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On Positive Solutions for Some Nonlinear Semipositone Elliptic Boundary Value Problems

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Abstract. This study concerns the existence of positive solutions to classes of boundary value problems of the form

$$
\begin{aligned}
-\Delta u &= g(x,u), && x \in \Omega,\\
u(x) &= 0, && x \in \partial\Omega,
\end{aligned}
$$

where Δ denote the Laplacian operator, Ω is a smooth bounded domain in R^N ($N \geq 2$) with $\partial\Omega$ of class C^2, and connected, and $g(x,0) < 0$ for some $x \in \Omega$ (semipositone problems). By using the method of sub-super solutions we prove the existence of positive solution to special types of $g(x,u)$.

Keywords: positive solutions, sub-super solution.

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1 Introduction

In this paper we consider the existence of positive solution to boundary value problems of the form

$$
\begin{aligned}
-\Delta u &= g(x,u), && x \in \Omega,\\
u(x) &= 0, && x \in \partial\Omega,
\end{aligned}
\tag{1}
$$

where Δ denote the Laplacian operator, Ω is a smooth bounded domain in R^N ($N \geq 2$) with $\partial\Omega$ of class C^2, and connected, and $g(x,0) < 0$ for some $x \in \Omega$ (semipositone problems). In particular, we first study the case when $g(x,u) =$

$a(x)\ u - b(x)\ u^2 - ch(x)$, where $a(x), b(x)$ are $C^1(\bar{\Omega})$ functions that $a(x)$ is allowed to be negative near the boundary of Ω, and $b(x) > b_0 > 0$ for $x \in \Omega$. Here $h \colon \bar{\Omega} \to R$ is a $C^1(\bar{\Omega})$ function satisfying $h(x) \geq 0$ for $x \in \Omega$, $h(x) \not\equiv 0$, and $\max_{x\in\bar{\Omega}} h(x) = 1$. We prove that there exists a $c_0 = c_0(\Omega, a, b) > 0$ such that for $0 < c < c_0$ there exists a positive solution.

The above equation arises in the studies of population biology of one species with u representing the concentration of the species or the population density, and $ch(x)$ representing the rate of harvesting (see [1]). The case when $a(x), b(x)$ are positive constants throughout $\bar{\Omega}$, has been studied in [1]. In [2] the authors studied the case when $c = 0$ (non-harvesting case), $b(x) \equiv 1$ for $\bar{\Omega}$ and $a(x)$ is apositive function throughout $\bar{\Omega}$. However the $c > 0$ case is a semipositone problem ($g(x, 0) < 0$) and studying positive solutions in this case is significantly harder. Here we consider the challenging semipositone case $c > 0$. Semipositone problems have been of great interest during the past two decades, and continue to pose mathematically difficult problems in the study of positive solutions (see [3–6]).

We next study the case when $g(x, u) = \lambda m(x) f(u)$, where the weight m satisfying $m \in C(\Omega)$ and $m(x) \geq m_0 > 0$ for $x \in \Omega$, $f \in C^1[0, \rho)$ is a nondecreasing function for some $\rho > 0$ such that $f(0) < 0$ and there exist $\alpha \in (0, \rho)$ such that $f(t)(t - \alpha) \geq 0$ for $t \in [0, \rho]$.

See [7] where positive solution is obtained for large λ when $m(x) \equiv 1$ for $x \in \Omega$ and f is sublinear at infinity. We are interested in the existence of a positive solution in a range of λ without assuming any condition on f at infinity. Our approach is based on the method of sub-super solutions, see [2, 8].

2 Existence results

We first give the definition of sub-super solution of (1). A super solution to (1) is defined as a function $z \in C^2(\bar{\Omega})$ such that

$$\begin{aligned} -\Delta z &\geq \lambda g(x, z), && x \in \Omega, \\ z &\geq 0, && x \in \partial\Omega. \end{aligned}$$

Sub solutions are defined similarly with the inequalities reversed and it is well known that if there exists a sub solution ψ and a super solution z to (1) such that

$\psi(x) \leq z(x)$ for $x \in \bar{\Omega}$, then (1) has a solution u such that $\psi(x) \leq u(x) \leq z(x)$ for $x \in \bar{\Omega}$. Further note that if $\psi(x) \geq 0$ for $x \in \Omega$ then $u \geq 0$ for $x \in \Omega$.

