Nonlinear Analysis: Modelling and Control, Vol. 24, No. 6, 870–883 ISSN: 1392-5113
https://doi.org/10.15388/NA.2019.6.2 eISSN: 2335-8963

New theorems on extended b-metric spaces under new contractions*

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Received: October 6, 2018 / **Revised:** March 21, 2019 / **Published online:** November 8, 2019

Abstract. The notion of extended b-metric space plays an important role in the field of applied analysis to construct new theorems in the field of fixed point theory. In this paper, we construct and prove new theorems in the filed of fixed point theorems under some new contractions. Our results extend and modify many existing results in the literature. Also, we provide an example to show the validity of our results. Moreover, we apply our result to solve the existence and uniqueness of such equations.

Keywords: fixed point, b-metric spaces, contraction.

1 Introduction

Fixed point theory is one of the main tools that used to solve many nonlinear problems in applied analysis. Many authors extended the metric spaces to many new spaces such as partial metric spaces, G-metric spaces, and cone metric spaces. For fixed point theorems in metric spaces, see [1, 2, 10, 12–16, 18, 22, 27, 30, 31].

Bakhtin introduces the notion of b-metric spaces in [11] as a generalization of metric spaces and proved many exciting fixed point theorem. For some interesting fixed point results in b-metric spaces, see [3–9, 19–21, 23–26, 29]. Kamran et al. in [17] introduced the definition of extended b-metric spaces as a generalization of b-metric spaces. Shatanawi et al. [28] considered more interesting results on extended b-metric spaces. However, in the results of Kamran et al. in [17], they assumed that the metric d_θ is continuous. We

*This research was supported by Prince Sultan University through research group NAMAM, group number RG-DES-2017-01-17.
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do not think the continuous hypothesis is necessary. This does not mean that the result is wrong, it just means that the hypothesis of these results can be weaker, which will make the their result stronger and more general, which will cover a lager class of self mappings in such space.

2 Mathematical preliminaries

First of all, we remind the reader of the definition of extended b-metric spaces.

Definition 1. (See [17].) Let X be a nonempty set and $\theta : X \times X \to [1, \infty)$. A function $d_\theta : X \times X \to [0, \infty)$ is called an extended b-metric if, for all $x, y, z \in X$, it satisfies:

(i) $d_\theta(x, y) = 0$ iff $x = y$;
(ii) $d_\theta(x, y) = d_\theta(y, x)$;
(iii) $d_\theta(x, z) \leqslant \theta(x, z)[d_\theta(x, y) + d_\theta(y, z)]$.

The pair (X, d_θ) is called an extended b-metric space.

The notions of a Cauchy sequence and a convergent sequence in extended b-metric spaces are defined as follows.

Definition 2. Let (X, d_θ) be an extended b-metric space. Let $\{x_n\}$ be a sequence in a X.

(i) A sequence $\{x_n\}$ in X is said to converge to $x \in X$ if, for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_n, x) < \epsilon$ for all $n \geqslant N$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is said to be Cauchy if, for every $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\theta(x_m, x_n) < \epsilon$ for all $m, n \geqslant N$.

As usual, an extended b-metric space (X, d_θ) is said to be complete if every Cauchy sequence in X is convergent. The continuity of a mapping with respect to an extended b-metric is defined as follows.

Definition 3. Let (X, d_θ) and (X', d'_θ) be two extended b-metric spaces. A mapping $T : X \to X'$ is called continuous if, for each sequence $\{x_n\}$ in X, which converges to $x \in X$ with respect to d_θ, $\{Tx_n\}$ converges to Tx with respect to d'_θ.

3 Main results

Now we are ready to prove our main results.

Theorem 1. *Let (X, d_θ) be a complete extended b-metric space, and let T be a self mapping on X. Assume there exists $k \in [0, 1)$ such that*

$$d_\theta(Tx, Ty) \leqslant k\theta(x, y)d_\theta(x, y) \tag{1}$$

for all $x, y \in X$. *Assume for* $x_0 \in X$, *we have*

$$\limsup_{n,m\to\infty} \theta(x_n, x_m)\theta(x_n, x_{n+1}) < \frac{1}{k},$$

where $x_j = T^j x_0$ *for all* $j \in \mathbb{N}$. *Moreover, assume* $\limsup_{n\to+\infty} \theta(x, x_n)$ *exists for any* $x \in X$. *Then* T *has a fixed point* x^*. *Moreover, assume that for any* $x, y \in X$, *we have*

$$\limsup_{n\to+\infty} \theta\bigl(T^n x, T^n y\bigr) < \frac{1}{k},$$

where $T^n x = T^{n-1}(Tx)$. *Then the fixed point of* T *is unique.*

Proof. Let $x_0 \in X$. Construct a sequence $\{x_n\}$ in X in the following way: $x_0,\ x_1 = Tx_0,\ x_2 = T^2 x_0,\ \dots$. Then

