| + \sum\limits_{i=1}^{m-1}|b_i| + \gamma|b_0|\Big] + \rho_\beta\right\} - 1 \\ \quad \text{if} \quad \varepsilon_b \le |b_m| \le \sum\limits_{i=1}^{n}|a_i| + \sum\limits_{i=1}^{m-1}|b_i| + \gamma|b_0| + \rho_b, \\ 0 \quad \text{if} \quad |b_m| \ge \max\Big(\varepsilon_b,\ \sum\limits_{i=1}^{n}|a_i| + \sum\limits_{i=1}^{m-1}|b_i| + \gamma|b_0| + \rho_b\Big), \end{cases} \tag{11b}$$

$$\beta' = \sum_{i=1}^{n}|a_i| + \sum_{i=1}^{m-1}|b_i| + \gamma|b_0| + \rho'_b + |b_m|,\ \gamma = \begin{cases} 0 & \text{if} \quad m = n, \\ 1 & \text{if} \quad m < n \end{cases} \tag{11c}$$

for prefixed given constants $\rho_\alpha \in (\rho'_\alpha, 1]$, $\rho'_\alpha \in (0,1)$, $\rho_\beta > 0$; $\rho'_\beta > 0$, and β and β' are calculated for all time for the implementation of the modification with and $\alpha = \rho_\alpha = 1$ if $\sum\limits_{i=1}^{n}|a_i| + \sum\limits_{i=0}^{m}|b_i| = 0$.

Remark 1. *Condition 1 guarantees that all the absolute values eigenvalues of the Sylvester matrix of the estimated model* (7) *are positive and upper-bounded by a finite constant. As a result, Condition 1 guarantees that* $\left|\mathrm{Det}\big(S(\theta_0)\big)\right|$ *is bounded away from zero. If it is violated Condition 2 guarantees that the Sylvester matrix is diagonally dominant and then nonsingular. The scheme is stated in terms of achieving similar absolute relative increments in the modified estimated model for each nonzero estimate distinct of* b_m*. This is a major difference with the modification scheme* (9), (10).

The reason of using pairs of distinct test values for checking those conditions is to avoid chattering at their switching points, i.e., when $\left\|S(\theta_0)\right\|_\infty \to \frac{1}{\varepsilon'_0\sqrt{n+m}}$ *and* $\left\|S(\theta_0)\right\|_1 \to \frac{\sqrt{n+m}}{\varepsilon_0}$ *simultaneously as time tends to infinity with either constant values* $(\varepsilon_{01}, \varepsilon'_{01})$ *or* $(\varepsilon_{02}, \varepsilon'_{02})$ *(Condition 1), or when* $|b_m| \to \varepsilon_{b2}$ *(Condition 2). The reason is that the unmodified estimates have finite limits depending on the initial conditions of the estimation algorithm so that each norm of the Sylvester matrix or* $|b_m|$ *cannot converge to two distinct values. A possible convergence to any of the switching points of the matrix norms and* $|b_m|$ *(which would imply chattering) is avoided with the use of Conditions 1, 2 in (10). The mechanism used is to switch the values of the constants after a large number* N *of consecutive switches have occurred with the same values of those constants over a prefixed arbitrarily large time interval* T*.*

Remark 2. *Note that the switches in the alternative modification scheme, equations* (9a), (9b) *and* (10), (11)*, automatically end in some finite time as it follows from the subsequent reasoning. Assume that the limits of the above norms and* $|b_m|$ *estimate are arbitrarily close to any of the switching points of Conditions 1, 2 after a large time because the unmodified estimates are very close to their limit points. The existence of these limits will follow rigorously from the properties of the estimation and modification algorithms proved in the subsequent section. Thus, the switching conditions change after extra finite time to their alternative values because of the structure of the modification rule. More switches cannot occur after extra time* νT *(some finite* ν*) since the (very close to its limit) unmodified estimates do not generate switches from Conditions 1, 2 for one of the two values of the* $\varepsilon_{(\cdot)b}$*-constants. A good practical strategy to apply coherently Condition 1 is the use of very large values for* ε'_{0i} *and very small ones for* ε_{0i} *and a sufficient (although small) values for* $|\varepsilon'_{02} - \varepsilon'_{01}|$ *and for a* $|\varepsilon_{02} - \varepsilon_{01}|$ *fast ending of the switches of the modification mechanism. As in the basic modification mechanism, a possibly existing "a priori" knowledge on the true plant parameters could be used to design the various constants so that Conditions 1, 2 hold for the true plant so that if the estimates converge to the true parameters, the modification mechanism is switched off automatically in finite time. However, the absence of that knowledge do not affect to the stability of the closed-loop system.*

The subsequent result is also proved in Appendix.

Proposition 2. *If Condition 1 holds then the estimated plant model obtained from the algorithm* (7) *is controllable and its associate Sylvester matrix is nonsingular. If it does not hold then the alternative modification scheme* (10), (11) *is controllable for all time and it does not exhibit chattering generated by switches related to Conditions 1, 2.*

Remark 3. *A simple motivation of Propositions 1, 2 can be obtained from the Perturbation Banach's Lemma from Numerical Analysis,* [14] *that establishes that small perturbations of nonsingular matrices yield to nonsingular matrices. In terms of Sylvester matrices, the modification rule* (9a) *implies that,* $S'(\bar{\theta}_0, \sigma_{(\cdot)}) = S'(\theta_0) + \alpha.\delta S'(\theta_0, \sigma_{(\cdot)})$*, when the modification takes* *place, where the superscript prime indicates than the first row and column of the Sylvester matrices have been deleted, since they are irrelevant for their determinants and*

$$\delta S'(\theta_0, \sigma_{(\cdot)}) = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & \sigma_0 & \cdots & 0 \\ \vdots & \sigma_1 & 0 & \vdots & \sigma_1 & \ddots & \vdots \\ \sigma_n & \vdots & \ddots & 0 & \vdots & \ddots & \sigma_0 \\ 0 & \sigma_n & & \sigma_1 & \sigma_m & & \sigma_1 \\ \vdots & 0 & \ddots & \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma_n & 0 & \cdots & \sigma_m \end{bmatrix}$$

is a $(n+m)$-square matrix with each σ_i potentially taking values in the set $\{0, -1, 1\}$ i the modification scheme (8), (9). *By simple inspection it is easy to see that $\delta S'(\theta_0, \sigma_{(\cdot)})$ can be built as being nonsingular for many of the choices of the $\sigma_{(\cdot)}$. (Constructions such like $\sigma_{n+i+1} = \sigma_i$ $(i = 1, 2, \ldots, n)$ for $m = n - 1$ have to be excluded since $\delta S'(\theta_0, \sigma_{(\cdot)})$ becomes singular). Thus, $S'(\bar{\theta}_0, \sigma_{(\cdot)})$ is nonsingular and*

$$\left\| S^{-1}(\bar{\theta}_0, \sigma_{(\cdot)}) \right\| \leq \frac{\alpha^{-1} \left\| \delta S'^{-1}(\theta_0, \sigma_{(\cdot)}) \right\|}{1 - \alpha^{-1} \left\| \delta S'^{-1}(\theta_0, \sigma_{(\cdot)}) \right\| \left\| S(\theta_0) \right\|}$$

for any matrix norm provided that $\alpha > \left\| \delta S'^{-1}(\theta_0, \sigma_{(\cdot)}) \right\| \left\| S(\theta_0) \right\|$ what follows if $\alpha > (n+m+1) \max \left(\sum_{i=1}^{n} a_i^2, \sum_{i=0}^{m} b_i^2 \right)^{1/2}$ by taking ℓ_1 matrix norms. Since $\delta_\alpha \geq \alpha$ from (8b). *That means that if α or δ_α is sufficiently large compared to a measure of the absolute values of the estimates, then the modified Sylvester matrix can be made nonsingular even if that prior to the modification is singular. A lower-bound for δ_α is given explicitly in the proof of Proposition 1. The modification rule* (10), (11) *is based on guaranteeing that either the unmodified Sylvester matrix is nonsingular and no modification is made or the modified Sylvester matrix is diagonally dominant and then nonsingular. For this case, $\delta S'(\theta_0) = \mathrm{Diag}(-\alpha, \ldots, -\alpha, \beta) S'(\theta_0)$ under modification for nonzero b_m and $\delta S'(\theta_0) = \mathrm{Diag}(-\alpha, \ldots, -\alpha, 0) S'(\theta_0) + \beta'$, otherwise.*

