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Asymptotic Stability, Orbital Stability of Hopf-Bifurcating Periodic Solution of a Simple Three-Neuron Artificial Neural Network with Distributed Delay*

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Abstract. A distributed delay model of a class of three-neuron network has been investigated. Sufficient conditions for existence of unique equilibrium, multiple equilibria and their local stability are derived. A closed interval for a parameter of the system is identified in which Hopf-bifurcating periodic solution occurs for each point of such interval. The orbital stability of such bifurcating periodic solution at the extreme points of the interval is ascertained. Lastly global bifurcation aspect of such periodic solutions is studied. The results are illustrated by numerical simulations.

Keywords: distributed delay, orbital stability, supercritical Hopf-bifurcation,snake of orbits.

1 Introduction

It is wellknown that human brain is made up of a large number of cells called neurons and their interaction. An artificial neural network is an information processing system that has certain characteristics in common with biological neural networks. In recent years, Hopfield neural network and their various generalisations have attracted the attention of many scientists due to their potential applications in areas such as classification, associative memory, pattern recognization, parallel computations, optimization [1–9].

In modelling artificial neural networks, it is necessary to incorporate the processing time of each neuron to make the models more realistic. Marcus and Westervelt [3] first introduced a single delay in Hopfield model and showed that symmetrically connected continuous time network oscillates as the delay crosses a critical value. There is an extensive research work on the dynamics of Hopfield neural networks with time delays [10–17].

The periodic nature of natural impulses is of fundamental significance in the control of regular dynamical functions such as breathing and heart beating. Neural networks

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involving persistent oscillations such as limit cycle may be applied to pattern recognization and associative memory. Thus it is important to understand the mechanism of neural networks that cause such periodic activities. However, neural networks are complex and large-scale nonlinear dynamical systems. But in most of the cases for simplicity,simple systems are studied. This is still useful, since complexity study may be carried over to large scale networks.

Several papers [18–21] are devoted to the existence and stability of periodic solutions of neural network models with two neurons with discrete delay. Olien and Belair [15] studeid stability and existence of Hopf-bifurcation of a two neuron system with discrete delay. Gopalsamy and Leung [16], N. C. Majee and A. B.Roy [17], Wei and Ruan [22], Liao, Wong, Wu [23] have investigated two neuron network models with transmission delay. Due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths, neural networks usually have a spatial nature. So it is desirable to model them by introducing distributed delays. S. Bernard, J. Belair, M. C. Mackey [24] studied the effect of distributed delay on stability condition of linear delay differential equations. G. S. K. Wolkowicz, S.Ruan [25,26] analyzed the global asymptotic behaviour and bifurcation aspect of a chemostat model with distributed delay. Gopalsamy and He [27], Sree Hari Rao, Phaneendra and Prameela [28] have studied a lot about the effect of distributed delays on neural network. Chunhua Feng, Réjean Plamandon [29] have investigated the stability of delayed neural networks including both discrete and distributed delay. Two Neuron systems with distributed delay have been studied in papers [30, 31]. Three neuron network with discrete delays has been studied in [32–35].

In this paper we have investigated local asymptotic stability, orbital stability of Hopf bifurcating periodic solution,existence of global bifurcation for a special kind of three-neuron model with distributed delay. In Section 2 the model is presented. In Section 3 some preliminary results about the existence of unique and multiple equilibria are shown. In Section 4 the system of integrodifferential equations has been converted to a system of six ordinary differential equations with the introduction of three auxiliary variables, then its linear analysis has been done. Routh-Hurwitz criteria have been applied to derive the sufficient condition (depending on synaptic weight and decay rate of kernel) for local asymptotic stability of its equilibria. In Section 5 considering the decay rate of kernel as bifurcating parameter, a closed interval of such decay rate has been obtained in which Hopf-bifurcation occurs. In Section 6 orbital stability of this Hopf-bifurcating periodic solution has been analyzed following the subcritical and supercritical approach proposed by Poore. In Section 7, existence of global Hopf-bifurcation has been studied. Most of the mathematical calculations are done by using symbolic computation with the help of software MATHEMATICA. Numerical simulation of the model that confirms the results obtained analytically is presented in Section 8. Finally a conclusion has been drawn in Section 9.

2 Model description

In this paper we consider a three neuron network with distributed delay described by following system of integrodifferential equations

$$\frac{dx_i}{dt} = -p\,x_i(t) + \sum_{j=1}^{3} a_{ij} \tanh\left[\int_{-\infty}^{t} k(t-s)\,x_j(s)ds\right], \quad i = 1, 2, 3, \tag{1}$$

where $p > 0$ is the decay rate of neurons, that is, it represents the rate with which a neuron will reset to its potential to the resisting state in isolation when disconnected from network. a_{ij} is the the weight of synaptic connections from neuron j to neuron i and k is the delay kernel assumed to satisfy the following conditions:

$$\begin{aligned} &\text{(i)} \quad k\colon [0,\infty) \to [0,\infty);\\ &\text{(ii)} \quad k \text{ is piecewise continuous;}\\ &\text{(iii)} \quad \int_0^\infty k(s)ds = 1, \quad \int_0^\infty sk(s)ds < \infty. \end{aligned} \tag{2}$$

It is assumed that system (1) is provided with initial conditions:

$$x_i(s) = \psi_i(s), \quad s \in (-\infty, 0] \quad \text{and}$$
$$\psi_i \text{ is bounded and continuous in } (-\infty, 0], \quad i = 1, 2, 3.$$

The general form of delay kernel $k(s)$ is as follows:

$$k(s) = \beta^{n+1} \frac{s^n\, e^{-\beta s}}{n!}, \quad s \in (0, \infty), \quad n = 0, 1, 2,$$

where β is a parameter denoting the rate of decay of the effects of past memories and it is a positive real number. It is also known as exponentially fading memory.

$n = 0$ represents weak kernel, whereas $n = 1$ represents strong kernel. In this paper we have studied the effect of weak kernel only, that is k is of the form:

$$k(s) = \beta e^{-\beta s}, \quad s \in (0, \infty). \tag{3}$$

Now (1) takes the form

$$\frac{dx_i}{dt} = -p\,x_i(t) + \sum_{j=1}^{3} a_{ij} \tanh\left[\beta \int_{-\infty}^{t} e^{-\beta(t-s)} x_j(s)ds\right], \quad t > 0, \quad i = 1, 2, 3. \tag{4}$$

Due to the complexity of system (4), a special case of general model has been considered here to make the calculation more tractable. We consider here a three-neuron model, in which there is no self-connection and there is a pair of neurons, which are not directly connected (shown in Fig. 1). To be specific the decay rate is taken as unity and synaptic weights are as follows:

$$p = 1, \;\; a_{11} = a_{22} = a_{33} = 0, \;\; a_{12} = a_{13} = b, \;\; a_{21} = a_{31} = a, \;\; a_{23} = a_{32} = 0.$$

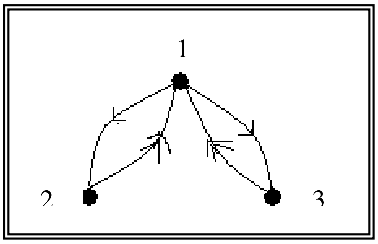


Fig. 1. Three-neuron network model.