$$\limsup_{n,m\to\infty} \theta(x_n, x_m)\theta(x_n, x_{n+1}) < \frac{1}{k},$$

and $\limsup_{n\to+\infty} \theta(x, x_n)$ exists for any $x \in X$. By using (1) n times, we get

$$\begin{aligned}
d_\theta(x_n, x_{n+1}) &= d_\theta(Tx_{n-1}, Tx_n) \\
&\leqslant k\theta(x_{n-1}, x_n) d_\theta(x_{n-1}, x_n) \\
&= k\theta(x_{n-1}, x_n) d_\theta(Tx_{n-2}, Tx_{n-1}) \\
&\leqslant k^2\theta(x_{n-1}, x_n)\theta(x_{n-2}, x_{n-1}) d_\theta(x_{n-2}, x_{n-1}) \\
&\leqslant \cdots \\
&\leqslant k^n\theta(x_{n-1}, x_n)\theta(x_{n-2}, x_{n-1})\cdots\theta(x_0, x_1) d_\theta(x_0, x_1) \\
&= k^n \prod_{i=1}^{n} \theta(x_{i-1}, x_i) d_\theta(x_0, x_1).
\end{aligned} \tag{2}$$

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
d_\theta(x_n, x_m) &\leqslant \theta(x_n, x_m)\bigl[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)\bigr] \\
&\leqslant \theta(x_n, x_m) d_\theta(x_n, x_{n+1}) \\
&\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d_\theta(x_{n+1}, x_{n+2}) \\
&\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m) d_\theta(x_{n+2}, x_m) \\
&\leqslant \theta(x_n, x_m) d_\theta(x_n, x_{n+1}) \\
&\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m) d_\theta(x_{n+1}, x_{n+2}) \\
&\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m) d_\theta(x_{n+2}, x_{n+3}) \\
&\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m) d_\theta(x_{n+2}, x_m) \\
&\leqslant \cdots \\
&\leqslant \sum_{i=n}^{m-1} \prod_{j=n}^{i} \theta(x_j, x_m) d_\theta(x_i, x_{i+1}).
\end{aligned} \tag{3}$$

Now, combining (2) and (3) together, we get

$$d_\theta(x_n, x_m) \leqslant \sum_{i=n}^{m-1} \prod_{j=n}^{i} \theta(x_j, x_m) \prod_{s=1}^{i} \theta(x_{s-1}, x_s) k^i d_\theta(x_0, x_1). \tag{4}$$

Let

$$a_i = \prod_{j=n}^{i} \theta(x_j, x_m) \prod_{s=1}^{i} \theta(x_{s-1}, x_s) k^i d_\theta(x_0, x_1).$$

Then

$$a_{i+1} = \prod_{j=n}^{i+1} \theta(x_j, x_m) \prod_{s=1}^{i+1} \theta(x_{s-1}, x_s) k^{i+1} d_\theta(x_0, x_1).$$

Thus,

$$\frac{a_{i+1}}{a_i} = \theta(x_{i+1}, x_m)\theta(x_i, x_{i+1})k.$$

Hence,

$$\limsup_{i\to+\infty} \frac{a_{i+1}}{a_i} = \limsup_{i,m\to+\infty} \theta(x_{i+1}, x_m)\theta(x_i, x_{i+1})k < 1.$$

Thus, we conclude that

$$\sum_{i=1}^{+\infty} \prod_{j=n}^{i} \theta(x_j, x_m) \prod_{s=1}^{i} \theta(x_{s-1}, x_s) k^i d_\theta(x_0, x_1) < +\infty.$$

Hence,

$$\left(\sum_{i=n}^{m-1} \prod_{j=n}^{i} \theta(x_j, x_m) \prod_{s=1}^{i} \theta(x_{s-1}, x_s) k^i d_\theta(x_0, x_1) \right)$$

is a Cauchy sequence in $\mathbb{R}$. By (4), we conclude that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to some $u \in X$. By triangular inequality and (1), we get

$$\begin{aligned} d_\theta(Tu, u) &\leqslant \theta(Tu, u)\big[d_\theta(Tu, x_{n+1}) + d_\theta(x_{n+1}, u)\big] \\ &= \theta(Tu, u)\big[d_\theta(Tu, Tx_n) + d_\theta(x_n, u)\big] \\ &\leqslant \theta(Tu, u)\big[k\theta(u, x_n)d_\theta(u, x_{n-1}) + d_\theta(x_n, u)\big]. \end{aligned}$$

