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Mathematical Modelling on $RLCG$ Transmission Lines*

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Abstract. A new model on $RLCG$ transmissions lines is presented in the paper. The model suits to be taken to directly simulating a circuit system in time-domain. Mathematically, a circuit system with distributed elements may be described by a special kind of nonlinear integral-differential-algebraic equations with multiple constant delays.

Keywords: $RLCG$ transmission lines, modelling, nonlinear circuits, integral-differential-algebraic equations with multiple delays, simulation in time-domain.

1 Introduction

It is known that the conductors of a circuit system should be regarded as transmission lines for theoretical analysis and practical design in the recent high-speed integrated circuit technology [1]. At relatively higher signal-speeds, transmission line models based on quasi-transverse electro-magnetic mode (TEM) assumptions are severely useful for circuit simulation [2]. The TEM approximation represents the ideal case. Often, from the system design point of view the solution to Maxwell's equations may be given by the so-called quasi-TEM modes, and it can be characterized by distributed parameters R, L, C, and G [3].

The simulation task is to compute the transient response of a circuit system consisting of nonlinear devices interconnected by transmission lines. In the short

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paper we establish a new model on $RLCG$ transmission lines in time-domain. A circuit system with distributed elements may be described by nonlinear integral-differential-algebraic equations (IDAEs) with multiple constant delays such that the general-purpose circuit simulators can be then used to solve the new equations in theory.

2 Distributed models of $RLCG$ transmission lines

In general, a transmission line is presented by Telegrapher's equations. For an $RLCG$ transmission line system shown in Fig. 1, at time t ($0 \le t \le T_e$) let $v(x,t)$ and $i(x,t)$ respectively be voltage and current at point x ($0 \le x \le d$). The basic equations are

$$\begin{aligned} \frac{\partial v(x,t)}{\partial x} + L\,\frac{\partial i(x,t)}{\partial t} &= -Ri(x,t), \\ \frac{\partial i(x,t)}{\partial x} + C\,\frac{\partial v(x,t)}{\partial t} &= -Gv(x,t), \end{aligned} \tag{1}$$

where R is resistance, L is inductance, C is capacitance, and G is conductance for unit length. The constants R, L, C, and G are distributed parameters for $RLCG$ transmission line. If $R = 0$ and $G = 0$, the transmission line is lossless, see [4].

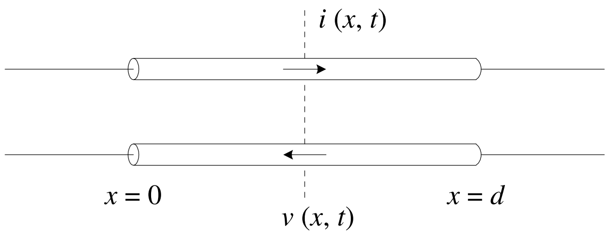

Fig. 1. An $RLCG$ transmission line.

Let $LC = 1/\nu^2$ where ν is velocity of signal propagation, $\tau = \frac{d}{\nu}$ which is the delay of a signal going from $x = 0$ to $x = d$, and $z_0 = \sqrt{L/C}$ which is the characteristic impedance. We now write (1) in matrix form as follows

$$\frac{\partial U(x,t)}{\partial x} + \widetilde{A}\,\frac{\partial U(x,t)}{\partial t} = \widetilde{B}\,U(x,t), \tag{2}$$

where $U(x,t) = \left[v(x,t), i(x,t)\right]^t$, and

$$\widetilde{A} = \begin{bmatrix} 0 & L \\ C & 0 \end{bmatrix}, \quad \widetilde{B} \begin{bmatrix} 0 & -R \\ -G & 0 \end{bmatrix}.$$

Obviously, we have

$$T\widetilde{A}T^{-1} = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix},$$

where $\lambda = \sqrt{LC}$ and

$$T = \frac{1}{2}\begin{bmatrix} 1 & z_0 \\ 1 & -z_0 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 1 \\ \frac{1}{z_0} & -\frac{1}{z_0} \end{bmatrix}. \tag{3}$$

