

Asymptotic Stability of an Abstract Delay Functional-Differential Equation

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Abstract. We study the exponential asymptotic stability of an abstract functional-differential equation with a mixed type of deviating arguments, namely: c which might represent the gestation period of the population and $r(u(t))$, a suitably defined function. The equation is reduced to its equivalent integral form and solved via Laplace transform method. An interesting connection of our study is with generalizations of populations with potentially complex (chaotic) dynamics.

Keywords: stability, resolvent, gestation, delay differential equation.

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1 Introduction

Over many decades, indeed for more than two centuries, abstract functional-differential equations have been the subject and a proving ground for a wealth of mathematical theories. An outstanding problem is the analysis of the linear and nonlinear extension of the abstract Cauchy problem. The former is well-known to be closely related to some population dynamics problems that can be fitted into it by some transformations.

Cooke [1] several years ago pointed out that the study of Functional-Differential Equations (FDE) or differential-difference equations with time lag began in the eighteenth century and intensely continued in the second half of the twentieth century. This class of equations contains equations in which the “lag” of the

argument is a function of the dependent variable, that is of the unknown solution itself. A simple example is the equation

$$\dot{u}(t) = u(t - u(t)), \quad (1)$$

in which u is an unknown function of the real variable t . Such an equation is self consistent, and $u(t)$ may represent the density of a population of plants in which the removed still contribute to the dynamics of the living [2]. Noticing that there is an urgent need to demonstrate the compromise between the conflicting demands of mathematical tractability and biological realism, Tchuente [2] posited that in order to get the dimension right, equation (1) above should take the form $\dot{u}(t) = u(t - \epsilon u(t))$, where ϵ is a scaling parameter. But, in what follows, we shall assume that $\epsilon \approx 1$.

In general, FDE contains equations in which the delay is an implicitly defined functional of the values of u over its domain of definition. Since the need for mathematical tractability imposes constraints on the number of factors that a single model can accommodate [3], this paper focusses attention on abstract functional-differential equations (AFDE) with mixed type of deviating arguments, namely: c which might represent the gestation period of the population and $r(u(t))$, a suitably defined non-linear function of u . This is relevant for a number of reasons, some fundamental (gestation is one of the key factors in population dynamics which have been the least favoured) and some associated with real, observable phenomena which are non-linear in nature. We would be pleased to know more about the combination of such time lags. In this connection, we extend the class of linear inhomogeneous abstract Cauchy problem studied in [4] via the Semigroup approach. A lot of studies addressed the potential benefits of Semigroup theory (see [5] and the references therein). The models considered in [1] can be fitted into the abstract formalism of this work and many more. This paper also generalizes and further extends the result of Liadi [6]. In the study of equilibrium states, most complex biological models derived in terms of system of ordinary or partial differential equations can be transformed into abstract forms (see for instance [7]).

Over the years, mathematics has played an important role in increasing our understanding of complex phenomena. G.H. Hardy¹ said of good mathematics

¹G. H. Hardy. A Mathematician's Apology, p. 85, Canto Edition (Cambridge Univ. Press.)

“Beauty is the first test: there is no permanent place in the world for ugly mathematics” and along side beauty he placed the criteria of economy and a measure of unexpectedness. Taking this as our measure, for concreteness and clarity, we will forego generality and make the context more approachable. The result is elegant and concise, though the method (which has not been applied to such problems before) is not new. The approach to obtain the a priori estimates of linear equations may be quite simple, but theories developed for non-linear model is somewhat not satisfactory. This forms the basis of our motivation, and it is in order to address this problem and the challenges posed by non-linear AFDEs that the author decided to carry out this study. Epidemic models with partial immunization or epidemics in which the resistance is weakened by the first infection are generally very complex due to increased susceptibility, and thus can be fitted into the proposed delay functional equations. Also, for models with multiple absorbing states as well as periodic emerging and re-emerging infectious diseases, the question of stability remains open and the objective of this study serves to draw attention to a class of AFDEs (which certainly are least studied) that can effectively handle such a problem after some suitable transformations.

2 The linear model

(a) Hypotheses and notations

Let $X = B \cup (\mathbb{R})$, the space of uniformly continuous and bounded functions on \mathbb{R} be a Banach space [8], denote by $L(X, X)$ the Banach Algebra of all linear continuous operators from X to itself.