Letting $n \to +\infty$ in above inequalities, we conclude that $d_\theta(Tu, u) \leqslant 0$. Hence, $d_\theta(Tu, u) = 0$. So u is a fixed point of T. To prove the uniqueness of the fixed point, let u and v be two distinct fixed points of T. Then

$$\begin{aligned} d_\theta(u, v) = d_\theta(Tu, Tv) &\leqslant k\theta(v, u)d_\theta(v, u) \\ &= k\theta\big(T^n u, T^n y\big)d_\theta(u, v). \end{aligned}$$

Letting $n \to +\infty$ in above inequality, we deduce that $d_\theta(u, v) < d_\theta(u, v)$, which is a contradiction. So we conclude that the fixed point of T is unique. □

We can remove the condition that $\limsup_{n\to+\infty}\theta(x,x_n)$ exists for any $x\in X$ in Theorem 1 if we assume that T is continuous, then we have the following result.

Theorem 2. *Let (X,d_θ) be a complete extended b-metric space. Let $T:X\to X$ be a mapping. Assume there exists $k\in[0,1)$ such that*

$$d_\theta(Tx,Ty)\leqslant k\theta(x,y)d_\theta(x,y)$$

for all $x,y\in X$. Suppose for any $x_0\in X$, we have

$$\limsup_{n,m\to\infty}\theta(x_n,x_m)\theta(x_n,x_{n+1})<\frac{1}{k},$$

where $x_n=T^nx_0=T(T^{n-1}x_0)$. If T is continuous, then T has a fixed point $u\in X$.

Proof. By using the same arguments as those given in Theorem 1, we generate a Cauchy sequence $\{x_n\}$ in X. Since X is complete, $\{x_n\}$ converges to some $u\in X$. Since T is continuous, we have $Tx_n\to Tu$.

Triangular inequality implies that

$$\begin{aligned}d_\theta(Tu,u)&\leqslant\theta(Tu,u)\bigl[d_\theta(Tx^*,x_{n+1})+d_\theta(x_{n+1},x^*)\bigr]\\&=\theta(Tx^*,x^*)\bigl[d_\theta(Tx^*,Tx_{n-1})+d_\theta(x_n,x^*)\bigr].\end{aligned}$$

Letting $n\to\infty$ in the above inequality, we get $d_\theta(Tx^*,x^*)=0$, and hence, x^* is a fixed point of T. □

Corollary 1. *Let (X,d) be a complete b-metric space with $b\geqslant1$. Let $T:X\to X$ be a mapping. Assume there exists $k\in[0,1)$ such that*

$$d(Tx,Ty)\leqslant kbd(x,y)$$

for all $x,y\in X$. Assume that $b^2k<1$. Then T has a unique fixed point x^*.

Proof. The proof follows from Theorem 1 by defining $\theta:X\times X\to[1,+\infty)$ via $\theta(x,y)=b$. □

Now we construct our second main result.

Theorem 3. *Let (X,d_θ) be a complete extended b-metric space such that d_θ is a continuous functional. Let $T:X\to X$ be a continuous mapping. Assume there exist $a,b\in[0,\infty)$ with*

$$\limsup_{n,m\to\infty}\theta(x_n,x_m)\frac{a\theta(x_n,x_{n-1})}{1-b\theta(x_{n+1},x_{n+2})}<1$$

and $\theta(x_n,x_{n+1})\leqslant1/b$, where $x_n=T^nx_0$ such that

$$d_\theta(Tx,Ty)\leqslant a\theta(x,Tx)d_\theta(x,Tx)+b\theta(y,Ty)d_\theta(y,Ty)\tag{5}$$

for all $x,y\in X$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be arbitrary. Define the iterative sequence $\{x_n\}$ by: $x_0,\ x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0,\ \dots,\ x_n = Tx_{n-1} = T^2x_{n-2} = \cdots = T^nx_0$.

Using (5), we obtain

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &= d_\theta(Tx_{n-1}, Tx_n) \\ &\leqslant a\theta(x_{n-1}, x_n)d_\theta(x_{n-1}, x_n) + b\theta(x_n, x_{n+1})d_\theta(x_n, Tx_{n+1}). \end{aligned}$$

Hence,

$$\begin{aligned} d_\theta(x_n, x_{n+1}) &\leqslant \frac{a\theta(x_{n-1}, x_n)}{1 - b\theta(x_n, x_{n+1})} d_\theta(x_{n-1}, x_n) \\ &\leqslant \frac{a\theta(x_{n-1}, x_n)}{1 - b\theta(x_n, x_{n+1})} \frac{a\theta(x_{n-2}, x_{n-1})}{1 - b\theta(x_{n-1}, x_n)} d_\theta(x_{n-2}, x_{n-1}) \\ &\leqslant \cdots \\ &\leqslant \prod_{i=1}^{n} \frac{a\theta(x_{i-1}, x_i)}{1 - b\theta(x_i, x_{i+1})} d_\theta(x_0, x_1). \end{aligned}$$

