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Eigenvalue Problem for the Second Order Differential Equation with Nonlocal Conditions

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Abstract. The paper deals with numerical methods for eigenvalue problem for the second order ordinary differential operator with variable coefficient subject to nonlocal integral condition. FD-method (functional-discrete method) is derived and analyzed for calculating of eigenvalues, particulary complex eigenvalues. The convergence of FD-method is proved. Finally numerical procedures are suggested and computational results are schown.

Keywords: eigenvalues, nonlocal condition, functional-discrete method, convergence, systems of symbolic mathematics.

1 Introduction

The article deals with the functional-discrete method for eigenvalue problem of ordinary differential operator with variable coefficients subject to integral nonlocal condition. The eigenvalue problems for differential operators with nonlocal conditions, except of a few separate articles, has been systematically investigated only over the past decade. Articles [1–7] deal with the eigenvalue problem subject to nonlocal condition including only boundary values (Ionkin-Samarsky conditions). The eigenvalue problem for one and two-dimensional differential operators subject to Bitsadze-Samarsky nonlocal condition are investigated in [8–11], while articles [12–18] deal with the same problem subject to integral nonlocal condition. We mention as well articles [19–22], where similar problems are investigated subject to other types of nonlocal conditions. Only few articles of mentioned above deal with the equations with variable coefficients (see [4, 6, 12, 14, 18]).

This article generalizes the results of [15]. The latter deals with a several eigenvalue problems, all of which are included as separate cases into the following more general formulation:

$$u''(x) + \big[\lambda - q(x)\big]u(x) = 0, \quad x \in (0,1), \tag{1}$$

$$u(0) = \int_0^1 u(x)d\sigma_0(x), \quad u(1) = \int_0^1 u(x)d\sigma_1(x), \tag{2}$$

where $\sigma_0(x),\ \sigma_1(x)$ are given functions with bounded variation, $q(x)$ is a given piecewise continuous function, and $q(x) \geq 0,\ \forall x \in [0,1]$.

For instance, [15] deals with $q(x) \equiv 0$ in the three follwing cases of boundary value problems (2):

1) $\sigma_0(x) \equiv 0, \qquad \sigma_1(x) = ax,$

2) $\sigma_0(x) \equiv 0, \qquad \sigma_1(x) = \begin{cases} 0, & 0 \leq x < \frac{1}{4}, \\ ax, & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 0, & \frac{3}{4} < x \leq 1, \end{cases}$

3) $\sigma_0(x) = a_1x, \quad \sigma_1(x) = a_2x,$

where $a,\ a_1,\ a_2$ are given parameters.

The eigenvalue problem (1), (2) arises in many cases when we consider the numerical solution for ordinary or parabolic differential equations with a nonlocal conditions. Although the differential equation (1) is very simple, the structure of the spectrum of a differential operator with a nonlocal condition can be rather complicated (see, f.e., [7,10]). The analysis of eigenvalues of a difference operator corresponding to differential problem (1), (2) permits one to analyze the stability of difference schemes [6, 13, 16] and justify the convergence of iterative methods for finite-difference equations [8]. The problem (1), (2) is also of interest in itself from mathematical point of view.

In this article we restrict ourselves to the following problem

$$u''(x) + \big[\lambda - q(x)\big]u(x) = 0, \quad x \in (0,1), \tag{3}$$

$$u(0) = 0, \quad u(1) = a \int\limits_{1/4}^{3/4} u(\xi)d\xi. \tag{4}$$

The method we apply for the problem (3), (4) has twofold importance for the investigation. Firstly, it allows to investigate qualitative properties of the solution of (3), (4), i.e., to determine whether eigenvalues are real numbers, presence of complex eigenvalues, multipleness of spectrum, etc. Secondly, the method under consideration gives an algorithm for numerical solution of the problem (3), (4) within any arbitrary accuracy. In a series of articles the method has been entitled as FD-method (functional-discrete method). The method was introduced in [23] to find an approximate solution for Sturm-Liouville problem with Dirichlet conditions. Further development it gained in [2, 4, 9, 22]. Specific feature of FD-method is that for a wide class of eigenvalue problems accuracy increases with the increase of the order number of the corresponding eigenvalue. Moreover, the method does not require to solve complete algebraic eigenvalue problem. Finally, it allows selective computations and as a result, parallelisation.

The text is organized in the following way. Section 2 introduces the algorithm of FD-method, the theorem s on convergence of the method, and the estimation of accuracy. Section 3 presents numerical experiment, as well as the test problem stated and implemented by means of computer application of symbolic mathematics (CASM) Maple [24].

2 FD-method. Estimation of accuracy

According to FD-method, solution of the problem (3), (4) has a form

$$\overset{m}{u}(x) = \sum_{j=0}^{m} u^{(j)}(x), \quad \overset{m}{\lambda} = \sum_{j=0}^{m} \lambda^{(j)}, \tag{5}$$

where summands of (5) are determined through the solutions of recurrent sequence of boundary value problems

$$\frac{d^2u^{(j+1)}(x)}{dx^2} + \big[\lambda^{(0)} - \overline{q}(x)\big]u^{(j+1)}(x)$$
$$= -\sum_{p=0}^{j} \lambda^{(j+1-p)}u^{(p)}(x) + \big[q(x) - \overline{q}(x)\big]u^{(j)}(x), \quad x \in (0,1), \quad (6)$$
$$u^{(j+1)}(0) = 0, \quad u^{(j+1)}(1) = a\int\limits_{1/4}^{3/4} u^{(j+1)}(\xi)d\xi, \quad j = 0, 1, \ldots, m-1, \quad (7)$$

where m is a rank of the method. There $\overline{q}(x)$ is a piecewise constant function, approximating $q(x)$:

$$\overline{q}(x) = \frac{1}{2}\big[q(x_{i-1}) + q(x_i)\big], \quad i = 1, \ldots, N, \quad x \in [x_{i-1}, x_i].$$