We let $\widetilde{U}(x,t) = TU(x,t)$ where $\widetilde{U}(x,t) = \left[\widetilde{U}_v(x,t), \widetilde{U}_i(x,t)\right]^t$, from (2) we have

$$\frac{\partial \widetilde{U}(x,t)}{\partial x} + \widetilde{D}\,\frac{\partial \widetilde{U}(x,t)}{\partial t} = \widetilde{E}\widetilde{U}(x,t),$$

where $\widetilde{D} = T\widetilde{A}T^{-1}$ and $\widetilde{E} = T\widetilde{B}T^{-1}$. Namely,

$$\begin{aligned} \frac{\partial \widetilde{U}_v(x,t)}{\partial x} + \lambda\,\frac{\partial \widetilde{U}_v(x,t)}{\partial t} &= \left(\widetilde{E}\widetilde{U}(x,t)\right)_1, \\ \frac{\partial \widetilde{U}_i(x,t)}{\partial x} - \lambda\,\frac{\partial \widetilde{U}_i(x,t)}{\partial t} &= \left(\widetilde{E}\widetilde{U}(x,t)\right)_2, \end{aligned} \tag{4}$$

where $\left(\widetilde{E}\widetilde{U}(x,t)\right)_1$ and $\left(\widetilde{E}\widetilde{U}(x,t)\right)_2$ are the two elements of $\widetilde{E}\widetilde{U}(x,t)$. The above partial differential equations (PDEs) can be further expressed as a form of integral equations with constant delay by the method of characteristics (MC), see [5].

First, we construct two characteristic lines as follows

$$\begin{aligned} l_+ &: \frac{dt}{dx} = \lambda, \\ l_- &: \frac{dt}{dx} = -\lambda. \end{aligned}$$

In other words, the lines l_+ and l_- are defined by $t - \lambda x = c$ and $t + \lambda x = c$, where c is some constant. For any point (x,t), we integrate the first equation and the second equation of (4) along the lines l_+ and l_- respectively from $(0, t-\lambda x)$ and $(0, t+\lambda x)$. Thus,

$$\begin{aligned} \widetilde{U}_v(x,t) &= \widetilde{U}_v(0, t-\lambda x) + \int\limits_{l_+} \big(\widetilde{E}\widetilde{U}(x,t)\big)_1 dl \\ &= f(t-\lambda x) + \int\limits_0^x \Big(\widetilde{E}\widetilde{U}\big(l, t-\lambda(x-l)\big)\Big)_1 dl, \\ \widetilde{U}_i(x,t) &= \widetilde{U}_i(0, t+\lambda x) + \int\limits_{l_-} \big(\widetilde{E}\widetilde{U}(x,t)\big)_2 dl \\ &= g(t+\lambda x) + \int\limits_0^x \Big(\widetilde{E}\widetilde{U}\big(l, t+\lambda(x-l)\big)\Big)_2 dl, \end{aligned} \tag{5}$$

where f and g are two continuously differentially initial functions.

Based on (5), by $\widetilde{E} = T\widetilde{B}T^{-1}$ and $\widetilde{U} = TU$ we have

$$TU(x,t) = \begin{bmatrix} f(t-\lambda x) \\ g(t+\lambda x) \end{bmatrix} + \int\limits_0^x \begin{bmatrix} \big(T\widetilde{B}U\big(l, t-\lambda(x-l)\big)\big)_1 \\ \big(T\widetilde{B}U\big(l, t+\lambda(x-l)\big)\big)_2 \end{bmatrix} dl. \tag{6}$$