Therefore,

$$\begin{aligned} d_\theta(x_n, x_m) &\leqslant \theta(x_n, x_m)\big[d_\theta(x_n, x_{n+1}) + d_\theta(x_{n+1}, x_m)\big] \\ &\leqslant \theta(x_n, x_m)d_\theta(x_n, x_{n+1}) \\ &\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)d_\theta(x_{n+1}, x_{n+2}) \\ &\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)d_\theta(x_{n+2}, x_m) \\ &\leqslant \theta(x_n, x_m)d_\theta(x_n, x_{n+1}) \\ &\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)d_\theta(x_{n+1}, x_{n+2}) \\ &\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)d_\theta(x_{n+2}, x_{n+3}) \\ &\quad + \theta(x_n, x_m)\theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)d_\theta(x_{n+2}, x_m) \\ &\leqslant \cdots \\ &\leqslant \sum_{i=n}^{m-1} \prod_{j=n}^{i} \theta(x_j, x_m)d_\theta(x_i, x_{i+1}). \end{aligned}$$

Hence,

$$d_\theta(x_n, x_m) \leqslant \sum_{i=n}^{m-1} \prod_{j=n}^{i} \theta(x_j, x_m) \prod_{s=1}^{i} \frac{a\theta(x_{s-1}, x_s)}{1 - b\theta(x_s, x_{s+1})} d_\theta(x_0, x_1).$$

Now, let

$$a_i = \prod_{j=n}^{i} \theta(x_j, x_m) \prod_{s=1}^{i} \frac{a\theta(x_{s-1}, x_s)}{1 - b\theta(x_s, x_{s+1})} d_\theta(x_0, x_1),$$

$$\limsup_{i\to+\infty} \frac{a_{i+1}}{a_i} = \limsup_{i,m\to+\infty} \theta(x_{i+1}, x_m) \frac{a\theta(x_i, x_{i-1})}{1 - b\theta(x_{i+1}, x_{i+2})} < 1.$$

Therefore, by the ratio test we deduce that $d_\theta(x_n, x_m)$ converges as $n, m \to \infty$.

Thus, $\{x_n\}$ is Cauchy sequence, and we know that X is complete, which implies that the sequence $\{x_n\}$ converges to some $u \in X$.

To show that u is a fixed point, we use the triangular inequality as follows:

$$\begin{aligned} d_\theta(Tu, u) &\leqslant \theta(Tu, u)\big[d_\theta(Tu, x_{n+1}) + d_\theta(x_{n+1}, u)\big] \\ &= \theta(Tu, u)\big[d_\theta(Tu, Tx_n) + d_\theta(x_n, u)\big]. \end{aligned}$$

Letting $n \to +\infty$ in above inequalities and given the fact that T is continuous, we deduce that $d_\theta(Tu, u) \leqslant 0$. Hence, $d_\theta(Tu, u) = 0$. Thus, $Tu = u$ and that is u is a fixed point of T as desired. Finally, Assume that there exist two fixed points of T in X, say u, v,

$$\begin{aligned} d_\theta(u, v) &= d_\theta(Tu, Tv) \\ &\leqslant a\theta(u, Tu)d_\theta(u, Tu) + b\theta(v, Tv)d_\theta(v, Tv) \\ &= a\theta(u, u)d_\theta(u, u) + b\theta(v, v)d_\theta(v, v) = 0. \end{aligned}$$

Thus, $u = v$ as required. □

The continuity of T in Theorem 3 can be dropped if add more suitable conditions.

Theorem 4. *Let (X, d_θ) be a complete extended b-metric space. Let $T : X \to X$ be a mapping. Assume there exist $a \in [0, \infty)$ and $b \in (0, +\infty)$ with*

$$\limsup_{n,m\to\infty} \theta(x_n, x_m)\frac{a\theta(x_n, x_{n-1})}{1 - b\theta(x_{n+1}, x_{n+2})} < 1$$

such that

$$d_\theta(Tx, Ty) \leqslant a\theta(x, Tx)d_\theta(x, Tx) + b\theta(y, Ty)d_\theta(y, Ty)$$

for all $x, y \in X$. If the following conditions are satisfied:

(i) $\theta^2(x, Tx) < 1/b$ *for all $x \in X$*

(ii) $\limsup_{n\to\infty} \theta(T^n x, T^{n+1}x)$ *exists,*

then T has a unique fixed point.