Initial condition of the problem $u^{(0)}(x),\ \lambda^{(0)}$ is determined by a solution of a basic problem

$$\begin{aligned} &\frac{d^2u^{(0)}(x)}{dx^2} + \big[\lambda^{(0)} - \overline{q}(x)\big]u^{(0)}(x) = 0, \quad x \in (0,1), \\ &u^{(0)}(0) = 0, \quad u^{(0)}(1) = a\int\limits_{1/4}^{3/4} u^{(0)}(\xi)d\xi. \end{aligned} \quad (8)$$

2.1 Case $\overline{q}(x) \equiv 0$

In this case the basic problem (8) takes the form

$$\begin{aligned} &\frac{d^2u^{(0)}(x)}{dx^2} + \lambda^{(0)}u^{(0)}(x) = 0, \quad x \in (0,1), \\ &u^{(0)}(0) = 0, \quad u^{(0)}(1) = a\int\limits_{1/4}^{3/4} u^{(0)}(\xi)d\xi. \end{aligned} \quad (9)$$

This problem was investigated in [15], as well. We note the principal properties of the eigenvalues and eigenfunctions of problem (9):

1^0 $\lambda^{(0)} = 0$ is an eigenvalue of problem (9), if and only if $a = 4$. The corresponding eigenfunction is $u^{(0)}(x) = x$.

2^0 Inequality $a > 4$ is a necessary and sufficient condition of existence of exactly one simple eigenvalue $\lambda^{(0)} = -\alpha^2$, where α is a single positive root of equation

$$\tanh\frac{\alpha}{2} = \frac{2\alpha}{a}\cosh\frac{\alpha}{4}.$$

The corresponding eigenfunction has a form

$$u^{(0)}(x) = \sinh\alpha x.$$

3^0 For any value of a problem (9) has infinitely many positive eigenvalues, depending on $a\colon \lambda_k = \alpha_k^2$, where α_k is the positive roots of equation

$$\cos\frac{\alpha}{2} = \frac{a}{\alpha}\sin\frac{\alpha}{4}.$$

Moreover, there exist infinitely many positive eigenvalues of the problem (9) not depending on a : $\lambda_k = (2k\pi)^2, k = 1, 2, 3, \ldots$.

In both cases corresponding eigenfunctions are

$$u_k(x) = \sin\sqrt{\lambda_k}x.$$

For instance, there was proved in [15], that problem (9) over the interval $[0, (8\pi)^2]$ has

1) Four distinct eigenvalues, provided $0 < a < 4$;

2) Three distinct positive eigenvalues and $\lambda_0 = 0$, provided $a = 4$;

3) Three distinct eigenvalues, provided $4 < a < a^* = 18.98891711977085\ldots$;

4) A simple eigenvalue and two complex conjugate eigenvalues, provided $a > a^*$ (complex conjugate eigenvalues were not mentioned in [15]);

5) A simple eigenvalue and one multiple eigenvalue of order 2: $\sqrt{\lambda_2} = \sqrt{\lambda_3} = 19.126182900683865\ldots$, provided $a = a^*$.

We note, that for a multiple eigenvalue there exists only one eigenfunction

$$u(x) = \sin\sqrt{\lambda}x.$$

In all case the eigenvalues are the roots of transcendent equation

$$\sin\sqrt{\lambda_n^{(0)}} = \frac{a}{\sqrt{\lambda_n^{(0)}}}\Big[\cos\frac{\sqrt{\lambda_n^{(0)}}}{4} - \cos\frac{3}{4}\sqrt{\lambda_n^{(0)}}\Big]. \tag{10}$$

Solution of boundary value problem (6), (7) has a form

$$u_n^{(j+1)}(x) = \int\limits_0^x \frac{\sin\sqrt{\lambda_n^{(0)}}(x-\xi)}{\sqrt{\lambda_n^{(0)}}}\Big[-\sum_{p=0}^{j}\lambda_n^{(j+1-p)}u_n^{(p)}(\xi)+q(\xi)u_n^{(j)}(\xi)\Big]d\xi, \tag{11}$$

provided $\lambda_n^{(j+1)}$ is determined by

$$\lambda_n^{(j+1)} = \frac{1}{\chi_n}\Big\{-a\int\limits_{1/4}^{3/4}\int\limits_0^x \frac{\sin\sqrt{\lambda_n^{(0)}}(x-\xi)}{\sqrt{\lambda_n^{(0)}}}\Big[\sum_{p=1}^{j}\lambda_n^{(j+1-p)}u_n^{(p)}(\xi)-q(\xi)u_n^{(j)}(\xi)\Big]d\xi dx + \int\limits_0^1 \frac{\sin\sqrt{\lambda_n^{(0)}}(1-\xi)}{\sqrt{\lambda_n^{(0)}}}\Big[\sum_{p=1}^{j}\lambda_n^{(j+1-p)}u_n^{(p)}(\xi)-q(\xi)u_n^{(j)}(\xi)\Big]d\xi\Big\}, \tag{12}$$

where

$$\chi_n = a\int\limits_{1/4}^{3/4}\int\limits_0^x \frac{\sin\sqrt{\lambda_n^{(0)}}(x-\xi)}{\sqrt{\lambda_n^{(0)}}}u_n^{(0)}(\xi)d\xi dx - \int\limits_0^1 \frac{\sin\sqrt{\lambda_n^{(0)}}(1-\xi)}{\sqrt{\lambda_n^{(0)}}}u_n^{(0)}(\xi)d\xi,$$

$$\|u_n^{(0)}\| = 1,$$

where $\|\cdot\|$ is the norm in the space $L_2[0,1]$.