2.4 Stabilizing adaptive control law

Introducing (9a) into (7a), we obtain:

$$\begin{aligned} D^n y_f &= e + \theta^T \varphi = e + (\bar{\theta}^T - \bar{\delta}^T)\varphi \\ &= e + A(D,t) y_f + B(D,t) u_f + \varepsilon_0^T(t) i_\varphi(t) \end{aligned} \tag{12}$$

with $A(D,t)$ and $B(D,t)$ being time-varying polynomials associated with the estimates obtained from (7), which define the estimated model of the plant prior to eventual modification, and whose adjustable parameters are the components of the "a priori" estimated vector θ. *The filtered and unfiltered control inputs are generated from the adaptive version of* (3), (4),

$$S(D,t) u_f = -R(D,t) y_f, \tag{13}$$

$$u = \big(E^*(D) - S(D,t)\big) u_f - R(D,t) y_f \tag{14}$$

so that the following closed-loop diophantine equation is satisfied by the controller polynomials $R(D)$ and $S(D)$ which are calculated from modified parameter estimates:

$$\bar{A}(D,t) S(D,t) + \bar{B}(D,t) R(D,t) = C^*(D) \tag{15a}$$

with

$$\bar{A}(D,t) = A(D,t) + \delta A(D,t), \quad \bar{B}(D,t) = B(D,t) + \delta B(D,t),$$
$$\delta A(D,t) = \sum_{i=1}^{n} \delta a_i D^{n-i} \quad \text{and} \quad \delta B(D,t) = \sum_{i=0}^{m} \delta b_i D^{m-i}.$$

The solution is unique since the modified plant parameter estimated model is controllable at all time what implies that the time-varying polynomials $\bar{A}(D,t)$ and $\bar{B}(D,t)$ are coprime for all time.

2.5 Calculation of the parameters of the adaptive stabilizer

The expression (15a) is equivalent to the following algebraic linear system

$$S(\bar{\theta}_0) v = c^* \tag{15b}$$

for all time with

$$v = [1, s_1, \ldots, s_n, r_0, r_1, \ldots, r_{m-1}]^T,$$
$$c^* = [1, c_1^*, c_2^*, \ldots, c_{n^*}^*]^T \tag{15c}$$

which is uniquely solvable with updated parameters at all time in $s_{(\cdot)}$ and $r_{(\cdot)}$ which are used to generate the filtered plant input (3) so that the reference closed-loop dynamics characteristic equation is $C^*(D) = 0$.

3 Stability results

The following assumption on some of the design constants is introduced to guarantee the stability of the closed-loop system under Estimates Modification.

Assumption 1. (a) The design constant ρ in (8a) is chosen sufficiently small according to the constraint $\rho < \dfrac{|\sigma|}{6(n+m)}$ in the Basic Modification Scheme of Subsection 2.2, equations (8), (9).

(b) The design constant ε_{01}' is sufficiently large and the design constants $\varepsilon_{02}, \varepsilon_{b2}$ and ρ_b' are sufficiently small so that $|\sigma| > \max(\bar{\delta}_1', \bar{\delta}_2')$ with $0 < \varepsilon_{01} + \bar{\varepsilon}_0 \leq \varepsilon_{02} < \sqrt{n+m}$, where

$$\bar{\delta}_1' = (1-\rho_\alpha) + \sqrt{2}\left(1 + \frac{1}{\varepsilon_{01}'\sqrt{n+m}} + \rho_\beta' + \varepsilon_{b2}\right),$$
$$\bar{\delta}_2' = \left(2 + \frac{\varepsilon_{02}}{\sqrt{n+m} - \varepsilon_{02}}\right)(1-\rho_\alpha)$$

in the implementation of the Alternative Modification Scheme of Subsection 2.4, equations (9a), (9b), (11).

Theorem 1. *The adaptive control law* (13), (14), *under the estimation scheme* (7)–(9) *(or* (7), (9a) *and* (10), (11)*) and* (15), *has the following properties when applied to the plant* (1) *provided that Assumption 1 holds:*

(i) $\theta, \bar{\theta}$ *and* P *are uniformly bounded and the modified estimated plant model is controllable at all time;*

(ii) e *and* P_φ *are in* L_2*;*

(iii) $\theta, P, \bar{\theta}, s_i$ *and* r_j $(i = 1, 2, \ldots, n, j = 0, 1, \ldots, m-1)$ *converge asymptotically to finite limits for any bounded initial conditions for the plant* *and the estimation algorithm . Also, the Sylvester determinants of the unmodified and modified parameter estimates converge asymptotically to finite limits;*

(iv) $D^i u_f, D^i y_f$ $(i = 0, 1, \ldots, n-1)$ *and* u *and* y *are uniformly bounded and converge asymptotically to zero.*

Note that $e \in L_2 \bigcap L_\infty$ from Theorem 1 ((i) and (iv)) so that $e \to 0$ as $t \to \infty$ and $\theta \in L_\infty$ and converges to a finite limit. Also, $\|\dot{\tilde{\theta}}\| \in L_\infty$ from (7b) since $P \in L_\infty$ and $\varphi \in L_\infty$. These properties guarantee that $\mathrm{Det}\big(S(\theta_0)\big)$ and θ_0 are bounded and converge to finite limits so that the modification $\bar{\delta}$ is bounded and converges for both proposed modification schemes (8), (9) and (9a) and (10), (11).

Remark 4. *Assume that the plant is not perfectly modelled and/or it is subject to bounded disturbances with the unmodelled dynamics being related to* u_f *by a exponentially stable transfer function. Thus, it is modelled after filtering as* $A^*(D)y_f = B^*(D)u_f + \eta_f + \varepsilon_0^T(t) i_\varphi(t)$ *with* $\eta_f \dfrac{1}{E^*(D)\eta(t)}$. *Assume that* (1) *is controllable when* $\eta \equiv 0$ *and that an overbounding measurable function* $\bar{\eta}_f(t) = \varepsilon_1 \rho(t) + \varepsilon_2 = \varepsilon_1 \operatorname*{Sup}_{0 \le \tau \le t} \left\{ \left\| \varphi(\tau) e^{-\sigma_0 (t-\tau)} \right\| \right\} + \varepsilon_2 \ge |\eta_f|$, *for some nonnegative real constants* ε_i $(i = 1, 2)$ *where* $v(t)$ *is a vector whose components are* $D^j u_f$ *and* $D^j u_f,\ j = 0, 1, \ldots, n-1$ *(*[7, 8] *and* [15]*). The estimation scheme of* (7) *is modified by premultiplying the right-hand-sides of* (7a), (7b) *by the normalizing factor* $b := \dfrac{gs}{1 + \gamma \phi^T P \phi}$, *where*

$$s := \begin{cases} 0 & \text{if} \quad t \in \mathbf{I_1} := \left\{ t \in \mathbf{R_0^+} : |e| < \mu \bar{\eta}_f \right\}, \\ f(\mu \bar{\eta}_f, e)/e & \text{otherwise (i.e., for } \mathbf{I_2} := \mathbf{R_0^+} - \mathbf{I_1}), \end{cases}$$
$$f(\sigma, e) := \begin{cases} e - \sigma & \text{if} \quad e > \sigma, \\ 0 & \text{if} \quad |e| \le \sigma, \\ e + \sigma & \text{if} \quad e < -\sigma \end{cases} \tag{16}$$

with g, γ *and* $\mu > 1$ *are prefixed positive constants. Note that* $b(t)$ *includes a relative dead zone for small prediction error related to the size of the unmodelled dynamics (see, for instance,* [7], [8] *and* [15]*). Thus, it can be proved that* $\theta \in L_\infty,\ \bar{\theta} \in L_\infty,\ P\varphi \in L_2$ *and* $b|\eta_f^2 - e^2| \in L_1 \bigcap L_\infty$ *and also that the filtered and unfiltered input and output signals are uniformly bounded. The proof is very similar to that of Theorem 1 and it is omitted by space reasons.*