Since

$$T\widetilde{B}U = \frac{1}{2} \begin{bmatrix} -z_0 G & -R \\ z_0 G & -R \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} z_0 G v + Ri \\ -z_0 G v + Ri \end{bmatrix}, \tag{7}$$

from (6) we have

$$\begin{aligned} f(t-\lambda x) &= \frac{1}{2}\big[v(x,t) + z_0 i(x,t)\big] \\ &\quad + \frac{1}{2}\left[z_0 G \int\limits_0^x v\big(l, t-\lambda(x-l)\big) dl + R \int\limits_0^x i\big(l, t-\lambda(x-l)\big) dl\right], \\ g(t+\lambda x) &= \frac{1}{2}\big[v(x,t) - z_0 i(x,t)\big] \\ &\quad - \frac{1}{2}\left[z_0 G \int\limits_0^x v\big(l, t+\lambda(x-l)\big) dl - R \int\limits_0^x i\big(l, t+\lambda(x-l)\big) dl\right]. \end{aligned} \tag{8}$$

Let $x = d$ in the second equation of (8), it follows

$$g(t+\lambda d) = \frac{1}{2}\big[v(d,t) - z_0 i(d,t)\big] - \frac{1}{2}\bigg[z_0 G \int_0^d v\big(l, t+\lambda(d-l)\big)dl - R\int_0^d i\big(l, t+\lambda(d-l)\big)dl\bigg].$$

Let $x = 0$ in the first equation of (8) and set $t = t - \lambda d$ in the above expression, by $\lambda d = \tau$ we can arrive at

$$\begin{aligned} f(t) &= \frac{1}{2}W_B(t), \\ g(t) &= \frac{1}{2}W_A(t-\tau) - \frac{1}{2}\bigg[z_0 G\int_0^d v(l, t-\lambda l)dl - R\int_0^d i(l, t-\lambda l)dl\bigg], \end{aligned} \tag{9}$$

where the new functions W_A and W_B are defined as

$$W_A(t) = v(d,t) - z_0 i(d,t), \quad W_B(t) = v(0,t) + z_0 i(0,t). \tag{10}$$

Now, by (6), (7), and the form of T^{-1} in (5) we know

$$\begin{aligned} v(x,t) &= f(t-\lambda x) + g(t+\lambda x) \\ &\quad - \frac{z_0 G}{2}\int_0^x \Big[v\big(l, t-\lambda(x-l)\big) - v\big(l, t+\lambda(x-l)\big)\Big]dl \\ &\quad - \frac{R}{2}\int_0^x \Big[i\big(l, t-\lambda(x-l)\big) + i\big(l, t+\lambda(x-l)\big)\Big]dl, \\ i(x,t) &= \frac{1}{z_0}\big[f(t-\lambda x) - g(t+\lambda x)\big] \\ &\quad - \frac{G}{2}\int_0^x \Big[v\big(l, t-\lambda(x-l)\big) + v\big(l, t+\lambda(x-l)\big)\Big]dl \\ &\quad - \frac{R}{2z_0}\int_0^x \Big[i\big(l, t-\lambda(x-l)\big) - i\big(l, t+\lambda(x-l)\big)\Big]dl. \end{aligned}$$

Then, by (9) it further deduces

$$\begin{aligned}
v(x,t) &= \frac{1}{2}W_A(t-\tau+\lambda x) + \frac{1}{2}W_B(t-\lambda x) \\
&\quad - \frac{z_0 G}{2}\Big[\int_0^x v\big(l, t-\lambda(x-l)\big)dl + \int_x^d v\big(l, t+\lambda(x-l)\big)dl\Big] \\
&\quad - \frac{R}{2}\Big[\int_0^x i\big(l, t-\lambda(x-l)\big)dl - \int_x^d i\big(l, t+\lambda(x-l)\big)dl\Big], \\
i(x,t) &= -\frac{1}{2z_0}W_A(t-\tau+\lambda x) + \frac{1}{2z_0}W_B(t-\lambda x) \\
&\quad - \frac{G}{2}\Big[\int_0^x v\big(l, t-\lambda(x-l)\big)dl - \int_x^d v\big(l, t+\lambda(x-l)\big)dl\Big] \\
&\quad - \frac{R}{2z_0}\Big[\int_0^x i\big(l, t-\lambda(x-l)\big)dl + \int_x^d i\big(l, t+\lambda(x-l)\big)dl\Big], \\
&0 \le x \le d, \quad 0 \le t \le T_e.
\end{aligned} \tag{11}$$