With reference to (11), (12), we build the estimates of $u_n^{(j+1)}(x)$ and $\lambda_n^{(j+1)}$.

Assuming $\lambda_n^{(0)} > 0$, we have

$$\big|u_n^{(j+1)}(x)\big| \le \frac{1}{\sqrt{\lambda_n^{(0)}}}\Big[\sum_{p=0}^{j}|\lambda_n^{(j+1-p)}|\int\limits_0^x\Big|\sin\sqrt{\lambda_n^{(0)}}(x-\xi)\Big|\cdot\big|u_n^{(p)}(\xi)\big|d\xi + \int\limits_0^x\Big|\sin\sqrt{\lambda_n^{(0)}}(x-\xi)\Big|\cdot\big|q(\xi)\big|\cdot\big|u_n^{(j)}(\xi)\big|d\xi\Big]$$

$$\leq \frac{\sqrt{\int\limits_0^x \sin^2 \sqrt{\lambda_n^{(0)}}(x-\xi)d\xi}}{\sqrt{\lambda_n^{(0)}}} \Big[\sum_{p=0}^{j} |\lambda_n^{(j+1-p)}| \cdot \|u_n^{(p)}\| + \|q\|_\infty \|u_n^{(j)}\| \Big].$$

Next,

$$\|u_n^{(j+1)}\| \leq \frac{1}{\sqrt{\lambda_n^{(0)}}} \Big[\int\limits_0^1 \int\limits_0^x \sin^2 \sqrt{\lambda_n^{(0)}}(x-\xi)d\xi dx \Big]^{1/2}$$

$$\times \Big[\sum_{p=0}^{j} |\lambda_n^{(j+1-p)}| \cdot \|u_n^{(p)}\| + \|q\|_\infty \|u_n^{(j)}\| \Big],$$

$$|\lambda_n^{(j+1)}| = \Big| -a \Big[\int\limits_{1/4}^{3/4} \int\limits_0^{1/4} \frac{\sin \sqrt{\lambda_n^{(0)}}(x-\xi)}{\sqrt{\lambda_n^{(0)}}} F_0^{(j+1)}(\xi) d\xi dx$$

$$+ \int\limits_{1/4}^{3/4} \int\limits_{1/4}^{x} \frac{\sin \sqrt{\lambda_n^{(0)}}(x-\xi)}{\sqrt{\lambda_n^{(0)}}} F_0^{(j+1)}(\xi) d\xi dx \Big]$$

$$+ \int\limits_0^1 \frac{\sin \sqrt{\lambda_n^{(0)}}(1-\xi)}{\sqrt{\lambda_n^{(0)}}} F_0^{(j+1)}(\xi) d\xi \Big| \cdot |\chi_n|^{-1}$$

$$= \Big| \int\limits_0^1 g_n(\xi) F_0^{(j+1)}(\xi) d\xi \Big| \cdot |\chi_n|^{-1} \leq \|g_n\| \cdot \big\| F_0^{(j+1)}(\xi) \big\| \cdot |\chi_n|^{-1},$$

where

$$g_n(\xi) = \frac{1}{\sqrt{\lambda_n^{(0)}}} \begin{cases} -a \int\limits_{1/4}^{3/4} \sin \sqrt{\lambda_n^{(0)}}(x-\xi)dx + \sin \sqrt{\lambda_n^{(0)}}(1-\xi), \ \xi \in [0, 1/4], \\ -a \int\limits_{\xi}^{3/4} \sin \sqrt{\lambda_n^{(0)}}(x-\xi)dx + \sin \sqrt{\lambda_n^{(0)}}(1-\xi), \ \xi \in [1/4, 3/4], \\ \sin \sqrt{\lambda_n^{(0)}}(1-\xi), \quad \xi \in [3/4, 1], \end{cases}$$

$$F_0^{(j+1)}(x) = \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)}(x) - g_n(x) u_n^{(j)}(x).$$

This implies the estimates

$$\|u_n^{(j+1)}\| \le M_n \Big[\sum_{p=0}^{j} |\lambda_n^{(j+1-p)}| \cdot \|u_n^{(p)}\| + \|q\|_\infty \|u_n^{(j)}\| \Big], \tag{13}$$

$$|\lambda_n^{(j+1)}| \le A_n \Big[\sum_{p=1}^{j} |\lambda_n^{(j+1-p)}| \cdot \|u_n^{(p)}\| + \|q\|_\infty \|u_n^{(j)}\| \Big], \tag{14}$$

where

$$\begin{aligned} M_n &= \frac{1}{\sqrt{\lambda_n^{(0)}}} \Big[\int_0^1 \int_0^x \sin^2 \sqrt{\lambda_n^{(0)}}(x-\xi) d\xi dx \Big]^{1/2}, \\ A_n &= \|g_n\| \cdot |\chi_n|^{-1}. \end{aligned} \tag{15}$$