4 Numerical example

A numerical example is now tested for a nominally unstable and inversely unstable plant (1) parametrized by $A^*(D) = D^4 + 0.75D^3 + 0.5D^2 + 0.25D + 0.25$ and $B^*(D) = 0.75D^3 + 2/3D^2 + 0.25D + 0.25$ with initial conditions $(-5, -7, 0, 0)^T$ with filter parameter $E^*(D) = (D + 6.93)^2$. The estimation algorithm used prior to modification is that of Remark 2. The unmodelled dynamics is defined by a second-order differential equation $\dot{\eta} + 0.12\eta - 7.8 = 7.8u$. The estimation-modification algorithm used is that of (7)–(9) with the replacement of (8a) with (10). The determinant threshold for parameter modification of the estimates is $\rho = 0.01$. The adaptive stabilizer satisfies the constraints $\deg\big(R(D)\big) = \deg\big(S(D)\big) - 1 = 1$. The initialization of the estimation algorithm is $b_0(0) = 1,\ b_1(0) = -0.008,\ b_2(0) = -0.003,\ a_1(0) = 0.005,\ a_2(0) = -0.005,\ a_3(0) = 0,\ a_4(0) = 0$. The parameter b_3^* is assumed known and deleted from the estimation algorithm . The estimates of the initial conditions of the plant (1) are zero. The covariance matrix is initialized to $P(0) = \mathrm{Diag}(10^6)$ and $g = \gamma = 1,\ \mu = 1.04$. The absolute overbounding of the unmodelled dynamics contribution is computed with constants $\varepsilon_1 = 1$ and $\varepsilon_2 = 10^{-5}$ and $\sigma_0 = 0.1$. The output and input versus time are shown on Fig. 1. Figure 2 show the absolute value of the Sylvester determinant related to the estimates and modified estimates, respectively.

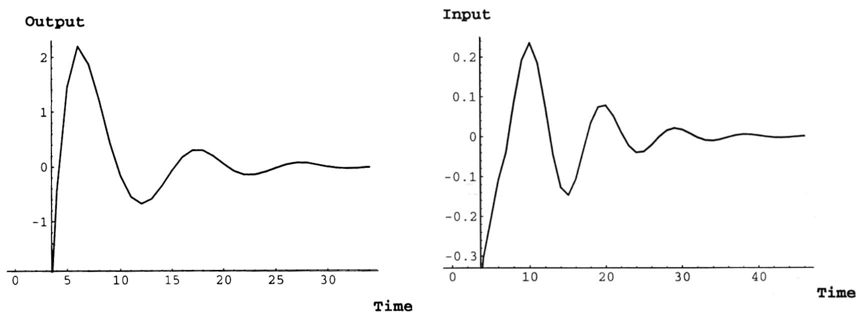

Fig. 1. Output and input versus time of the closed-loop system.

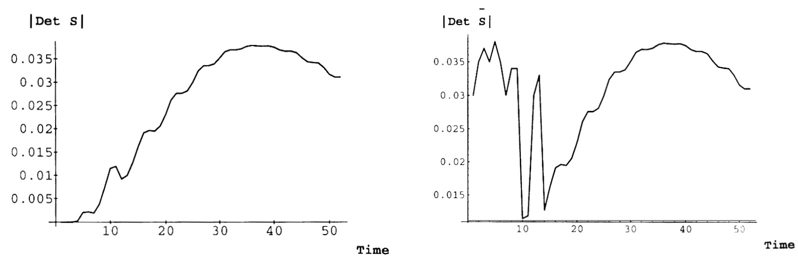

Fig. 2. Absolute Sylvester determinants of the a priori and modified estimatin schemes.

5 Adaptive stabilization with a continuous-time controller of a first-order plant under saturated input

The inputs to physical systems usually present saturation phenomena which limit the amplitudes which excite the linear dynamics. Also, the adaptive stabilization and control of linear continuous and discrete systems has been successfully investigated in the last years. Classically, the plant is assumed to be inversely stable and its relative degree and its high-frequency gain sign are assumed to be known together with an absolute upper-bound for that gain in the discrete case. Attempts of relaxing such assumptions have been made for continuous systems. The assumption on the knowledge of the order can be relaxed by assuming a known nominal order and considering the exceeding modes and unmodelled dynamics. The problem has been solved by using either excitation of the plant signals or by exploiting the properties of the standard least-squares covariance matrix combined with an estimation modification rule based upon the use of a hysteresis switching function. Such an estimates modification technique guarantees that the modified estimated plant model is controllable at all time provided that the plant is controllable. *This paper presents an adaptive stabilization algorithm for first-order continuous-time systems with a zero which can be either stable or unstable under saturated input. The saturating device is modelled by a sigmoidal function.* Such an approach is a very good approximation to the common saturations usually modelled as piecewise-continuous functions. Also, it is an exact model for saturations inherent to practical MOS-type amplifiers. The adaptive scheme uses a parameter modification rule which guarantees that the absolute value of the determinant of the Sylvester matrix associated with the modified parameter estimates is bounded from below by a positive threshold and, thus, the estimated model is guaranteed to be controllable. That feature is the main contribution of this manuscript. The results are then extended to the case when an adaptive stabilizer, which re-updates at sampling instants the plant estimates, modified estimates and controller parameters, is used for the above continuous-time plant. This strategy results in a hybrid closed-loop system because of the discrete nature of the updating procedure of the parametrical estimation/modification.

5.1 Plant, estimation/modification scheme and adaptive stabilization law

Consider the following continuous-time first-order controllable system under saturated input:

$$\dot{y} + a^* y = b_0^* \dot{u} + b_1^* u', \tag{17a}$$

$$u' = \mathrm{sat}_{v^*}(u) = \mathrm{than}(v^* u) = \frac{1 - e^{-2v^* u}}{1 + e^{-2v^* u}} \tag{17b}$$

where the saturated input u' to the plant (17a) is modelled by a sigmoidal function [16]. To simplify the writing, the argument (t) is omitted and all the constants are denoted by superscripts by "$*$". Equation (17a) can be rewritten as

$$\dot{Y} = -A^* y + b_0^* \dot{u} + b_1^* u + b_0^* (\dot{u}' - \dot{u}) + b_1^* (u - u). \tag{18}$$

Note that the equivalence between (17a) and (18) is an identity where positive and negative terms concerned with the unsaturated input and its time-derivative are cancelled in the right-hand-side of (18). Define filtered signals

$$\dot{u}_f = -d^* u_f + u, \quad \dot{u}'_f = -d^* u'_f + u', \quad \dot{y}_f = -d^* y'_f + y' \tag{19}$$

for some scalar $d^* > 0$ so that one gets from (18) for filtered signals

$$\dot{y}_f = \theta^{*T} \varphi = -a^* y_f + b_0^* \dot{u}_f + b_1^* u'_f + \varepsilon_0^* e^{-d^* t}, \tag{20a}$$

$$\dot{y}_f = -a^* y_f + b_0^* \dot{u}_f + b_1^* u'_f + b_0^* (\dot{u}'_f - \dot{u}_f) + b_1^* (u'_f - u_f) + \varepsilon_0^* e^{-d^* t} \tag{20b}$$

where

$$\theta^* = [b_0^*, b_1^*, a^*, b_0^*, b_1^*, \varepsilon_0^*]^T, \tag{21a}$$

$$\varphi = [\dot{u}_f,\, u_f,\, -y_f,\, \dot{u}'_f - \dot{u}_f,\, u'_f - u_f,\, e^{-d^* t}]^T \tag{21b}$$

where $\varepsilon_0^* = y_f(0) - u'_f(0)$ has been included in θ^{*T} to obtain (4) without neglecting the exponentially decaying term due to initial conditions of the filters $1/(s + d^*)$ used in (20). Also, the over-parametrization of (21a), (21b), in the sense that the coefficients of the numerator polynomial are estimated twice with different regressors, allows describing (20a) as driven by u_f and $u'_f - u_f$. This idea will be then exploited for the stability analysis of the adaptive stabilizer. The parameter vector θ^{*T} can now be estimated by using the least-squares algorithm

$$e = \dot{y}_f - \theta^T \varphi, \tag{22}$$

$$\dot{\theta} = P\varphi e, \tag{23}$$

$$\dot{P} = -P\varphi\varphi^T P, \quad P(0) = P^T(0) > 0 \tag{24}$$

where e is the prediction error, $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)^T$ is the estimate of θ^*, defined in (21a), and P is the covariance matrix. The use of (20b) into (6) yields

$$\dot{y}_f = \theta_1 \dot{y}_f + \theta_2 u_f - \theta_3 y_f + \theta_4(\dot{u}'_f - u_f) + \theta_5(u'_f - u_f) + \theta_6 e^{-d^* t} + e. \tag{25}$$

