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On a Fluid Outflow from a Bottle Turned Upside-Down

V. Skakauskas[1], **P. Katauskis**[1], **G. Simeonov**[2]

[1]Vilnius University, 24 Naugarduko str., 03225 Vilnius, Lithuania
vladas.skakauskas@maf.vu.lt; pranas.katauskis@takas.lt

[2]Institute of Mechanics of Bulgarian Academy of Sciences
Bl. 4 Acad. G. Bonchev str., 1113 Sofia, Bulgaria
gsimeonov48@yahoo.co.uk

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Abstract. An incompressible viscous as well as nonviscous fluid outflow from an axially symmetric bottle turned upside-down is considered. This problem relates the gravity acceleration and air bubbles inflow into the bottle and in the mathematical sense presents a very complicated task. The simplified setting of problem based on a one-dimensional approximation of the fluid flow is proposed and results of numerical experiments are discussed.

Keywords: fluid dynamics, outflow, dynamics of dropper.

1 Introduction

In the private life or in a medicine practice (e.g., a fluid outflow from a dropper) we often meet with a fluid outflow from a bottle turned upside-down. Authors do not know any paper related to this essentially nonstationary problem. Numerical modelling of the potential steady flow through a horizontal bottle-neck is given in [1]. Some problems of the one-dimensional nonstationary flow (including a water clock) are solved analytically in [2] (Part 1, Chapter 2, pp. 75–82 and Chapter 4, pp. 124–129).

In the present paper, by using numerical experiments, we consider the outflow of the incompressible nonviscous as well as viscous fluid from an axially symmetric bottle (see Fig. 1). The parameter h_s in Fig. 1 determines the neck beginning point of the bottle.

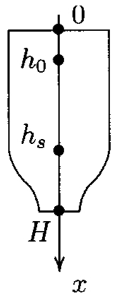

Fig. 1. The axial section of the bottle.

The paper is organized as follows. In Section 2 we consider the outflow of a nonviscous fluid while Section 3 is devoted to the viscous fluid flow. Numerical results are discussed in Section 4 and some remarks in Section 5 conclude the paper.

2 Nonviscous fluid flow

The outflow of a liquid from the bottle turned upside-down relates the air bubbles inflow into the bottle and in the mathematical sense is a very complicated problem. To simplify solving of this problem we use the one-dimensional approximation of the fluid flow [2, 4]. This means [4] that *the axial velocity component* v_x *and pressure* p *are uniform over any cross section normal to the axis of symmetry.* More strictly, we mean a flow in which the rate of change of v_x and p along the normal to the axis of symmetry direction is negligible small compared with the rate of change along the axis x. To describe the flow we use the continuity and momentum equations (see, e.g., [2–4])

$$\operatorname{div} v = 0, \tag{1}$$

$$\partial_t v + v \cdot \nabla v = \widetilde{g} x^0 - \rho^{-1} \nabla p. \tag{2}$$

Here ρ, v, $\widetilde{g}$, p, x^0, t, and ∂_t mean the density, velocity, modulus of the gravity acceleration, pressure, unit vector parallel to the gravity direction, time, and partial derivative, respectively, while div and ∇ mean the divergence and gradient operators.

We *assume* that the fluid occupies the domain lying between the sections $x = h(t)$ and $x = H$ and that *the axial component* v_x *of the vector* v *at the* $x = h(t)$

is equal to h'. Here and in what follows the prime indicates the differentiation.

We *use equation* (1) *only in the integrated over the domain lying between sections* $h(t)$ *and* x *form.* Taking into account the fact that there is no flow through the boundary, $r = f(x)$, of the bottle,(i.e., $v_n\big|_{r=f(x)} = 0$ with v_n being the normal component of the vector v), we get

$$v_x(t, x) = f^{-2}(x) f^2(h) h'. \tag{3}$$

Excluding function v_x from equation (2) written for the component v_x, $\partial_t v_x + v_x \partial_x v_x = \widetilde{g} - \rho^{-1}\partial_x p$, and integrating it over $[h(t), H]$, we derive the equation for h,

$$\begin{gathered} h'' + h'^2 \Big\{ 2f^*(h)/f(h) + \Big(\big(f(h)/f(H) \big)^4 - 1 \Big) \Big(2f^2(h) \int\limits_h^H f^{-2}(x)dx \Big)^{-1} \Big\} \\ = \Big(f^2(h) \int\limits_h^H f^{-2}(x)dx \Big)^{-1} \Big\{ \widetilde{g}(H-h) + \rho^{-1}\big(p(h) - p(H) \big) \Big\}, \\ h(0) = h_0, \quad h'(0) = 0. \end{gathered} \tag{4}$$

Here $f^*(h) = df/dx\big|_{x=h}$.