Proof. Following the proof of Theorem 3 word by word, we construct a Cauchy sequence $\{x_n\}$ in X defined by $x_n = T^n x_0$. By completeness of X, the sequence $\{x_n\}$ converges to some $u \in X$.

To show that u is a fixed point of T, we consider the following:

$$\begin{aligned} d_\theta(u, Tu) &\leqslant \theta(u, Tu)\big[d_\theta(u, x_{n+1}) + d_\theta(x_{n+1}, Tu)\big] \\ &= \theta(u, Tu)\big[d_\theta(u, x_{n+1}) + d_\theta(Tx_n, Tu)\big] \\ &\leqslant \theta(u, Tu)\big[d_\theta(u, x_{n+1}) + a\theta(x_n, x_{n+1})d_\theta(x_n, x_{n+1}) \\ &\quad + b\theta(u, Tu)d_\theta(u, Tu)\big] \\ &\leqslant \theta(u, Tu)\big[d_\theta(u, x_{n+1}) + a\theta(x_n, x_{n+1})\theta(x_n, x_{n+1})(d_\theta(x_n, u) \\ &\quad + d_\theta(u, x_{n+1})) + b\theta(u, Tu)d_\theta(u, Tu)\big]. \end{aligned}$$

Taking the limit as $n \to \infty$, we obtain

$$d_\theta(u, Tu) \leqslant \theta(u, Tu)\big[b\theta(u, Tu)d_\theta(u, Tu)\big]. \qquad (*)$$

Since $\theta^2(u, Tu) < 1/b$, we deduce that $d_\theta(u, Tu) = 0$, which implies that $Tu = u$ as desired.

Finally, assume that there exist two fixed points of T in X, say u, v,

$$\begin{aligned} d_\theta(u, v) &= d_\theta(Tu, Tv) \\ &\leqslant a\theta(u, Tu)d_\theta(u, Tu) + b\theta(v, Tv)d_\theta(v, Tv) \\ &= a\theta(u, u)d_\theta(u, u) + b\theta(v, v)d_\theta(v, v) = 0. \end{aligned}$$

Thus, $u = v$ as required. □

Corollary 2. *Let (X, d) be a complete b-metric space with $s \geqslant 1$. Let $T : X \to X$ be a continuous mapping. Assume there exist $a, b \in [0, \infty)$ with $as^2 + bs < 1$ such that*

$$d(Tx, Ty) \leqslant asd_\theta(x, Tx) + bsd(y, Ty)$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. The proof follows from Theorem 3 by defining the mapping $\theta : X\times X \to [1, +\infty)$ via $\theta(x, y) = s$ and noting that T satisfies all conditions of Theorem 3. □

Corollary 3. *Let (X, d) be a complete b-metric space with $s \geqslant 1$. Let $T : X \to X$ be a mapping. Assume there exist $a \in [0, \infty)$ and $b \in (0, +\infty)$ with $as^2 + bs < 1$ and $bs^2 \leqslant 1$ such that*

$$d(Tx, Ty) \leqslant ad_\theta(x, Tx) + bsd(y, Ty)$$

for all $x, y \in X$. Then T has a unique fixed point.

Proof. The proof follows from Theorem 4 by defining the mapping $\theta : X\times X \to [1, +\infty)$ via $\theta(x, y) = s$ and noting that T satisfies all conditions of Theorem 4. □

Now, we introduce an example on an extended b-metric space, which not a b-metric space, and our contraction is meaningful.

Example 1. Let $X = \big[0, +\infty\big]$. Define $T : X \to X$ by $Tx = x/8$. Also, define $d_\theta : X \times X \to [0, +\infty)$ by

$$d_\theta(x, y) = \begin{cases} 0 & \text{if } x = y; \\ \frac{x}{1+x} & \text{if } x \neq 0,\ y = 0; \\ \frac{y}{1+y} & \text{if } x = 0,\ y \neq 0; \\ x + y & \text{if } 0 \neq x \neq y \neq 0, \end{cases}$$

and $\theta : X \times X \to \mathbb{R}^+$ by $\theta(x, y) = 2x + 2y + 2$. Then:

(i) (X, d_θ) is an extended b-metric space, which is not a b-metric space.
(ii) (X, d_θ) is complete.
(iii) For $x_0 \in X$, we have

$$\limsup_{n,m\to\infty} \theta(x_n, x_m)\theta(x_n, x_{n+1}) < 8,$$

where $x_n = T^n x_0$.
(iv) For $x \in X$, $\limsup_{n\to+\infty} \theta(x, x_n)$ exists.
(v) For any $x, y \in X$, we have

$$d_\theta(Tx, Ty) \leqslant \frac{1}{8}\theta(x, y)d_\theta(x, y).$$

(vi) For any $x, y \in X$, we have

$$\limsup_{n\to+\infty} \theta(T^n x, T^n y) < 8.$$

Proof. We leave the proof of d_θ is an extended b-metric space to the reader. To prove that d_θ is not a b-metric space, assume the contrary. So d_θ is a b-metric space with base $s > 1$. So, for any $0 \neq x$, we have

$$\begin{aligned} 2x + 1 = d(x, x+1) &\leqslant s\big(d(x, 0) + d(0, x+1)\big) \\ &= s\left(\frac{x}{1+x} + \frac{x+1}{2+x}\right). \end{aligned}$$

Letting $x \to +\infty$ in above inequality, we get $+\infty \leqslant 2s$, which is a contradiction. So d_θ is not a b-metric space with base s.