Now (4) assumes the form

$$\begin{aligned}\frac{dx}{dt} &= -x(t) + b\tanh\left\{\beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}y(s)ds\right\} + b\tanh\left\{\beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}z(s)ds\right\},\\ \frac{dy}{dt} &= -y(t) + a\tanh\left\{\beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}x(s)ds\right\},\\ \frac{dz}{dt} &= -z(t) + a\tanh\left\{\beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}x(s)ds\right\}.\end{aligned} \tag{5}$$

3 Some preliminary results

Fargue (1973) has shown that if the kernel function is a solution of a homogeneous linear differential equation with constant coefficients, that is, if it is a polynomial multiplied by an exponential function, the integrodifferential equation is equivalent to a system of ordinary differential equations of higher dimension [36].

In this section system (5) of integrodifferential equations has been converted to a set of ordinary differential equations with the help of three auxiliary variables defined by

$$\begin{aligned}u(t) &= \beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}x(s)ds,\\ v(t) &= \beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}y(s)ds,\\ w(t) &= \beta\int\limits_{-\infty}^{t} e^{-\beta(t-s)}z(s)ds, \quad t > 0.\end{aligned} \tag{6}$$

Now combining (5) and (6) it can be written as

$$\begin{aligned}
\frac{dx}{dt} &= -\,x(t) + b\tanh[v(t)] + b\tanh[w(t)],\\
\frac{dy}{dt} &= -\,y(t) + a\tanh[u(t)],\\
\frac{dz}{dt} &= -\,z(t) + a\tanh[u(t)],\\
\frac{du}{dt} &= \beta\big(x(t) - u(t)\big),\\
\frac{dv}{dt} &= \beta\big(y(t) - v(t)\big),\\
\frac{dw}{dt} &= \beta\big(z(t) - w(t)\big), \quad t > 0.
\end{aligned} \tag{7}$$

Now the system (7) can be rewritten in the following form (only for notational, and computational simplicity):

$$\begin{aligned}
\frac{dx_1}{dt} &= -x_1 + b\tanh(x_5) + b\tanh(x_6) = f_1(x_1, x_5, x_6),\\
\frac{dx_2}{dt} &= -x_2 + a\tanh(x_4) = f_2(x_2, x_4),\\
\frac{dx_3}{dt} &= -x_3 + a\tanh(x_4) = f_3(x_3, x_4),\\
\frac{dx_4}{dt} &= \beta(x_1 - x_4) = f_4(x_1, x_4),\\
\frac{dx_5}{dt} &= \beta(x_2 - x_5) = f_5(x_2, x_5),\\
\frac{dx_6}{dt} &= \beta(x_3 - x_6) = f_6(x_3, x_6).
\end{aligned} \tag{8}$$

The system (5) and (7) have a common set of bounded solutions and therefore system (7) can be studied instead of system (5) to investigate its dynamics [36].

Lemma 1. *The system* (8)*, that is system* (5) *has unique equilibrium if* $ab \leq 0.5$*. Otherewise it has three equilibria.*

Proof. Let $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3, \widehat{x}_4, \widehat{x}_5, \widehat{x}_6)$ be an equilibrium of (8), then we have the following relations:

$$\begin{aligned}
&\widehat{x}_1 = b\tanh[\widehat{x}_5] + b\tanh[\widehat{x}_6], \quad &&\widehat{x}_2 = a\tanh[\widehat{x}_4], \quad &&\widehat{x}_3 = a\tanh[\widehat{x}_4],\\
&\widehat{x}_1 = \widehat{x}_4, &&\widehat{x}_2 = \widehat{x}_5, &&\widehat{x}_3 = \widehat{x}_6.
\end{aligned}$$

That is $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_2, \widehat{x}_1, \widehat{x}_2, \widehat{x}_2)$ is an equilibrium of (8), where $\widehat{x}_1, \widehat{x}_2$ are the solutions of following equations

$$x_1 = 2b\tanh[x_2], \tag{9}$$

$$x_2 = a\tanh[x_1] \tag{10}$$

obviously $(0,0,0,0,0,0)$ is an equilibrium of (8). Both the isoclines (9) and (10) have $(0,0)$ as the point of inflexion. Curve (9) is concave with respect to x_1-axis while curve (10) is convex with respect to x_1-axis. Both the curves are monotonic and bounded. They will not intersect at nonzero points if

$$2b \leq \frac{1}{a},$$

(9) and (10) will intersect at two non-zero points if $2b > \frac{1}{a}$.

Therefore $(0,0,0,0,0,0)$ is unique equilibrium of (8) if

$$2ab \leq 1 \quad \Rightarrow \quad ab \leq 0.5.$$

If $ab > 0.5$ origin E_0 is an equilibrium together with two non-zero equilibria E_1 and E_2 as shown in Fig. 2. From symmetrical nature of the vector field of (8), it is clear that if E_1 is of the form $(\widehat{x}_1, \widehat{x}_2, \widehat{x}_2, \widehat{x}_1, \widehat{x}_2, \widehat{x}_2)$ then E_2 is $(-\widehat{x}_1, -\widehat{x}_2, -\widehat{x}_2, -\widehat{x}_1, -\widehat{x}_2, -\widehat{x}_2)$.

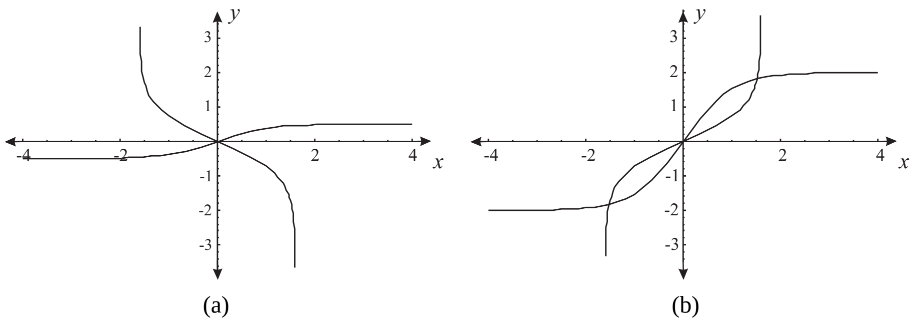


Fig. 2. (a) shows the existence of unique equilibrium, here $ab < 0.5$; (b) shows the existence of three equilibria, here $ab > 0.5$.

4 Linear analysis

Now we are to study the local asymptotic stability of equilibria of system (5), which is similar in nature to that of system (7).