It remains to get the equation for $p(h)$. Let at the initial moment $t = 0$ the pressure above the fluid in the bottle ($x \in (0, h_0]$) be $p(h_0) = p(H) - \beta\rho\widetilde{g}(H - h_0)$ with a given constant $\beta \in [0, 1)$. From equation (4) it follows that $h''(0) > 0$ and the outflow starts. At the same time the volume above the fluid, $\pi \int\limits_0^h f^2(x)dx$, grows and the air pressure in this volume decreases. We assume that it is uniform over this volume and is equal to $p(h)$. By the Boile–Mariotte law we have

$$p(h) = p(h_0) \int\limits_0^{h_0} f^2(x)dx \Big/ \int\limits_0^h f^2(x)dx. \tag{5}$$

Since $p(h)$ decreases as h grows, the function $\widetilde{g}(H-h) + \rho^{-1}(p(h) - p(H))$ decreases, too, and after some time it and, consequently, h'' become negative. Hence, at a moment $t_1 > 0$, such that $h'(t_1) = 0$, the outflow stops. Since $p(h(t_1)) < p(H)$, the air bubbles begin move into the bottle and, after some time, h'' becomes positive and outflow starts again. In what follows, *we do not take into* *account time necessary for bubbles to come into the bottle by crossing the liquid layer and assume that at the moment t_1 the pressure above the fluid in the bottle changes by the jump*

$$p\big(h(t_1)\big) - p(H) = -\beta\rho\widetilde{g}\big(H - h(t_1)\big).$$

Hence, formula (5) is valid for $h \in [h_0, h_1),\ h_1 = h(t_1),\ h'(t_1) = 0.$

Because the outflow recurs, we will use the formula

$$p(h) = p(h_k)\int_0^{h_k} f^2(x)dx \Big/ \int_0^{h} f^2(x)dx \tag{6}$$

for $h \in [h_k, h_{k+1})$, $h_k = h(t_k),\ h'(t_k) = 0$,

$$p(h_k) = p(H) - \beta\rho\widetilde{g}(H - h_k), \quad \beta \in [0, 1), \quad k = 0, 1, 2, \ldots, \tag{7}$$

where $h_{k+1} < H$.

As a result we have to solve system (4), (6), and (7).

Note that (4) can be written in the form

$$dh'^2/dh + 2a_1 h'^2 = 2a_2,$$

$$a_1(h) = 2f^*(h)/f(h) + \Big(\big(f(h)/f(H)\big)^4 - 1\Big)\Big(2f^2(h)\int_h^H f^{-2}(x)dx\Big)^{-1},$$

$$a_2(h) = \Big(f^2(h)\int_h^H f^{-2}(x)dx\Big)^{-1}\Big\{\widetilde{g}(H - h) + \rho^{-1}\big(p(h) - p(H)\big)\Big\}.$$

Hence,

$$h' = \Big\{2\int_{h_k}^{h} a_2(y)\exp\Big\{-2\int_y^h a_1(x)dx\Big\}dy\Big\}^{1/2}, \quad h \in [h_k, h_{k+1}] \tag{8}$$

$$t = t_k + \int_{h_k}^{h}\Big\{2\int_{h_k}^{z} a_2(y)\exp\Big\{-2\int_y^z a_1(x)dx\Big\}dy\Big\}^{-1/2} dz. \tag{9}$$

From (8) we get the equation for h_{k+1},

$$\int_{h_k}^{h_{k+1}} a_2(y)\exp\Big\{-2\int_y^{h_{k+1}} a_1(x)dx\Big\}dy = 0.$$

Knowing h_{k+1}, from equation (9) we determine t_{k+1}. Since the integral in equation (9) posses a weak singularity at h_{k+1}, we prefer to solve system (4)–(7) by the Runge-Kutta scheme. In all numerical calculations we used the function

$$f(x) = \begin{cases} f(h_s), & 0 \le x \le h_s, \\ f(h_s)\exp\big\{-b(x-h_s)^\gamma\big\}, & x \in [h_s, H] \end{cases}$$

with $b = (H - h_s)^{-\gamma} \ln\big(f(h_s)/f(H)\big)$ and a constant $\gamma > 1$. Numerical results are discussed in Section 4.

