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On Nonlinear Vekua Type Equations*

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Abstract. Nonlinear Vekua-Bers type differential equations are studied on the base of certain methods of nonlinear analysis. A survey of recent results in the area is presented.

Keywords: nonlinear Vekua-Bers type equations, integral representations, Banach-Cacciopolli fixed point theorem, Newton-Kantorovich method, implicit function theorem, superposition operator.

1 Introduction

The theory of generalizations of analytic functions goes back to the early fifties (see, e.g., the survey by Tutschke [1] and references therein). An interest to the developing of this area is connected first of all with different type applications of these functions called usually *generalized analytic functions* (see [2–4]). The most known constructions are those generalized analytic functions of Vekua type (see [4]) defined as a solution to elliptic systems of differential equations generalizing the Cauchy-Riemann system, or pseudo-analytic functions of Bers type determined by ordinary differential equations in complex domains containing so called (F, G)-derivatives (see [5]). Different methods are developed for the study of the corresponding differential equations and the corresponding boundary value problems. One can mention here the method of integral representations which was started by Vekua himself (see [4], and [6]), related to it method of (singular)

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integral equations (see [2]), topological methods (see [7]), as well as certain methods of functional analysis (see [1, 5, 8–10]).

Let us represent the most interesting results in this direction.

Any solution of the inhomogeneous differntial equation

$$w_{\overline{z}} = f, \quad z \in G, \tag{1}$$

can be represented ([4], p. 29) in the following form

$$w = \phi + \mathbf{T}_G f, \tag{2}$$

where ϕ is a function analytic in G, and

$$\mathbf{T}_G f(z) = -\frac{1}{\pi} \iint_G f(\zeta) \frac{d\xi d\eta}{\zeta - z} \tag{3}$$

is the so called Pompeiu integral operator.

Generalized Beltrami equation

$$w_{\overline{z}} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} = f, \quad z \in G, \tag{4}$$

with coefficients satisfying the relation $|\mu_1(z)| + |\mu_2(z)| \leq q_0 < 1$ has a solution of the form (see [6], p. 225)

$$w = \mathbf{T}_G \rho, \tag{5}$$

where ρ is a unique solution to the singular integral equation

$$\rho + \mu_1 \mathbf{\Pi}_G \rho + \mu_1 \overline{\mathbf{\Pi}_G \rho} + a\mathbf{T}_G \rho + b\overline{\mathbf{T}_G \rho} = f, \tag{6}$$

where

$$\mathbf{\Pi}_G f(z) = -\frac{1}{\pi} \iint_G f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2} \tag{7}$$

is the Calderon-Zygmund type singular integral operator.

The Riemann-Hilbert boundary value problem

$$\mathrm{Re}\left[\overline{\lambda(z)} w\right] = \gamma(z), \quad z \in \partial G \equiv \Gamma, \tag{8}$$

in the class of generalized analytic functions, i.e. those satisfying inhomogeneous differential equation

$$Lw = \partial_{\overline{z}} w + A(z)w + B(z)\overline{w} = F(z), \quad z \in G, \tag{9}$$

with $A, B, F \in L_{p,2}(G), p > 2$; $\Gamma \in \mathcal{C}^1_{\mu,\nu_1,\ldots,\nu_k}$; $\lambda, \gamma \in \mathcal{C}_\nu(\Gamma)$, $|\lambda| = 1$, has the solution of the following form

$$w = w_0 + \widetilde{w}, \tag{10}$$

where w_0 is a solution to the homogeneous equation $Lw = 0$

$$w_0(z) = \Phi(z)e^{\omega(z)}, \quad \omega(z) = -\mathbf{T}_G\Big(A + B\frac{\overline{w_0}}{w}\Big), \tag{11}$$

$$\widetilde{w}(z) = -\frac{1}{\pi}\iint\limits_G \Omega_1(z,\zeta)F(\zeta)d\xi d\eta - \frac{1}{\pi}\iint\limits_G \Omega_2(z,\zeta)\overline{F(\zeta)}d\xi d\eta, \tag{12}$$

and $\Omega_1(z,t,G), \Omega_2(z,t,G)$ are the principal kernels of the class $\mathcal{N}_{p,2}(A,B,G)$, $p > 2$, which is the class of all regular solutions to the linear differential equations $Lw = 0$ with coefficients $A, B \in L_{p,2}(G)$ for a fixed p.

In the next sections we will consider certain ideas of nonlinear analysis which can be applied at the study nonlinear generalizations of Vekua (or Bers) type differential equations and/or nonlinear boundary value problems for these equations. Some results of such a type were obtained by different authors (see, e.g., [8, 10, 11], and references therein).