To precisely state our existence result we consider the eigenvalue problem

$$\begin{aligned} -\Delta\phi &= \lambda\phi, \qquad x \in \Omega, \\ \phi &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{2}$$

Let $\phi_1 \in C^1(\bar{\Omega})$ be the eigenfunction corresponding to the first eigenvalue λ_1 of (3) such that $\phi_1(x) > 0$ in Ω, and $\|\phi_1\|_\infty = 1$. It can be shown that $\frac{\partial\phi_1}{\partial n} < 0$ on $\partial\Omega$. Here n is the outward normal. This result is well known (see, e.g., [9]), and hence, depending on Ω, there exist positive constants k, η, μ such that

$$\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \leq -k, \quad x \in \bar{\Omega}_\eta, \tag{3}$$

$$\phi_1 \geq \mu, \quad x \in \Omega_0 = \Omega \setminus \bar{\Omega}_\eta, \tag{4}$$

with $\bar{\Omega}_\eta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \eta\}$. Further assume that there exists a constants $a_0, a_1 > 0$ such that $a(x) \geq -a_0$ in $\bar{\Omega}_\eta$ and $a(x) \geq a_1$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$.

We will also consider the unique solution, $\zeta \in C^1(\bar{\Omega})$, of the boundary value problem

$$\begin{aligned} -\Delta\zeta &= 1, \qquad x \in \Omega, \\ \zeta &= 0, \qquad x \in \partial\Omega, \end{aligned}$$

to discuss our existence result. It is known that $\zeta > 0$ in Ω and $\frac{\partial\zeta}{\partial n} < 0$ on $\partial\Omega$.

First we obtain the existence of positive solution of (1) in the case when $g(x,u) = a(x)u - b(x)u^2 - ch(x)$.

Theorem 1. *Suppose that* $a_0 < 2k$ *and* $2\lambda_1 < a_1\mu^2$. *Then there exists* $c_0 = c_0(\Omega, a_0, a_1, b) > 0$ *such that if* $0 < c < c_0$ *then the problem* (1) *has a positive solution* u.

Proof. To obtain the existence of positive solution to problem (1) we constructing a positive subsolution ψ and supersolution z. We shall verify that $\psi = \delta\phi_1^2$ is a subsolution of (1), where $\delta > 0$ is small and specified later (note that $\|\psi\|_\infty \leq \delta$). Since $\nabla\psi = 2\delta\phi_1\nabla\phi_1$, a calculation shows that

$$-\Delta\psi = -\delta\Delta\phi_1^2 = -2\delta\big(|\nabla\phi_1|^2 + \phi_1\Delta\phi_1\big) = 2\delta\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big).$$

Then ψ is a subsolution if

$$2\delta\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big) \leq a(x)\psi - b(x)\psi^2 - ch(x),$$

Now $\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \leq -k$ in $\bar{\Omega}_\eta$, and therefore

$$2\delta\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big) \leq -2k\delta \leq -a_0\delta - \|b\|_\infty\delta^2 - c,$$

if

$$\delta < \theta_1 = \frac{2k - a_0}{\|b\|_\infty},$$
$$c \leq \widehat{c}(\delta) = \delta\big(2k - a_0 - \|b\|_\infty\delta\big).$$

Clearly $\widehat{c}(\delta) > 0$.

Furthermore, we note that $\phi_1 \geq \mu > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, also in Ω_0 we have

$$2\delta\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big) \leq 2\lambda_1\delta \leq a_1\delta\phi_1^2 - \|b\|_\infty\delta^2 - c,$$

if

$$\delta < \theta_2 = \frac{a_1\mu^2 - 2\lambda_1}{\|b\|_\infty},$$
$$c \leq \bar{c}(\delta) = \delta\big(a_1\mu^2 - 2\lambda_1 - \|b\|_\infty\delta\big).$$

Clearly $\bar{c}(\delta) > 0$. Choose $\theta = \min\{\theta_1, \theta_2\}$ and $\delta = \theta/2$. Then simplifying, both $\widehat{c}$ and $\bar{c}$ are greater than $(\frac{\theta}{2})^2\|b\|_\infty$. Hence if $c \leq (\frac{\theta}{2})^2\|b\|_\infty = c_0(\Omega, a_0, a_1, b)$ then ψ is a subsolution.

