Nonlinear Analysis: Modelling and Control, 2008, Vol. 13, No. 2, 169–179

Transient Mixed Convection Flow of a Second-Grade Visco-Elastic Fluid over a Vertical Surface

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Received: 28.02.2007 **Revised:** 23.01.2008 **Published online:** 02.06.2008

Abstract. The viscoelastic boundary layer flow and mixed convection heat transfer near a vertical isothermal surface have been examined in this paper. The governing equations are formulated and solved numerically using an explicit finite difference technique. The velocity and temperature profiles, boundary layer thicknesses, Nusselt numbers and the local skin friction coefficients are shown graphically for different values of the viscoelsatic parameter. In general, it is found that the velocity decreases inside the boundary layer as the viscoelsatic parameter is increased and consequently, the local Nusselt number decreases. This is due to higher tensile stresses between viscoelsatic fluid layers which has a retardation effects on the motion of these layers and consequently, on the heat transfer rates for the mixed convection heat transfer problem under investigation. A Comparison with available published results on special cases of the problem shows excellent agreement.

Keywords: viscoelastic fluids, transient, mixed convection.

Nomenclature

A_1, A_2	first two Rivlin-Ericksen tensor
C_f	local coefficient of friction
C_p	specific heat of the fluid at constant pressure
g	magnitude of acceleration due to gravity
Gr	Grashof number, $g\beta(T_w - T_\infty)L^3/\nu^2$
h	heat transfer coefficient
k	thermal conductivity
k_0	elastic parameter
k_l^*	dimensionless viscoelsatic parameter, $(k_0/L^2)Gr^{1/2}$
L	characteristic length of plate
M^*	mixed convection parameter, $\nu^2/[g\beta(T_w - T_\infty)L^3]^{1/2}$

Nu_x	local Nusselt number	T_w	wall temperature
PI	spherical stress	T_∞	ambient fluid temperature
Pr	Prandtl number, ν/α	u, v	dimensionless velocity components along x- and y-axes respectively
τ	dimensionless time		
T	temperature	x, y	dimensionless coordinates

Greek symbols

α	thermal diffusivity	μ	dynamic viscosity
$\alpha_1, \alpha_2, \alpha_3$	material moduli	ν	kinematic viscosity
β	coefficient in the density	ρ	fluid density
Θ	non-dimensional temperature	Γ	Cauchy stress tensor

Subscripts

w	wall surface
∞	free stream condition

Supercripts

$*$	dimensional variables

1 Introduction

Numerous applications of viscoelsatic fluids in several manufacturing processes have led to renewed interest among researchers to investigate viscoelsatic boundary layer flow over a stretching plastic sheet, Rajagopal et al. [1, 2], Dandapat and Gupta [3], Rollins and Vajravelu [4], Anderson [5], Lawrence and Rao [6], Char [7] and Rao [8]. Some of the physical applications of such study are polymer sheet extrusion from a dye, glass fiber and paper production, drawing of plastic film etc. The viscoelastic fluid model used by Hassanien was a simplified version of the so-called second-grade fluid [9]. Like Sakiadis [10], Hassanien relied on boundary layer approximation [11] for simplifying the governing equations. The final equation was in the form of a forth-order non-linear ordinary differential equation that can not be solved analytically, or even numerically, due to the lack of sufficient boundary conditions. To circumvent this problem, Hassanien utilized perturbation technique [12] to reduce the governing equation into a system of two third-order differential equations which could be solved with the available boundary conditions. Hassanien reported results for elasticity or Deboch numbers up to 0.2 and concluded that the wall skin friction coefficient is increased whenever a fluid exhibits elasticity, a prediction which is undesirable from an industrial standpoint because it translates into a larger driving force or torque to withdraw the surface.

There is no doubt that Hassanien work is of fundamental importance for it relies on boundary layer theory and thus can be regarded as a step forward towards answering the still unresolved issue of what would be the effects of fluids elasticity on the characteristics of its boundary layer [13–15]. In spite of its relevance, however, Hassanien work has certain drawbacks. One of the major drawbacks of his work is in the use of perturbation theory to solve the governing equation. That is, due to the inherent limitation of this theory, results could be obtained only for small values of Deborah numbers whereas in most processes of practical interest this number is of order one or even larger. That is to say the range of applicability of Hassanien work is quite limited. Another shortcoming of Hassanien work is in the use of the second grade model to represent viscoelsatic fluids. That is, a second grade fluid is the first deviation from a Newtonian behavior and can not be expected to render meaningful results for highly elastic fluids such as polymer melts and solutions, even for fluids of low elasticity, the use of this model is not recommended in rapid flows.