2 Nonlinear Vekua-Bers type equations

Let us described the situation which leads to the problems we would like to discuss here. Consider the most simple nonlinear analog of the Cauchy-Riemann equations in one of the following forms

$$w_{\overline{z}} = F_1(z, w), \tag{13}$$

or

$$w_{\overline{z}} = F_2(z, w, \overline{w}). \tag{14}$$

For the simplicity we will study either equations on the unit disc $\mathbb{D}$. Moreover, for the beginning we will not fix any class for the solutions of (13) and (14) and will not pose any boundary value problem for these equations. These moments will be specified later.

We will not consider more general equations of the type

$$w_{\overline{z}} = F_3(z, w, \overline{w}, w_z), \tag{15}$$

or

$$F_4(z, w, \overline{w}, w_z, w_{\overline{z}}) = 0, \tag{16}$$

since it leads to the problems on the type of these differential equations.

2.1 Banach-Cacciopolli fixed point principle

One can apply to each of the equations (13), (14) the same ideas as to the inhomogeneous equation (1), i.e. rewrite these equations as integral equations

$$w + \frac{1}{\pi}\iint\limits_G F_1\big(\zeta, w(\zeta)\big)\frac{d\xi d\eta}{\zeta - z} = 0, \tag{17}$$

$$w + \frac{1}{\pi}\iint\limits_G F_2\big(\zeta, w(\zeta), \overline{w(\zeta)}\big)\frac{d\xi d\eta}{\zeta - z} = 0, \tag{18}$$

or in operator form

$$(\mathbf{I} - \mathbf{T}_{\mathbb{D}}\mathbf{F}_j)w = 0, \quad j = 1, 2. \tag{19}$$

Here $\mathbf{F}_j,\ j = 1, 2$, are so called superposition operators (or Nemytsky operators) defined by one of the following formulas

$$\mathbf{F}_1 \colon w(z) \mapsto F_1\big(z, w(z)\big), \tag{20}$$

$$\mathbf{F}_2 \colon w(z) \mapsto F_2\big(z, w(z), \overline{w(z)}\big). \tag{21}$$

These operators (mostly in real-valued settings) are intensively studied recently. We have to mention here the book [12] which is devoted to the general questions of the theory of operators of type (20) as well as to the study of these operators in the classical functional spaces. The book [13] deals with the set-valued operators (20) (or (21)), i.e. when the values of functions $\mathbf{F}_j$ are sets but not numbers. These operators can be applied e.g. in the case of the discontinuous right-hand sides in (13) and (14) by using concepts of sub- and super-solutions. At last in the book [14] the superposition operators are studied in Sobolev type spaces. The corresponding results are applicable to the study of weak (generalized or distributional type) solutions of the equations on the discussion.

Let us return to the operator equation (19). Denoting by

$$\mathbf{P}_j \equiv \mathbf{T}_{\mathbb{D}}\mathbf{F}_j, \quad j = 1, 2, \tag{22}$$

we arrive at so called fixed point equation (see [15], p. 510)

$$w = \mathbf{P}_j w. \tag{23}$$

The idea of Banach-Cacciopolli approach is well-known. We will get a solution of (23) in a metric space $(\mathcal{X}, \rho)$ if the operator $\mathbf{P}_j$ is contructive, i.e. if there exists $\alpha \in (0, 1)$ such that

$$\rho\big(\mathbf{P}_j(w_1), \mathbf{P}_j(w_2)\big) \le \alpha\rho(w_1, w_2), \quad w_1, w_2 \in \mathcal{X}. \tag{24}$$

More delicate forms of this conditions are presented in the books [16, 17]. One of the most simple generalizations is the following: let w_0 be a given point in the space $\mathcal{X}$. Denote by $B = B(w_0, R)$ the ball of radius R centered at w_0. Equation (23) has a solution in the ball B if inequality (24) is valid for all points $w_1, w_2 \in B$. If the existence of a solution to (23) is known then the solution itself can be obtained by the limit of the successive approximations (w_n):

$$w_{n+1} = \mathbf{P}_j(w_n), \quad n = 0, 1, \dots. \tag{25}$$

The speed of convergence is usually described in terms of elements of the estimates of the type (24).