Let A be the infinitesimal generator of a well-defined C_0 -semigroup $T(t) = e^{tA}$, which is bounded [9], (differential operators are as a rule closed [10]), then, $A: D(A) \subset X \rightarrow X$ and $u(t) \in D(A)$, where $D(A) = \{u: u \in X, \dot{u} \in X\}$, (it is assumed that the derivative of u exists) is dense in X . It is well-known that if the domain of a given differential operator is specified, then it is sufficient to write down a problem with boundary conditions by a single equation [9]. The norm in $L(X, X)$ will be denoted by $\|\cdot\|_{L(X, X)}$ [11]. Frequently, we will omit the index $L(X, X)$, which distinguish the various norms from each other, simply because the context unambiguously stipulates which norm is meant.

(b) Problem setting

Systems of first order partial differential equations can be fitted into abstract Cauchy problem with some modifications [3, 12, 13] as well as ordinary differential equations [7]. The profit of transforming such systems is technical in character. Many authors based on the fact that the explicit solutions of most systems are analytically difficult to construct, i.e., the systems are almost intractable, have used the abstract formulation to at least ascertain the existence and possibly uniqueness of weak solutions [14–16].

Let $u(t)$ describe the state of some physical system at time t . Suppose that the time rate of change of $u(t)$ is given by some function A of the state of the system. The initial data u_0 is also given. Thus, $\dot{u}(t) = Au(t)$. Nearly as simple is the case of standard Cauchy problem with or without delay. In the present paper, a theoretic generalization is studied.

By introducing two families of operators B and E , the extended abstract Cauchy problem becomes an AFDE given here by:

$$\dot{u}(t) = Au(t) + Bu(t) + Eu(t - c), \quad u(0) = u_0, \quad (2)$$

an equation in which the linearity is considered in the lag, and where A and E are linear operators, B is not necessarily linear, and $u_0 \in X$ is known as point Cauchy data [17]. It could also be assumed that $u_0 \in H_0^1$, a Sobolev space [8]. Some systems of differential equations can be fitted into (2). For simplicity, let $\|B\|$ be bounded and measurable in $(0, t_0)$. We assume that this equation is of parabolic type because in many instances, “parabolic” is stronger than “hyperbolic” [18]. Hence, (2) is a parabolic equation with constant delay c . Evolution equations with two types of delays namely; constant and variable have been considered by Dautrey and Lions [17].

Let

$$u(t) \in C^1(\mathbb{R}_+), \quad u \in L^1(X, X), \quad t > 0,$$

then, pretend A is a real number, then solving (2) by freshman calculus and writing e^{At} in lieu of $T(t)$ [19], we have :

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-\tau)} (Eu(\tau - c) + Bu(\tau)) d\tau, \quad (3)$$

which can also be obtained by the variation of constants formula. For a proof of the variation of parameters formula, see [10]. The solution $u(t)$ is always required to be continuous [20], for $0 < t < t_0$ and satisfies (3) as well as continuous differentiability (for a proof, see Proposition 4.10 in [21]). The integral in (3) will be Bochner if the operators are sequences of simple functions with support in a measurable set X . Instead of using the local or global existence Theorem which requires the Lipschitz continuity of the operators with a generic constant, we employ another powerful tool, namely, the theory of resolvent.

Let z_e be a steady state of (2), then writing $u = z_e + z$, (2) can be put into the following form

$$\dot{z}(t) = Az(t) + Ez(t-c) + h(t, z(t)), \quad z(0) = z_0, \quad (4)$$

where $h(t, z(t))$ represents the nonlinear term, and

$$z(t) = e^{At}z_0 + \int_0^t e^{A(t-\tau)} (Eu(\tau-c) + h(\tau, u(\tau))) d\tau. \quad (5)$$

Equation (4) is an abstract setting of the typical Cauchy initial value problem for first order differential operator. By taking the Laplace transform [22] of (5) with ξ as the transform variable, \mathcal{L} the transform operator and \hat{z} the transform of z , we get

$$\begin{aligned} \mathcal{L}\{z(t)\} = \hat{z}(\xi) &= \mathcal{L}\{e^{At}\}z_0 + \mathcal{L}\{e^{At}\} [E\mathcal{L}\{u(t-c)\} + \mathcal{L}\{h(t, u(t))\}] \\ &= (\xi I - A)^{-1} [z_0 + Ee^{-c\xi}\hat{z}(\xi) + h(\xi, u(\xi))]. \end{aligned} \quad (6)$$