The variational matrix of system (7) about any point $P(x, y, z, u, v, w)$ is given by

$$A(P) = \begin{bmatrix} -1 & 0 & 0 & 0 & b\,\mathrm{sech}^2(v) & b\,\mathrm{sech}^2(w) \\ 0 & -1 & 0 & a\,\mathrm{sech}^2(u) & 0 & 0 \\ 0 & 0 & -1 & a\,\mathrm{sech}^2(u) & 0 & 0 \\ \beta & 0 & 0 & -\beta & 0 & 0 \\ 0 & \beta & 0 & 0 & -\beta & 0 \\ 0 & 0 & \beta & 0 & 0 & -\beta \end{bmatrix}. \tag{11}$$

Case 1. First we shall analyse the local asymptotic stability of unique equilibrium $(0,0,0,0,0,0)$. The characteristic equation of system (7) about $(0,0,0,0,0,0)$ is given by

$$\begin{vmatrix} -1-\lambda & 0 & 0 & 0 & b & b \\ 0 & -1-\lambda & 0 & a & 0 & 0 \\ 0 & 0 & -1-\lambda & a & 0 & 0 \\ \beta & 0 & 0 & -\beta-\lambda & 0 & 0 \\ 0 & \beta & 0 & 0 & -\beta-\lambda & 0 \\ 0 & 0 & \beta & 0 & 0 & -\beta-\lambda \end{vmatrix} = 0 \tag{12}$$

$$\implies \quad \lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0,$$

where

$$\begin{aligned} a_1 &= 3(1+\beta), \\ a_2 &= 3[(1+\beta)^2 + \beta], \\ a_3 &= (1+\beta)[(1+\beta)^2 + 6\beta], \\ a_4 &= 3\beta(1+\beta)^2 + \beta^2(3-2ab), \\ a_5 &= \beta^2(3-2ab)(1+\beta), \\ a_6 &= \beta^3(1-2ab). \end{aligned}$$

To derive some sufficient conditions of local asymptotic stability of $(0,0,0,0,0,0)$ Routh-Hurwitz criteria has been applied.

According to this criteria all the roots of an equation

$$y^k + \alpha_1 y^{k-1} + \alpha_2 y^{k-2} + \ldots + \alpha_{k-1} y + \alpha_k = 0$$

have negative real part if

$$D_m = \begin{vmatrix} \alpha_1 & \alpha_3 & \alpha_5 & - & - & - & - \\ 1 & \alpha_2 & \alpha_4 & - & - & - & - \\ 0 & \alpha_1 & \alpha_3 & \alpha_5 & - & - & - \\ 0 & 1 & \alpha_2 & \alpha_4 & - & - & - \\ \vdots & & & & & & \\ 0 & - & - & - & - & - & \alpha_m \end{vmatrix} > 0 \quad \forall m = 1, 2, \ldots, k.$$

In case of equation (12)

$$\begin{aligned} D_1 &= 3(1+\beta) > 0, \\ D_2 &= (1+\beta)\big(8\beta^2 + 19\beta + 8\big) > 0, \\ D_3 &= 4(1+\beta)^2\big[2\beta^4 + 14\beta^3 + 3(8+ab)\beta^2 + 14\beta + 2\big], \end{aligned}$$

$$\begin{aligned}
D_4 &= 4\beta(1+\beta)^2\big[6\beta^6 + (42+8ab)\beta^5 + (120+29ab)\beta^4 + \big(168+45ab-4a^2b^2\big)\beta^3 \\
&\qquad\qquad + (120+29ab)\beta^2 + (42+8ab)\beta + 6\big], \\
D_5 &= 16\beta^3(1+\beta)^3\big[2\beta^2 + 4\beta + (2-ab)\big]\big[(2-ab)\beta^2 + 4\beta + 2\big] \\
&\qquad\qquad \times \big[\beta^2 + 2(1+ab)\beta + 1\big], \\
D_6 &= (1-2ab)\beta^3 D_5.
\end{aligned}$$

Now if origin is unique equilibrium, that is $ab \le 0.5$, then

$$\begin{aligned}
&D_3 > 0 \quad \forall\beta > 0 \quad \text{if} \quad (8+ab) > 0 \quad \Longrightarrow \quad -8 < ab \le 0.5, \\
&D_4 > 0 \quad \forall\beta > 0 \quad \text{if} \quad (42+8ab) > 0, \ (120+29ab) > 0, \\
&\qquad \big(168+45ab-4a^2b^2\big) > 0 \quad \Longrightarrow \quad \big(45-\sqrt{4713}\big)/8 \simeq -2.96 < ab \le 0.5, \\
&D_5 > 0, \quad D_6 > 0 \quad \forall\beta > 0 \quad \text{if} \quad -2 \le ab < 0.5.
\end{aligned}$$

Combining above results it can be concluded that origin is locally asymptotically stable $\forall\beta > 0$ if $-2 \le ab < 0.5$.

Let us now consider the case $-2.96 < ab < -2$.

Then $D_5, D_6 > 0$ if $[\beta^2 + 2(1+ab)\beta + 1] > 0 \Longrightarrow (\beta-\beta_1)(\beta-\beta_2) > 0$, where

$$\begin{aligned}
\beta_1 &= -(1+ab) - \sqrt{ab(ab+2)}, \\
\beta_2 &= -(1+ab) + \sqrt{ab(ab+2)}.
\end{aligned} \tag{13}$$

Hence if $-2.96 < ab < -2$, then origin is locally asymptotically stable $\forall\beta\epsilon(0,\beta_1)$ or (β_2,∞) and not locally asymptotically stable for any $\beta\epsilon[\beta_1,\beta_2]$, where β_1 and β_2 are given by (13).

Case 2. Now we shall study the case when $ab > 0.5$.

If $ab > 0.5$, then D_5 and D_6 are of opposite sign. Therefore origin E_0 is not locally asymptotically stable.

The characteristic equation of system (7) about non-zero equilibrium $E_1(\widehat{x},\widehat{y},\widehat{y},\widehat{x},\widehat{y},\widehat{y})$ is given by

$$\begin{vmatrix}
-1-\lambda & 0 & 0 & 0 & b\,\mathrm{sech}^2(\widehat{y}) & b\,\mathrm{sech}^2(\widehat{y}) \\
0 & -1-\lambda & 0 & a\,\mathrm{sech}^2(\widehat{x}) & 0 & 0 \\
0 & 0 & -1-\lambda & a\,\mathrm{sech}^2(\widehat{x}) & 0 & 0 \\
\beta & 0 & 0 & -\beta-\lambda & 0 & 0 \\
0 & \beta & 0 & 0 & -\beta-\lambda & 0 \\
0 & 0 & \beta & 0 & 0 & -\beta-\lambda
\end{vmatrix}$$

$$\Longrightarrow \quad \lambda^6 + c_1\lambda^5 + c_2\lambda^4 + c_3\lambda^3 + c_4\lambda^2 + c_5\lambda + c_6 = 0,$$

where

$$
\begin{aligned}
c_1 &= 3(1+\beta),\\
c_2 &= 3\big[(1+\beta)^2+\beta\big],\\
c_3 &= (1+\beta)\big[(1+\beta)^2+6\beta\big],\\
c_4 &= 3\beta(1+\beta)^2+\beta^2\big(3-2ab\,\mathrm{sech}^2(\widehat{x})\mathrm{sech}^2(\widehat{y})\big),\\
c_5 &= \beta^2\big(3-2ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big)(1+\beta),\\
c_6 &= \beta^3\big(1-2ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big).
\end{aligned}
$$