How one can apply Banach-Cacciopolli principle to equation (23) with the operators $\mathbf{P}_j$ of the form (22)? Consider solvability of (23) on the whole space $\mathcal{X}$. First property which the operators $\mathbf{P}_j$ have to satisfy is the following:

($*$) the operators $\mathbf{P}_j$ should act on the space $\mathcal{X}$, i.e.

$$\mathbf{P}_j\,(\mathcal{X}) \subset \mathcal{X}. \tag{26}$$

It really gives certain additional relations to be checked (or assumed). E.g. it is known [4], pp. $38, 47, 51$ that for $f \in L_p\left(\overline{\mathbb{D}}\right)$ we have

1) $\mathbf{T}_{\mathbb{D}} f \in \mathcal{C}^{\alpha}(\mathbb{C}),\ \alpha = \frac{p-2}{2p} \quad \mathit{if} \quad p > 2;$

2) $\mathbf{T}_{\mathbb{D}} f \in L^a_\gamma\left(\overline{\mathbb{D}}\right),\ 1 < \gamma < \frac{2p}{2-p},\ a = \frac{1}{\gamma} - \frac{2-p}{2p} \quad \mathit{if} \quad 1 \le p \le 2.$

The requirement (26) leads then to certain conditions on the superposition operators $\mathbf{F}_j$. Since it is not always possible (and sometimes even impossible) to have $\mathbf{F}_j\colon \mathcal{X} \to \mathcal{X}$ then our question is to find suitable metric space $\mathcal{Y}$ such that

$$\mathbf{F}_j\colon \mathcal{X} \to \mathcal{Y}, \quad \mathbf{T}_{\mathbb{D}}\colon \mathcal{Y} \to \mathcal{X}. \tag{27}$$

These are acting properties of the superposition operators $\mathbf{F}_j$ and of the integral operator $\mathbf{T}_{\mathbb{D}}$. Since the latters are known and described in the literature then the choice of the space $\mathcal{Y}$ is more or less clear. But one have to be careful even in a very simple cases. Thus for the case of acting of superposition operators in the Lebesgue spaces the well-known Krasnosel'skii theorem should be taken into account. In real setting this result reads (see, e.g., [12]): if the operator

$$\mathbf{F}\colon x(t) \mapsto F\big(t, x(t)\big)$$

maps $L_p(\Omega)$ into $L_q(\Omega)$, where Ω is a bounded subset of $\mathbb{R}^n$ and $q \ge p$, then the function $F(t,u)$ is linear with respect to the second variable, i.e.

$$F(t,u) = a(t) + b(t)u.$$

The second condition we have to consider is a form of contructness

$(**)$ the operators $\mathbf{P}_j$ should be contructive on the space $\mathcal{X}$, i.e. for certain $\alpha < 1$

$$\rho\big(\mathbf{P}_j(w_1), \mathbf{P}_j(w_2)\big) \le \alpha\rho(w_1, w_2). \tag{28}$$

Since the operator $\mathbf{T}_{\mathbb{D}}$ is linear then in those pairs of spaces $(\mathcal{X}, \mathcal{Y})$ in which $\mathbf{T}_{\mathbb{D}}$ is bounded (and thus continuous) the requirement $(**)$ is simply the condition on the superposition operators $\mathbf{F}_j$. Usually it is enough to suppose that the operators $\mathbf{F}_j$ are Lipschitz-continuous with respect to functional variables and the Lipschitz constant satisfies certain inequality involving the norm of $\mathbf{T}_{\mathbb{D}}$ as an element of $\mathcal{L}(\mathcal{X},\mathcal{Y})$. For $j = 1$ it is quite predictible that the Lipschitz-continuity of the operator $\mathbf{F}_1$ follows from the differentiability of the function F_1 with respect to the second variable. The cases $j = 2, 3, 4$ are less studied and the corresponding conditions on the functions $\mathbf{F}_j, j = 2, 3, 4$, have to be found.

At last the difference between the above discussed case (solution on the whole space $\mathcal{X}$) and another case (solution in a ball $B = B(w_0, R) \subset \mathcal{X}$) is not too essential. First of all one have to choose the starting point of the consideration (the center of a ball B). Then, since we are looking for the solution in the ball we need to check the condition

$$\rho(w, w_0) < R,$$

which means that the operators $\mathbf{P}_j$ possess their values in the ball $B(w_0, R)$. The standard procedure is the following: to find a ball $B(\psi_0, r) \subset \mathcal{Y}$, $\psi_0 = \mathbf{F}_j w_0$ such that

$$\begin{cases} \rho_{\mathcal{Y}}(\mathbf{F}_j w, \psi_0) < r, & \forall w \in B(w_0, R), \\ \rho_{\mathcal{X}}(\mathbf{T}_{\mathbb{D}}\psi, w_0) < r, & \forall \psi \in B(\psi_0, r). \end{cases} \tag{29}$$