3 The viscous fluid flow

In this section we consider a viscous fluid outflow. We *assume that the fluid flow is irrotational, the pressure p is homogeneous over any section normal to the axis x, and the velocity component v_x varies with the radius r according to the relation*

$$v_x(t,x,r) = 2v(t,x)\big(1 - r^2/f^2(x)\big), \tag{10}$$

where the average velocity $v(t,x)$ is defined by the formula

$$v(t,x) = 2f^{-2}(x) \int\limits_0^{f(x)} r v_x(t,x,r)\,dr. \tag{11}$$

Formula (10) means that at each time t the flow is similar to that in the Poiseuile flow (see, e.g., [2, 3]). Note that, because of the cylindrical symmetry of motion,

$$v_r\big|_{r=0} = 0 \tag{12}$$

with v_r being the radial component of vector v. By using these formulas and integrating the continuity equation written in cylindrical coordinates [1],

$$\partial_x r v_x + \partial_r r v_r = 0,$$

we determine the radial component v_r of the velocity v,

$$v_r(x,r) = -r^{-1} \int\limits_0^r \partial_x r v_x dr = -\partial_x\big(v(r - r^3 f^{-2}(x)/2)\big). \tag{13}$$

Assuming that $v(t,h(t)) = h'$, by the same argument as in Section 2, we get

$$2\int_0^{f(x)} r v_x(t,x,r)\,dr = f^2(h)h'$$

and then, by equation (11),

$$v(t,x) = f^{-2}(x)f^2(h)h'. \tag{14}$$

It is easy to see that $v_r = 0$ at the $r = f(x)$.

To get the equation for h we use the momentum equation written in the cylindrical coordinates [2]. More precisely, we use only the equation for the component v_x,

$$\partial_t v_x + v_x\partial_x v_x + v_r\partial_r v_x = \widetilde{g} - \rho^{-1}\partial_x p + \mu\rho^{-1}\big(\partial^2_{xx}v_x + r^{-1}\partial_r(r\partial_r v_x)\big),$$

in the following *average sense*

$$\begin{aligned}\int_h^H 2f^{-2}(x)dx \int_0^{f(x)} r\Big\{&\partial_t v_x + v_x\partial_x v_x + v_r\partial_r v_x - \widetilde{g} + \rho^{-1}\partial_x p \\ &- \mu\rho^{-1}\big(\partial^2_{xx}v_x + r^{-1}\partial_r(r\partial_r v_x)\big)\Big\}dr = 0.\end{aligned} \tag{15}$$

Here μ means the viscosity coefficient. Using equations (10), (13), and (14) and performing simple calculations, we get

$$\begin{aligned}
&\partial_r v_x = -4v(t,x)f^{-2}r,\\
&\partial_x v_x = 2\partial_x v(t,x)(1 - f^{-2}r^2) + 4v(t,x)f^{-3}r^2 f'_x(x),\\
&\int_0^{f(x)} r v_r\partial_r v_x dr = 4f^{-2}(x)v(t,x)\int_0^{f(x)} \partial_x\big(v(r^3 - r^5 f^{-2}/2)\big)dr\\
&\qquad = 4f^{-2}v\int_0^{f(x)} \big\{\partial_x v(r^3 - r^5 f^{-2}/2) + f^{-3}f'_x r^5 v\big\}dr\\
&\qquad = (2/3)vf^{-2}\{f^4\partial_x v + f^3 v f'_x\} = (2/3)\big(\partial_x(vf^2) - vff'_x\big)v\\
&\qquad = -(2/3)v^2 ff'_x = -(2/3)f^{-3}(x)f'_x(x)f^4(h)(h')^2,
\end{aligned}$$

$$\int_0^{f(x)} r\partial_t v_x dr = \partial_t \int_0^{f(x)} 2rv(1-r^2 f^{-2})dr = \partial_t\big(vf^2(x)/2\big) = \big(f^2(h)h'\big)'/2,$$

$$\int_0^{f(x)} r\partial_x(v_x^2/2)dr = \partial_x\Big(\int_0^{f(x)} rv_x^2/2dr\Big)$$
$$= \partial_x \int_0^{f(x)} 2rv^2(1-f^{-2}r^2)^2 dr = (1/3)\partial_x(v^2f^2)$$
$$= -(2/3)f^4(h)(h')^2 f^{-3}(x) f'_x(x),$$

$$\int_0^{f(x)} r\big(\partial_{xx}^2 v_x + (1/r)\partial_r(r\partial_r v_x)\big)dr = -4v + \partial_x\int_0^{f(x)} r\partial_x v_x dr - f' f\partial_x v_x\big|_{r=f(x)}$$
$$= -4v\big(1+(f')^2\big) + (1/2)\partial_{xx}^2(vf^2) = -4v(t,x)\big(1+\big(f'(x)\big)^2\big)$$
$$= -4f^{-2}(x)f^2(h)h'\big(1+\big(f'(x)\big)^2\big),$$

$$\int_0^{f(x)} r\big(g-(1/\rho)\partial_x p\big)dr = (1/2)f^2(x)\big(\widetilde{g}-(1/\rho)\partial_x p\big).$$

