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Quasistatic Adhesive Contact of Piezoelectric Cylinders

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Abstract. We consider two mathematical models which describe the antiplane shear deformation of a piezoelectric cylinder in adhesive contact with a rigid foundation. The material is assumed to be electro-viscoelastic in the first model and electro-elastic in the second one. In both models the process is quasistatic, the foundation is electrically conductive and the adhesion is described with a surface variable, the bonding field. We derive a variational formulation of the models which is given by a system coupling two variational equations for the displacement and the electric potential fields, respectively, and a differential equation for the bonding field. Then we prove the existence of a unique weak solution to each model. We also investigate the behavior of the solution of the electro-viscoelastic problem as the viscosity converges to zero and prove that it converges to the solution of the corresponding electro-elastic problem.

Keywords: antiplane shear, quasistatic process, electro-elastic material, electro-viscoelastic material, contact process, adhesion, fixed point, weak solution.

1 Introduction

The present paper is devoted to the study of quasistatic antiplane contact problems with adhesion for piezoelectric cylinders. Our interest is to present two problems in which both antiplane shear, contact, adhesion and piezoelectric effect are involved, to prove their unique solvability, and to study their link by providing a convergence result.

Antiplane shear deformations are one of the simplest examples of deformations that solids can undergo: in antiplane shear of a cylindrical body, the displacement is parallel to the generators of the cylinder and is independent of the axial coordinate. For this reason, considerable attention has been paid to the modelling of such kind of problems, see for instance [1–3]. Antiplane frictional contact problems were used in geophysics in order to describe pre-earthquake evolution of the regions of hight tectonic activity, see for instance [4, 5] and the references therein. The mathematical analysis of various models for antiplane frictional contact problems can be found in [6–8] and in the recent monograph [9].

Piezoelectric materials are characterized by the coupling between the mechanical and electrical properties, see [10–12] and the references therein. This coupling leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied. Piezoelectric materials for which the mechanical properties are elastic are called electro-elastic materials and those for which the mechanical properties are viscoelastic are called electro-viscoelastic materials. Antiplane contact problems for piezoelectric materials were considered in [13–16]. In [13, 15, 16] the contact was assumed to be frictional and in [14] is was assumed to be adhesive.

Processes of adhesion are important in many industrial settings where parts, usually nonmetallic, are glued together. For this reason the adhesive contact between bodies has recently received increased attention in the literature. General models can be found in [17, 18] and the mathematical analysis of various adhesive contact problems can be found in [19–23]. Existence and uniqueness results in the study of mathematical models which describes the adhesive contact of piezoelectric materials were obtained recently in [24, 25], in the three-dimensional framework.

The present paper represents a continuation of [14]. There, a mathematical model which describes the antiplane shear deformation of a piezoelectric cylinder in adhesive contact with a rigid foundation was considered. The material was assumed to be electro-viscoelastic and the process was assumed to be mechanically dynamic. An existence and uniqueness result of the solution to the model was obtained by using arguments of evolution equations with monotone operators and fixed point. Unlike [14], in the present paper we model the material's behavior by an electro-viscoelastic constitutive law or by an electro-elastic constitutive law; also, we neglect the inertial term in the equation of motion and, therefore, we assume that the process is mechanically quasistatic. This leads to consider two mathematical models, different from that studied in [14], which represents the first trait of novelty of this paper. We derive the variational formulation of the models and then we prove the existence of a unique weak solution, for each model. In addition, we study the link of the two models and provide a converge result, which consists the second trait of novelty of this paper.

The rest of the paper is structured as follows. In Section 2 we present the models for the antiplane adhesive contact of piezoelectric cylinders. Then we introduce the notation, list the assumptions on problem's data and derive the variational formulation of each model. In Section 3 we study the electro-viscoelastic problem for which we state and prove an existence and uniqueness result, Theorem 1. In Section 4 we state and prove an existence and uniqueness result for the electro-elastic problem, Theorem 2. The proof of both theorems are carried out in several steps by constructing intermediate problems for the displacement field, the electric potential and the bonding field. We prove the unique solvability of the intermediate problems, then we consider a contraction mapping whose unique fixed point leads us to construct the solution of the original problem. Finally, in Section 5 we provide a convergence result, Theorem 3. It states that the solution of the electro-viscoelastic problem converges to the solution of the electro-elastic problem as the viscosity converges to zero.