Hence,

$$(I - (\xi I - A)^{-1}Ee^{-c\xi})z(\xi) = (\xi I - A)^{-1}(z_0 + h(\xi, u(\xi))).$$

If $(\xi I - A)^{-1}$ exists whenever $Re(\xi) < \alpha < 0$, i.e., $(\xi I - A)^{-1} \in L(X, X)$, (this is true if $\|T(t)\| \leq Ke^{wt}$, $t > 0$, $K > 0$ and $w \in R^1$ [11]), then we assume that \mathcal{H}^* is the unique resolvent of $(\xi I - A)^{-1}Ee^{-c\xi}$, so that

$$\begin{aligned} \hat{z}(\xi) &= (\xi I - A)^{-1}(z_0 + h(\xi, u(\xi))) \\ &\quad + \mathcal{H}^* \{(\xi I - A)^{-1}z_0 + (\xi I - A)^{-1}h(\xi, z(\xi))\}. \end{aligned} \quad (7)$$

Let $(\xi I - A)^{-1} E e^{-c\xi} = \frac{E e^{-c\xi}}{\xi} + O\left(\frac{1}{\xi^2}\right) = \mathcal{H}$, as $|\xi| \rightarrow \infty$ with $e^{-c\xi} = O(1)$. Then,

$$\mathcal{H}^* = \mathcal{H} + \mathcal{H}^* \mathcal{H} = \mathcal{H} + O(\mathcal{H}^2) = \frac{E e^{-c\xi}}{\xi} + O\left(\frac{1}{\xi^2}\right), \quad \text{as } \|\mathcal{H}\| \rightarrow 0,$$

$|\xi|$ being large enough. We will now make use of some concepts and assumptions that will ease the burden of the work. One good and easy bargain is to assume that $\|u(t - c)\| \leq H(t - c)$, where H represents the Heaviside unit function [18]. In this case,

$$\begin{aligned} z(t) = e^{\alpha t} z_0 + \int_0^t e^{\alpha(t-\tau)} h(\tau, z(\tau)) d\tau + \int_0^t \bar{\mathcal{H}}^*(\tau) e^{\alpha(t-\tau)} z_0 d\tau \\ + \int_0^t \bar{\mathcal{H}}^*(\tau) e^{\alpha(t-\tau)} h(\tau, z(\tau)) d\tau, \end{aligned} \quad (8)$$

where we have assumed Laplace invertibility of equation (7) with $\bar{\mathcal{H}}^*$ as the inverse transform of \mathcal{H}^* . Substituting for $\bar{\mathcal{H}}^*$ in (8), we have

$$\begin{aligned} z(t) = e^{\alpha t} z_0 + e^{\alpha t} \int_0^t e^{-\alpha\tau} h(\tau, z(\tau)) d\tau + E e^{\alpha t} z_0 \int_0^t u(\tau - c) e^{-\alpha\tau} d\tau \\ + E e^{\alpha t} \int_0^t h(\tau - c, u(\tau - c)) e^{-\alpha\tau} d\tau. \end{aligned} \quad (9)$$

The pointers to possible simplifications are found in the analysis of an idealized case which helps elucidate the mathematical structures inherent in structured population dynamics problems.

Lemma 1. *Let (i) $\|h(t, u(t))\| \leq M\|u(t)\|$ and (ii) $\|u(t - c)\| \leq H(t - c)$, then the steady state solution of (2) is exponentially asymptotically stable (e.a.s) provided $M + M\|E\| + \alpha < 0$.*

Proof. From (9), using certain straightforward estimates one can show that:

$$\begin{aligned}
 \|z(t)\|e^{-\alpha t} &\leq \|z_0\| + M \int_0^t e^{-\alpha\tau} \|u(\tau)\| d\tau + \|E\| \|z_0\| \int_0^c e^{-\alpha\tau} d\tau \\
 &\quad + \|E\| M \int_0^t \|u(\tau)\| e^{-\alpha\tau} d\tau \\
 &= \|z_0\| \left(1 + \frac{\|E\|}{\alpha} (1 - e^{-\alpha c}) \right) \\
 &\quad + M + \left(1 + \|E\| \int_0^t e^{-\alpha\tau} \|u(\tau)\| d\tau \right),
 \end{aligned}$$

and applying the classical Gronwall's Lemma [11], we obtain

$$\|z(t)\| \leq \|z_0\| \left(1 + \frac{\|E\|}{\alpha} (1 - e^{-\alpha c}) \right) e^{(M+M\|E\|+\alpha)t}. \quad (10)$$