Gathering these equalities and using equation (15) after simple calculations, we get the equation

$$h'' + h'^2\Big\{2f^*(h)/f(h) + 2\big(\big(f(h)/f(H)\big)^4 - 1\big)\Big(3f^2(h)\int_h^H f^{-2}(x)dx\Big)^{-1}\Big\}$$
$$= \Big(f^2(h)\int_h^H f^{-2}(x)dx\Big)^{-1}\Big\{\widetilde{g}(H-h) + \rho^{-1}\big(p(h)-p(H)\big) \qquad (16)$$
$$- 8\mu\rho^{-1}f^2(h)h'\int_h^H f^{-4}(x)\big(1+\big(f^*(x)\big)^2\big)dx\Big\},$$
$$h(0) = h_0, \quad h'(0) = 0$$

with $f^* = df/dx\big|_{x=h}$. We add equations (6) and (7) for pressure $p(h)$. System (6), (7), and (16) were solved numerically by the Runge-Kutta scheme. Numerical results are exhibited in Section 4.

4 Numerical results

Systems (4)–(7) and (6), (7), and (16) were solved numerically by the fourth order Runge–Kutta scheme. The basic time step is $\Delta t = 0.001$. Near the points $t_k,\ k = 1, 2, \ldots, \Delta t$ is reduced until the condition $|h'(t_k + \Delta t)| < 0.01,\ k = 0, 1, 2, \ldots$, be satisfied.

Equations (4) and (16) show that the bottle wall resistance includes the term which in the case of the viscous fluid flow is 4/3 times larger than that for the nonviscous fluid flow.

In what follows we use $\mu = 0$ for nonviscous liquid, $\mu = 0.018$ for water at the temperature $30\,^{\circ}$C, and $\mu = 13.93$ for glycerine at the temperature $18\,^{\circ}$C that are measured g/(s cm). Parameters $H, h_s, h_0, f(H), f(h_s)$ are measured in centimeters, $p(H) = 1.01325\ 10^6 g/(s^2 cm)$, and $\gamma = 1.1$.

Figs. 2–5 exhibit the outcoming mass rate $Q = \pi h' f^2(h)$ for $H = 30$, $h_s = 20,\ f(H) = 1.5,\ f(0) = f(h_s) = 5$, and the same volume of liquid 1650.131 cm^3. Figs. 2, 4, and 5 illustrate Q versus h, while Fig. 3 represent Q versus time t. Figures show that Q possess a maximal value at some time t^* and some position h^*. This position decreases when viscosity coefficient μ increases, e.g., $\max Q = 599.27$ at $h^* = 10.469$ (Fig. 2), $\max Q = 577.44$ at $h^* = 9.875$ (Fig. 4) and $\max Q = 439.1$ at $h^* = 8.307$ (Fig. 5). Time t^* posses the similar behavior. Fig. 3 shows that $\max Q = 577.44$ at $t^* = 1.593$. We can also observe the decrease of Q as μ increases.

Figs. 6–9 illustrate volume of drops $V_k = \int\limits_{t_{k-1}}^{t_k} Q(t)\,dt$ versus t_k such that $h'(t_k) = 0$ for $k = 1, 2, \ldots$.

Fig. 6 represents graphs of V_k for the solution of (4)–(7). We see that V_k for $\beta = 0.1$ is less then that for the nonviscous liquid until some k_1. But, when $k > k_1$, V_k behaves vice versa. Time of the total outflow increases together with β.

Fig. 7 illustrates the behavior of V_k for nonviscous liquid flow. Graphs show that, for fixed $f(H)$, V_k increases together with h_s until some value k_2 and behaves vice versa as $k > k_2$. Time of the total outflow is smaller for the bottle with greater h_s and the same $f(H)$. We can also see that, for fixed h_s, V_k grows together with $f(H)$ until some k_3 and behaves vice versa as $k > k_3$. For fixed h_s, time of the total outflow increases as $f(H)$ decreases.

Fig. 8 illustrates the behavior of V_k for $\mu = 0.018$. The behavior is similar to that exhibited in Fig. 7.

Fig. 9 represents the behavior of V_k for the nonviscous fluid and liquid with $\mu = 0.018$ and 13.93. We see that V_k decreases as μ increases until some $k_4(\mu)$ and behaves vice versa for $k > k_4(\mu)$.