Since in reality most of the fluids considered in industrial applications are more non-Newtonian in nature, especially of viscoelsatic type than viscous type, we extend the mixed convection heat transfer work to viscoelsatic fluids flow and heat transfer. The governing equations for this investigation are written in dimensionless form using a set of dimensionless variables and solved numerically using the MackCormak's technique. Numerical results for the velocity, and temperature profiles as well as the local coefficient of friction and local Nusselt number under the effect of viscoelsatic parameter, are presented.

2 Problem formulation

The viscoelastic fluid model used in this work is the so-called second-order, or more commonly second-grade model. This rheological model was first introduced by Rivlin and Ericksen [16] and is generally regarded as one of the simplest viscoelastic fluid models available. For an incompressible homogenous fluid of a second-grade type, the Cauchy stress, Γ is related to the deformation field through:

$$\Gamma = -PI + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3, \tag{1}$$

where $-PI$ is the isotropic part of the stress tensor, α_1, α_2 and α_3 are the material moduli, and A_1, A_2 and A_3 are kinematical tensors defined by [17, 18]:

$$A_1 = (\nabla V) + (\nabla v)^T, \tag{2}$$

$$A_2 = \frac{DA_1}{Dt} + A_1(\nabla V) + (\nabla V)^T A_1, \tag{3}$$

$$A_3 = A_1^2, \tag{4}$$

where ∇V denotes the velocity gradient tensor, and D/Dt is the familiar time derivative. Based on the response of a second-grade fluid to steady shear flow, α_1 is in fact the same as the coefficient of viscosity, μ. Similarly, α_2 and α_3 can be related to the first and second normal stress differences, N_1 and N_2, respectively. Experimental data available for a large number of viscoelastic fluids suggest that N_1 is positive. On the other hand, N_2 is often found to be either positive or negative or zero. Also when N_2 is measured to be non-zero, it is usually found to be much smaller than N_1. This means that for a second-grade fluid to comply with experimental observations, one should have $\alpha_2 > 0$ and $\alpha_3 \leq 0$. Having this in mind, it should be mentioned that there are some controversies around this rheological model, particularly about the sign of α_2 and the size of α_3. Fosdick and Rajagopal [19] argue that for a second-grade rheological model to be thermodynamically compatible, the

Clasius-Duhem inequality should hold together with the Helmoltz free energy being at its minimum whenever the fluid is locally at rest. These thermo dynamical constraints put some severe restrictions on the sign and magnitude of the material moduli:

$$\alpha_1 \geq 0, \quad \alpha_2 \geq 0; \quad \text{and} \quad \alpha_2 + \alpha_3 = 0. \tag{5}$$

The sign proposed above for α_2 is tantamount to saying that N_1 is negative. If this sign is accepted for α_2 then based on equation (5) should be positive. Both signs are in direct contradiction with experimental data available for viscoelastic fluids. The last relationship in equation (5) also suggests the absolute values of N_1 and N_2 are equal to each other which simply cannot confirmed experimentally. Obviously, there certain important issues still unresolved about this controversial rheological model, for a critical review of the second grade model the reader is referred to Dunn [20]. In this work we have decided to take $\alpha_2 < 0$ and to let $\alpha_2 + \alpha_3 \neq 0$ in our second grade fluid. We still go one step further and in accordance with the so-called Weissenberg hypothesis [22], assume that the second normal stress difference is zero for our fluids; i.e., we set $\alpha_3 = 0$ in our model. With the above arguments in mind, we use the deformation rate tensor, 2d, in place of the kinematical tensor A_1 and write the deviatoric part of the stress tensor, τ_{ij} as Beard and Walters [23]:

$$\tau_{ij} = 2\alpha_1 d_{ij} + \alpha_2 \frac{\delta}{\delta t} d_{ij} = 2\alpha_1 \left(d_{ij} + \frac{\alpha_2}{2\alpha_1} \frac{\delta}{\delta t} d_{ij} \right) = 2\mu \left(d_{ij} - \lambda \frac{\delta}{\delta t} d_{ij} \right), \tag{6}$$