□

A similar result was obtained by Liadi in [7] with $M + M\|E\|c + \alpha < 0$. Since the restriction $\|u(t-c)\| \leq H(t-c)$ is very severe and not realistic in most cases, we consider:

Proposition 1. *Let $\|u(t-c)\| \leq \|u(t)\|$ in (9), then (10) reduces to*

$$\|z(t)\| \leq \|z_0\| e^{(M+M\|E\|+\|E\|\|z_0\|+\alpha)t}. \quad (11)$$

Proof. The proof is straightforward, since by substituting $\|u(t-c)\| \leq \|u(t)\|$ in (9) and by freshman computations, we have:

$$\begin{aligned}
 \|z(t)\|e^{-\alpha t} &\leq \|z_0\| + M \int_0^t e^{-\alpha\tau} \|u(\tau)\| d\tau + \|E\| \|z_0\| \int_0^t \|u(\tau)\| e^{-\alpha\tau} d\tau \\
 &\quad + \|E\| M \int_0^t \|u(\tau)\| e^{-\alpha\tau} d\tau \\
 &= \|z_0\| + (M\|E\| + \|z_0\|\|E\| + M) \int_0^t e^{-\alpha\tau} \|u(\tau)\| d\tau,
 \end{aligned}$$

and on applying Gronwall's Lemma yields (11). □

Corollary 1. *Let $\|h(t, u(t))\| \leq M\|u(t)\|$ and $\|u(t - c)\| \leq \|u(t)\|$, then (2) is e.a.s. for $\operatorname{Re}(\xi) < \alpha < 0$, and α large enough provided*

$$M + M\|E\| + \|E\|\|z_0\| + \alpha < 0.$$

Remark 1. *If $c \equiv 0$ or large enough in (10), then the expression $1 - e^{-\alpha c}$ vanishes identically. But in reality, c measured in years is small such that $1 - e^{-\alpha c} > 0$.*

3 The nonlinear case

We have dealt with equation (2) and the method used will then be modified and carried over to

$$\begin{aligned} \dot{u}(t) &= Au(t) + Bu(t - c) + Eu(t - r(u(t))) + h(t, u(t)), \\ u(0) &= u_0, \end{aligned} \tag{12}$$

in which another nonlinearity occurs in the lag itself. Equation (12) is of course more difficult to treat and more interesting in the context of this paper.

Lemma 2. *Let $r(u(t)) > 0 \rightarrow 0$ as $t \rightarrow \infty$ be continuous for $t \geq t_0$, and assume that the following conditions are satisfied.*

- (i) $r(u(t)) \rightarrow 0$ as $t \rightarrow \infty$,
- (ii) $\int_{t_0}^{\infty} r(u(t)) dt < \infty$, $r(0) = 0$.

Then, equation (12) is asymptotic to

$$\dot{u}(t) = Au(t) + Bu(t - c) + Eu(t) + h(t, u(t)). \tag{13}$$

Proof. The proof of this Lemma is as sketched below.

For simplicity of discussion and without restriction to generality, let $r(u(t)) \rightarrow r_0$, where r_0 is a non-negative constant, then, it is natural that equation (12) is equivalent to

$$\dot{u}(t) = Au(t) + Bu(t - c) + Eu(t - r_0) + h(t, u(t)), \tag{14}$$

but from (ii), $r_0 = 0$. Thus, equation (14) becomes

$$\dot{u}(t) = Au(t) + Bu(t - c) + Eu(t-) + h(t, u(t)).$$

Condition (ii) implies that $r(u(t))$ is integrable over the domain of definition of its variable. \square

Remark 2. $r(u(t)) \in L^1(\mathbb{R}^+; \mathbb{R}^+)$ and if $r(\cdot)$ is not integrable, the main difficulty to extend this method is in obtaining a suitable integral equation with which to work [23] if $r(u(t))^{2-\gamma}$ ($\gamma > 0$) is integrable. A suitable function $r(u(t))$ satisfying Lemma 2 is $r(u(t)) = 1 - e^{-u(t)}$, where $u(\infty)$ is large enough. Also $r(u(t)) = u(t)e^{-u(t)}$ will do.