Making in (13), (14) the substitution

$$\begin{aligned} v_j &= \big(\|q\|_\infty M_n\big)^{-j} \|u_n^{(j)}\|, \\ \lambda_j &= |\lambda_n^{(j)}| \big(\|q\|_\infty M_n\big)^{-j} M_n, \end{aligned} \tag{16}$$

yields

$$\begin{aligned} v_{j+1} &\le \sum_{p=0}^{j} \lambda_{j+1-p}\, v_p + v_j, \\ \lambda_{j+1} &\le A_n \Big[\sum_{p=1}^{j} \lambda_{j+1-p} v_p + v_j \Big]. \end{aligned} \tag{17}$$

It can be easily shown, that the solution of the recurrent system of equations

$$V_{j+1} = \sum_{p=0}^{j} \Lambda_{j+1-p} V_p + V_j, \quad V_0 = 1,$$

$$\Lambda_{j+1} = A_n \Big[\sum_{p=1}^{j} \Lambda_{j+1-p} V_p + V_j \Big], \quad j = 0, 1, \ldots$$

is a majorant of the solution of previous system of inequalities.

Applying the method of generating functions [2], [9], the solution of the previous system of equations, as well as the corresponding estimates take the form

$$\begin{aligned} V_j &= \frac{1}{2}\left(\frac{1+A_n}{\beta_n}\right)^j \frac{(2j-3)!!}{(2j)!!} \\ &\quad\times\left\{1+\beta_n^{2j}-\frac{(2j)!!}{(2j-3)!!}\sum_{p=1}^{j-1}\frac{(2p-3)!!}{(2p)!!}\frac{(2j-2p-3)!!}{(2j-2p)!!}\beta_n^{2p}\right\} \\ &\le \frac{1}{2}\left(\frac{1+A_n}{\beta_n}\right)^j \frac{(2j-3)!!}{(2j)!!}, \\ \Lambda_j &= \frac{1}{2\beta_n}\left(\frac{1+A_n}{\beta_n}\right)^{j-1} \frac{(2j-3)!!}{(2j)!!} \\ &\quad\times\left\{1+\beta_n^{2j}-\frac{(2j)!!}{(2j-3)!!}\sum_{p=1}^{j-1}\frac{(2p-3)!!}{(2p)!!}\frac{(2j-2p-3)!!}{(2j-2p)!!}\beta_n^{2p}\right\} \\ &\le \frac{1}{2\beta_n}\left(\frac{1+A_n}{\beta_n}\right)^{j-1} \frac{(2j-3)!!}{(2j)!!}, \end{aligned}$$

where

$$\beta_n = \left[1+2A_n+2\sqrt{A_n(1+A_n)}\right]^{-1}.$$

Utilising substitution (2.12) we obtain the following estimates:

$$\begin{aligned} \|u_n^{(j)}\| &\le \frac{1}{2}\big(\|q\|_\infty M_n(1+A_n)\beta_n^{-1}\big)^j \frac{(2j-3)!!}{(2j)!!}, \\ |\lambda_n^{(j)}| &\le \frac{\|q\|_\infty}{2\beta_n}\Big(\|q\|_\infty M_n(1+A_n)\beta_n^{-1}\Big)^{j-1}\frac{(2j-3)!!}{(2j)!!}. \end{aligned} \tag{18}$$

The following propositions are true:

Theorem 1. *Suppose that* $\lambda_n^{(0)} > 0$ *– is a eigenvalue of problem* (8) *with* $\overline{q}(x) \equiv 0$ *and the following condition holds:*

$$r_n = M_n\|q\|_\infty(1+A_n)\beta_n^{-1} < 1, \tag{19}$$

where M_n, A_n *are determined in* (15)*. Then, using the FD-method, the solution of* *the problem* (3), (4) *under the integral conditions can be represented as the series*

$$\begin{aligned} u_n(x) = u_n\big(x, q(\cdot)\big) = \sum_{j=0}^{\infty} u_n^{(j)}(x) \equiv \sum_{j=0}^{\infty} u_n^{(j)}(x, 0), \\ \lambda_n = \lambda_n\big(q(\cdot)\big) = \sum_{j=0}^{\infty} \lambda_n^{(j)} \equiv \sum_{j=0}^{\infty} \lambda_n^{(j)}(0), \end{aligned} \tag{20}$$

which converge not slower than a geometric progression with the ratio r_n, *and the following prior-posterior accuracy estimates hold:*

$$\begin{aligned} |\lambda_n - \overset{m}{\lambda}_n| = \Big|\lambda_n - \sum_{j=0}^{m} \lambda_n^{(j)}\Big| \le \frac{\|q\|_\infty}{2\beta}\,\frac{(r_n)^m}{1-r_n}\,\frac{(2m-1)!!}{(2m+2)!!}, \\ \|u_n - \overset{m}{u}_n\| = \Big\|u_n\big(x, q(\cdot)\big) - \sum_{j=0}^{m} u_n^{(j)}(x)\Big\| \le \frac{1}{2}\,\frac{(r_n)^{m+1}}{1-r_n}\,\frac{(2m-1)!!}{(2m+2)!!}. \end{aligned} \tag{21}$$

The theorem is proved in the same way as in [2, 4, 9]. Theorem 1 entails the following theorem.

Theorem 2. *If* $\lambda_n^{(0)}$ *is a real eigenvalue and condition* (19) *is satisfied, then the corresponding exact eigenvalue* λ_n *of problem* (1), (2) *– is also real. If all* $\lambda_n^{(0)},\ n = 1, 2, \ldots$, *– are real eigenvalues and condition* (19) *holds for each* n, *then all eigenvalues of problem* (3), (4) *are also real.*

In the case $\operatorname{Im}\lambda_n^{(0)} \ne 0$ the following proposition analogous to Theorem 1 holds, if M_n is considered to be

$$M_n = \frac{1}{\Big|\sqrt{\lambda_n^{(0)}}\Big|}\bigg(\int_0^1\int_0^x \Big|\sin\sqrt{\lambda_n^{(0)}}(x-\xi)\Big|^2 d\xi\, dx\bigg)^{1/2},$$

and in r_n expression $\lambda_n^{(0)}$ is substituted with $|\lambda_n^{(0)}|$.