2 Statement of the problems

We consider a piezoelectric body which occupies a region $\mathcal{B} \subset \mathbb{R}^3$, in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axes with a cross-section which is a regular domain Ω in the x_1, x_2 plane, $Ox_1x_2x_3$ being a cartesian coordinate system. The cylinder is assumed to be sufficiently long so that the end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times (-\infty, +\infty)$. The cylinder is acted upon by body forces and electric charges. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2 and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand, such that the one-dimensional measure of Γ_1 and Γ_a, denoted by $meas\,\Gamma_1$ and $meas\,\Gamma_a$, are positive. The cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and therefore the displacement field vanishes there. We assume that surface tractions act on $\Gamma_2 \times (-\infty, +\infty)$, the electrical potential vanishes on $\Gamma_a \times (-\infty, +\infty)$ and a surface electrical charge is prescribed on $\Gamma_b \times (-\infty, +\infty)$. Also, the cylinder is in contact over $\Gamma_3 \times (-\infty, +\infty)$ with a conductive obstacle, the so called foundation; the contact is adhesive and it is modelled with a surface internal variable, the bonding field. We assume that the process is mechanically quasistatic, i.e. we neglect the inertial term in the equation of motion; moreover, we consider the antiplane context described in [14], in which the evolution of the cylinder's state does not depend on the axial coordinate and is described by functions defined on the x_1, x_2 plane.

We denote by $T > 0$ the time interval of interest; everywhere in this paper the dot above represents the derivative with respect to the time, i.e. $\dot{u} = \frac{\partial u}{\partial t}$, and the index that follows a comma represents the partial derivative with respect to the corresponding spatial variable, i.e. $u_{,i} = \frac{\partial u}{\partial x_i}$, $i = 1, 2$. We denote by ν_1, ν_2 the components of the unit normal on Γ and we use the notation

$$
\begin{aligned}
&\operatorname{div}\boldsymbol{\tau} = \tau_{1,1} + \tau_{2,2} \quad \text{for } \boldsymbol{\tau} = \big(\tau_1(x_1,x_2,t), \tau_2(x_1,x_2,t)\big),\\
&\nabla v = (v_{,1}, v_{,2}), \quad \partial_\nu \nu = v_{,1}\nu_1 + v_{,2}\nu_2 \quad \text{for } v = v(x_1,x_2,t).
\end{aligned}
$$

For the first problem we assume that the material is electro-viscoelastic. Then, following the arguments in [14], it follows that the problem can be formulated as follows.

Problem P**.** *Find a displacement field* $u\colon \Omega \times [0,T] \to \mathbb{R}$, *an electric potential* $\varphi\colon \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\beta\colon \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

$$\operatorname{div}(\theta\nabla\dot{u} + \mu\nabla u + e\nabla\varphi) + f_0 = 0 \quad \textit{in} \quad \Omega \times (0,T), \tag{1}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) = q_0 \quad \textit{in} \quad \Omega \times (0,T), \tag{2}$$

$$u = 0 \quad \textit{on} \quad \Gamma_1 \times (0,T), \tag{3}$$

$$\varphi = 0 \quad \textit{on} \quad \Gamma_a \times (0,T). \tag{4}$$

$$\theta\partial_\nu\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi = f_2 \qquad \textit{on}\ \ \Gamma_2 \times (0,T), \tag{5}$$
$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_b \qquad \textit{on}\ \ \Gamma_b \times (0,T), \tag{6}$$
$$-\left(\theta\partial_\nu\dot{u} + \mu\partial_\nu u + e\partial_\nu\varphi\right) = p(\beta)R(u) \qquad \textit{on}\ \ \Gamma_3 \times (0,T), \tag{7}$$
$$e\partial_\nu u - \alpha\partial_\nu\varphi = k(\varphi - \varphi_F) \qquad \textit{on}\ \ \Gamma_3 \times (0,T), \tag{8}$$
$$\dot{\beta} = -\big(\gamma\beta R(u)^2 - \epsilon_a\big)_+ \qquad \textit{on}\ \ \Gamma_3 \times (0,T), \tag{9}$$
$$u(0) = u_0 \qquad \textit{in}\ \ \Omega, \tag{10}$$
$$\beta(0) = \beta_0 \qquad \textit{on}\ \ \Gamma_3. \tag{11}$$

We now describe problem (1)–(11) and provide a brief explanation of the equations and the boundary conditions. More details can be found in [14] where the dynamic version of Problem P was considered.