To prove that the metric d_θ is complete, let $\{x_n\}$ be a Cauchy sequence in X. Then $d_\theta(x_n, x_m) \to 0$ as $n, m \to +\infty$. We divide it into two cases:

Case 1: $x_n = x$ for some $x \in X$ for all but finitely many. In this case, $d_\theta(x_n, x) = 0$ for all but finitely many. So $x_n \to x$ in (X, d_θ).

Case 2: $x_n \neq 0$ for all but finitely many. Then $d_\theta(x_n, x_m) = x_n + x_m \to 0$. Thus, $d(x_n, 0) = x_n/(1 + x_n) \to 0$ and (x_n) converges to 0 in (X, d_θ). Thus, (X, d_θ) is complete.

To prove (iii), let $x \in X$. Then $T^n x = x/8^n$. Then

$$\begin{aligned} &\limsup_{j,m\to+\infty} \theta(x_j, x_m)\theta(x_j, x_{j+1}) \\ &\quad = \limsup_{j,m\to+\infty} \theta\left(\frac{1}{8^j}x, \frac{1}{8^m}x\right)\theta\left(\frac{1}{8^j}x, \frac{1}{8^{j+1}}x\right) \\ &\quad = \limsup_{j,m\to+\infty} \left(\frac{2}{8^j}x + \frac{2}{8^m}x + 2\right)\left(\frac{2}{8^j}x + \frac{2}{8^{j+1}}x + 2\right) \\ &\quad = 4 \leqslant 8. \end{aligned}$$

To prove (iv), given $x \in X$. Then

$$\limsup_{n\to+\infty} \theta(x, x_n) = \limsup_{n\to+\infty} \theta\left(x, \frac{1}{8^n}x\right) = \limsup_{n\to+\infty} \left(2x + \frac{1}{8^n}x + 2\right)$$
$$= 2x + 2 \quad \text{exists.}$$

To prove (v), given $x, y \in X$. We divide the proof into the following cases:

Case 1: $x = y$. Here we have

$$d_\theta(Tx, Ty) = 0 \leqslant \frac{1}{8}\theta(x, y)d_\theta(x, y).$$

Case 2: $x = 0$ or $y = 0$. Without loss of generality, we may assume that $x \neq 0$ and $y = 0$. In this case, we have

$$d_\theta(Tx, T0) = d_\theta\left(\frac{1}{8}x, 0\right) = \frac{x}{x+8} \leqslant \frac{1}{8}x \leqslant \frac{1}{8}(2x+2)\left(\frac{x}{1+x}\right)$$
$$= \frac{1}{8}\theta(x, 0)d_\theta(x, 0).$$

Case 3: $0 \neq x \neq y \neq 0$. Here we have

$$d_\theta(Tx, Ty) = d_\theta\left(\frac{1}{8}x, \frac{1}{8}y\right) = \frac{1}{8}x + \frac{1}{8}y \leqslant \frac{1}{8}(2x+2y+2)(x+y)$$
$$= \frac{1}{8}\theta(x, y)d_\theta(x, y).$$

To prove item (vi), given $x, y \in X$. Then

$$\limsup_{n\to+\infty} \theta\big(T^n x, T^n y\big)$$
$$= \limsup_{n\to+\infty} \theta\left(\frac{1}{8^n}x, \frac{1}{8^n}y\right) = \limsup_{n\to+\infty} \left(\frac{2}{8^n}x + \frac{2}{8^n}y + 2\right) = 2.$$

Note that T satisfies all the hypotheses of Theorem 1 with $k = 1/8$. Thus, T has a unique fixed point. Here 0 is the unique fixed point of T. □

4 Application

One of the most important application of our results is to prove the existence and uniqueness of such nontrivial equations. The following theorem is an application of our main result.

Theorem 5. *For each natural number $m \geqslant 2$, the equation*

$$x^m + 1 = \big(m^2 - 1\big)x^{m+1} + m^2 x \tag{6}$$

has a unique real solution.

Proof. Note that if $x \in \mathbb{R}$ with $|x| > 1$, then x is not a solution for equation (6).