Applying Routh-Hurwitz criteria as in Case 1 we get

$$
\begin{aligned}
D_1 &= 3(1+\beta)>0,\\
D_2 &= (1+\beta)\big(8\beta^2+19\beta+8\big)>0,\\
D_3 &= 4(1+\beta)^2\big[2\beta^4+14\beta^3+3\big\{8+ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big\}\beta^2+14\beta+2\big]>0,\\
D_4 &= 4\beta(1+\beta)^2\big[6\beta^6+\{42+8ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\}\beta^5\\
&\quad+\big\{120+29ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big\}\beta^4\\
&\quad+\big\{168+45ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})-4a^2b^2\,\mathrm{sech}^4(x)\,\mathrm{sech}^4(\widehat{y})\big\}\beta^3\\
&\quad+\big\{120+29ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big\}\beta^2+\big\{42+8ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big\}\beta+6\big]\\
D_5 &= 16\beta^3(1+\beta)^3\big[2\beta^2+4\beta+\big\{2-ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big\}\big]\\
&\quad\times\big[\big\{2-ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big\}\beta^2+4\beta+2\big]\\
&\quad\times\big[\beta^2+2\big\{1+ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big\}\beta+1\big],\\
D_6 &= \big[1-2ab\,\mathrm{sech}^2(\widehat{x})\,\mathrm{sech}^2(\widehat{y})\big]\beta^3 D_5.
\end{aligned}
$$

Using $0<\mathrm{sech}(x)<1$, it can be shown that $D_4>0, D_5>0, D_6>0\ \ \forall\beta>0$. Hence E_1 is locally asymptotically stable $\forall\beta>0$. Similarly E_2 can be shown to be locally asymptotically stable $\forall\beta>0$. Hence from discussions in Case 1 and Case 2 we have the following theorem:

Theorem 1. *For system* (5)*:*

(i) *If* $-2\le ab<0.5$ *, then unique equilibrium* $(0,0,0,0,0,0)$ *is locally asymptotically stable* $\forall\beta>0$*;*

(ii) *If* $-2.96<ab<-2$ *, then unique equilibrium* $(0,0,0,0,0,0)$ *is locally asymptotically stable* $\forall\beta\in(0,\beta_1)$ *or* (β_2,∞)*, where* β_1 *and* β_2 *are given by* (13)*;*

(iii) *If* $-2.96<ab<-2$ *, then unique equilibrium* $(0,0,0,0,0,0)$ *is not locally asymptotically stable for any* $\beta\in[\beta_1,\beta_2]$*, where* β_1 *and* β_2 *are given by* (13)*;*

(iv) *If* $ab>0.5$ *, then origin* E_0 *is not locally asymptotically stable for any* $\beta>0$*, where other two non-zero equilibria* E_1 *and* E_2 *are locally asymptotically stable* $\forall\beta>0$.

Stability region in $ab-\beta$ parametric space is shown in Fig. 3.

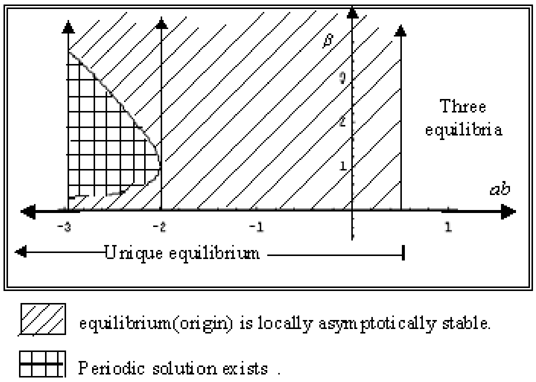


Fig. 3. Stability region in $ab - \beta$ parametric space.

5 Existence of Hopf bifurcating periodic solution in case (iii) of Theorem 1

From Theorem 1 it is clear that if $-2.96 < ab < -2$, then a breakdown of stability occurs at $\beta = \beta_1$ and $\beta = \beta_2$.

First it will be proved that when $\beta = \beta_1$ or $\beta = \beta_2$ then equation (12) has a pair of purely imaginary roots.

Let $\lambda = i\omega, \omega > 0$ be a root of equation (12). Substituting $\lambda = i, \omega$ in (12) and then separating real and imaginary parts we get

$$\begin{aligned} &a_6 - a_4\omega^2 + a_2\omega^4 - \omega^6 = 0, \\ &a_5\omega - a_3\omega^3 + a_1\omega^5 = 0. \end{aligned} \tag{14}$$

Eliminating ω from equations in (14) and then substituting values of $a_1, a_2, a_3, \ldots, a_6$ after simplification it is obtained as

$$\begin{aligned} &16(2ab-1)\beta^6(1+\beta)^3\left[\beta^2+2(ab+1)\beta+1\right]\left[2\beta^2+4\beta+(2-ab)\right] \\ &\times\left[(2-ab)\beta^2+4\beta+2\right] = 0. \end{aligned} \tag{15}$$

This is possible only when $\beta^2 + 2(ab+1)\beta + 1 = 0$.

Thus when $\beta = \beta_1$, the characteristic equation (12) has a pair of purely imaginary roots $\pm i\omega_1$ (say). From (14)

$$\begin{aligned} \omega_1^2 &= \left[\frac{a_5a_1a_2 - a_5a_3 - a_1^2a_6}{a_1a_5 - a_1^2a_4 + a_1a_2a_3 - a_3^2}\right]_{\beta=\beta_1} \\ &= \frac{-\beta_1^2\left[(4ab-6)+(5ab-12)\beta_1+(4ab-6)\beta_1^2\right]}{2+14\beta_1+(3ab+24)\beta_1^2+14\beta_1^3+2\beta_1^4} = \beta_1 \\ \text{as}\quad &\beta_1^2 + 2(ab+1)\beta_1 + 1 = 0. \end{aligned} \tag{16}$$

Now applying Descarte's rule and relation between roots and coefficients of a polynomial it can be shown that, apart from $\pm i\omega_1$ other 4 roots of equation (12) at $\beta = \beta_1$, have negative real part. Similarly it can be concluded that at $\beta = \beta_2$ equation (12) has two purely imaginary roots $\pm i\omega_2$ and other 4 roots have negative real part.