In case condition (19) is not satisfied, the general rutine of *FD*-method (6)–(8) should be applied with $\overline{q}(x) \ne 0$.

2.2 General case: $\overline{q}(x) \not\equiv 0$

In order to estimate the accuracy of FD-method we introduce Cauchy function $K_n(x,\xi)$, provided the solution of the problem (6), (7) is put to the form

$$u_n^{(j+1)}\big(x,\overline{q}(\cdot)\big) = \int_0^x K_n(x,\xi)$$
$$\times \Big\{ \sum_{p=0}^{j} \lambda_n^{j+1-p}\big(\overline{q}(\cdot) u_n^{(p)}\big(\xi,\overline{q}(\cdot)\big)\big) + \big[q(\xi) - \overline{q}(\xi)\big] u_n^{(j)}\big(\xi,\overline{q}(\cdot)\big) \Big\} d\xi, \quad (22)$$

$$K_n(x,x) = 0, \quad \Big[\frac{d}{dx} K_n(x,\xi)\Big]_{\xi=x} = 1. \quad (23)$$

This function $K_n(x,\xi)$ plays the same role as the function

$$\frac{\sin\sqrt{\lambda_n^{(0)}(0)}(x-\xi)}{\sqrt{\lambda_n^{(0)}(0)}} \quad (24)$$

in the formula (11) with $\overline{q}(x) \equiv 0$. It can be easily verified, that the following estimates will take place instead of (13), (14):

$$\big\|u_n^{(j+1)}\big(\cdot,\overline{q}(\cdot)\big)\big\| \le M_n\big(\overline{q}(\cdot)\big)\Big[\sum_{p=0}^{j} \big|\lambda_n^{(j+1-p)}\big(\overline{q}(\cdot)\big)\big| \cdot \big\|u_n^{(p)}\big(\cdot,\overline{q}(\cdot)\big)\big\|$$
$$+ \|q-\overline{q}\|_\infty \big\|u_n^{(j)}\big(\cdot,\overline{q}(\cdot)\big)\big\|\Big],$$
$$\big|\lambda_n^{(j+1)}\big(\overline{q}(\cdot)\big)\big| \le A_n\big(\overline{q}(\cdot)\big)\Big[\sum_{p=1}^{j} \big|\lambda_n^{(j+1-p)}\big(\overline{q}(\cdot)\big)\big| \cdot \big\|u_n^{(p)}\big(\cdot,\overline{q}(\cdot)\big)\big\|$$
$$+ \|q-\overline{q}\|_\infty \big\|u_n^{(j)}\big(\cdot,\overline{q}(\cdot)\big)\big\|\Big],$$

where

$$M_n = \Big\{ \int_0^1 \int_0^x \big|K_n(x,\xi)\big|^2 d\xi dx \Big\}^{1/2},$$
$$A_n = \big\|g_n\big(\cdot,\overline{q}(\cdot)\big)\big\| \cdot \big|\chi_n\big(\overline{q}(\cdot)\big)\big|^{-1},$$

$$\chi_n\big(\overline{q}(\cdot)\big) = a\int\limits_{1/4}^{3/4}\int\limits_{0}^{x} K_n(x,\xi)u_n^{(0)}\big(\xi,\overline{q}(\cdot)\big)d\xi dx - \int\limits_{0}^{1} K_n(1,\xi)u_n^{(0)}\big(\xi,\overline{q}(\cdot)\big)d\xi,$$

$$g_n\big(\xi,\overline{q}(\cdot)\big) = \begin{cases} -a\int\limits_{1/4}^{3/4} K_n(x,\xi)dx + K_n(1,\xi), & \xi\in[0,1/4],\\ -a\int\limits_{\xi}^{3/4} K_n(x,\xi)dx + K_n(1,\xi), & \xi\in[1/4,3/4],\\ K_n(1,\xi), & \xi\in[3/4,1]. \end{cases}$$

To determine the Cauchy function $K_n(x,\xi)$, we introduce two functions $v_\alpha(x)$, $\alpha = 1,2$ in the form of the solutions of the following problems:

$$\begin{aligned} v''_\alpha(x) + \big[\lambda_n^{(0)}(\overline{q}(\cdot)) - \overline{q}(x)\big]v_\alpha(x) = 0, \quad x\in(0,1), \quad \alpha = 1,2,\\ v_1(0) = 0, \quad v'_1(0) = 1,\\ v_2(0) = 1, \quad v'_2(0) = 0. \end{aligned} \tag{25}$$