The following modification rule of the parameter estimates is used to guarantee the controllability of the estimated plant model

$$\bar{\theta} = \theta + P\beta \tag{26}$$

with β being a vector which can be chosen to be equal to one of the following vectors

$$\beta_1 = [0, 0, \overbrace{\ldots}^{6}, 0]^T, \quad \beta_2 = v, \quad \beta_3 = -\beta_2, \tag{27a}$$

$$\beta_4 = p_1 - p_4 + p_3, \quad \beta_5 = -\beta_4, \quad \beta_6 = p_1 - p_4 - p_3, \tag{27b}$$

$$\beta_7 = -(p_1 - p_4) + p_3, \quad v = (\theta_1 - \theta_4)p_3 + \theta_3(p_1 - p_4) - (p_2 - p_5) \tag{27c}$$

and whose current value is selected from a hysteresis switching function which is defined by the following rule. Define

$$c(\beta) = \left|(\bar{\theta}_1 - \bar{\theta}_4)\bar{\theta}_3 - (\bar{\theta}_2 - \bar{\theta}_5)\right| = \left|\text{Det}\begin{bmatrix} 1 & 0 & 0 \\ \bar{\theta}_3 & 1 & \bar{\theta}_1 - \bar{\theta}_4 \\ 0 & \bar{\theta}_3 & \bar{\theta}_2 - \bar{\theta}_5 \end{bmatrix}\right|$$

which is the absolute value of the Sylvester matrix of the modified parameter estimates associated with the estimation of the plant numerator and denominator polynomials obtained from (23), (24) and (25)–(27). Assume that $\beta(t^-) = \beta_i(t^-)$ and $c\big(\beta_j(t^+)\big) \geq c\left(\beta_m(t^+)\right)$ for some $j = 1, 2, \ldots, 7$ with $j \neq i$ and all $m = 1, 2, \ldots, 7$. Thus, for some prefixed design scalar $\alpha^* \in (0, 1]$:

$$\beta(t^+) = \begin{cases} \beta_j(t^+) & \text{if} \quad c\big(\beta_j(t^+)\big) \geq (1 + \alpha^*)c\big(\beta_i(t^+)\big), \\ \beta_i(t^+) & \text{otherwise} \end{cases} \tag{28}$$

where p_i denotes the i-th column of P. This modification strategy guarantees that the parametrical error lies in the image of the of P while allowing that the diophantine equation, which will be then used for the synthesis of the adaptive stabilizer, will have no cancellations at any time. It will be then shown that the two following conditions are satisfied:

1) β converges,
2) $c\,(\beta) \geq \delta^* > 0$

which will be then required in the proofs of convergence and stability. Equation (25) can be rewritten as dependent of the modified estimates (26)–(28) as follows :

$$\begin{aligned} \dot{y}_f &= \bar{\theta}_1 \dot{u}_f + \bar{\theta}_2 u_f - \bar{\theta}_3 y_f + \bar{\theta}_4(\dot{u}'_f - \dot{u}_f) \\ &\quad + \bar{\theta}_5(u'_f - u_f) + \bar{\theta}_6 e^{-d^* t} + e - \beta^T P \varphi. \end{aligned} \tag{29}$$

The filtered control input u_f to the saturating device and its unfiltered version u are generated as follows:

$$\dot{u}_f = -s_1 u_f - r_0 y_f, \quad u = d^* u_f + \dot{u}_f = (d^* - s_1)u_f - r_0 y_f \tag{30}$$

with the parameters r_0 and s_1 of the adaptive stabilizer being calculated for all time from the diophantine polynomial equation

$$(D+\bar{\theta}_3)(D+s_1)+\big[(\bar{\theta}_1-\bar{\theta}_4)D+(\bar{\theta}_2-\bar{\theta}_5)\big]r_0 = C^*(D) \underset{\text{def}}{=} D^2+c_1^*D+c_2^* \tag{31}$$

with $D = d/dt$ in (15a) and $C^*(D)$ being a strictly Hurwitz polynomial that defines the suited nominal closed-loop dynamics.

5.2 Convergence and stability results

They are summarized in the following main result whose proof is omitted.

Theorem 2. *Consider the plant* (17) *subject to the estimation scheme* (22)–(24)*, the modification scheme* (26)*,* (27) *and the control law* (30)*. Assume that either* $a^* > 0$ *(i.e., the open-loop plant is stable) or* $\left|y(0)\right| \leq \dfrac{b_1^* - a^* b_0^*}{a^*}$ *if* $a^* < 0$ *(i.e., the initial condition is sufficiently small if the plant is unstable).*

Thus, the resulting closed-loop scheme has the following properties:

(i) *The modified estimated plant model is controllable for all time for the chosen* β *in such a way that* $c\,(\beta) \geq \delta^* > 0$;

(ii) $\widetilde{\theta} = \theta - \theta^* \in L_\infty$ *and* e *and* $P\varphi$ *are in* $L_\infty \bigcap L_2$;

(iii) θ, P, β, $\bar{\theta}$, s_1 *and* r_0 *are uniformly bounded and converge asymptotically to finite limits. Also, the number of switches in* β *is finite. Also,* $\dot{\theta} \in L_2 \bigcap L_\infty$;

(iv) *The signals* u, u' *and* y *and their corresponding filtered signals are in* $L_\infty \bigcap L_2$*. The signals* u, u', u_f, u'_f, y *and* y_f *converge to zero and their time-derivatives are in* $L_\infty \bigcap L_2$ *so that they converge to zero asymptotically.*

Note that the requirement of the initial conditions being sufficiently small when the plant is unstable is a usual requirement for stabilization in the presence of input saturation since it is impossible to globally stabilize an open-loop unstable system with saturated input. This avoids the closed-loop system trajectory to explode. Such a phenomenon occurs when the initial time-derivative of the state vector is positive and continues to be positive for all time because its sign cannot be modified for any input value within the allowable input range. Note also that Theorem 2 (i)–(iii) imply that Conditions 1, 2 for the $\beta_{(\cdot)}$-functions of the modification scheme are fulfilled. Finally, note that the controllability of the modified estimation scheme allows to keep coprime the modified estimates of the polynomials for zeros and poles. Thus, the diophantine equation (31) associated with the controller synthesis is solvable for all time without any singularities.

The mechanism which is used to ensure local stability for unstable plants and global one for stable ones is to guarantee the boundedness of all the unsaturated filtered and unfiltered signals from the regressor bondedness while the saturated ones are bounded by construction. This also ensures the identification (or adaptation) error to be bounded for all sampling time since the unmodified and modified plant parameter estimates as well as those of the adaptive controller are all bounded.