Differentiating equation (12) with respect to β implicitly,

$$\frac{d\lambda}{d\beta} = -\frac{3(1-2ab)\beta^2 + \big[(6-4ab)\beta + (9-6ab)\beta^2\big]\lambda}{(1+\beta+2\lambda)\big[6\beta\lambda(1+\lambda)^2 + 3\lambda^2(1+\lambda)^2 - \beta^2\big(2ab - 3(1+\lambda)^2\big)\big]} - \frac{\big[3+2(9-2ab)\beta+9\beta^2\big]\lambda^2 + 3\big(3+6\beta+\beta^2\big)\lambda^3 + (9+6\beta)\lambda^4 + 3\lambda^5}{(1+\beta+2\lambda)\big[6\beta\lambda(1+\lambda)^2 + 3\lambda^2(1+\lambda)^2 - \beta^2\big(2ab - 3(1+\lambda)^2\big)\big]}.$$

Since $\omega_1^2 = \beta$ and $\beta_1^2 = -1 - 2(ab+1)\beta_1$

$$\left(\frac{d\lambda}{d\beta}\right)_{\lambda=i\omega_1} = -\frac{\beta_1 - i\omega_1}{(1+\beta_1)\beta_1 + 2i\omega_1\beta_1} = -\frac{(\beta_1 - i\omega_1)[(1+\beta_1)\beta_1 - 2i\omega_1\beta_1]}{(1+\beta_1)^2\beta_1^2 + 4\omega_1^2\beta_1^2}$$

$$\mathrm{Re}\left(\frac{d\lambda}{d\beta}\right)_{\lambda=i\omega_1} = -\frac{(1+\beta_1)\beta_1^2 - 2\beta_1^2}{(1+\beta_1)^2\beta_1^2 + 4\beta_1^3} = \frac{1-\beta_1}{(1+\beta_1)^2 + 4\beta_1} \neq 0$$

$$\mathrm{Re}\left(\frac{d\lambda}{d\beta}\right)_{\lambda=i\omega_2} = \frac{1-\beta_2}{(1+\beta_2)^2 + 4\beta_2} \neq 0.$$

That is all the sufficient conditions for Hopf bifurcation [37–39] are satisfied.

Hence we have the following theorem:

Theorem 2. *If* $-2.96 < ab < -2$ *and* $\beta = \beta_1$ *or* $\beta = \beta_2$*, then there exists Hopf-bifurcating periodic solutions with* β *as bifurcation parameter, about the unique equilibrium of the system* (5)*.*

6 Stability of bifurcating periodic solution

Now to investigate the orbital stability of such Hopf-bifurcating periodic solution Poore's condition has been followed [40].

According to Poore's sufficient condition, the supercritical and subcritical nature of Hopf-bifurcating periodic solution is determined by positive and negative sign of real part of magic number Φ, where Φ is given by:

$$\Phi = -u_l \frac{\partial^3 f_l}{\partial x_j \partial x_m \partial x_s} v_j v_m \overline{v}_s + 2u_l \frac{\partial^2 f_l}{\partial x_j \partial x_m} v_j \big[(A^{-1})_{mr}\big] \frac{\partial^2 f_r}{\partial x_p \partial x_q} v_p \overline{v}_q + u_l \frac{\partial^2 f_l}{\partial x_j \partial x_m} \overline{v}_j \big(\big[(A - 2i\omega_c I)^{-1}\big]_{mr}\big) \frac{\partial^2 f_r}{\partial x_p \partial x_q} v_p v_q. \tag{17}$$

Here repeated suffixes indicate sum notation and all the derivatives are evaluated at the equilibrium (origin). A is the variational matrix of (8) calculated at origin. $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}, \mathbf{u_4}, \mathbf{u_5}, \mathbf{u_6})$ and $\mathbf{v} = (\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}, \mathbf{v_6})$ are the left and right eigen vectors respectively of A with respect to eigen values $\pm i\omega_j$ $(j = 1, 2)$ at the points $\beta = \pm\beta_j (j = 1, 2)$ and it is important to remember that $\mathbf{u}$, $\mathbf{v}$ are normalised with requirement $\mathbf{u} \cdot \mathbf{v} = \mathbf{1}$ [Th 4.1] [40]. $\overline{\mathbf{u}} = (\overline{u}_1, \overline{u}_2, \overline{u}_3, \overline{u}_4, \overline{u}_5, \overline{u}_6)$ and $\overline{\mathbf{v}} = (\overline{v}_1, \overline{v}_2, \overline{v}_3, \overline{v}_4, \overline{v}_5, \overline{v}_6)$ are the complex conjugates of $\mathbf{u}$ and $\mathbf{v}$ respectively.

Now at origin

$$\begin{aligned}
&\frac{\partial f_1}{\partial x_1} = -1, \quad \frac{\partial^2 f_1}{\partial x_1^2} = 0, \quad \frac{\partial^3 f_1}{\partial x_1^3} = 0,\\
&\frac{\partial f_1}{\partial x_5} = b, \quad \frac{\partial^2 f_1}{\partial x_5^2} = 0, \quad \frac{\partial^3 f_1}{\partial x_5^3} = -2b,\\
&\frac{\partial f_1}{\partial x_6} = b, \quad \frac{\partial^2 f_1}{\partial x_6^2} = 0, \quad \frac{\partial^3 f_1}{\partial x_6^3} = -2b,\\
&\frac{\partial f_2}{\partial x_2} = -1, \quad \frac{\partial^2 f_2}{\partial x_2^2} = 0, \quad \frac{\partial^3 f_2}{\partial x_2^3} = 0,\\
&\frac{\partial f_2}{\partial x_4} = a, \quad \frac{\partial^2 f_2}{\partial x_4^2} = 0, \quad \frac{\partial^3 f_2}{\partial x_4^3} = -2a,\\
&\frac{\partial f_3}{\partial x_3} = -1, \quad \frac{\partial^2 f_3}{\partial x_3^2} = 0, \quad \frac{\partial^3 f_3}{\partial x_3^3} = 0,\\
&\frac{\partial f_3}{\partial x_4} = a, \quad \frac{\partial^2 f_3}{\partial x_4^2} = 0, \quad \frac{\partial^3 f_3}{\partial x_4^3} = -2a.
\end{aligned} \tag{18}$$

Simplifying (17) and substituing values of above partial derivatives

$$\begin{aligned}
\Phi = &- u_1(-2b)v_5v_5\overline{v}_5 - u_1(-2b)v_6v_6\overline{v}_6\\
&- u_2(-2a)v_4v_4\overline{v}_4 - u_3(-2a)v_4v_4\overline{v}_4.
\end{aligned} \tag{19}$$

Now we calculate the right and left eigen vectors of variational matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & b & b\\ 0 & -1 & 0 & a & 0 & 0\\ 0 & 0 & -1 & a & 0 & 0\\ \beta & 0 & 0 & -\beta & 0 & 0\\ 0 & \beta & 0 & 0 & -\beta & 0\\ 0 & 0 & \beta & 0 & 0 & -\beta \end{bmatrix}.$$