Both conditions means boundedness of the corresponding operators. These conditions are described e.g. in [4,12]. A little bit more delicate is to find a variant of Lipschitz condition in a ball. Since we do not know in advance the possible values of the solution then we have to assume the fulfillment of the Lipschitz condition on the set which surely cover the set of all possible values of the solution, e.g. for the operator $\mathbf{F}_1$ this condition has a form

$$\big|F_1(z, w_1) - F_1(z, w_2)\big| \leq L|w_1 - w_2|,$$
$$\forall z \in \mathbb{D},\ \forall w_1, w_2 \in E = \big\{\omega \in \mathbb{C}\colon |\omega| \leq sup_{z\in\overline{\mathbb{D}}}\big|w_0(z)\big| + R\big\}.$$

2.2 Differentiation and implicit functions theorem

The above discussed approach deals mainly with the case of "small nonlinearity" $\mathbf{F}_j$. It means that the Lipschitz constant L have to be small enough. It leads immediately to smallness of $\mathbf{F}_j$ with respect to possible changes of solutions $w(z)$.

In order to study more general situation one can linearize the operators $\mathbf{P}_j$ in one of the following forms

$$\mathbf{P}_j w = A_j w + \widetilde{\mathbf{P}_j} w, \tag{30}$$

or

$$\mathbf{P}_j(w, \overline{w}) = A_j w + B_j \overline{w} + \widetilde{\mathbf{P}_j}(w, \overline{w}), \tag{31}$$

where the operators $\widetilde{\mathbf{P}_j}$ are small in norm with respect to their variables (e.g. $\|\widetilde{\mathbf{P}_j} w\| = o(\|w\|)$ in the case of representation (30)). The representation (30) (or (31)) is valid if the operator $\mathbf{P}_j$ is differentiable with respect to the wariable w (respectively, with respect to both variables w and $\overline{w}$). Both cases are possible but leed to different meaning of the operators $\mathbf{P}_j$. In order to get the representation (30) one needs to suppose that $\mathbf{P}_j$ depends only on z and the functional variable w. These are "complex variable" $\mathbf{P}_j$-operators. In the second case we have to separate the dependence with respect to w and $\overline{w}$. Such situation is quite common in the theory of generalized analytic functions. We deals with "real variable" $\mathbf{P}_j$-operators, i.e. $\mathbf{P}_j$ depends in fact in variables u, v (where $u + iv = w$).

Since the operator $\mathbf{T}_{\mathbb{D}}$ is linear, all we need to discover either (30) or (31) is to get the differentiation of the superposition operators $\mathbf{F}_j$ in one of the above said sense. The corresponding conditions are described in [12, 14].

The most straightforward way to study the problem (23) in this case is to find coefficients $A_j(z), B_j(z)$ explicitly (or describe them as completely as possible), and to inverse then the linearized equation

$$w_{\overline{z}} + A_j(z) w + B_j(z) \overline{w} = C(z, w, \overline{w}), \tag{32}$$

supposing for the moment that the right-hand side C depends solely in z. After that we arrive at the following equation

$$w(\cdot) - \mathbf{Q} C\big(\cdot, w(\cdot), \overline{w}(\cdot)\big) = 0, \tag{33}$$

where $\mathbf{Q}$ is an inverse operator to the left-hand side of (32). The last equation can be also considered as the fixed point equation. Anyway it is quite difficult to formulate the corresponding conditions in terms of the initial functions F_j.

There are some ways to study the operator equations (33) in differentiable case. One of them is connected with application of a form of the Implicit Function Theorem (in fact, we can apply such theorems already to the starting equation (23)). Certain variants of implicit function theorems are presented in the survey [18]. The most suitable for considered equations is the following

Theorem 1. ([18], Thm. 15) *Let $\mathcal{Z}$ be an ideal space such that $\mathcal{Z} \subseteq L_2 \subseteq \mathcal{Z}'$, where $\mathcal{Z}'$ is the associate space to $\mathcal{Z}$.*

Let $F_j\colon \Lambda \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a Caratheodory functions satisfying the following inequality:

$$\big\langle F_j(\lambda, s, u_1) - F_j(\lambda, s, u_2), u_1 - u_2 \big\rangle \geq \gamma |u_1 - u_2|^2 \quad (u_1, u_2 \in \mathbb{R}^n). \quad (34)$$

Denote by $\gamma(F_j)$ the supremum of all constants γ for which inequality (34) *holds.*

Let the linear integral operator $\mathbf{T}_{\mathbb{D}}$ be bounded from $\mathcal{Z}'$ into $\mathcal{Z}$ having finite number of negative eigenvalues of finite multiplicity. Denote by $\varkappa(\mathbf{T}_{\mathbb{D}})$ the smallest of these eigenvalues. Let at last $\varkappa(\mathbf{T}_{\mathbb{D}})\gamma(F_j) < -1$.