Now let $X = [-1, 1]$. Define $\theta : X \times X \to [1, +\infty)$ by $\theta(x, y) = |x| + |y| + 1$ and $d_\theta : X \times X \to [0, +\infty)$ by $d_\theta(x, y) = |x - y|$. Then (X, d_θ) is a complete extended b-metric space.

Also, define the mapping $T : X \to X$ by

$$Tx = \frac{x^m + 1}{(m^2 - 1)x^m + m^2}.$$

Now, we study the following cases:

Case I: $x = y$. Then

$$d_\theta(Tx, Ty) = 0 \leqslant \frac{1}{m^3}\theta(x, y)d_\theta(x, y).$$

Case II: $x \neq y$. Without lose of generality, we assume that $x > y$. Then

$$\begin{aligned}
d_\theta(Tx, Ty) &= d_\theta\left(\frac{x^m + 1}{(m^2 - 1)x^m + m^2}, \frac{y^m + 1}{(m^2 - 1)y^m + m^2}\right) \\
&= \left|\frac{x^m + 1}{(m^2 - 1)x^m + m^2} - \frac{y^m + 1}{(m^2 - 1)y^m + m^2}\right| \\
&= \frac{|x^m - y^m|}{((m^2 - 1)x^m + m^2)((m^2 - 1)y^m + m^2)} \\
&\leqslant \frac{1}{m^3}(x - y) \leqslant \frac{1}{m^3}\big(|x| + |y| + 1\big)|x - y| \\
&\leqslant \frac{1}{m^3}\theta(x, y)d_\theta(x, y).
\end{aligned}$$

For any $x_0, x \in X$, we note that

$$x_n = T^n x_0 \leqslant \frac{2}{m^2}.$$

Thus,

$$\limsup_{n\to+\infty} \theta(x, x_n) \leqslant |x| + \frac{2}{m^2} + 1.$$

Therefore, $\limsup_{n\to+\infty} \theta(x, x_n)$ exists. On the other hand, we have

$$\begin{aligned}
&\limsup_{n,t\to+\infty} \theta(x_n, x_t)\theta(x_n, x_{n+1}) \\
&\quad= \limsup_{n,t\to+\infty} \big(|x_n| + |x_t| + 1\big)\big(|x_n| + |x_{n+1}| + 1\big) \leqslant \left(\frac{4}{m^2} + 1\right)^2 \\
&\quad< m^3.
\end{aligned}$$

Moreover, for any $x, y \in X$, we have

$$\limsup_{n\to+\infty} \theta\big(T^n x, T^n y\big) \leqslant \frac{4}{m^2} + 1 < m^3.$$

Thus, T satisfies all conditions of Theorem 1 with $k = 1/m^3$. Therefore, T has a unique fixed point. Note that the unique fixed point of T is the unique solution of (6). □

Example 2. The equation

$$x^{100} + 1 = 9999x^{101} + 10000x$$

has a unique real solution.

Proof. The proof follows from Theorem 5 by taking $m = 100$. □

5 Conclusion

In this paper, we utilized the concept of generalized b-metric spaces to formulate and prove some fixed point theorems. Our results generalized many exciting results known in the literature. An example and an application are included to show the useability of our paper. As future work, we propose for the reader to formulate and prove fixed and common fixed point theorems for multi-valued mappings in the setting of extended b-metric spaces.

Acknowledgment. We would like to express our sincere thanks for the editor and the reviewers for their valuable comments on this paper, which made our paper complete and more significant.

References

1. T. Abdeljawad, J.O. Alzabut, A. Mukheimer, Y. Zaidan, Banach contraction principle for cyclical mappings on partial metric spaces, *Fixed Point Theory Appl.*, **2012**:154, 2012.
2. T. Abdeljawad, J.O. Alzabut, A. Mukheimer, Y. Zaidan, Best proximity points for cyclical contraction mappingswith boundedly compact decompositions, *J. Comput. Anal. Appl.*, **15**(4): 678–685, 2013.
3. A. Aghajani, M. Abbas, J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, *Math. Slovaca*, **64**(4):941–960, 2014.
4. M.U. Ali, T. Kamran, M. Postolache, Fixed point theorems for multivalued G-contractions in Hausdorff b-gauge spaces, *J. Nonlinear Sci. Appl.*, **8**(5):847–855, 2015.
5. M.U. Ali, T. Kamran, M. Postolache, Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem, *Nonlinear Anal. Model. Control*, **22**(1):17–30, 2017.
6. H. Alsamir, M. Salmi, Md. Noorani, W. Shatanawi, F. Shaddad, Generalized Berinde-type $(\eta, \xi, \vartheta, \theta)$ contractive mapping in b-metric spaces with an application, *J. Math. Anal.*, **7**(6): 1–12, 2016.
7. E. Ameer, M. Arshad, W. Shatanawi, Common fixed point results for generalized α_*-ψ-contraction multivalued mappings in b-metric spaces, *J. Fixed Point Theory Appl.*, **19**(2):1–18, 2017.
8. H. Aydi, M.-F. Bota, E. Karapınar, S. Mitrović, A fixed point theorem for set valued quasi-contractions in b-metric spaces, *Fixed Point Theory Appl.*, **2012**:88, 2012.