A right eigen vector $(x_1 + iy_1, x_2 + iy_2, \ldots, x_6 + iy_6)^T$ of matrix A is given by

$$A \begin{bmatrix} x_1 + iy_1\\ x_2 + iy_2\\ \vdots\\ x_6 + iy_6 \end{bmatrix} = i\omega \begin{bmatrix} x_1 + iy_1\\ x_2 + iy_2\\ \vdots\\ x_6 + iy_6 \end{bmatrix}.$$

Proceeding in above way and solving the set of equations our required right eigen vector $\mathbf{v}$ and left eigen vector $\mathbf{u}$ are obtained in the following form (if they are calculated at bifurcation point $\beta = \beta_1$ corresponding to eigen value $i\omega = i\omega_1$):

$$\mathbf{v} = \frac{a(c+ik)}{(1+\beta_1)^2}\begin{bmatrix} 2b\omega_1(i+\omega_1) \\ (1+\beta_1)+(1-i\omega_1) \\ (1+\beta_1)+(1-i\omega_1) \\ (1+\beta_1)^2/a \\ -i\omega_1(1+\beta_1) \\ -i\omega_1(1+\beta_1) \end{bmatrix} \tag{20}$$

$$\begin{aligned}\mathbf{u} = \frac{\eta_0 b(c+ik)}{\beta_1(1+\beta_1)}\bigg[&\frac{\beta_1(1+\beta_1)}{b}, -i\omega_1\beta_1, -i\omega_1\beta_1, \frac{-2a\omega_1 i(\beta_1-i\omega_1)}{(1+\beta_1)}, \\ &(\beta_1-i\omega_1), (\beta_1-i\omega_1)\bigg],\end{aligned} \tag{21}$$

where c, k are real numbers and η_0 is a complex number to be determined using the condition $\mathbf{u}\cdot\mathbf{v} = 1$. Now

$$\begin{aligned}&\mathbf{u}\cdot\mathbf{v} = 1 \quad \Longrightarrow \quad \eta_0\frac{-4\omega_1 ab}{\beta_1(1+\beta_1)^2}\big[(c+ik)^2(\omega_1+i\beta_1)\big] = 1 \\ &\Longrightarrow \quad \eta_0 = \frac{-(1+\beta_1)}{4\omega_1 ab\big(c^2+k^2\big)^2}\Big[\Big((c^2-k^2)\omega_1-2ck\beta_1\Big)-i\Big(2ck\omega_1+\beta_1(c^2-k^2)\Big)\Big].\end{aligned}$$

That is if η_0 is expressed in the form $\eta_0 = \phi_1 + i\phi_2$, then

$$\phi_1 = M\big[(c^2-k^2)\omega_1 - 2ck\beta_1\big], \quad \phi_2 = -M\big[2ck\omega_1 + \beta_1\big(c^2-k^2\big)\big], \tag{22}$$

where

$$M = -\frac{(1+\beta_1)}{4\omega_1 ab\big(c^2+k^2\big)^2} > 0$$

Now from (19)

$$\begin{aligned}(\Phi)_{\beta=\beta_1} &= 2b\big[u_1v_5|v_5|^2 + u_1v_6|v_6|^2\big] + 2a\big[u_2v_4|v_4|^2 + u_3v_4|v_4|^2\big] \\ &= 4b\big[u_1v_5|v_5|^2\big] + 4a\big[u_2v_4|v_4|^2\big] \\ &= -\frac{4ab\omega_1\eta_0\big(c^2+k^2\big)i(c+ik)^2}{(1+\beta_1)}\left[\frac{\omega_1^2a^2}{(1+\beta_1)^2}+1\right] \\ &= -P\beta_1\big[(a-b)^2-b^2\big]\Big[\Big(2ck\phi_1+\phi_2\big(c^2-k^2\big)\Big)+i\Big(2ck\phi_2-\phi_1\big(c^2-k^2\big)\Big)\Big] \\ &\text{as} \quad v_5 = v_6, \ \ u_2 = u_3,\end{aligned}$$

where $P = -\frac{4ab\omega_1(c^2+k^2)}{(1+\beta_1)^3} > 0$. Hence

$$\mathrm{Re}\,[\phi]_{\beta=\beta_1} = P\beta_1^2 M\big[(a-b)^2-b^2\big]\big[c^2+k^2\big]^2 > 0.$$

Proceeding in the same manner, calculating the left and right eigen vectors of matrix A corresponding to the eigen value $i\omega_2$ at the bifurcation point $\beta = \beta_2$, it is obtained that

$$\mathrm{Re}\,[\phi]_{\beta=\beta_2} > 0$$

From above discussion we get the following result:

Theorem 3. *If* $-2.96 < ab < -2,$ *there exists stable periodic orbits for* $\beta \in [\beta_1, \beta_2]$ *and the system* (5) *undergoes a supercritical Hopf-bifurcation as* β *is increased through* β_1 *and also as* β *is decreased through* β_2*, where* β_1, β_2 *are given by* (13)*.*

7 Global Hopf-bifurcation

In this section we shall consider the continuation of the bifurcating periodic solutions as the bifurcation parameter β increases and varies over the interval (β_1, β_2). To study this we shall use the technique of Alexander and Auchmuty [41].

Let P^1 denote the space of all $x\colon \mathbb{R} \to \mathbb{R}^6$ which are periodic with period 2π. The space P^1 is a Banach space with the norm

$$\|x\|_\infty^{(1)} = \max_{1\le i\le 6} \max_{0\le t\le 2\pi} \left[|x_i(t)| + \left| \frac{dx_i(t)}{dt} \right| \right].$$

Let Λ denote the open interval $(0, \infty)$. Let $L(P^1)$ denote the set of all continuous linear maps of P^1 into itself with the induced norm topology. Let $F\colon P^1 \times \Lambda \to P^1$ be continuous and let us consider the problem of finding the solutions $(y, \beta, \omega) \in P^1 \times \Lambda \times (0, \infty)$ of the equation

$$\omega \frac{dy}{dt} = F(y, \beta). \tag{23}$$

If $y(t)$ is a solution of equation (23), and if $x(t) = y(\omega t)$, then $x(t)$ is a solution of

$$\frac{dx}{dt} = F(x, \beta), \tag{24}$$

where x is periodic with period $T = 2\pi/\omega$.

Now if we denote the system (8) by

$$\frac{dX}{dt} = F_1(X) \tag{25}$$

then (25) can be rewritten as

$$\omega \frac{dZ}{dt} = A(\beta)Z + R(Z, \beta), \tag{26}$$

where $A(\beta)$ is variational matrix of (8) about (0,0,0,0,0,0) and $R(Z, \beta) = F_1(Z, \beta) - A(\beta)Z$.

Now we shall use the following result whose proof is in Alexander and Achmuty [41].