Equations (1) and (2) represent the balance equations in which θ is a viscosity coefficient, μ is the the Lamé coefficient, α is the electric permittivity constant and e is a piezoelectric coefficient. Here f_0 and q_0 represent the axial component of the body force and the electric charge density, respectively. We note that equation (1) is obtained from the equation of motion by neglecting the inertial term and we use it since the process is assumed to be mechanically quasistatic. Conditions (3) and (4) represent the boundary conditions for the displacement and the electrical potential field and prescribe that these variables vanish on Γ_1 and Γ_a, respectively, during the process. Conditions (5) and (6) represent the traction and electrical condition on Γ_2 and Γ_b, respectively, in which f_2 and q_b represent the densities of the axial component of the traction force and the electric charge, respectively.

Condition (7) represents the traction condition on the contact surface Γ_3 and we use it since the contact is adhesive. Here p is a given function and R is the real valued function defined by

$$R(v) = \begin{cases} -L & \text{if } \ v < L, \\ \ \ v & \text{if } \ |v| \le L, \\ \ \ L & \text{if } \ v > L. \end{cases} \tag{12}$$

with $L > 0$ being a characteristic length of the bonds, see e.g. [18]. It follows from (7) that the shear of the contact surface depends on the bonding field and on the tangential displacement, but only up to the bond length L. The frictional tangential traction is assumed to be much smaller than the adhesive one and, therefore, omitted.

Condition (8) represents the electrical conductivity on the contact surface, where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. This condition shows that the normal component of the electric displacement field is proportional to the difference between the potential on the foundation and the body's surface. We use it since the foundation is electrically conductive and the shear is antiplane, which implies that there is no loss of the contact during the process.

The differential equation (9) describes the evolution of the bonding field in which γ and ϵ_a are given adhesion coefficients, R is defined by (12) and $r_+ = \max\{r, 0\}$. In

(9) and below we use the simplified notation $R(u)^2$ for the square of $R(u)$, i.e. $R(u)^2 = (R(u))^2$. We note that the adhesive process described by (9) is irreversible; indeed, once debonding occurs, bonding cannot be reestablished, since $\dot{\beta} \le 0$. Considering a condition which allows the adhesive process for rebonding would represent an important extension of the results in this paper.

Finally, (10) and (11) represent the initial conditions in which u_0 and β_0 are the prescribed initial displacement and bonding fields, respectively.

For the second problem we assume that the material is electro-elastic, i.e. the viscosity coefficient vanishes. Therefore, we remove the initial condition for the displacement field and take $\theta = 0$ in Problem P, to obtain the following problem.

Problem Q**.** *Find a displacement field* $u\colon \Omega \times [0,T] \to \mathbb{R}$*, an electric potential* $\varphi\colon \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\beta\colon \Gamma_3 \times [0,T] \to \mathbb{R}$ *such that*

$$
\begin{aligned}
&\mathrm{div}\,(\mu\nabla u + e\nabla\varphi) + f_0 = 0 && \textit{in}\ \ \Omega\times(0,T), && (13)\\
&\mathrm{div}\,(e\nabla u - \alpha\nabla\varphi) = q_0 && \textit{in}\ \ \Omega\times(0,T), && (14)\\
&u = 0 && \textit{on}\ \ \Gamma_1\times(0,T), && (15)\\
&\varphi = 0 && \textit{on}\ \ \Gamma_a\times(0,T), && (16)\\
&\mu\partial_\nu u + e\partial_\nu\varphi = f_2 && \textit{on}\ \ \Gamma_2\times(0,T), && (17)\\
&e\partial_\nu u - \alpha\partial_\nu\varphi = q_b && \textit{on}\ \ \Gamma_b\times(0,T), && (18)\\
&-(\mu\partial_\nu u + e\partial_\nu\varphi) = p(\beta)R(u) && \textit{on}\ \ \Gamma_3\times(0,T), && (19)\\
&e\partial_\nu u - \alpha\partial_\nu\varphi = k(\varphi-\varphi_F) && \textit{on}\ \ \Gamma_3\times(0,T), && (20)\\
&\dot{\beta} = -(\gamma\beta R(u)^2 - \epsilon_a)_+ && \textit{on}\ \ \Gamma_3\times(0,T) && (21)\\
&\beta(0) = \beta_0 && \textit{on}\ \ \Gamma_3. && (22)
\end{aligned}
$$