9. H. Aydi, M.-F. Bota, E. Karapınar, S. Moradi, A common fixed point for weak ϕ-contractions on b-metric spaces, *Fixed Point Theory appl.*, **13**(2):337–346, 2012.
10. H. Aydi, M. Jellali, E. Karapinar, On fixed point results for α-implicit contractions in quasimetric spaces and consequences, *Nonlinear Anal. Model. Control*, **21**(1):40–56, 2016.
11. I.A. Bakhtin, The contraction mapping principle in almost metric spaces, *Funct. Anal., Gos. Ped. Inst., Unianowsk*, **30**:26–37, 1989.
12. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, *Nonlinear Anal., Theory Methods Appl.*, **65**(7):1379–1393, 2006.
13. B.S. Choudhury, N. Metiya, M. Postolache, A generalized weak contraction principle with applications to coupled coincidence point problems, *Fixed Point Theory Appl.*, **2013**:152, 2013.
14. L.B. Ćirić, A generalization of Banach's contraction principle, *Proc. Am. Math. Soc.*, **45**(2): 267–273, 1974.
15. T. Kamran, M.U. Ali, M. Postolache, A. Ghiura, M. Farheen, Best proximity points for a new class of generalized proximal mappings, *Int. J. Anal. Appl.*, **13**(2):198–205, 2017.
16. T. Kamran, M. Postolache, Fahimuddin, M.U. Ali, Fixed point theorems on generalized metric space endowed with graph, *J. Nonlinear Sci. Appl.*, **2016**(9):4277–4285, 2016.
17. T. Kamran, M. Samreen, Q. UL Ain, A generalization of b-metric space and some fixed point theorems, *Mathematics*, **19**(5), 2017.
18. M.A. Miandaragh, M. Postolache, S. Rezapour, Some approximate fixed point results for generalized α-contractive mappings, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **75**(2):3–10, 2013.
19. N. Mlaiki, N.Y. Özgür, A. Mukheimer, N. Taş, A new extension of the M_b-metric spaces, *J. Math. Anal.*, **9**(2):118–193, 2018.
20. A. Mukheimer, α-ψ-ϕ-contractive mappings in ordered partial b-metric spaces, *J. Nonlinear Sci. Appl.*, **7**(3):168–179, 2014.
21. A. Mukheimer, Common fixed point theorems for a pair of mappings in complex valued b-metric spaces, *Advances in Fixed Point Theory*, **4**(3):344–354, 2014.
22. M. Nazam, M. Arshad, M. Postolache, Coincidence and common fixed point theorems for four maps satisfying (α_s, f)-contractions, *Nonlinear Anal. Model. Control*, **23**(5):664–690, 2018.
23. V. Parvaneh, J.R. Roshan, S. Radenović, Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations, *Fixed Point Theory Appl*, **2013**:130, 2013.
24. J.R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, W. Shatanawi, Common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered b-metric spaces, *Fixed Point Theory Appl.*, **2013**:159, 2013.
25. M. Samreen, T. Kamran, M. Postolache, Extended b-metric space, extended b-comparison function and nonlinear contractions, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **80**(4):21–28, 2018.
26. W. Shatanawi, Fixed and common fixed point for mapping satisfying some nonlinear contraction in b-metric spaces, *J. Math. Anal.*, **7**(4):1–12, 2016.

27. W. Shatanawi, Common fixed points for mappings under contractive conditions of (α, β, ψ)-admissibility type, *Mathematics*, **6**:261, 2018.
28. W. Shatanawi, K. Abodayeh, A. Mukheimer, Some fixed point theorems in extended b-metric spaces, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **80**(4):71–78, 2018.
29. W. Shatanawi, A. Pitea, Lazovic, Contraction conditions using comparison functions on b-metric spaces, *Fixed Point Theory Appl.*, **2014**:135, 2014.
30. W. Shatanawi, M. Postolache, A.H. Ansari, W. Kassab, Common fixed points of dominating and weak annihilators in ordered metric spaces via C-class functions, *J. Math. Anal.*, **8**(3):54–68, 2017.
31. T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, *Proc. Am. Math. Soc.*, **136**(5):1861–1869, 2008.