Based on the second equation of (11), by $\lambda d = \tau$ we also have

$$\begin{aligned}
i(0,t) &= -\frac{1}{2z_0}W_A(t-\tau) + \frac{1}{2z_0}W_B(t) \\
&\quad + \frac{G}{2}\int_0^d v(l, t-\lambda l)dl - \frac{R}{2z_0}\int_0^d i(l, t-\lambda l)dl, \\
i(d,t) &= -\frac{1}{2z_0}W_A(t) + \frac{1}{2z_0}W_B(t-\tau) \\
&\quad - \frac{G}{2}\int_0^d v(l, t-\tau+\lambda l)dl - \frac{R}{2z_0}\int_0^d i(l, t-\tau+\lambda l)dl.
\end{aligned}$$

By use of the expressions on $W_A(t)$ and $W_B(t)$ in (10), for $t \in [0, T_e]$ we obtain the currents at the near and far ends of the line as

$$\begin{aligned} i(0,t) &= \frac{1}{z_0}v(0,t) - \frac{1}{z_0}W_A(t-\tau) \\ &\quad + G\int_0^d v(l,t-\lambda l)dl - \frac{R}{z_0}\int_0^d i(l,t-\lambda l)dl, \\ i(d,t) &= -\frac{1}{z_0}v(d,t) + \frac{1}{z_0}W_B(t-\tau) \\ &\quad - G\int_0^d v(l,t-\tau+\lambda l)dl - \frac{R}{z_0}\int_0^d i(l,t-\tau+\lambda l)dl. \end{aligned} \tag{12}$$

This is a basic characteristic for $RLCG$ transmission lines.

From (12), for $t \in [0, T_e]$ we also have the voltages at the near and far ends of the line as

$$\begin{aligned} v(0,t) &= z_0 i(0,t) + W_A(t-\tau) \\ &\quad - z_0 G\int_0^d v(l,t-\lambda l)dl + R\int_0^d i(l,t-\lambda l)dl, \\ v(d,t) &= -z_0 i(d,t) + W_B(t-\tau) \\ &\quad - z_0 G\int_0^d v(l,t-\tau+\lambda l)dl - R\int_0^d i(l,t-\tau+\lambda l)dl. \end{aligned} \tag{13}$$

To combine (10) and (13), we further arrive at

$$\begin{aligned} W_A(t) &= 2v(d,t) - W_B(t-\tau) \\ &\quad + z_0 G\int_0^d v(l,t-\tau+\lambda l)dl + R\int_0^d i(l,t-\tau+\lambda l)dl, \\ W_B(t) &= 2v(0,t) - W_A(t-\tau) \\ &\quad + z_0 G\int_0^d v(l,t-\lambda l)dl - R\int_0^d i(l,t-\lambda l)dl. \end{aligned}$$

3 A general form of circuit equations with distributed elements

We first see two simple circuits with $RLCG$ transmission lines. For the basic circuit with distributed elements shown in Fig. 2, its circuit equations are