We turn now to the variational formulation of the Problems P and Q. To this end we introduce the function spaces

$$V = \big\{v \in H^1(\Omega)\colon\ v = 0 \ \text{ on } \ \Gamma_1\big\}, \quad W = \big\{\psi \in H^1(\Omega)\colon\ \psi = 0 \ \text{ on } \ \Gamma_a\big\},$$

where, here and below, we write w for the trace on Γ of a function $w \in H^1(\Omega)$. Since $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$
\begin{aligned}
(u,v)_V &= \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x \quad \forall u,v\in V,\\
(\varphi,\psi)_W &= \int_\Omega \nabla\varphi\cdot\nabla\psi\,\mathrm{d}x \quad \forall\varphi,\psi\in W.
\end{aligned}
$$

Moreover, the associated norms

$$\|v\|_V = \|\nabla v\|_{L^2(\Omega)^2} \ \ \forall v\in V, \quad \|\psi\|_W = \|\nabla\psi\|_{L^2(\Omega)^2} \ \ \forall\psi\in W \tag{23}$$

are equivalent with the usual norm $\|\cdot\|_{H^1(\Omega)}$. Also, by Sobolev's trace theorem we deduce that there exists positive constants $c_V > 0$, $c_W > 0$ such that

$$\|v\|_{L^2(\Gamma_3)} \leq c_V \|v\|_V \ \ \forall v \in V, \quad \|\psi\|_{L^2(\Gamma_3)} \leq c_W \|\psi\|_W \ \ \forall \psi \in W. \tag{24}$$

For a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $1 \leq p \leq \infty$, $k = 1, 2, \ldots$; we also denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in X, respectively, with the norms

$$\|u\|_{C([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X,$$
$$\|u\|_{C^1([0,T];X)} = \max_{t\in[0,T]} \|u(t)\|_X + \max_{t\in[0,T]} \|\dot{u}(t)\|_X.$$

Finally, we use the set

$$\mathcal{Z} = \big\{\theta \in C\big([0,T]; L^2(\Gamma_3)\big) : \ 0 \leq \theta(t) \leq 1 \ \ \forall t \in [0,T], \ \text{ a.e. on } \Gamma_3\big\}$$

and we recall that if X is reflexive, then $W^{1,\infty}(0,T;X)$ is the space of Lipschitz continuous functions defined on $[0,T]$ with values in X.

We list now the assumptions on the problem's data. We assume that the viscosity coefficient and the electric permittivity coefficient satisfy

$$\theta \in L^\infty(\Omega) \ \text{ and there exists } \ \theta^* > 0 \ \text{ such that } \ \theta(\boldsymbol{x}) \geq \theta^* \ \text{ a.e. } \ \boldsymbol{x} \in \Omega, \tag{25}$$
$$\alpha \in L^\infty(\Omega) \ \text{ and there exists } \ \alpha^* > 0 \ \text{ such that } \ \alpha(\boldsymbol{x}) \geq \alpha^* \ \text{ a.e. } \ \boldsymbol{x} \in \Omega. \tag{26}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \quad \text{and} \quad \mu(\boldsymbol{x}) \geq 0 \ \text{ a.e. } \ \boldsymbol{x} \in \Omega, \tag{27}$$
$$e \in L^\infty(\Omega). \tag{28}$$

The tangential function p is such that

$$\begin{cases} \text{(a)} \ p \colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b)} \ \text{There exists } L_p > 0 \text{ such that} \\ \qquad |p(\boldsymbol{x},\beta_1) - p(\boldsymbol{x},\beta_2)| \leq L_p|\beta_1 - \beta_2| \ \ \forall \beta_1, \beta_2 \in \mathbb{R}, \quad \text{a.e. } \ \boldsymbol{x} \in \Gamma_3. \\ \text{(c)} \ \text{There exists } \ M > 0 \ \text{ such that } \ |p(\boldsymbol{x},\beta)| \leq M \ \ \forall \beta \in \mathbb{R}, \quad \text{a.e. } \ \boldsymbol{x} \in \Gamma_3. \\ \text{(d)} \ \text{The mapping } \ \boldsymbol{x} \mapsto p(\boldsymbol{x},\beta) \ \text{ is measurable on } \ \Gamma_3 \ \ \forall \beta \in \mathbb{R}. \end{cases} \tag{29}$$