Theorem 4. *Let F be a Frechet differentiable map of $P^1 \times \Lambda$ into P^1. There is a global bifurcation of 2π-periodic solutions of equation (23) from a solution (y^*, β_0, ω_0) provided:*

(i) $A(\beta) \in L(P^1)$ *for* $\beta \in \Lambda$. *The mapping* $\beta \to A(\beta)$ *is continuous and 0 is not in the spectrum of* $A(\beta_0)$.

(ii) *The number of linearly independent solutions in* P^1 *of* $\omega_0 \frac{dW}{dt} = A(\beta)W$ *is finite and congruent to 2 mod 4.*

(iii) *There are positive* δ *and* ε *such that if* $\lambda(\beta)$ *is in the spectrum of* $A(\beta)$ *and* $\operatorname{Re}\lambda(\beta) = \alpha(\beta)$, *then* $|\alpha(\beta)| > \varepsilon|\beta - \beta_0|$ *for* $|\beta - \beta_0| < \delta$.

We will verify the above sufficient conditions of global Hopf-bifurcation for the system (26). We consider the linearized system

$$\omega_0 \frac{dW}{dt} = A(\beta)W \tag{27}$$

and let us suppose that it has a periodic solution of period 2π; let it be

$$W = \sum_{k=-\infty}^{\infty} \mathbf{d}_k e^{ikt}.$$

The coefficients $\mathbf{d}_k$ are solutions of the linear system $\omega_0 ik\mathbf{d}_k = A(\beta)\mathbf{d}_k$, $k = 0, \pm 1, \pm 2, \ldots$. Nontrivial periodic solutions of period 2π exist if $ik\omega_0$is an eigen value of $A(\beta)$. That is $ik\omega_0$ is a solution of equation (12). From previous discussion it is clear that non-trivial periodic solutions exist only for $k = \pm 1$ at $\beta = \beta_1$ and $\beta = \beta_2$, and hence there are only two periodic solutions of period 2π for the linearized system (27).

It can be shown that

$$\operatorname{Re}\left(\frac{d\lambda}{d\beta}\right)_{\beta=\beta_1} > 0.$$

It follows that if $\alpha(\beta) = \operatorname{Re}[\lambda(\beta)]$ then

$$\lim_{\beta \to \beta_1} \frac{\alpha(\beta) - \alpha(\beta_1)}{\beta - \beta_1} > 0$$

which implies that there exists $\varepsilon > 0, \delta > 0$ such that

$$\begin{aligned} &\Longrightarrow \quad \left|\frac{\alpha(\beta) - \alpha(\beta_1)}{\beta - \beta_1}\right| > \varepsilon \quad \text{if} \quad |\beta - \beta_1| < \delta, \\ &\Longrightarrow \quad |\alpha(\beta)| > \varepsilon|\beta - \beta_1| \quad \text{for} \quad |\beta - \beta_1| < \delta \end{aligned} \tag{28}$$

as $\quad \alpha(\beta_1) = 0.$

Also $A(\beta)$ is continuous in β. Thus all the conditions of Theorem 4 are satisfied. Therefore there is a global bifurcation of 2π periodic solutions from the point $(0, \beta_1, \omega_1)$.

The global Hopf-bifurcation has been established in a product space of the phase space,parameter space and the frequency space. For applications it will be informative to consider bifurcation by projecting the globally bifurcating arc on the product space of the phase space and parameter space as it is usually done in local Hopf-bifurcation. So here we will consider the frequency component of globally bifurcating arc.

It is known from work of Yorke [42, 43] that the period of any periodic solution of (25) (that is of (8)) remains bounded below away from zero due to the Lipschitzian nature of vector field in equation $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, where A is variational matrix of (8) about equilibrium $(0, 0, 0, 0, 0, 0)$. Therefore by relation between frequency and period of any periodic solution, it can be concluded that ω-component of any bifurcating arc is bounded above. Again by Lemma 1 in the range $-2.96 < ab < -2$ origin is the unique equilibrium of system (8) and as the equation (12) has no positive real root (proved in Section 5), the unique equilibrium point is not a saddle point. Consequently, there is no homoclinic orbit and hence the period of periodic solutions of equation (8) is bounded above and so its frequency is bounded below.

So it can be concluded that ω-component of bifurcating arc of global Hopf-bifurcation is bounded between two positive numbers.

Now we are to calculate the center index Φ_1 introduced by Mallet-Paret and Yorke [44] at the two bifurcation points $(0, \beta_1), (0, \beta_2)$ given by

$$\Phi_1(0, \beta_1) = \frac{1}{2}\left[E(\beta_1+) - E(\beta_1-)\right](-1)^{E(\beta_1)}, \tag{29}$$

where $E(\beta)$ denotes the sum of the multiplicities of the eigen values of $A(\beta)$ having strictly positive real parts. $E(\beta_1+)$ and $E(\beta_1-)$ denote the right and left hand limit of $E(\beta)$ at β_1, β_2 respectively.

From our previous discussion in Section 2, it is clear that $E(\beta_1-) = 0$, $E(\beta_1) = 0$, $E(\beta_1+) = 2$ and hence by (29) $\Phi_1(0, \beta_1) = 1$. Similar calculation shows that $\Phi_1(0, \beta_2) = -1$.

Thus the bifurcation point $(0, \beta_1)$ is a source, where $(0, \beta_2)$ is a sink. By results of [45], each source is connected to a sink by an oriented one parameter *“snake”* of orbits.

8 Numerical simulation

In Section 4 Fig. 3 is drawn to show the number of equilibrium and stability character of unique equilibrium in $ab - \beta$ parametric space.

Then to simulate the above described model numerically, system (7) has been solved by fourth-order Runga-Kutta method and corresponding phase potraits in xyz plane are drawn.

First the values of a, b has been considered in such a way that $-2 < ab < 0.5$ and in Fig. 4 it has been shown that for this value of ab origin is locally asymptotically stable.

Then to illustrate Hopf-bifurcation it has been taken that $a = 1.2,\ b = -2$. Substituting this value of ab in equation (13) we get $\beta_1 = 0.42$; $\beta_2 = (1.4 + \sqrt{.96}) \simeq 2.38$. Then for different values of β corresponding figures are drawn.

Then for $ab = -2.5$ (that is $-2.96 < ab < -2$) Fig. 8, Fig. 9, Fig. 10 has been drawn with $\beta = 0.5, 1.1, 2.2$ respectively and it has been shown that in the interval $[\beta_1, \beta_2]$ (β_1, β_2 are calculated by (13)) period of Hopf-bifurcating periodic solution first increases and then decreases.

Here Theorem 1 is verified by Fig. 4, Fig 5, Fig 6 and Fig 7. Figs. 8–10 illustrates a *"snake"* of orbits in the interval (β_1, β_2).

At last it has been taken $a = 3, b = 2$, that is $ab > 0.5$. Then for $\beta = 1$ Fig. 11 has been drawn. From Fig. 11 it is clear that origin E_0 is a repeller and other two non-zero equilibria E_1 and E_2 are sink.