where μ is the viscosity, and $-\lambda$ has been used in the place of the ratio $\alpha_2/2\alpha_1$. Since this ratio is a negative number with dimension of time, λ is positive and can be interpreted as the relaxation time of the fluid. The time derivative $\delta/\delta t$ appearing in the above equation is the so-called "upper convective term derivative". This time derivative when applied to the deformation rate tensor reads as Larson [24]:

$$\frac{\delta}{\delta t} d_{ij} = \frac{D}{Dt} d_{ij} + L_{ki} d_{kj} + L_{kj} d_{ik}, \tag{7}$$

where L_{ij} are the components of the velocity gradient tensor ∇V or $\partial u_i/\partial x_j$. Now, assuming the fluid to be incompressible and the flow to be laminar and two-dimensional, the x^*- and y^*- momentum equations are written as:

$$\begin{aligned} \rho \left(\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} \right) &= -\frac{\partial p}{\partial x^*} + \frac{\partial \tau_{x^* x^*}}{\partial x^*} + \frac{\partial \tau_{x^* y^*}}{\partial y^*}, \\ \rho \left(\frac{\partial v^*}{\partial t} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) &= -\frac{\partial p}{\partial y^*} + \frac{\partial \tau_{y^* x^*}}{\partial x^*} + \frac{\partial \tau_{y^* y^*}}{\partial y^*}. \end{aligned} \tag{8}$$

The boundary layer approximations can now be invoked; i.e $x^* = O(1)$, $u^* = O(1)$, $v^* = O(\delta)$. With these orders of magnitudes, the y^* momentum equation reduces to $\partial p/\partial y^* = 0$. Consider laminar mixed convection boundary layer flow of a viscoelsatic fluid over an isothermal vertical flat plate which is heated in an unsteady manner. The problem is described in a rectangular coordinate system attached to the plate such that the

x^*-axis lies along the plate surface and the y-axis is normal to the plate. It is assumed that at time $t \leq 0$, the temperatures of the plate and the viscoelsatic fluid are maintained at the constant temperature T_∞, and at time $t > 0$, the temperature of the plate is impulsively increased to the constant value T_w such that $T_w > T_\infty$. The continuity, momentum and energy equations under the boundary layer and Boussinesq approximations can be written as Cortell [25]:

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0, \tag{9}$$

$$\frac{\partial u^*}{\partial t} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \gamma \frac{\partial^2 u^*}{\partial y^{*2}} - k_0 \begin{pmatrix} u^* \frac{\partial^3 u^*}{\partial x^* \partial y^{*2}} + v^* \frac{\partial^3 u^*}{\partial y^{*3}} \\ - \frac{\partial u^*}{\partial y^*} \frac{\partial^2 u^*}{\partial x^* \partial y^*} + \frac{\partial u^*}{\partial y^*} \frac{\partial^2 u^*}{\partial y^{*2}} \end{pmatrix} + g\beta(T - T_\infty), \tag{10}$$

$$\frac{\partial T}{\partial t} + u^* \frac{\partial T}{\partial x^*} + v^* \frac{\partial T}{\partial y^*} = \alpha \frac{\partial^2 T}{\partial y^{*2}}. \tag{11}$$

Here u^* and v^* are the velocity components in x^* and y^* directions respectively, ν is the kinematic coefficient of viscosity, $k_0 = -\alpha_1/\rho$ is the elastic parameter. Hence in the case of a second order fluid k_0 takes positive values as α_1, where $\alpha = k/\rho c_p$ is the thermal diffusivity and other quantities have their usual meanings.

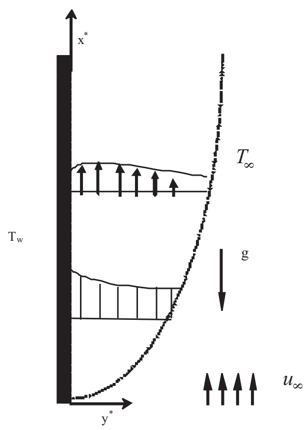

Fig. 1. The transient free convection model for a viscoelastic fluid near a vertical wall.