Fig. 10 exhibits the behavior of the span $w_k = t_k - t_{k-1}$ for $\mu = 0.018$. Graphs show that w_k possess a maximal value at some k_5 depending on the bottle geometry. For fixed $f(H)$, $\max_k w_k$ grows with h_s increasing. For fixed h_s, it increases as $f(H)$ decreases.

Note that every fixed $(t_k, h_k, p(h_k))$ with $p(h_k)$ given by formula (7) may be treated as a initial datum. The influence of this datum on the behavior of Q for $t > t_k (h > h_k)$ can be examined by Figs. 2, 4, and 5, too. These figures show that Q gets a maximal value at the $h = h_0$ and $Q(h_0) < Q(h^*)$ for $h_0 > h^*$.

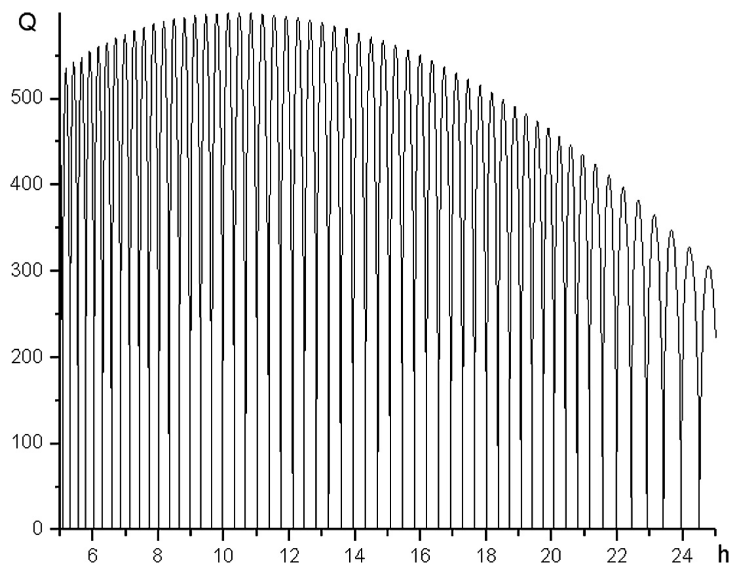

Fig. 2. The graph of Q versus h for the solution of (4)–(7) with $\beta = 0$ and $H = 30, h_s = 20, h_0 = 3, f(0) = 5, f(H) = 1.5$.

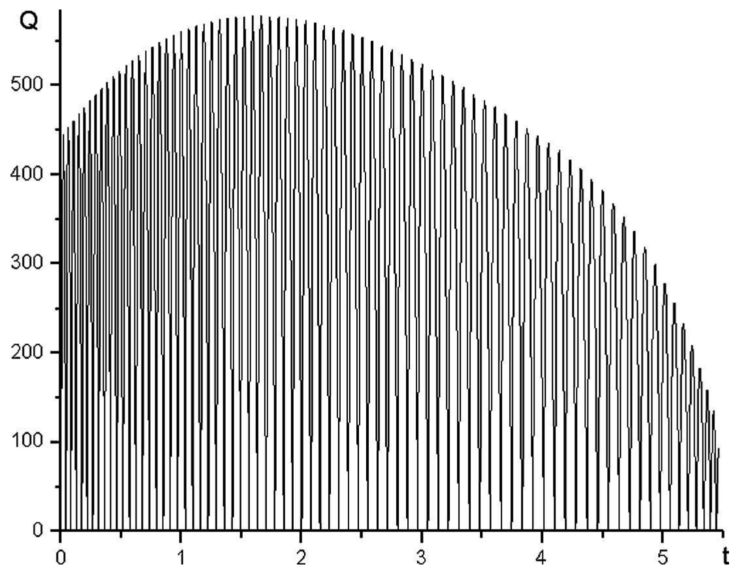

Fig. 3. The graph of Q versus t for the solution of (6),(7), and (16) with $\beta, H, h_s, f(0), f(H), h_0$ as in Fig. 3 and $\mu = 0.018$.