Since $r(u) \in C(\mathbb{R}^+)$, let $r(0) = 0$, then $r(u)$ will be nearly zero over the interval $C(\mathbb{R}^+)$, and $u(t - r(u))$ will differ little from $u(t)$. Hence, the estimate

$$\|u(t - r(u))\| \leq \|u(t)\|$$

is reasonable. The integral form of (12) is expressed by the variation of parameters formula as

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-\tau)} \{Bu(\tau - c) + Eu(\tau - u(\tau)) + h(\tau, u(\tau))\} d\tau, \quad (15)$$

where for brevity and without any ambiguity, $r(u)$ is simply written as $u(t)$.

Theorem 1. Let conditions (i) of Lemma 1, and $\|u(t - (u))\| \leq \|u(t)\|$ be satisfied. Then, equation (12) is exponentially asymptotically stable, if we can find constant M, c , and w , such that $M + \|B\|\|u_0\|e^{-wc} + \|u_0\|\|E\| + w < 0$.

Proof. The plan of the proof is similar to that of Lemma (2). Taking the Laplace transform of $u(t - u(t))$, gives:

$$\begin{aligned} \mathcal{L}\{u(t - u(t))\} &= \int_0^\infty e^{-\xi t} u(t - u(t)) dt = \int_{-u_0}^\infty e^{-\xi(s+u(t))} u(s) (ds - \dot{u} dt) \\ &= \tilde{u}(\xi) H(t + u_0) e^{-\xi u(t)} - e^{-\xi s} u(s) \int_{-u_0}^\infty e^{-\xi u(\alpha)} \dot{u}(\alpha) d\alpha. \end{aligned} \quad (16)$$

Now, $e^{-\xi u(t)} \simeq \xi - \xi u(t) + O(\xi u)^2$. Denote $u(\infty)$ by N . Let $u(-u_0) = 0$ for u_0 large enough (negative populations are biologically irrelevant), then,

$$\begin{aligned} \int_{-u_0}^{\infty} (\xi - \xi u(t)) u(t) dt &= \lim_{t \rightarrow \infty} u(t) \left(\xi - \frac{\xi}{2} u(t) \right) \\ &= \gamma \left(N - \frac{N^2}{2} \right) \quad \text{for } \operatorname{Re}(\xi) < \gamma < 0. \end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} \mathcal{L}\{u(t - u(t))\} &= \tilde{u}(\xi) H(t + u_0) e^{-\xi u(t)} - \gamma e^{-\xi s} u(s) \left(N - \frac{N^2}{2} \right) \\ &\leq \tilde{u}(\xi) H(t + u_0) e^{-\xi u(t)}. \end{aligned} \quad (18)$$

Let $P = e^{-c\xi}$, $Q = e^{-\xi u(t)}$, then

$$\tilde{u}(\xi) [I - (\xi I - A)^{-1} (BP + EQ)] = (\xi I - A)^{-1} (u_0 + h(\xi, u(\xi))). \quad (19)$$

Let $T(t)$, $t > 0$ be the C_0 -semigroup generated by the infinitesimal generator A , \mathcal{H}^* the unique resolvent of $(\xi I - A)^{-1} (BP + EQ)$, then

$$\tilde{u}(\xi) = (\xi I - A)^{-1} (u_0 + h(\xi, u(\xi))) + \mathcal{H}^* (\xi I - A)^{-1} (z_0 + h(\xi, u(\xi))), \quad (20)$$

and

$$\begin{aligned} u(t) &= T(t)u_0 + \int_0^t T(t-s)h(s, u(s))ds + \int_0^t \tilde{\mathcal{H}}^*(s)T(t-s)z_0ds \\ &\quad + \int_0^t \tilde{\mathcal{H}}^*(s)T(t-s)h(s, u(s))ds \\ &= T(t)u_0 + \int_0^t T(t-s)h(s, u(s))ds + \int_0^t T(t-s)Bu(s-c)u_0ds \\ &\quad + \int_0^t T(t-s)Eu(t-u(t))u_0ds \\ &\quad + \int_0^t T(t-s)h(s, u(s))Bu(t-c)ds \\ &\quad + \int_0^t T(t-s)h(s, u(s))Eh(s-u(s), u(t-u(t))). \end{aligned} \quad (21)$$