3 Boundary conditions on velocity

A critical review on the boundary conditions and the existence and uniqueness of the solution has been given by Rajagopal et al. [1]. Most of available literature on boundary layer flow of a viscoelsatic over linearly stretching sheets deal with the three boundary conditions on velocity, which are one less than the number required to solve the problem uniquely, Rollins and Vajravelu [26], Anderson [5], Cortell [25] and Mahapatra and Gupta [27]. Troy et al. [28] derived a unique solution of the problem containing exponential terms of similarity variables. In view of the above discussions on boundary conditions the physical initial and boundary conditions for this problem are given by:

$$t \le 0, \qquad u^* = 0,\ v^* = 0,\ T = T_\infty \qquad \text{for all} \quad x^* \ge 0,\ y^* \ge 0,$$
$$t > 0, \qquad \begin{cases} u^* = 0,\ v^* = 0,\ T = T_\infty & \text{for} \quad x^* = 0,\ y^* \ge 0, \\ u^* = 0,\ v^* = 0,\ T = T_w & \text{for} \quad y^* = 0,\ x^* \ge 0, \\ u^* = u_\infty,\ \partial u^*/\partial y^* = 0,\ T = T_\infty & \text{for} \quad y^* \to \infty, \end{cases} \tag{12}$$

where $(\partial u^*/\partial y^*)_{y^* \to \infty} = 0$ is taken as a boundary layer condition in order to determine boundary layer thicknesses. Defining the non-dimensional variables such that

$$\tau = Gr^{1/2}(\nu/L^2)t, \quad x = x^*/L, \quad y = Gr^{1/4}(y^*/L), \tag{13}$$
$$u = Gr^{-1/2}(\nu/L)u^*, \quad v = Gr^{-1/4}(\nu/L)v^*, \quad \Theta = T - T_\infty/T_w - T_\infty, \tag{14}$$

where L is the characteristic length of the plate and $Gr = g\beta(T_w - T_\infty)L^3/\nu^2$ is the Grashof number and then substituting equations (13) and (14) into equations (9)–(11) yields the following dimensionless equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{15}$$
$$\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} - k_l^*\left(u\frac{\partial^3 u}{\partial x \partial^2 y} + v\frac{\partial^3 u}{\partial^3 y} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial^2 y} - \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y}\right) + \Theta, \tag{16}$$
$$\frac{\partial \Theta}{\partial t} + u\frac{\partial \Theta}{\partial x} + v\frac{\partial \Theta}{\partial y} = \frac{1}{Pr}\frac{\partial^2 \Theta}{\partial y^2}, \tag{17}$$

where $k_l^* = (k_0/L^2)Gr^{1/2}$ is the modified viscoelsatic parameter and $Pr = \mu c_p/k$ is the Prandtl number. It is noted that for the special case of $k_l^* = 0$ the fluid is again a Newtonian fluid. The corresponding dimensionless initial and boundary conditions can be written as

$$t \le 0, \qquad u = 0,\ v = 0,\ \Theta = 1 \qquad \text{for all} \quad x \ge 0,\ y \ge 0,$$
$$t > 0, \qquad \begin{cases} u = 0,\ v = 0,\ \Theta = 1 & \text{for} \quad x = 0,\ y \ge 0, \\ u = 0,\ v = 0,\ \Theta = 1 & \text{for} \quad y = 0,\ x \ge 0, \\ u = M^*,\ \partial u/\partial y = 0,\ \Theta = 0 & \text{for} \quad y \to \infty, \end{cases} \tag{18}$$

where $M^* = \nu^2/[g\beta(T_w - T_\infty)L^3]^{1/2}$ is the mixed convection parameter. The dimensionless skin friction coefficient of friction C_f and local Nusselt number Nu_x are important physical parameters for this type of flow and heat transfer situation Khan and Sajayanad [29] and Sadeghy and Sharifi [30]. They can be defined in dimensionless form as:

$$C_f Gr^{3/4} = (\partial u/\partial y)_{(x,0,t)} - 2k_l^*(\partial u/\partial y)_{(x,0,t)}(\partial v/\partial y)_{(x,0,t)}, \tag{19}$$