The the expression holds:

$$K_n(x,\xi) = v_1(x)v_2(\xi) - v_1(\xi)v_2(x). \tag{26}$$

Now we show that estimate of function $K_n(x,\xi)$ with respect to $\lambda_n^{(0)}(0)$ behaves analogously to the estimate of function (24). To show that, we denote

$$d_n(x) = \lambda_n^{(0)}(0) - \lambda_n^{(0)}(\overline{q}(\cdot)) + \overline{q}(x) \tag{27}$$

and put the problems (25) into equivalent form

$$\begin{aligned} v_1(x) &= \frac{\sin\sqrt{\lambda_n^{(0)}(0)}x}{\sqrt{\lambda_n^{(0)}(0)}} + \int\limits_0^x \frac{\sin\sqrt{\lambda_n^{(0)}(0)}(x-\xi)}{\sqrt{\lambda_n^{(0)}(0)}} d_n(\xi)v_1(\xi)d\xi,\\ v_2(x) &= \cos\sqrt{\lambda_n^{(0)}(0)}x + \int\limits_0^x \frac{\sin\sqrt{\lambda_n^{(0)}(0)}(x-\xi)}{\sqrt{\lambda_n^{(0)}(0)}} d_n(\xi)v_2(\xi)d\xi. \end{aligned} \tag{28}$$

For $\lambda_n^{(0)}(0) > 0$ in accordance to (28), the Gronwalls lemma yields the estimates

$$\begin{aligned} \left|v_1(x)\right| &\le \frac{1}{\sqrt{\lambda_n^{(0)}(0)}} \exp\left[\frac{d_n}{\sqrt{\lambda_n^{(0)}((0))}}\, x\right], \\ \left|v_2(x)\right| &\le \exp\left[\frac{d_n}{\sqrt{\lambda_n^{(0)}(0)}}\, x\right], \end{aligned} \tag{29}$$

where $d_n = \max_{x\in[0,1]} |d_n(x)|$.

In accordance to (26), (29), we get

$$\left|K_n(x,\xi)\right| \le \frac{2}{\sqrt{\lambda_n^{(0)}(0)}} \exp\left[\frac{2d_n}{\sqrt{\lambda_n^{(0)}(0)}}\, x\right].$$

We show, that analogously to Theorems 1, 2, the following propositions hold.

Theorem 3. *Suppose that* $\lambda_n^{(0)}(\overline{q}(\cdot)) > 0$ *is a eigenvalue of problem* (8) *and the following condition holds:*

$$r_n\big(\overline{q}(\cdot)\big) = M_n \|q - \overline{q}\|_\infty \big[1 + A_n\big(\overline{q}(\cdot)\big)\big] \beta_n\big(\overline{q}(\cdot)\big)^{-1} < 1. \tag{30}$$

Then, using the FD*-method, the solution of the problem* (3), (4) *can be represented as the series*

$$\begin{aligned} u_n\big(x, q(\cdot)\big) &= \sum_{j=0}^{\infty} u_n^{(j)}\big(x, \overline{q}(\cdot)\big), \\ \lambda_n\big(q(\cdot)\big) &= \sum_{j=0}^{\infty} \lambda_n^{(j)}\big(\overline{q}(\cdot)\big), \end{aligned} \tag{31}$$

with converge not slower than a geometric progression with the ratio $r_n(\overline{q}(\cdot))$*, and the following prior-posterior accuracy estimates hold:*

$$\begin{aligned} \left|\lambda_n\big(q(\cdot)\big) - \overset{m}{\lambda}_n\big(\overline{q}(\cdot)\big)\right| &\le \frac{\|q - \overline{q}\|_\infty}{2\beta\big(\overline{q}(\cdot)\big)} \frac{\big[r_n\big(\overline{q}(\cdot)\big)\big]^m}{1 - r_n\big(\overline{q}(\cdot)\big)} \frac{(2m-1)!!}{(2m+2)!!}, \\ \left\|u_n\big(\cdot, q(\cdot)\big) - \overset{m}{u}_n\big(\cdot, \overline{q}(\cdot)\big)\right\| &\le \frac{1}{2} \frac{\big[r_n\big(\overline{q}(\cdot)\big)\big]^m}{1 - r_n\big(\overline{q}(\cdot)\big)} \frac{(2m-1)!!}{(2m+2)!!}. \end{aligned} \tag{32}$$

3 Numerical experiment

Let $q(x) = x,\ a = 20$. Then the exact solution of differential equation (1) is of a form

$$u_n(x,\lambda_n) = C_n\sqrt{\lambda_n - x}\,\Bigg(J_{1/3}\Big(\frac{2}{3}(\lambda_n - x)^{3/2}\Big)\cdot J_{-1/3}\Big(\frac{2}{3}\lambda_n^{3/2}\Big) - J_{-1/3}\Big(\frac{2}{3}(\lambda_n - x)^{3/2}\Big)\cdot J_{1/3}\Big(\frac{2}{3}\lambda_n^{3/2}\Big)\Bigg), \tag{33}$$

where $J_{\pm 1/3}(z)$ are either cylindrical or Bessel functions of first type (see [25]). We notice, that in (33) the first boundary value condition in (4) is taken into account, and C_n is an arbitrary constant.

The eigenvalue λ_n can be found using the second boundary value condition in (4), i.e., solving equation

$$\Delta\left(\lambda_n\right) \equiv u_n(1,\lambda_n) - a\int\limits_{1/4}^{3/4} u_n\left(x,\,\lambda_n\right)dx = 0. \tag{34}$$

Numbering of eigenvalues $\lambda_n^{(0)},\ n = 1,2,\ldots$ is arranged in the following way. The eigenvalues $\lambda_n^{(0)}$ precede $\lambda_{n+1}^{(0)}$, if

$$\operatorname{Re}\lambda_n^{(0)} < \operatorname{Re}\lambda_{n+1}^{(0)}.$$

Table 1 gives the exact eigenvalues λ_n of problem (3), (4), i.e., real roots of transcendental equation (34) with respect to unknown λ_n within the accuracy of 12 decimal places. We computed the eigenvalues using systems Maple 9.5 with the greatest possible accuracy of interim calculations, which is 24 significant digits of mantissa (Digits:=24).