5.3 Hybrid approach

Now, the continuous-time plant (17) is subject to the given control law above in Section 5, under the saturating sigmoidal function (17b), but the estimation algorithm only updates parameters at the sampling instants $t_{k+1} = t_k + h = (k+1)h$ of the sampling period h while the regressor is evaluated at all time for re-updating the various estimates at sampling instants only. That scheme lies in the class of the so-called hybrid systems, [7] and [16]–[19]. The estimation modification and calculation of the controller parameters are also updated at sampling instants. The discrete-time parameter estimation and inverse of the covariance matrix adaptation laws are:

$$\theta_k = \theta_{k-1} + \Delta\theta_{k-1}\theta_{k-1} - P_k \frac{\int_0^h \left|\varphi\left[(k-1)h+\tau\right]\right|^2 \varphi\left[(k-1)h+\tau\right]\varphi^T\left[(k-1)h+\tau\right]d\tau}{c_k\left(1+\int_0^h \varphi^T\left[(k-1)h+\tau\right]\varphi\left[(k-1)h+\tau\right]d\tau\right)}\widetilde{\theta}_{k-1}, \tag{32a}$$

$$P_{k+1}^{-1} = P_k^{-1} + \Delta\widetilde{P}_k^{-1} = P_k^{-1} + \frac{\int_0^h \left\|\varphi\left[(k-1)h+\tau\right]\right\|^2 \varphi\left[(k-1)h+\tau\right]\varphi^T\left[(k-1)h+\tau\right]d\tau}{c_k\left(1+\int_0^h \varphi^T\left[(k-1)h+\tau\right]\varphi\left[(k-1)h+\tau\right]d\tau\right)}\widetilde{\theta}_{k-1}, \tag{32b}$$

$$c_k \geq c_{k0} \underset{\text{def}}{=} \lambda_{max}^2(P_k)\frac{\int_0^h \left\|\varphi\left[(k-1)h+\tau\right]\right\|^4 d\tau}{1+\int_0^h \left\|\varphi\left[(k-1)h+\tau\right]\right\|^4 d\tau} \tag{32c}$$

with $P(0) = P^T(0) > 0$ and $\widetilde{\theta}_k = \theta_k - \theta^*$ for all integer $k \geq 0$. The main result of this section is ennounced in the following. Its proof, which is very close to that of Theorem 1, is omitted.

Theorem 3. *Consider the plant* (17) *under the same estimation/modification scheme as in Section 5, with the estimation being updated only at $t = kh$, and the same stabilizing control law. Thus, the resulting closed-loop scheme fulfils the same properties of Theorem 1 under the same assumptions.*

Appendix

Proof of Proposition 1. Firstly, note that the first-order derivatives of the determinant with respect to any parameter estimate are calculated as follows from elementary algebra (see, for instance, [13]):

$$\frac{\partial}{\partial \theta_1} \mathrm{Det}\big(S(\theta_0)\big) = \mathrm{Trace}\left(\frac{\partial S(\theta_0)}{\partial \theta_i} \widetilde{S}(\theta_0)\right) \tag{$A.1$}$$

which holds when taking derivatives of determinants with respect to any value of the parameter estimate θ_i for $i = 1, 2, \ldots, n+m+1$. The derivatives are evaluated at θ_0. However, it is clear from (8e) that $S_{\theta_{i_1},\ldots,\theta_{i_k}} = \dfrac{\partial^k S(\theta_0)}{\partial \theta_{i_1}^k, \ldots, \theta_{i_k}^k} = 0$, $k = 2, 3, \ldots, n+m+1$ with all the partial derivatives being evaluated at θ_0. Also, since $\widetilde{S}(\theta_0)$ is a matrix of cofactors, it contains products of at most $(n+m)$ parameters at each one of its entries so that $\widetilde{S}_{\theta_{i_1},\ldots,\theta_{i_k}}(\theta_0) = 0$ if $k > n+m$ for any integers $i_j \geq 1$ for $j = 1, 2, \ldots, k$. Now, $\mathrm{Det}\big(S(\bar{\theta}_0)\big)$ is expanded in Taylor series around $\mathrm{Det}\big(S(\theta_0)\big)$ by taking successive derivatives with respect to parameter components evaluated at θ_0 by starting with $(A.1)$ while zeroing any derivatives of higher-order than (n+m). One obtains directly

$$\mathrm{Det}\big(S(\bar{\theta}_0)\big) = \mathrm{Det}\big(S(\theta_0)\big) + \Delta(\theta_0, \bar{\theta}_0) \tag{$A.2a$}$$

with

$$\Delta(\theta_0, \bar{\theta}_0) = \sum_{k=1}^{n+m} \sum_{i_1, i_2, \ldots, i_k = 1}^{n+m+1} \frac{1}{k!} \mathrm{Trace}\big(S_{\theta_{i_1}}(\theta_0) \widetilde{S}_{\theta_{i_1},\ldots,\theta_{i_k}}(\theta_0)\big) \prod_{j=i_1}^{i_k} (\bar{\theta}_j - \theta_j) \tag{$A.2b$}$$

being the maximum absolute achievable increment between the modified and unmodified determinants. Now, it is proved by contradiction that

$$\begin{aligned}&\mathrm{Trace}\big(S_{\theta_{i_1}}(\theta_0)\widetilde{S}_{\theta_{i_1},\ldots,\theta_{i_k}}(\theta_0)\big)=0\\ &\quad\text{for all}\quad i_k\in\{1,\ldots,n+m+1\},\ k=1,2,\ldots,n+m\end{aligned}\tag{A.3}$$

is impossible since $(A.3)$ depends on the estimates of the plant parameters irrespective of the modification scheme. Now, assume that $\big|\mathrm{Det}S(\theta_0)\big|\neq\zeta<\rho$ with $\zeta>0$. Then, note from the definition of $S(\bar{\theta}_0)$ that $\big|\mathrm{Det}S(\bar{\theta}_0)\big|=\zeta$ with arbitrary nonzero ζ if the subsequent modification rule is used after estimation: $\delta a_i=-a_i,\ \delta b_j=-b_j$ and $\delta b_m=\pm\zeta^{\frac{1}{n}}-b_m$ for $i=1,2,\ldots,n,\ j=0,$ $1,\ldots,m$. Assume that $(A.3)$ holds. Thus, one has the impossible relationships $\zeta=\big|\mathrm{Det}S(\bar{\theta}_0)\big|=\big|\mathrm{Det}S(\theta_0)\big|\neq\zeta$ by using a Taylor series expansion in the parameter space of the modified estimates around the estimated ones obtained from (7) according to $(A.2)$. Thus, $(A.3)$ is false, since all the derivatives used in $(A.2)$ are not dependent on the modification scheme. Then, there is at least one parameter component θ_i of θ_0 for which $\mathrm{Trace}\big(S_{\theta_{i_1}}(\theta_0)\widetilde{S}_{\theta_{i_1},\ldots,\theta_{i_k}}(\theta_0)\big)\neq 0$ and then $\bar{C}$ in (8d), (8e) is nonzero. Thus, $\mathrm{Det}\big(S(\theta_0)\big)$ is not constant for all the values of the components of θ_0 belonging to arbitrary real intervals and a modification $\theta_0\to\bar{\theta}_0$ can be carried out to guarantee that $\big|\mathrm{Det}\big(S(\bar{\theta}_0)\big)\big|\geq\rho$. If δ_α is discontinuous at t then $\big|\delta_\alpha(t^+)\big|\geq\dfrac{2\rho}{|\bar{C}|}$ if $h(t^+)=\rho$ and $h(t^-)=\rho'$ and $\big|\delta_\alpha(t^+)\big|\geq\dfrac{2\rho'}{|\bar{C}|}$ if $h(t^+)=\rho'$ and $(t^-)=\rho$. In any of the above situations, $\alpha(t)\neq 0$. The switches in $h(t)$ make this eventual discontinuities to occur only at isolated time instants. Direct calculations yield:

$$\begin{aligned}\big|\mathrm{Det}\big(S(\bar{\theta}_0)\big)\big|&=\big|\mathrm{Det}\big(S(\theta_0)\big)+\Delta(\theta_0,\bar{\theta}_0)\big|\geq|\delta_\alpha||\bar{C}|-\big|\mathrm{Det}\big(S(\theta_0)\big)\big|\\ &\geq 3\rho-\mathrm{Det}\big(S(\bar{\theta}_0)\big)\mathrm{Sign}(\bar{C})-\big|\mathrm{Det}\big(S(\theta_0)\big)\big|>\rho>0.\end{aligned}\tag{A.4}$$