$$Nu_x Gr^{-1/4} = -(\partial \Theta/\partial y)_{(x,0,t)}. \tag{20}$$

4 Results and discussion

The transient boundary layer equations represented by equations (15)–(17) are solved subject to the initial and boundary conditions given by Equation (18) using the MacCormack's method which is an explicit finite-difference technique of second-order accuracy in space and time. The details of this method of solution are clearly explained by Anderson [31]. The employed numerical solution is a time marching technique giving the downstream velocity, micro-rotation and temperature profiles using the known upstream profiles. In the present work, the above quantities have been calculated by obtaining explicitly the flow field variables at grid point (i, j) at time $t + \Delta T$ from the known flow field variables at grid points $(i, j), (i + 1, j), (i - 1, j), (i, j - 1)$, and $(i, j + 1)$, at time t. The flow field variables at all other grid points at time are obtained in like fashion. Once the velocity and temperature fields are obtained at a given time, then the local coefficient of friction and local Nusselt number are calculated from equations (19) and (20). In order to verify the accuracy of the present method, comparison of results with the similarity solutions obtained by Oosthuizen and Naylor [32] for the steady laminar free convection over a vertical isothermal impermeable plate of Newtonian fluids is performed and is shown in Table 1. As is clear from Table 1, the results are found to be in excellent agreement. This favorable comparison lends confidence in the numerical results to be reported in the next section.

Table 1. Values of steady state heat transfer coefficient $h(\infty, x, 0)$ along stream-wise direction

$(\partial u/\partial y)_{(\infty,x,0)}$		
$k_l^* = 0,\ Pr = 7.0,\ t = \infty$		
x	Present Results	Oosthuizen and Naylor [32]
0.1	1.09978	1.10400
0.2	0.98760	0.92310
0.4	0.87653	0.84325
0.6	0.82341	0.80214
0.8	0.789234	0.79023
1.0	0.700231	0.72145

The viscoelastic fluid effects on this problem are found to be proportional to dimensionless viscoelastic parameter. The dimensionless viscoelastic parameter $k_l^* = (k_0/L^2)Gr^{1/2}$ is found to be directly proportional to the elasticity of the fluid and Grashof number. It is noted that the influence of the viscoelastic parameter increases as the value of Gr or the buoyancy effect increases for the transient free convection heat transfer problem under consideration.

Figs. 2 and 3 show the transient velocity $u(x, y, \tau)$ and temperature profiles $\Theta(x, y, \tau)$ against dimensionless time $\tau = 0.5, 1, 2, 4, 6, \infty$ for selected values of $Pr = 7.0$, $k_l^* = 0.1$ until steady state solution are obtained. Note the momentum and energy storage inside boundary layers until steady state solutions are obtained. The steady state velocity profiles $u(0.5, y, \infty)$ and temperatures profiles $u(0.5, y, \infty)$ at a distance midway of the plate against viscoelsatic parameter $k_l^* = 0, 0.4, 0.8, 1.2$ and for selected $Pr = 70$ are shown in Figs. 4 and 5. Increasing the viscoelsatic parameter decreases the velocity inside boundary layer and broadens the temperature distribution. This is due to the fact that a higher viscoelastic parameter means a higher tensile stress between fluid layers and consequently, higher resistance to motion which broadens the temperature distribution.

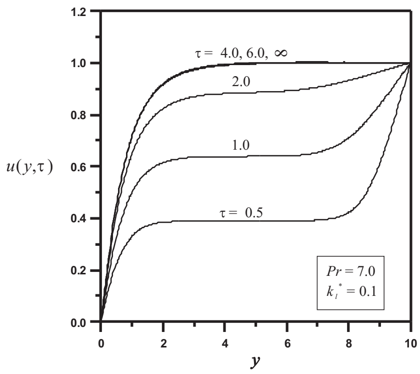

Fig. 2. Dimensionless velocity profiles with dimensionless time.

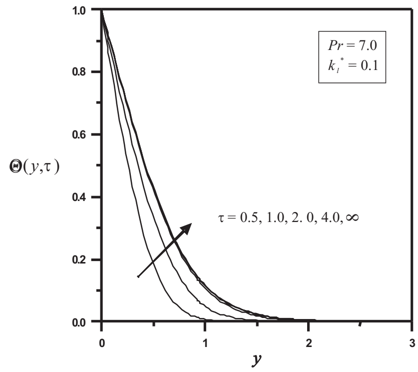

Fig. 3. Dimensionless temperature profiles with dimensionless time.