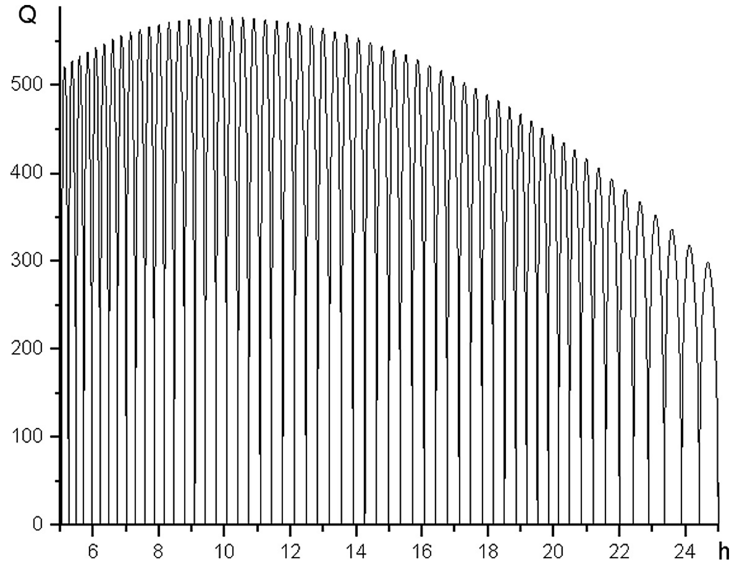

Fig. 4. The graph of Q versus h for the solution of (6),(7), and (16) with $\beta, H, h_s, f(0), f(H), h_0$, and μ as in Fig. 3.

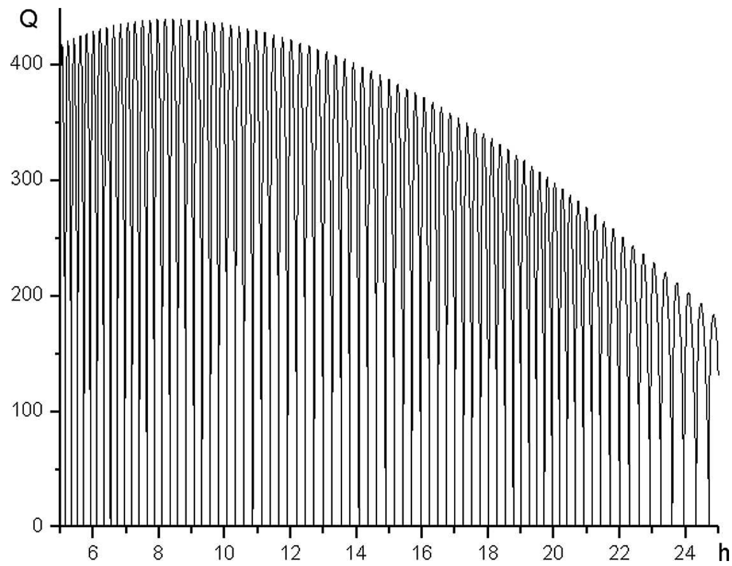

Fig. 5. The graph of Q versus h for the solution of (6), (7), and (16) with $\mu = 13.93$ and $\beta, H, h_s, f(0), f(H)$, and h_0 as in Fig. 3.

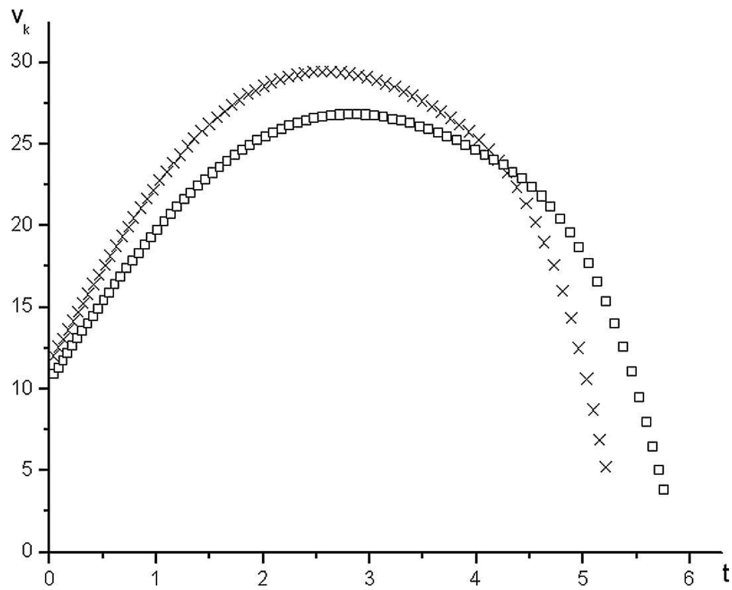

Fig. 6. Graphs of volume V_k of drops for the solution of (4)–(7) with $H, h_s, h_0, f(0)$, and $f(H)$ as in Fig. 2. $\times$ — $\beta = 0$, $\square$ — $\beta = 0.1$