Assume also that one of the following two conditions is satisfied:

(i) *the operator $\mathbf{T}_{\mathbb{D}}$ is compact;*

(ii) *the set $\{\mathbf{F}_j(\lambda)x\colon \lambda \in \Lambda, \|x\| \leq r\}$ $(r > 0)$ admits for any $\varepsilon > 0$ a U-bounded ε-net in $\mathcal{Z}'$.*

Then for any sufficiently small $\varepsilon > 0$ one can find a $\delta > 0$ such that for each λ with $|\lambda - \lambda_0| \leq \delta$, equation (19) *has a unique solution $x = x(\lambda, t)$ with $\|x - x_0\| \leq \varepsilon$.*

It has to be noted that the differentiation condition on the functions F_j with respect to (complex!) functional variables is in fact fairly strong and leads immediately to the analyticity of the nonlinear functions F_j. From this point of view it is better to apply the Implicit Function Theorem approach directly to the equation (23).

2.3 Newton-Kantorovich method

In the above described method one needs usually to have a starting point for consideration of the solvability (the center of the suitable ball in the local Banach-Cacciopolli method or the starting point of extension in the case of the implicit function theorem approach). As the procedures which allow us to find such points one can take those of Newton or Newton-Kantorovich method.

The Newton method is applied to the solution of the equation

$$\mathbf{G}x = 0 \tag{35}$$

in a Banach space $\mathcal{X}$ with the operators $\mathbf{G}$ acting from the Banach space $\mathcal{X}$ into another Banach space $\mathcal{Y}$. Its procedure is described by the equation

$$x_{n+1} = x_n - \big[\mathbf{G}'(x_n)\big]^{-1}\big(\mathbf{G}(x_n)\big), \tag{36}$$

with the initial point x_0 chosen under certain conditions. In the case of the Newton-Kantorovich method one have to replace all derivatives $\mathbf{G}'(x_n)$ are changed for the derivative at the initial point x_0.

The standard assumptions of the Newton and Newton-Kantorovich methods are described in the literature (see, e.g., [15, 16, 19]). One of the most essential condition is an existence of the Frechet derivatives of the operator $\mathbf{G}$ in a neighbourhood of a point x_0 and invertibility of the corresponding linear operator.

In the considered case $\mathbf{G} = \mathbf{I} - \mathbf{P}_j = \mathbf{I} - \mathbf{T}_{\mathbb{D}}\mathbf{F}_j$, and such a condition means again the analyticity of the superposition operators with respect to functional variables. It restricts in a sense the applicability of such approach to the arbitrary nonlinear generalizations of the Vekua-Bers type equations.

3 Boundary value problems for nonlinear Vekua-Bers type equations

The above discussed methods deals with the general solution to the nonlinear differential equations (13), (14). The aim of this discussion is either to find an integral representation for a class of the solutions or to show the way how to obtain the general solution of these equations.

The most interesting for applications is to solve the corresponding boundary value problems for the above equations. Let us briefly outline certain results for the solution of boundary value problems for the complex differential equations.

An existence for the Schwarz boundary value problem for the homogeneous equation (9) in the case of Wiener type domains is shown in [20]. The Riemann-Hilbert-type boundary value problem for the same equation (9) but in the case of singular coefficients is studied in [21] (see also [22]).

In [8] by means of the contraction principle are considered two examples from ordinary differential equation and elliptic differential equations in complex form. The right-hand sides of differential equations are supposed to be Lipschitz continuous.

The paper [23] is devoted to the solution in the class of μ, ν-generalized analytic functions of the Cauchy problem

$$\partial_t u = f(t, z, u, \partial_z u),\ u(0, z) = u_0(z).$$

The main idea is to associate this problem with the boundary value problem for linear differential equation and to apply Cauchy-Kovalevski theorem.

Boundary value problems of the conjugation type are studied in [24] for so called generalized q-analytic functions, i.e. the solutions of the equation

$$\partial_{\overline{z}} w - q(z)\partial_z w + A(z)w + B(z)\overline{w} = 0, \quad \big|q(z)\big| \leq q_0 < 1.$$

Existence and uniquenes are established in [25] for nonlinear boundary value problems of Riemann-Hilbert type for generalized analytic functions. Contrary to former investigations the nonlinearity in the boundary conditions need not to be Lipschitz continuous with a small constant.

Conditions of existence of periodic solutions of class $W_p^1(\Omega),\ p > 2$, to the inhomogeneous equation (9) are found in [26]. The solutions are represented via Weierstrass elliptic functions.

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