For simplicity, let $K = 1$, and $\|T(t)\| \leq e^{wt}$; $w \leq 0$, then

$$\begin{aligned}
 & \|u(t)\|e^{-wt} \\
 & \leq \|u_0\| + M \int_0^t e^{-ws} \|u(s)\| ds + \|B\| \|u_0\| e^{-wc} \int_0^t e^{-ws} \|u(s)\| ds \\
 & \quad + \|u_0\| \|E\| \int_0^t e^{-ws} \|u(s)\| ds + M \|B\| e^{-wc} \int_0^t e^{-ws} \|u(s)\|^2 ds \\
 & \quad + M^2 \|E\| \int_0^t e^{-ws} \|u(s)\|^2 ds. \tag{22}
 \end{aligned}$$

Now, assume that

$$\int_0^t e^{-ws} (u(s))^2 ds \leq \frac{\|u_0\|^2}{w}, \tag{23}$$

then,

$$\begin{aligned}
 \|u(t)\|e^{-wt} & \leq \|u_0\| + \frac{M}{w} \|u_0\|^2 (M\|E\| + \|B\|e^{-wc}) \\
 & \quad + (M + \|B\| \|u_0\| e^{-wc} + \|u_0\| \|E\|) \int_0^t e^{-ws} \|u(s)\| ds, \tag{24}
 \end{aligned}$$

and by the well-known Gronwall's Inequality [24],

$$\begin{aligned}
 \|u(t)\| & \leq \|u_0\| \left[1 + \frac{M}{w} \|u_0\| (M\|E\| + \|B\|e^{-wc}) \right] \\
 & \quad \times e^{(M + \|B\| \|u_0\| e^{-wc} + \|u_0\| \|E\| + w)t}. \tag{25}
 \end{aligned}$$

□

Remark 3. The estimate in (23) is reasonable. For example if

$$\int_0^t e^{-ws} \|u(s)\|^2 ds \leq \delta \int_0^t \|u(s)\| e^{-ws} ds,$$

δ large enough, then (25) becomes

$$\|u(t)\| \leq \|u_0\| e^{[M(1+M\delta\|E\|+\delta\|B\|e^{-wc})+\|u_0\|(\|E\|+\|B\|e^{-wc})+w]t}.$$

Furthermore for

$$\int_0^t e^{-ws} \|u\|^2 ds \leq \delta e^{-wt} \|u(t)\|,$$

we have

$$\|u(t)\| \leq \frac{\|u_0\|}{1 + M\delta\|B\|e^{-wc} + M^2\delta\|E\|} e^{(\|u_0\|\|E\| + \|B\|\|u_0\|e^{-wc} + M + w)t}$$

provided $1 + M\delta\|B\|e^{-wc} + M^2\delta\|E\| \neq 0$.

Before we conclude, we wish to point out that there is a relationship between equations (2) and (3).

Lemma 3. *There is a one-to-one correspondence between solutions of the retarded FDE (2) and its abstract integral form (3).*

Proof. It is sufficient to show that (2) implies (3). Let x be a solution of (2). Define $u(t) = x_t$ with

$$x_t = \begin{cases} u_0, & t = 0, \\ \langle \delta, u(t) \rangle, & t > 0, \end{cases}$$

where δ is indeed the Dirac δ at zero. Then,

$$\begin{aligned} u(t)(\sigma) = x_{(t+\sigma)} &= \begin{cases} u_0 + \int_0^{t+\sigma} (Eu(\tau - c) + Bu(\tau)) d\tau, \\ u(t + \sigma), \end{cases} \\ &= (T(t)u_0)(\sigma) + \int_0^{\max(0, t+\sigma)} (Bu(\tau) + Eu(\tau - c)) d\tau, \\ &= ((T(t)u_0)(\sigma) + \left(\int_0^t T(t - \tau)(Bu(\tau) + Eu(\tau - c)) d\tau \right)(\sigma)), \end{aligned}$$

where $T(t)$ is a Co-semigroup generated by its infinitesimal generator A . Thus, (3) holds. \square

4 Conclusion

We have considered perturbations which change an equation from one type to another (see equations (2) and (4)), and we believe a worthwhile overall objective is the study of the stability of properties under such perturbations. Most equations of population with complex dynamics can easily be fitted into AFDE of the form considered in this paper, which paves the way into population dynamics so that we can only select a particular case, fit it into the AFDE and work out the details without tears. A noteworthy feature of the above result is the applications and we end by pointing out the desirability of extending the results heretofore obtained to periodic solution of FDE in a subsequent paper, since pluriformity is a condition *sine qua non* for the advancement of any field of scientific activity.

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