Next, we construct a supersolution z of (1). We denote $z = N\zeta(x)$, where the constant $N > 0$ is large and to be chosen later. We shall verify that z is a supersolution of (1). A calculation shows that

$$-\Delta z = N(-\Delta\zeta) = N.$$

Thus z is a supersolution if

$$N \geq a(x)z - b(x)z^2 - ch(x),$$

and therefore if $N \geq N_0$ where $N_0 = \sup_{[0,\|a\|_\infty/b_0]}(\|a\|_\infty v - b_0 v^2)$, we have

$$-\Delta z \geq a(x)z - b(x)z^2 - ch(x),$$

and hence z is supersolution of (1). Since $\zeta > 0$ and $\partial\zeta/\partial n < 0$ on $\partial\Omega$, we can choose N large enough so that $\psi \le z$ is also satisfied. Hence Theorem 1 is proven. □

Now, we obtain the existence of positive solution of (1) in the case when $g(x,u) = \lambda m(x) f(u)$. Assume that there exist positive constants $r_1, r_2 \in (\alpha, \rho]$ satisfying:

(H.1) $\dfrac{r_2}{r_1} \ge \max\Big\{\dfrac{2\lambda_1\|\zeta\|_\infty}{\mu^2}, \dfrac{2\lambda_1\|\zeta\|_\infty\|m\|_\infty f(r_2)}{m_0\mu^2 f(r_1)}\Big\}$,

(H.2) $kf(r_1) > \lambda_1\big|f(0)\big|$.

Theorem 2. *Let* (H.1), (H.2) *hold. Then there exist* $\lambda_* < \widetilde{\lambda}$ *such that* (1) *has a positive solution for* $\lambda \in [\lambda_*, \widetilde{\lambda}]$.

Proof. Let $\lambda_1, \phi_1, k, \mu$ and $\zeta(x)$ are the same as in the proof of Theorem 1. We now construct our positive subsolution. Let $\psi = r_1(\phi_1/\mu)^2$. Using a calculation similar to the one in the proof of Theorem 1, we have

$$-\Delta\psi = \frac{2r_1}{\mu^2}\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big). \tag{5}$$

Thus ψ is a subsolution if

$$\frac{2r_1}{\mu^2}\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big) \le \lambda m(x) f(\psi),$$

Now $\lambda_1\phi_1^2 - |\nabla\phi_1|^2 \le -k$ in $\bar{\Omega}_\eta$, and therefore

$$\frac{2r_1}{\mu^2}\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big) \le -\frac{2kr_1}{\mu^2} \le \lambda m(x) f(\psi),$$

if

$$\lambda \le \widehat{\lambda} = \frac{2kr_1}{\mu^2 m_0\big|f(0)\big|}.$$

Furthermore, we note that $\phi_1 \ge \mu > 0$ in $\Omega_0 = \Omega \setminus \bar{\Omega}_\eta$, and therefore

$$\psi = r_1(\phi_1/\mu)^2 \ge r_1(\mu/\mu)^2 = r_1,$$

thus $f(\psi) \geq f(r_1)$. Hence if

$$\lambda \geq \lambda_* = \frac{2\lambda_1 r_1}{\mu^2 m_0 f(r_1)},$$

we have

$$\frac{2r_1}{\mu^2}\big(\lambda_1\phi_1^2 - |\nabla\phi_1|^2\big) \leq \frac{2\lambda_1 r_1}{\mu^2} \leq \lambda m_0 f(r_1) \leq \lambda m(x) f(\psi).$$

We get $\lambda_* < \widehat{\lambda}$ by using (H.2). Therefore if $\lambda_* \leq \lambda \leq \widehat{\lambda}$, then ψ is subsolution.

Next, we construct a supersolution z of (1) such that $z \geq \psi$. We denote $z = \frac{r_2}{\|\zeta\|_\infty}\zeta(x)$. We shall verify that z is a super solution of (1). We have

$$-\Delta z = \frac{r_2}{\|\zeta\|_\infty}. \tag{6}$$

Thus z is a super solution if

$$\frac{r_2}{\|\zeta\|_\infty} \geq \lambda m(x) f(z).$$

But $f(z) \leq f(r_2)$ and hence z is a super solution if

$$\lambda \leq \bar{\lambda} = \frac{r_2}{\|\zeta\|_\infty \|m\|_\infty f(r_2)}.$$

We easily see that $\lambda_* < \bar{\lambda}$, by using (H.1). Finally, using (5), (6) and the comparison principle, we see that $\psi \leq z$ in Ω when (H.1) is satisfied. Therefore (1) has a positive solution for $\lambda \in [\lambda_*, \widetilde{\lambda}]$, where $\widetilde{\lambda} = \min\{\widehat{\lambda}, \bar{\lambda}\}$. This completes the proof of Theorem 2. □

Remark 1. *Theorem* 2 *holds no matter what the growth condition of f is, for large u. Namely, f could satisfy superlinear, sublinear or linear growth condition at infinity.*

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