$$
\begin{aligned}
c_1\frac{dv_1(t)}{dt} &= g_1(e-v_1)(t) - \frac{1}{z_0}v_1(t) + \frac{1}{z_0}W_A(t-\tau)\\
&\quad - G\int_0^d v(l,t-\lambda l)dl + \frac{R}{z_0}\int_0^d i(l,t-\lambda l)dl,\\
c_2\frac{dv_2(t)}{dt} &= -g_2(v_2)(t) - \frac{1}{z_0}v_2(t) + \frac{1}{z_0}W_B(t-\tau)\\
&\quad - G\int_0^d v(l,t-\tau+\lambda l)dl - \frac{R}{z_0}\int_0^d i(l,t-\tau+\lambda l)dl,\\
W_A(t) &= 2v_2(t) - W_B(t-\tau)\\
&\quad + z_0G\int_0^d v(l,t-\tau+\lambda l)dl + R\int_0^d i(l,t-\tau+\lambda l)dl,\\
W_B(t) &= 2v_1(t) - W_A(t-\tau)\\
&\quad + z_0G\int_0^d v(l,t-\lambda l)dl - R\int_0^d i(l,t-\lambda l)dl,\\
v(x,t) &= \frac{1}{2}W_A(t-\tau+\lambda x) + \frac{1}{2}W_B(t-\lambda x)\\
&\quad - \frac{z_0G}{2}\Big[\int_0^x v\big(l,t-\lambda(x-l)\big)dl + \int_x^d v\big(l,t+\lambda(x-l)\big)dl\Big]\\
&\quad - \frac{R}{2}\Big[\int_0^x i\big(l,t-\lambda(x-l)\big)dl - \int_x^d i\big(l,t+\lambda(x-l)\big)dl\Big],\\
i(x,t) &= -\frac{1}{2z_0}W_A(t-\tau+\lambda x) + \frac{1}{2z_0}W_B(t-\lambda x)\\
&\quad - \frac{G}{2}\Big[\int_0^x v\big(l,t-\lambda(x-l)\big)dl - \int_x^d v\big(l,t+\lambda(x-l)\big)dl\Big]\\
&\quad - \frac{R}{2z_0}\Big[\int_0^x i\big(l,t-\lambda(x-l)\big)dl + \int_x^d i\big(l,t+\lambda(x-l)\big)dl\Big],
\end{aligned}
$$

$$v(0,t) = v_1(t), \quad v(d,t) = v_2(t), \quad 0 \le x \le d, \quad 0 \le t \le T_e,$$

where g_1 and g_2 are nonlinear functions.

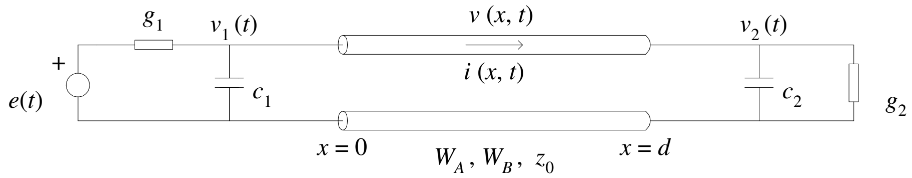

Fig. 2. A circuit with $RLCG$ transmission lines.

Another distributed circuit is shown in Fig. 3. Its circuit equations are described by IDAEs with multiple constant delays,