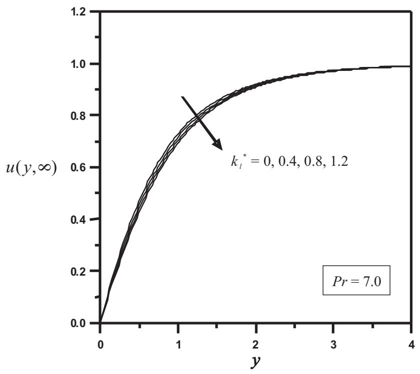

Fig. 4. Dimensionless steady state velocity profiles for different values of the viscoelastic parameter.

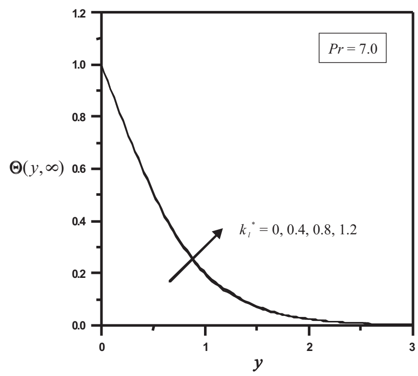

Fig. 5. Dimensionless steady state temperature profiles for selected values of the viscoelastic parameter.

The transient coefficient of friction $C_f Gr^{3/4}$ and local Nusselt numbers $NuGr^{-1/4}$ are drawn in Fig. 6 for different values of the viscoelsatic parameter $k_l^* = 0, 0.4$. It is found that the increasing of the viscoelsatic parameter decreases the local coefficient of friction due to lower velocities of fluid layers and consequently, decreases the local Nusselt numbers. This figure also shows the progress of transient local coefficient of friction and local Nusselt numbers until steady state conditions are reached.

Fig. 7 shows the representative values of the local coefficient of friction $C_f Gr^{3/4}$ and the local Nusselt numbers $NuGr^{-1/4}$ for different values of the viscoelsatic parameter $k_l^* = 0, 0.4$ where both values are decreased as the viscoelsatic parameter is increased. This is due to higher tensile stresses between fluid layers which retard motion and consequently, decreases heat transfer rates.

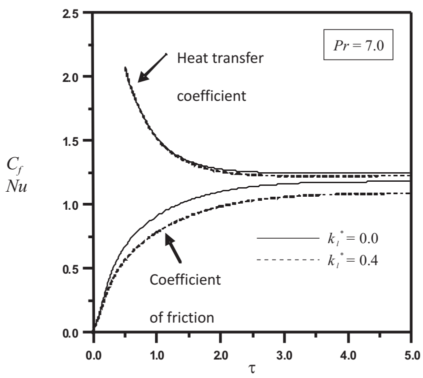

Fig. 6. Transient local coefficient of friction and local Nusselt numbers for different viscoelastic parameters.

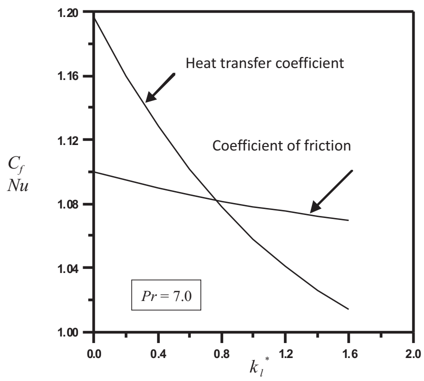

Fig. 7. Steady state local coefficient of friction and local Nusselt numbers for different viscoelastic parameters.

5 Conclusions

The transient laminar mixed convection heat transfer from a vertical surface for a viscoelastic fluid were studied. The governing equations were written in dimensionless form using a set of variables and then solved using an explicit finite-difference technique. Comparisons with previously published work were performed and found to be in excellent agreement. It was found that as the viscoelastic parameter increased, the local coefficient of friction and local Nusselt numbers at any specific time decreased. On the other hand, as the velocities are decreased and temperatures are increased due to favorable tensile stresses between fluid layers which had lowered coefficient of heat transfer.

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