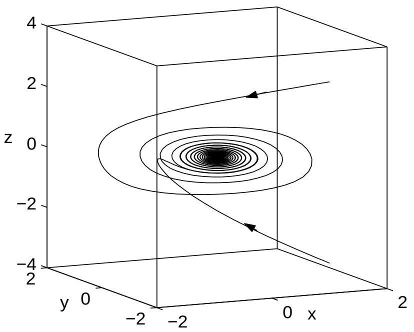


Fig. 4. $-2 \le ab < 0.5$. Here origin is locally asymptotically stable.

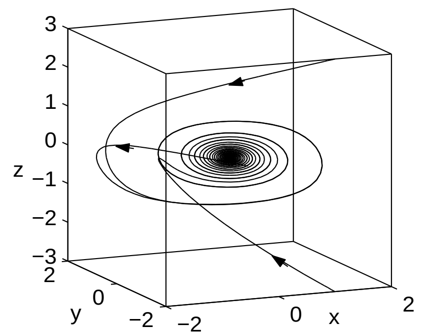


Fig. 5. $-2.96 < ab < -2,\ \beta = 0.3$. Here origin is locally asymptotically stable.

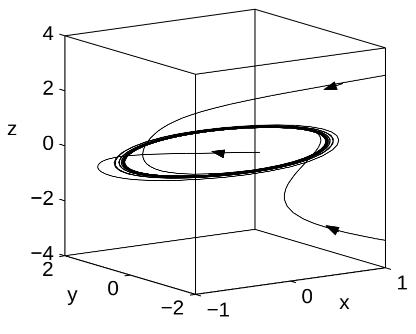


Fig. 6. $-2.96 < ab < -2,\ \beta = 1$. Origin is not locally asymptotically stable and there exists a periodic solution.

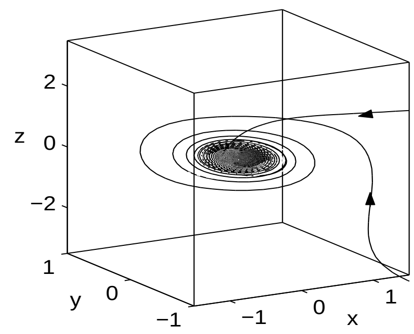
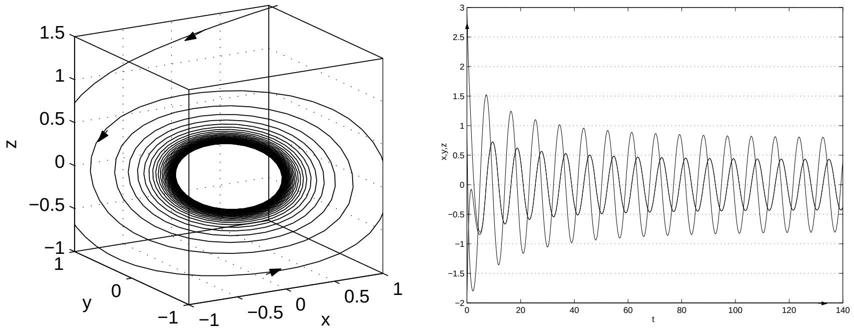
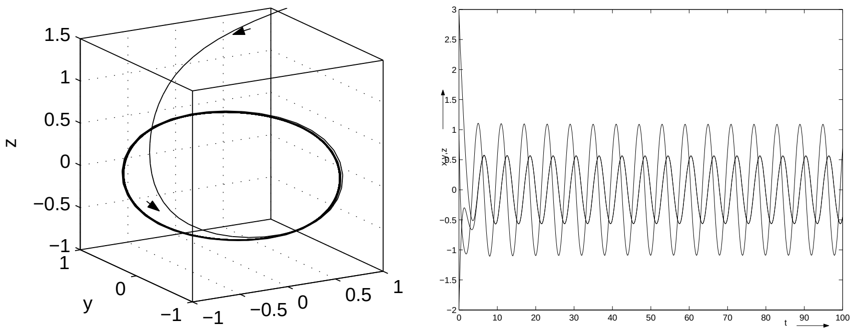


Fig. 7. $-2.96 < ab < -2,\ \beta = 2.5$. Here origin is locally asymptotically stable.



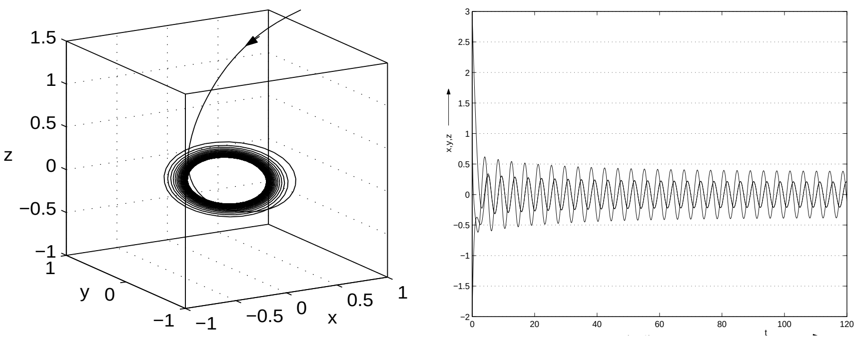
(a) (b)

Fig. 8. $-2.96 < ab < -2,\ \beta = 0.5$. (a) phase potrait; (b) waveform plot.



(a) (b)

Fig. 9. $-2.96 < ab < -2,\ \beta = 1.1$. (a) phase potrait; (b) waveform plot.



(a) (b)

Fig. 10. $-2.96 < ab < -2,\ \beta = 2.2$. (a) phase potrait; (b) waveform plot.

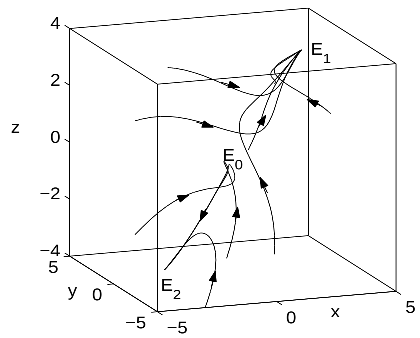


Fig. 11. Existence and stability character of multiple equilibria.

9 Conclusion

In this paper we analysed a system, composed of three neurons, with distributed delay. The system was assumed to be without self-connection and consisting of two neurons, which are not directly connected. Sufficient conditions (depending on values of synaptic weight and decay rate of kernel) for local asymptotic stability of equilibria and for existence of Hopf-bifurcating periodic solution have been derived. A closed interval of decay rate has been obtained, where Hopf-bifurcating periodic solution is orbitally stable and supercritical in nature. Global Hopf-bifurcation aspect has also been studied following the theorem of Alexander and Achmuty.

In subsequent work attempt will be taken to study a three-neuron network with distributed delay when the kernel is of second order.

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