Note that $\delta_\alpha=\alpha$ if $\delta_\alpha\geq 1$ (what implies that $\alpha^j\geq\alpha$ for $j\geq 1$) and $\delta_\alpha=\alpha^{n+m}$ if $\delta_\alpha<1$ (what implies that $\alpha^j<\alpha$ for $j>1$) with δ_α and α being chosen according to (8a), (8b). Such a constraint establishes the first inequality in $(A.4)$ since $\big|\Delta(\theta_0,\bar{\theta}_0)\big|\geq|\delta_\alpha\bar{C}|$ from $(A.2b)$. Thus, the first part of Proposition 1 has been proved. The absence of chattering follows directly since the α – function is continuous at $\delta_\alpha\bar{C}=1$ since $\delta_\alpha\bar{C}=$

$(\delta_\alpha \bar{C})^{\frac{1}{n+m}}\Big]_{\delta_\alpha \bar{C}=1}$. The eventual discontinuities in the determinant test (8b) are isolated at any time what is guaranteed by the switches switches in $h(t)$ given by (8a).

Proof of Proposition 2. One has from the definitions and properties of the ℓ_2, ℓ_1 and ℓ_∞ matrix norms (see, for instance, [20, 21])

$$\begin{aligned}(n+m)^{-\frac{1}{2}}\big\|S(\theta_0)\big\|_2 &= (n+m)^{-\frac{1}{2}}\big|\lambda_{max}^{1/2}\big(S^T(\theta_0)S(\theta_0)\big)\big| \\ &= \left|\frac{1}{\lambda_{max}^{1/2}\big(S^T(\theta_0)S(\theta_0)^{-1}\big)}\right| \le (n+m)^{\frac{1}{2}}\big\|S(\theta_0)\big\|_1\end{aligned} \qquad (A.5)$$

where $\big|\lambda_{max}(\cdot)\big|$ and $\big|\lambda_{min}(\cdot)\big|$ denote the maximum and minimum module of the eigenvalues of the $(\cdot)$-matrix, respectively. Thus, the two following inequalities follow directly from $(A.5)$

$$\begin{aligned}\Big|\lambda_{min}^{1/2}\big(S^T(\theta_0)S(\theta_0)^{-1}\big)\Big| &= \frac{1}{\Big|\lambda_{max}^{1/2}\big(S^T(\theta_0)S(\theta_0)\big)\Big|} = \frac{1}{\big\|S(\theta_0)\big\|_2} \\ &\ge \frac{1}{(n+m)^{\frac{1}{2}}\big\|S(\theta_0)\big\|_1} = \frac{1}{(n+m)^{\frac{1}{2}}\max\Big(1+\sum\limits_{i=1}^{n}|a_i|, \sum\limits_{i=0}^{m}|b_i|\Big)},\end{aligned} \qquad (A.6a)$$

$$\begin{aligned}\Big|\lambda_{max}^{1/2}\big(S^T(\theta_0)S(\theta_0)^{-1}\big)\Big| &= \frac{1}{\Big|\lambda_{min}^{1/2}\big(S^T(\theta_0)S(\theta_0)\big)\Big|} \\ &\le \frac{1}{\Big|\lambda_{max}^{1/2}\big(S^T(\theta_0)S(\theta_0)\big)\Big|} = \frac{1}{\big\|S(\theta_0)\big\|_2} \le \frac{1}{(n+m)^{-\frac{1}{2}}\big\|S(\theta_0)\big|_\infty} \\ &= \frac{1}{(n+m)^{-\frac{1}{2}}\Big(\sum\limits_{i=1}^{n-1}|a_i| + \sum\limits_{i=1}^{m-1}|b_i| + \max\big(1+|b_0|, |a_n|+|b_m|\big)\Big)},\end{aligned} \qquad (A.6b)$$

which imply

$$0 < \frac{1}{\varepsilon_0'} \le \Big|\lambda_{min}^{1/2}\big(S^T(\theta_0)S(\theta_0)\big)\Big| \le \Big|\lambda_{max}^{1/2}\big(S^T(\theta_0)S(\theta_0)\big)\Big| \le \frac{1}{\varepsilon_0} < \infty \quad (A.7)$$

if Condition 1 holds. Thus, Condition 1 guarantees that is nonsingular (i.e., the estimated plant model is controllable) and a parameter modification is not performed in (10), (11). If Condition 1 does not hold then $S(\theta_0)$ is not guaranteed to be nonsingular accordingly to the test of $(A.7)$. Thus, the estimation modification procedure of (9a), (10), (11) when Condition 1 does not hold guarantees that

$$1 > \sum_{i=1}^{n} |\bar{a}_i| + \sum_{i=0}^{m-1} |\bar{b}_i|, \quad |\bar{b}_m| > \sum_{i=1}^{n} |\bar{a}_i| + \sum_{i=1}^{m-1} |\bar{b}_i| + \gamma|\bar{b}_0|. \qquad (A.8)$$

Now, note that if $(A.8)$ holds then the modified $S(\bar{\theta}_0)$ is diagonally dominant what follows directly by inspection from its definition since for such a matrix structure, it suffices to guarantee diagonal dominance for the n-th and $(n+1)$-th rows. Since all diagonally dominant matrix is nonsingular, [20], $S(\bar{\theta}_0)$ is nonsingular and the modified estimated plant model is controllable. The proof of nonsingularity has been completed. The absence of chattering follows from the use of two possible values of all the $\varepsilon_{(\cdot)}$-constants in Conditions 1, 2, the fact that those values are modified after N consecutive switches with the same values of the constants over finite intervals of length T and the feature that the estimates prior to the modification have finite limits (see also Remarks 1, 2.

Proof of Theorem 1. The subsequent proof applies for both modification schemes (8), (9), and (9a), (9b), (10), (11).

(i), (ii) Note that $\dot{P}^{-1} = -P^{-1}\dot{P}P^{-1} = \varphi\varphi^T$ from (7c). Define the Lyapunov function candidate $V = \widetilde{\theta}^T P^{-1}\widetilde{\theta}$ where $\widetilde{\theta} = \widehat{\theta} - \theta^*$ is the parametrical error before modification of the estimates. Thus, (7a) can be rewritten as $e = -\widetilde{\theta}^T\varphi$ and $\dot{V} = -(\widetilde{\theta}^T\varphi)^2 = -e^2 \leq 0$ after direct calculations with V and (7), [5]. Thus, $e \in L_2$ and $\infty > \widetilde{\theta}^T P^{-1}\widetilde{\theta} \geq \lambda_{min}(P^{-1})\widetilde{\theta}^T\widetilde{\theta}$, with $\lambda_{min}(P^{-1})$ being the minimum eigenvalue of P^{-1} so that $\widetilde{\theta}$ is uniformly bounded since the maximum eigenvalue of $P, \lambda_{max}(P)$, is upper-bounded by a positive finite constant and then $\lambda_{min}(P^{-1}) = \lambda_{max}^{-1}(P) > 0$ for all $t \geq 0$. Thus, P, θ is uniformly bounded $\|P\|, \|\theta\|$ and $\|\bar{\delta}\|$ are in L_∞ from (9) since $\theta = (\theta_0^T, \varepsilon_0^T)^T$ and θ_0 and $\mathrm{Det}\big(S(\theta_0)\big)$ are uniformly bounded for all $t \geq 0$. Thus, the modified parameter vector $\bar{\theta} = (\bar{\theta}_0^T, \varepsilon_0^T)^T$ is also uniformly bounded for all $t \geq 0$. The modified estimated plant model is controllable since $\infty > \big|\mathrm{Det}\big(S(\bar{\theta}_0)\big)\big| \geq \rho > 0$ from (8), (9) and the fact that $\bar{\theta}_0$ is uniformly bounded for all $t \geq 0$. On the other hand, $P\varphi \in L_2$ since $\mathrm{tr}(\dot{P}) = -\|P\varphi\|_2^2 \in L_1$ from (7c) with $\|\cdot\|_2$ denoting the spectral (or Euclidean) vector norm. Thus, propositions (i), (ii) have been proved.