$$\begin{aligned}
c_1\frac{dv_1(t)}{dt} &= -\Big(\frac{1}{R_1}+\frac{1}{z_{01}}\Big)v_1(t) + \frac{1}{z_{01}}W_{A1}(t-\tau_1) \\
&\quad - G_{W1}\int_0^{d_1} v_{W1}(l, t-\lambda_1 l)dl + \frac{R_{W1}}{z_{01}}\int_0^{d_1} i_{W1}(l, t-\lambda_1 l)dl + \frac{e(t)}{R_e}, \\
c_2\frac{dv_2(t)}{dt} &= -\Big(\frac{1}{z_{01}}+\frac{1}{z_{02}}\Big)v_2(t) + \frac{1}{z_{01}}W_{B1}(t-\tau_1) + \frac{1}{z_{02}}W_{A2}(t-\tau_2) \\
&\quad - G_{W1}\int_0^{d_1} v_{W1}(l, t-\tau_1+\lambda_1 l)dl - \frac{R_{W1}}{z_{01}}\int_0^{d_1} i_{W1}(l, t-\tau_1+\lambda_1 l)dl \\
&\quad - G_{W2}\int_0^{d_2} v_{W2}(l, t-\lambda_2 l)dl + \frac{R_{W2}}{z_{02}}\int_0^{d_2} i_{W2}(l, t-\lambda_2 l)dl, \\
c_3\frac{dv_3(t)}{dt} &= -\Big(\frac{1}{R_f}+\frac{1}{z_{02}}\Big)v_3(t) + \frac{1}{z_{02}}W_{B2}(t-\tau_2) \\
&\quad - G_{W2}\int_0^{d_2} v_{W2}(l, t-\tau_2+\lambda_2 l)dl - \frac{R_{W2}}{z_{02}}\int_0^{d_2} i_{W2}(l, t-\tau_2+\lambda_2 l)dl, \\
W_{A1}(t) &= 2v_2(t) - W_{B1}(t-\tau_1) \\
&\quad + z_{01}G_{W1}\int_0^{d_1} v_{W1}(l, t-\tau_1+\lambda_1 l)dl + R_{W1}\int_0^{d_1} i_{W1}(l, t-\tau_1+\lambda_1 l)dl,
\end{aligned}$$

$$
\begin{aligned}
W_{B1}(t) &= 2v_1(t) - W_{A1}(t-\tau_1) \\
&\quad + z_{01}G_{W1}\int_0^{d_1} v_{W1}(l, t-\lambda_1 l)dl - R_{W1}\int_0^{d_1} i_{W1}(l, t-\lambda_1 l)dl, \\
W_{A2}(t) &= 2v_3(t) - W_{B2}(t-\tau_2) \\
&\quad + z_{02}G_{W2}\int_0^{d_2} v_{W2}(l, t-\tau_2+\lambda_2 l)dl + R_{W2}\int_0^{d_2} i_{W2}(l, t-\tau_2+\lambda_2 l)dl, \\
W_{B2}(t) &= 2v_2(t) - W_{A2}(t-\tau_2) \\
&\quad + z_{02}G_{W2}\int_0^{d_2} v_{W2}(l, t-\lambda_2 l)dl - R_{W2}\int_0^{d_2} i_{W2}(l, t-\lambda_2 l)dl, \\
v_{W1}(x,t) &= \frac{1}{2}W_{A1}(t-\tau_1+\lambda_1 x) + \frac{1}{2}W_{B1}(t-\lambda_1 x) \\
&\quad - \frac{z_{01}G_{W1}}{2}\Big[\int_0^{x} v_{W1}\big(l, t-\lambda_1(x-l)\big)dl + \int_x^{d_1} v_{W1}\big(l, t+\lambda_1(x-l)\big)dl\Big] \\
&\quad - \frac{R_{W1}}{2}\Big[\int_0^{x} i_{W1}\big(l, t-\lambda_1(x-l)\big)dl - \int_x^{d_1} i_{W1}\big(l, t+\lambda_1(x-l)\big)dl\Big], \\
i_{W1}(x,t) &= -\frac{1}{2z_{01}}W_{A1}(t-\tau_1+\lambda_1 x) + \frac{1}{2z_{01}}W_{B1}(t-\lambda_1 x) \\
&\quad - \frac{G_{W1}}{2}\Big[\int_0^{x} v_{W1}\big(l, t-\lambda_1(x-l)\big)dl - \int_x^{d_1} v_{W1}\big(l, t+\lambda_1(x-l)\big)dl\Big] \\
&\quad - \frac{R_{W1}}{2z_{01}}\Big[\int_0^{x} i_{W1}\big(l, t-\lambda_1(x-l)\big)dl + \int_x^{d_1} i_{W1}\big(l, t+\lambda_1(x-l)\big)dl\Big], \\
v_{W2}(y,t) &= \frac{1}{2}W_{A2}(t-\tau_2+\lambda_2 y) + \frac{1}{2}W_{B2}(t-\lambda_2 y) \\
&\quad - \frac{z_{02}G_{W2}}{2}\Big[\int_0^{y} v_{W2}\big(l, t-\lambda_2(y-l)\big)dl + \int_y^{d_2} v_{W2}\big(l, t+\lambda_2(y-l)\big)dl\Big] \\
&\quad - \frac{R_{W2}}{2}\Big[\int_0^{y} i_{W2}\big(l, t-\lambda_1(y-l)\big)dl - \int_y^{d_2} i_{W2}\big(l, t+\lambda_2(y-l)\big)dl\Big],
\end{aligned}
$$