In the same table we give the initial approximations $\lambda_n^{(0)}(0)$ of the considered problem for $\overline{q}(x) \equiv 0$, i.e., the solutions of the basic problem (9), that is, the real roots of equation (10) obtained by FD-method, as well as the corresponding values of the ratio r_n of geometric progression in (19), that are used to derive a priori estimate of convergence of FD method.

Table 2 gives real solutions $\lambda_n^{(j)}(0)$ of the sequence of problems (6), (7) for $j = 1,2,3$, correspondingly, as well as the deviations of approximations of rank 3

$\overset{3}{\lambda}_n(0)$ (see (5), (6)) of eigenvalues of problem (3), (4) with respect to exact values λ_n.

Table 1. The exact values λ_n of the problem (3), (4) with $a = 20$; $\lambda_n^{(0)}(0)$ – the initial approximations ($\overline{q}(x) \equiv 0$); r_n – the ratio of geometric progresion in (19)

n	λ_n	$\lambda_n^{(0)}(0)$	r_n
0	-25.428345153905	-26.115663574029	1.680456
1	39.778088064844	39.478417604357	1.380912
2	120.939503405441	120.613292811770	1.666367
3	158.508180719231	157.913670417430	1.354561
4	352.652341253466	355.305758439217	51.64523
7	632.130954512830	631.654681669719	0.355239
8	754.145646810721	753.661340439576	0.303225
9	987.386349218273	986.960440108936	0.210369
10	1241.468978417228	1240.993990218464	0.209989
11	1421.733567796926	1421.223033756868	0.192481
12	1736.658987400562	1736.203921780346	0.202347

Table 2. The solution of the first three boundary value problems (6), (7): $\lambda_n^1(0), \lambda_n^2(0), \lambda_n^3(0)$

n	$\lambda_n^{(1)}(0)$	$\lambda_n^{(2)}(0)$	$\lambda_n^{(3)}(0)$	$\lambda_n - \overset{3}{\lambda}_n(0)$
0	0.687677663347	-0.000357922508	$-1.31498\text{E}-6$	$-5.74\text{E}-9$
1	0.299498512336	0.000171313339	$6.27818\text{E}-7$	$6.99\text{E}-9$
2	0.325898623398	0.000313517863	$-1.56314\text{E}-6$	$1.55\text{E}-8$
3	0.594988609665	-0.000480592565	$2.30150\text{E}-6$	$-1.68\text{E}-8$
4	-2.906396462699	0.254017485398	$6.57684\text{E}-3$	$-7.62\text{E}-3$
7	0.476252847584	0.000019945174	$5.0325\text{E}-8$	$2.78\text{E}-11$
8	0.484257011559	0.000049360091	$-0.5220\text{E}-9$	$1.69\text{E}-11$
9	0.425883029502	0.000026081247	$-1.4130\text{E}-9$	$1.04\text{E}-12$
10	0.475008873007	-0.000020698259	$2.4025\text{E}-8$	$-9.25\text{E}-12$
11	0.510554289963	0.000009761608	$-1.1519\text{E}-8$	$7.23\text{E}-12$
12	0.455061661246	0.000003945897	$1.3086\text{E}-8$	$-1.36\text{E}-11$

For the presented eigenvalues, beginning with the 7th one, proposition of Theorem 1 holds and therefore convergence of FD-method is secured. For the eigenvalues No. 0–3 the proposition of the theorem is not true, but despite that, Table 2 shows that the convergence of the approximations $\lambda_n^{(j)}(0)$ is preserved.

However, for $n = 4$ the value $r_4 \gg 1$ (see Table 1), and FD-method diverges. Computations extended to the 4th rank give correction $\lambda_4^{(4)}(0) = -0.009517$, which absolute value exceeds $\lambda_4^{(3)}(0)$, even accuracy of approximations remains sufficiently sharp ($\overset{4}{\lambda}_4(0)$=352.650439, $\lambda_4 - \overset{4}{\lambda}_4(0) = 0.001902$).

To ensure convergence, we select on the uniform grid

$$\Omega_N = \Big\{x_i = \frac{i}{N}, \quad i = 0, \ldots, N\Big\}$$

an approximating function $\overline{q}(x)$ in the shape of piecewise constant function with number of constant pieces equal to N

$$\overline{q}(x) = \frac{1}{2}\big[q(x_{i-1}) + q(x_i)\big], \quad i = 1, \ldots, N, \quad x \in [x_{i-1}, x_i]. \tag{35}$$

Solution of basic problem (8) is given in the form

$$u_n^{(0)}\big(x, \overline{q}(\cdot)\big) = \begin{cases} c_1 \sin\sqrt{\lambda - \overline{q}_1}x, & 0 \le x \le x_1,\ i = 1, \\ c_{2i-2} \sin\sqrt{\lambda - \overline{q}_i}x + c_{2i-1} \cos\sqrt{\lambda - \overline{q}_i}x, & x_{i-1} \le x \le x_i,\ 2 \le i \le N. \end{cases}$$

Here with, initial approximations $\lambda_n^{(0)}(\overline{q}(\cdot))$ and unknowns c_i $(i = 1, \ldots, 2N - 1)$ were found based upon existence condition of nontrivial solution of homogeneous system of linear algebraic equations

$$\begin{cases} u_n^{(0)}(x_i, \overline{q}_i) = u_n^{(0)}(x_i, \overline{q}_{i+1}), & i = \overline{1, N-1}, \\ \big[u_n^{(0)}(x, \overline{q}_i)\big]'_{x=x_i} = \big[u_n^{(0)}(x, \overline{q}_{i+1})\big]'_{x=x_i}, & i = \overline{1, N-1}, \\ u_n^{(0)}(1, \overline{q}_N) = a \int\limits_{1/4}^{3/4} u_n^{(0)}\big(\xi, \overline{q}(\cdot)\big) d\xi, \end{cases} \tag{36}$$

corresponding conditions of continuity and continuous differentialbility of eigenfunction $u_n^{(0)}(x, \overline{q}(x, \cdot))$ in the interior points of the grid Ω_N, as well as the second boundary value condition in (8).