(iii) It is standard to prove that P and θ converge asymptotically from (7b) and the fact that $\lim_{t\to\infty}\Big(\int_0^t \|\dot\theta\|d\tau\Big) \le \frac{1}{2}\Big[\big(\|P\varphi\|^2 d\tau\big) + \lim_{t\to\infty}\Big(\int_0^t e^2 d\tau\Big)\Big] < \infty$ since $P\varphi \in L_2$ and $e \in L_2$ what implies $\dot\theta \in L_1$ and the θ converges from (ii) (see [22]). Also, θ_0 converges since θ converges and thus $\mathrm{Det}\big(S(\theta_0)\big)$ converge to a finite constant values as time tends to infinity. From the fact that θ_0 converges, the possible switches in (8a), (8b) end in finite time since there exists a large finite time $\bar t_0$ such that θ and $\mathrm{Det}\big(S(\theta_0)\big)$ are close to their limits and the piecewise-constant h-function maintains a constant value (ρ or $\rho' \ge \rho$) for all time $t > \bar t_0$ (see (15a), (15b)). As a result $\alpha, \sigma_{(\cdot)}, \bar\sigma_{(\cdot)}$ and $\bar C$ converge. Thus, the modified parameter vector $\bar\theta$, and then $\mathrm{Det}\big(S(\bar\theta_0)\big)$, converge asymptotically to finite limits. As a result, each controller parameter, namely, each coefficient of $R(D,t)$ and $S(D,t)$, converges to a finite limit value and (iii) has been proved.

(iv) Note that direct calculation from (12) yields for $m \le n-1$:

$$D^n y_f = e + (\bar\theta^T - \bar\delta^T)\varphi = e + \sum_{i=0}^{n} \bar b_i D^{m-i} u_f - \sum_{i=1}^{n} \bar a_i D^{n-i} u_f - \bar\delta_0^T \varphi_0$$

and the substitution $D^n u_f$ obtained explicitly from (13) into (12) yields for $m = n$:

$$\begin{aligned} D^n y_f =& e - \bar b_0\Big[\sum_{i=1}^{n} s_i D^{n-i} u_f + \sum_{i=0}^{n-1} r_i D^{n-i-1} y_f\Big] \\ &+ \Big[\sum_{i=1}^{n} b_i D^{n-i} u_f + \sum_{i=0}^{n-1} \bar a_{i+1} D^{n-i-1} y_f\Big] - \bar\delta_0^T \varphi_0. \end{aligned}$$

Thus, the substitution of the above identities together with (13) yield the following extended auxiliary dynamic system which describes the combination of the closed-loop dynamics and control law:

$$\dot x = Ax + w, \qquad (A.9a)$$

$$\dot z = Az + w_1 \qquad (A.9b)$$

with

$$w = [e + \varepsilon_0^T i_\varphi - \bar{\delta}_0^T \varphi_0, 0]^T = \bar{w} + w_1,$$
$$\bar{w} = [-\bar{\delta}_0^T \varphi_0, 0]^T, \quad w_1 = [e + \varepsilon_0^T i_\varphi, 0]^T, \tag{A.10a}$$

$$A(t) = \begin{bmatrix} & \bar{p}^T & \\ I_{n-1} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ & \bar{v}^T & \\ 0 & \vdots & I_{n-1} \end{bmatrix}, \quad \bar{p} = \begin{cases} \bar{p}^{(1)} & \text{if} \quad m \le n-1, \\ \bar{p}^{(2)} & \text{if} \quad m = n, \end{cases} \tag{A.10b}$$

$$\bar{p}^{(1)T} = [-\bar{a}_1, -\bar{a}_2, \ldots, -\bar{a}_n \vdots 0, \overbrace{\cdots}^{n-m-1}, 0 \vdots \bar{b}_0, \bar{b}_1, \ldots, -\bar{b}_m], \tag{A.10c}$$

$$\bar{p}^{(2)T} = \big[-(\bar{a}_1 + \bar{b}_0 r_0), -(\bar{a}_2 + \bar{b}_0 r_1), \ldots, -(\bar{a}_n + \bar{b}_0 r_{n-1}) \vdots$$
$$(\bar{b}_1 + \bar{b}_0 s_1), (\bar{b}_2 + \bar{b}_0 s_2), \ldots, (\bar{b}_n + \bar{b}_0 s_n)\big], \tag{A.10d}$$

$$\bar{v}^T = [r_0, r_1, \ldots, r_n, \vdots s_1, s_2, \ldots, s_n] \tag{A.10e}$$

with $x(0) = z(0) = x_0$, $x = (D^{n-1}y_f, \ldots, Dy_f, y_f, D^{n-1}u_f, \ldots, Du_f, u_f)^T$ and $\varphi_0(D^{n-1}y_f, \ldots, Dy_f, y_f, D^n u_f, D^{n-1}u_f, \ldots, Du_f, u_f)^T$. The proof of boundedness and convergence to zero of the input, output, their filtered versions and the time-derivatives of those ones up till $(n-1)$-th order of the closed-loop system is immediate by first proving that $(A.9b)$ is asymptotically stable in the large. Thus, by vector construction, $|D^n u_f| \le K'\|x\|$ from the controller equation (13) and, then, $\|\varphi_0\| \le \max\big(|D^n u_f|, \|x\|\big) \le K\|x\|$ with $K = 1 + K'$. Note from $(A.9b)$ and (15a) that all the eigenvalues of $A(t)$ are less than or equal to $(-\sigma)$ for some real constant $\sigma > 0$ which is less than or equal to the minimum absolute value of the roots of the strictly Hurwitz $C^*(D)$-polynomial for all $t \ge 0$ (equality applies when both roots are distinct, [21, 22]). Also, $A(t)$ is uniformly bounded and, furthermore, $\int_t^{t+T_0} \big\|\dot{A}(\tau)\big\| d\tau \le \mu T_0 + \mu_0'$ for positive constants μ and μ_0, all $t \ge 0$ and some finite T_0. This follows directly in the absence of modification on the integration interval since the time-derivative of the estimates and controller parameters are bounded as follows from Theorem 1. Assume that there are $\infty > s_t \ge 0$ modification switches on $[t, t+T_0]$. Their number is finite since the integration interval is finite and $\left|\mathrm{Det}\big(S(\theta_0)\big)\right|$ is a continuous function of time so that existing switches are isolated (i.e., there is no accumulation point of modification switches). Also, their associate discontinuities in $A(t)$ are given by bounded steps whose norms are upper-bounded by a positive finite constant $\bar{k}$ from Theorem 1(i) since $\theta \in L_\infty$. As a result,

$$\int\limits_{t^-}^{(t+T_0)^+} \|\dot{A}(\tau)\| d\tau \leq \sum_{i=1}^{s_t} \bar{k} \int\limits_{t_i^-(t)}^{t_i^+(t)} \partial\big((\tau - t_i(t)\big) d\tau + \int\limits_{\bigcup_{j=0}^{s_t} I_j(t)} \big\|\dot{A}(\tau)\big\| d\tau \leq \mu T_0 + \mu_0$$

with $\mu_0 = \mu_0' + \bar{s}\bar{k} < \infty$ where $\infty > \bar{s} = \operatorname*{Sup}_{t \geq 0}(s_t)$ where s_t is a nonnegative integer number and $\delta(\tau)$ is the Dirac-delta function at $\tau = 0$. The $t_{(\cdot)}(t)$ instants are the s_t separated instants within $(t, t + T_0)$ where the modification switches take place, $I_0(t) = \big(t, t_1(t)\big),\ I_i(t) = \big(t_i(t), t_{i+1}(t)\big)$, and $I_{s_t}(t) = \big(t_{s_t}(t), t + T_0\big)$ for $i \neq 0, s_t$ are s_{t+1} open intervals where the time-derivative of the modified estimates exist. (If $s_t = 0$ then $I_0(t) = (t, t + T_0]$).