$$i_{W2}(y,t) = -\frac{1}{2z_{02}}W_{A2}(t-\tau_2+\lambda_2 y) + \frac{1}{2z_{02}}W_{B2}(t-\lambda_2 y)$$
$$-\frac{G_{W2}}{2}\left[\int_0^y v_{W2}\big(l,t-\lambda_2(y-l)\big)dl - \int_y^{d_2} v_{W2}\big(l,t+\lambda_2(y-l)\big)dl\right]$$
$$-\frac{R_{W2}}{2z_{02}}\left[\int_0^y i_{W2}\big(l,t-\lambda_2(y-l)\big)dl + \int_y^{d_2} i_{W2}\big(l,t+\lambda_2(y-l)\big)dl\right],$$

$$v_{W1}(0,t) = v_1(t), \quad v_{W1}(d,t) = v_2(t),$$
$$v_{W2}(0,t) = v_2(t), \quad v_{W2}(d,t) = v_3(t),$$
$$0 \le x \le d_1, \quad 0 \le y \le d_2, \quad 0 \le t \le T_e,$$

where $R_{Wj}, L_{Wj}, C_{Wj}, G_{Wj}$ $(j = 1,2)$ are respectively the distributed parameters of the first and second lines, and $\lambda_j = \sqrt{L_{Wj}C_{Wj}}$ $(j = 1,2)$.

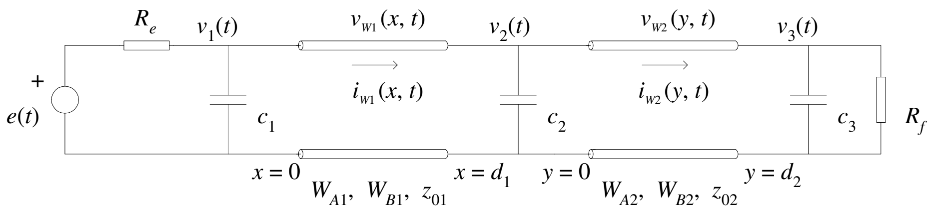

Fig. 3. A circuit with multiple $RLCG$ transmission lines.

Thus, the general form of equations on a circuit system with $RLCG$ transmission lines should be a system of nonlinear IDAEs with multiple constant delays as follows

$$C(t)\frac{dx(t)}{dt} + G\big(x(t),t\big) + DW(t-\tau) + E\int_0^d y(l,t-\lambda l)dl + F\int_0^d y(l,t-\tau+\lambda l)dl = b(t), \tag{14}$$

$$\begin{aligned}
&Ax(t) + W(t) + BW(t-\tau) \\
&\qquad + H\int_0^d y(l, t-\lambda l)dl + L\int_0^d y(l, t-\tau+\lambda l)dl = 0, \qquad (15)\\
&y(l,t) + PW(t-\lambda l) + QW(t-\tau+\lambda l) \\
&\qquad + M\int_0^l y\big(r, t-\lambda(l-r)\big)dr + N\int_l^d y\big(r, t+\lambda(l-r)\big)dr = 0, \qquad (16)\\
&y(0,t) = S_1 x(t),\ \ y(d,t) = S_2 x(t),\ \ y(l,\theta) = \psi(l,\theta),\ \text{for } -\tau \le \theta < 0,\\
&x(0) = x_0,\ \ W(\theta) \equiv \varphi(\theta),\ \text{for } -\tau \le \theta < 0,\\
&0 \le l \le d,\ \ 0 \le r \le l,\ \ t \in [0, T_e],
\end{aligned}$$