In Table 3 for $N = 2$ we present the refined results of calculation of approximations of the rank $0 - 2$ of the eigenvalues with the order number $n = 0, \ldots, 4$.

It is clearly seen, that sequences of approximations $\lambda_n^{(j)}(\overline{q}(\cdot))$ converge to zero, though for a order number $n = 4$ denominator $r_n(\overline{q}(\cdot))$ remains sufficiently larger than one. Therefore, for that number we run extra calculations for $N = 4, 8$, where initial approximations $\lambda_n^{(0)}(\overline{q}(\cdot))$ show better accuracy compared to the case $N = 2$ (see Tables 3, 4). Moreover, FD-method allows us to find a single complex conjugate roots of equation (10), i.e., initial approximations (that are solutions of basic problem (9)), as well as the following solutions of problems (6), (7) for $j = \overline{0,\ 4}$. The results are shown in Table 5. However, we failed to find exact complex values, since expansion of modified Bessel functions involves very large exponents in exponential functions. To complete the search procedure *fsolve* of the roots of the system of two equations of the form (34) with respect to variables corresponding real and imaginary parts of the eigenvalues, it is necessary to set sufficienly high accuracy of calculations ($Digits \gg 400$). This usually causes a breakdown of Maple.

Table 3. The results of the calculation of the eigenvalues $\overset{m}{\lambda}_n$ for different rank $m = 0, 1, 2$

n	$\lambda_n^{(0)}(\overline{q}(\cdot))$	$\lambda_n^{(1)}(\overline{q}(\cdot))$	$\lambda_n - \overset{1}{\lambda}_n$	$\lambda_n^{(2)}(\overline{q}(\cdot))$	$\lambda_n - \overset{2}{\lambda}_n(\overline{q})$	$r_n(\overline{q})$
0	-25.43947506	0.01118416	-0.000054	-0.00005424	-0.00000001	–
1	39.76261117	0.01558898	-0.000112	-0.00011209	0.00000001	0.347
2	120.85653350	0.08263282	0.000337	0.00033536	0.00000173	0.410
3	158.66526344	-0.15684773	-0.000235	-0.00023326	-0.00000173	0.332
4	354.59171687	-2.03575052	0.096375	0.09789554	-0.00152064	11.59

Table 4. The results of the calculation of the approximations of eigenvalues for different values N

N	$\lambda_4^{(0)}(\overline{q}(\cdot))$	$\lambda_4 - \lambda_4^{(0)}(\overline{q}(\cdot))$	$\lambda_{5,6}^{(0)}(\overline{q}(\cdot))$	$\lvert\Delta(\lambda^{(0)}(\overline{q}))\rvert$
4	353.17709311	-0.52475186	367.744727$\pm$28.795461 i	2.28E-6
8	352.59903452	0.09330673	368.0447082$\pm$8.947414 i	2.60E-7

Table 5 indicates the convergence of a sequence of modules $|\lambda_{5,6}^{(j)}(0)|$ to zero. Still, to endorse reliability and accuracy of the solution, analogously to the case with the order number $n = 4$, we choose an approximating function $\bar{q}(x)$ in the form of (35) and calculate the approximations of rank 0–2 for different values of N (see Table 4). It will be observed, that calculation of double integrals of eigenfunctions, that are in turn integrals with the variable upper limit, requires a series of simplifying procedures *simplify, expand, fnormal, evalc*. Next, in case $N = 8$ when using piecewise constant function $\overline{q}x$ it is necessary to find a nontrivial solution of the system (36), which exists when determinant is equal to zero. However, the package LinearAlgebra of Maple 9.5 has no means to calculate the zeroes of a polynomial expansion of determinant of 15×15 matrix, because of insufficient operational memory, presumably. Since this matrix has a 7-diagonal structure, the problem was successfully solved by a modified method of LU-factorization, created for sparse systems [26]. Moreover, we succeed in calculation of deviations of the residual $|\Delta(\lambda_n)|$ in (34) in case of complex conjugate eigenvalues by setting sufficiently large number of significant digits of mantissa (Digits:=800).

Table 5. The results of convergence of the sequence of modulus $|\lambda_{5,6}^{(j)}(0)|$ to zero

j	$\lambda_{5,6}^{(j)}(0)$	$\|\lambda_{5,6}^{(j)}(0)\|$	$\overset{j}{\lambda}_{5,6}(0)$	$\|\Delta(\overset{j}{\lambda}_{5,6})\|$
0	–	–	365.854296$\pm$28.318952 i	1.63E-5
1	2.291311$\pm$0.526362 i	2.350992	368.145607$\pm$28.845314 i	1.14E-6
2	-0.126955$\pm$0.105863 i	0.165301	368.018652$\pm$28.951178 i	1.80E-7
3	-0.003288$\mp$0.026500 i	0.026703	368.015363$\pm$28.924678 i	3.52E-8
4	0.004759$\pm$0.002318 i	0.005294	368.020122$\pm$28.926996 i	7.70E-9

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