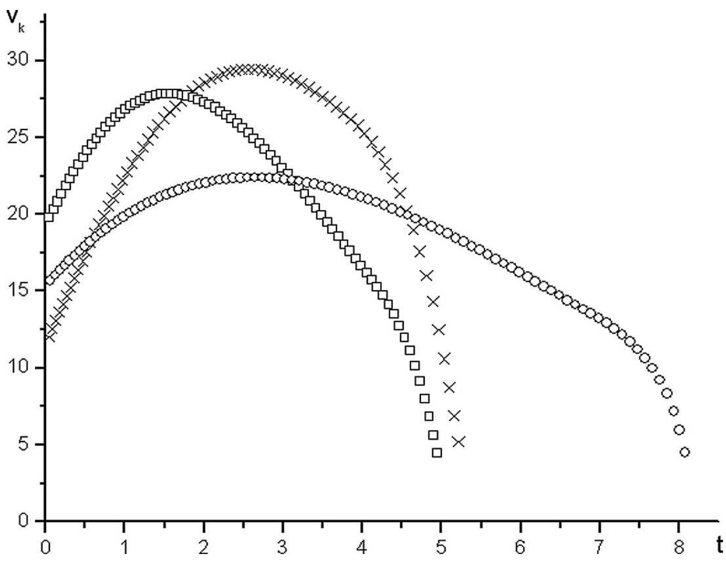

Fig. 7. Graphs of V_k for the solution of (4)–(7) with β, H, and $f(0)$ as in Fig. 2. $\times$ — $(h_s = 20, f(H) = 1.5, h_0 = 3)$, $\square$ — $(h_s = 25, f(H) = 1.5, h_0 = 5.99)$, and $\circ$ — $(h_s = 25, f(H) = 1, h_0 = 5.6)$.

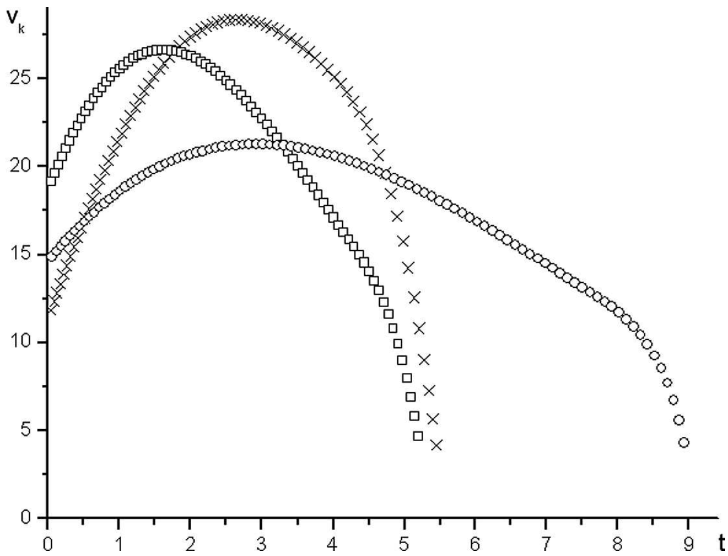

Fig. 8. Graphs of V_k for the solution of (6),(7), and (16) with β, H, and $f(0)$ as in Fig. 2 and μ as in Fig. 3. $\times$ — $(h_s = 20, f(H) = 1.5, h_0 = 3)$, $\square$ — $(h_s = 25, f(H) = 1.5, h_0 = 5.99)$, and $\circ$ — $(h_s = 25, f(H) = 1, h_0 = 5.6)$.

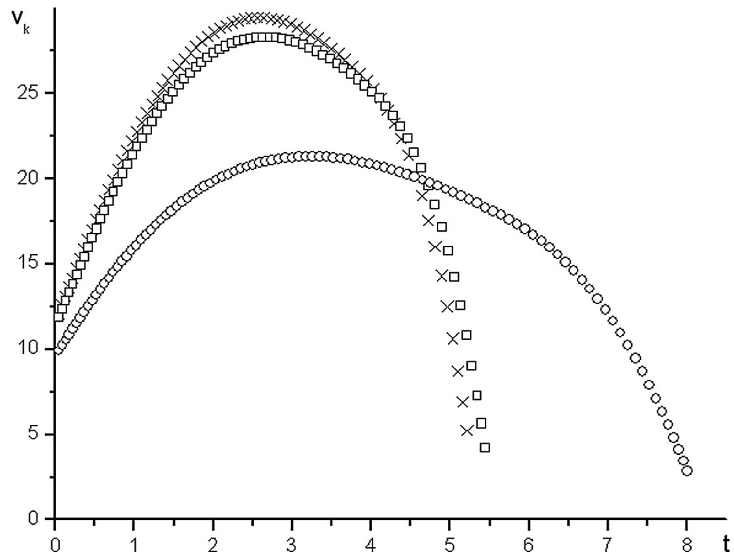

Fig. 9. Graphs of V_k for the solution of (4)–(7) (mark $\times$) and for the solution of (6), (7), and (16) (mark $\square$ for $\mu = 0.018$ and $\circ$ for $\mu = 13.93$) with $\beta, H, h_s, f(0), f(H), h_0$ as in Fig. 2.