where $C(\cdot) \in \mathbf{R}^{n\times n}$ is a matrix-valued function, $A, S_1, S_2 \in \mathbf{R}^{2m\times n}$, $D, E, F \in \mathbf{R}^{n\times 2m}$, $B, H, L, P, Q, M, N \in \mathbf{R}^{2m\times 2m}$, $G(\cdot,\cdot) \in \mathbf{R}^n$ is a nonlinear function, and for any t and l the functions $x(t) \in \mathbf{R}^n$, $y(l, t-\tau), W(t-\tau) \in \mathbf{R}^{2m}$ are to be computed in which

$$\begin{aligned}
y(l, t-\tau) &= \big[v_1(l_1, t-\tau_1), i_1(l_1, t-\tau_1),\\
&\qquad \cdots, v_m(l_m, t-\tau_m), i_m(l_m, t-\tau_m)\big]^t,\\
W(t-\tau) &= \big[W_{A1}(t-\tau_1), W_{B1}(t-\tau_1),\\
&\qquad \cdots, W_{Am}(t-\tau_m), W_{Bm}(t-\tau_m)\big]^t,
\end{aligned}$$

where $l = [l_1, \cdots, l_m]^t$ and $\tau = [\tau_1, \cdots, \tau_m]^t$. Further, $\lambda = [\lambda_1, \cdots, \lambda_m]^t > 0$, $d = [d_1, \cdots, d_m]^t > 0$, $\tau = \lambda d = [\tau_1, \cdots, \tau_m]^t > 0$, where $\tau_j = \lambda_j d_j$ ($j = 1, \cdots, m$), $r = [r_1, \cdots, r_m]^t$, and $\lambda l = [\lambda_1 l_1, \cdots, \lambda_m l_m]^t$.

For the above circuit system, $b(\cdot) \in \mathbf{R}^n$ is a known input vector function, x_0 is an initial value, and $\varphi(\theta)$ and $\psi(l,\theta)$ are initial states of the $RLCG$ transmission line system such that

$$\begin{aligned}
\varphi(\theta) &= \big[W_{A1}(\theta_1), W_{B1}(\theta_1), \cdots, W_{Am}(\theta_m), W_{Bm}(\theta_m)\big]^t,\\
\psi(l,\theta) &= \big[v_1(l_1,\theta_1), i_1(l_1,\theta_1), \cdots, v_m(l_m,\theta_m), i_m(l_m,\theta_m)\big]^t,
\end{aligned}$$

in which $-\tau_j \le \theta_j < 0$ $(1 \le j \le m)$. In practical application the initial values

$x_0, W(0)$ are consistent, that is,

$$Ax_0 + W(0) + BW(-\tau) + H\int_0^d y(l, -\lambda l)dl + L\int_0^d y(l, -\tau + \lambda l)dl = 0.$$

Moreover, by invoking the property of transmission lines we should also assume that the form of B in (15) is a block diagonal matrix such that

$$B = \begin{bmatrix} I_d & & 0 \\ & \ddots & \\ 0 & & I_d \end{bmatrix} \in \mathbf{R}^{2m\times 2m},$$

where $I_d = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. For the lossless case ($R = G = 0$), the mathematical model and its relaxation solutions in function space are provided in [4].

4 Summary

We have presented a new time-domain model on $RLCG$ transmission lines. The circuit system with distributed elements is described by nonlinear integral-differential-algebraic equations with multiple constant delays. In theory, the new approach directly leads to solution of the circuit system in time-domain and the general-purpose circuit simulators can be then used to solve the system.

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