Thus, the common unforced version of both time-varying systems (103) is exponentially stable in the large ([8, 15]). Now, direct calculus with the differential systems ($A.9a$) and ($A.9b$) yields that their solutions are related as follows:

$$x(t) = z(t) + \int\limits_0^t \Psi(t, \tau)\bar{w}(\tau) d\tau \tag{A.11}$$

with $\Psi(t, \tau)$ being the fundamental matrix of the unforced system of (9a) and (9b), i.e., $x(t) = z(t) = \Psi(t, 0)x_0$ for all $t \geq 0$ if $w \equiv w_1 \equiv 0$. Since such a system is exponentially stable in the large, one has for any matrix norm that $\big\|\Psi(t, \tau)\big\| \leq K_\Psi e^{-\sigma(t-\tau)}$ for any t and τ fulfilling $t \geq \tau \geq 0$. In particular, one has $\big\|\Psi(t, \tau)\big\|_2 \leq e^{-\sigma(t-\tau)}$ (i.e.,$K_\Psi = 1$) if the spectral matrix norm is used. Since $A(t)$ is exponentially stable and, furthermore, $w_1 \in L_\infty \bigcap L_2$ from (i), (ii) $z \in L_\infty \bigcap L_2$, $\dot{z} \in L_\infty \bigcap L_2$ and z converges exponentially to zero for any bounded initial condition (see [22]). Thus, by taking spectral vector and matrix norms in $(A.11)$, one gets directly from the definition of $\bar{w}$ in $(A.9a)$:

$$\left\|x(t)\right\|_2 = \|z(t)\|_2 + \int_0^t e^{-\sigma(t-\tau)}\|\bar{\delta}_0\|_2\left\|x(\tau)\right\|_2 d\tau. \qquad (A.12)$$

Now, define

$$\bar{z}_{t_j,e} = \operatorname*{Sup}_{t_j \le t \le T}\left\|z(t)\right\|_2 \quad \text{and} \quad \bar{z} = \operatorname*{Sup}_{t\in R_0^+}\left\|z_{t,e}\right\|_2 = \operatorname*{Sup}_{T\in R_0^+}\left(\operatorname*{Sup}_{0\le t\le T}\left\|z(t)\right\|_2\right)$$

for all finite $t_j \in Z_0^+$ and $T \in R_0^+$ where Z_0^+ and R_0^+ are the sets of nonnegative integer and real numbers, respectively. Since $z \in L_\infty \bigcap L_2$ there exists a sequence of time instants $T_s = \{t_k,\ k \ge 0\}$ with t_0 sufficiently large (but finite) such that $\bar{z}_{t_{k+1},e} < \bar{z}_{t_k,e} \le \bar{z} < \infty$ and since $\bar{z}_{t_k} \to 0$ as $k \to \infty$ since T_s is a monotonically increasing sequence and $z(t)$ converges to zero asymptotically since it is in $L_\infty \bigcap L_2$. Now, if the Basic Modification Scheme (8), (9) is used, it follows from

$$\begin{aligned}&\left\|x(t_k+\tau)\right\|_2 = \bar{z}_{t_k,e}e^{(-\sigma+\bar{\delta}_0')\tau} < \infty\\ &\qquad \text{for all } t_k \in T_s,\ 0 \le \tau \le t_{k+1}-t_k\end{aligned} \qquad (A.13)$$

where $\bar{\delta}_0' = \operatorname*{Sup}_{T\in R_0^+}\left(\operatorname*{Sup}_{0\le t\le T}\left\|\bar{\delta}_0(t)\right\|_2\right) < 1$, since $\rho' \ge 2\rho$, by applying Bellman-Gronwall's Lemma to $(A.12)$ (see [21]), provided that $\rho < \dfrac{|\sigma|}{3(n+m)}$. Thus, $\left\|x(t_k+\tau)\right\|_2 < \infty$ and $\left\|\dot{x}(t_k+\tau)\right\|_2 < \infty$ are uniformly bounded from $(A.13)$, boundedness of both the estimation error and $\bar{\delta}_0'$ and (103), (104). One has, in addition, from $(A.13)$ that $x(t_k+\tau) \to 0$ for all $\tau \in [t_k, t_{k+1})$ as $k \to \infty$ since $\bar{\delta}_0'$ is bounded from (i). As a result, $x \in L_\infty$, $\dot{x} \in L_\infty$ and $x \to 0$ and $\dot{x} \to 0$ as $t \to \infty$. Thus, the proof of (iv) follows for the Basic Modification Scheme from the calculation of the solution $x\colon [0,\infty) \to \mathbf{R^{2n}}$ to $(A.9a)$ for any initial conditions.

If the Alternative Modification Scheme of (9a)–(9c) is used, then a new upper-bound for $\bar{\delta}_0'$ has to be fixed as follows. Direct calculations with (10), (11) yields $\|\bar{\delta}_0\|_2 \le \left((1-\alpha)^2\left(\sum_{i=1}^{n} a_i^2 + \sum_{i=0}^{m} b_i^2\right) + \delta b_m^2\right)^{\frac{1}{2}}$ if Condition 1 does not hold with or $\delta b_m = \beta b_m$ or $\delta b_m = \beta b'_m$. By taking also into account (10), (11), one gets

$$\begin{aligned}
\|\bar{\delta}_0\|_2 &\leq \left(\frac{(1-\rho_\alpha)^2}{\Big(\sum\limits_{i=1}^{n}|a_i| + \sum\limits_{i=0}^{m}|b_i|\Big)^2} \Big(\sum_{i=1}^{n}|a_i| + \sum_{i=0}^{m}|b_i|\Big)^2 + \beta^2 b_m^2 \right)^{\frac{1}{2}} \\
&\leq (1-\rho_\alpha) + \beta|b_m| \\
&= 1 - \rho_\alpha + (1-\alpha)\Big(\sum_{i=1}^{n}|a_i| + \sum_{i=0}^{m}|b_i|\Big)(1-\alpha)\rho_\beta - |b_m| \\
&\leq 2(1-\rho_\alpha) + \frac{1-\rho_\alpha}{\sum\limits_{i=1}^{n}|a_i| + \sum\limits_{i=0}^{m}|b_i|}\rho_\beta,
\end{aligned} \qquad (A.14a)$$

if $\delta b_m = \beta b_m$, and

$$\begin{aligned}
\|\bar{\delta}_0\|_2 &\leq \left((1-\alpha)^2 \Big(\sum_{i=1}^{n} a_i^2 + \sum_{i=0}^{m} b_i^2\Big) + \beta'^2 \right)^{\frac{1}{2}} \\
&\leq \left((1-\alpha)^2 + 2\Big(\sum_{i=1}^{n}|a_i| + \sum_{i=0}^{m}|b_i|\Big)^2 + 2\big(\rho'_b + |b_m|\big)^2 \right)^{\frac{1}{2}} \qquad (A.14b) \\
&\leq \left((1-\rho_\alpha)^2 + 2\Big(\sum_{i=1}^{n}|a_i| + \sum_{i=0}^{m}|b_i|\Big)^2 + 2(\rho'_b + \varepsilon_b)^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

Also, if Condition 1 does not hold then the following inequalities hold:

$$\begin{aligned}
\frac{(n+m)^{\frac{1}{2}} - \varepsilon_0}{\varepsilon_0} &< \big\|S(\theta_0)\big\|_1 - 1 \leq \sum_{i=1}^{n}|a_i| + \sum_{i=0}^{m}|b_i| \\
&< 1 + \big\|S(\theta_0)\big\|_\infty \leq 1 + \frac{(n+m)^{-\frac{1}{2}}}{\varepsilon'_0}.
\end{aligned} \qquad (A.15)$$

The remaining of the proof follows from $(A.13)$ after substituting $(A.15)$ into (105) to calculate upper-bounds $\bar{\delta}'_i$ $(i = 1, 2)$ of $\operatorname{Sup}\limits_{t \geq 0} \left(\|\bar{\delta}_0\|_2\right)$.

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