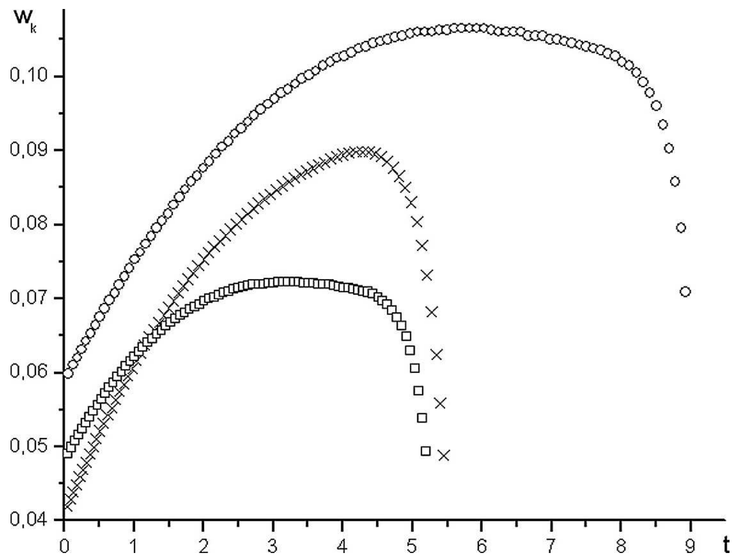

Fig. 10. Graphs of W_k for the solution of (6), (7), and (16) with $\mu = 0.018$, $H = 30, f(0) = 5$. $\times$ — $(h_s = 20, f(H) = 1.5, h_0 = 3)$, $\square$ — $(h_s = 25, f(H) = 1.5, h_0 = 5.99)$, $\circ$ — $(h_s = 25, f(H) = 1, h_0 = 5.6)$.

5 Concluding remarks

The problem of a fluid outflow from a bottle turned upside-down causes the air bubbles inflow into the bottle and therefore forms a very complicated problem in the mathematical sense. The numerical investigation in a simplified setting of this problem is given. For simplification, it *is assumed that:*

(i) the velocity component v_x and pressure p for the nonviscous fluid flow are uniform over any cross section normal to the axis of symmetry and the continuity equation is valid only in the integrated over the liquid layer form;

(ii) in the viscous liquid case, only pressure p is uniform over any cross section normal to the axis of symmetry, the Poiseuile type formula (10) is valid for the component v_x, and the momentum equation for v_x is valid in the average form (15);

(iii) $v_x(t, h(t)) = h'$ and $v(t, h(t)) = h'$ for nonviscous and viscous fluid flow, respectively;

(iv) the pressure p varies by a jump (see formula (6)) at the points t_k such that $h'(t_k) = 0$. Of course, it is possible to fix a constant bubbles inflow velocity v_b and ascribe the value $p(t_k)$ defined by equation (7) for the moment $\widetilde{t}_k = t_k + (H - h_k)/v_b$. Then using the linear approximation of $p(t)$ for $t \in (t_k, \widetilde{t}_k)$, we get a continuous function p. But this modification does not change essentially the solution of differential equations.

Parameter β in formula (7) shows how many times the difference between the external pressure, $p(H)$, and the pressure, $p(h_k)$, above the liquid layer in the bottle at the moment t_k conditioned by the bubbles inflow is less than the weight, $\rho\widetilde{g}(H - h_k)$, of the liquid layer.

Numerical results show that the outcoming mass rate, Q, volume of drops, V_k, and the span of drops, w_k, possess the maximal values which depend on the bottle geometry and liquid viscosity. The total outflow time depends on the bottle geometry and liquid viscosity, as well.

References

1. Comsol. Flow through a bottle-neck, `http://www.comsol.se/products/pde/demos/bottle.php`.

2. N. E. Kochin, I. A. Kibel, N. V. Roze, *Theoretical hydromechanics*, Parts 1, 2, Fizmatgiz, Moscow, 1963 (in Russian).

3. H. Schlichting, *Grenzschicht-Theorie*, Verlag G. Braun, Karsruhe, 1965.

4. A. H. Shapiro, *The dynamics and thermodynamics of compressible fluid flow*, Vol. 1, The Ronald Press Company, New York, 1953.