The adhesion coefficients γ and ϵ_a satisfy the conditions

$$\gamma \in L^\infty(\Gamma_3) \quad \text{and} \quad \gamma(\boldsymbol{x}) \geq 0 \ \text{ a.e. } \ \boldsymbol{x} \in \Gamma_3, \tag{30}$$
$$\epsilon_a \in L^2(\Gamma_3) \quad \text{and} \ \ \epsilon_a(\boldsymbol{x}) \geq 0 \ \text{ a.e. } \ \boldsymbol{x} \in \Gamma_3. \tag{31}$$

The forces, tractions, volume and surface free charges densities have the regularity

$$f_0 \in W^{1,\infty}\big(0,T;L^2(\Omega)\big), \quad f_2 \in W^{1,\infty}\big(0,T;L^2(\Gamma_2)\big), \tag{32}$$
$$q_0 \in W^{1,\infty}\big(0,T;L^2(\Omega)\big), \quad q_b \in W^{1,\infty}\big(0,T;L^2(\Gamma_b)\big), \tag{33}$$

and the electric conductivity coefficient satisfies

$$k \in L^\infty(\Gamma_3) \quad \text{and} \quad k(\boldsymbol{x}) \geq 0 \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{34}$$

Finally, we assume that the electric potential of the foundation and the initial data are such that

$$\varphi_F \in W^{1,\infty}\big(0,T;L^2(\Gamma_3)\big), \tag{35}$$
$$u_0 \in V, \tag{36}$$
$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0(\boldsymbol{x}) \leq 1 \ \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \tag{37}$$

Next, we define bilinear forms $a_\theta \colon V \times V \to \mathbb{R}$, $a_\mu \colon V \times V \to \mathbb{R}$, $a_e \colon V \times W \to \mathbb{R}$, $a_e^* \colon W \times V \to \mathbb{R}$ and $a_\alpha \colon W \times W \to \mathbb{R}$ by equalities

$$a_\theta(u,v) = \int\limits_\Omega \theta \nabla u \cdot \nabla v \, \mathrm{d}x, \tag{38}$$
$$a_\mu(u,v) = \int\limits_\Omega \mu \nabla u \cdot \nabla v \, \mathrm{d}x, \tag{39}$$
$$a_e(u,\varphi) = \int_\Omega e \nabla u \cdot \nabla \varphi \, \mathrm{d}x = a_e^*(\varphi,u), \tag{40}$$
$$a_\alpha(\varphi,\psi) = \int\limits_\Omega \alpha \nabla \varphi \cdot \nabla \psi \, \mathrm{d}x + \int\limits_{\Gamma_3} k\, \varphi \psi \, \mathrm{d}x, \tag{41}$$

for all $u,\ v \in V$, $\varphi,\ \psi \in W$. Assumptions (25)–(28) and (34) imply that the integrals above are well defined and, using (23) and (24), it follows that the forms a_θ, a_μ, a_e, a_e^* and a_α are continuous; moreover, the forms a_θ, a_μ and a_α are symmetric and, in addition, the form a_θ is V-elliptic and the form a_α is W-elliptic, i.e.

$$a_\theta(v,v) \geq \theta^* \|v\|_V^2 \quad \forall v \in V, \tag{42}$$
$$a_\alpha(\psi,\psi) \geq \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{43}$$

We also define the mappings

$$f \colon [0,T] \to V, \quad q \colon [0,T] \to W \quad \text{and} \quad j \colon L^2(\Gamma_3) \times V \times V \to \mathbb{R},$$

respectively, by

$$\big(f(t),v\big)_V = \int_\Omega f_0(t)v\,\mathrm{d}x + \int_{\Gamma_2} f_2(t)v\,\mathrm{d}a, \tag{44}$$

$$\big(q(t),\psi\big)_W = \int_\Omega q_0(t)\psi\,\mathrm{d}x - \int_{\Gamma_b} q_b(t)\psi\,\mathrm{d}a + \int_{\Gamma_3} k\,\varphi_F\psi\,\mathrm{d}a, \tag{45}$$

$$j(\beta,v,w) = \int_{\Gamma_3} p(\beta)R(v)w\,\mathrm{d}a, \tag{46}$$

for all $v,\ w\in V$, $\psi\in W$, $\beta\in L^2(\Gamma_3)$ and $t\in[0,T]$. The definition of f and q is based on Riesz's representation theorem; moreover, it follows from (32)–(35) that

$$f\in W^{1,\infty}(0,T;V), \tag{47}$$

$$q\in W^{1,\infty}(0,T;W). \tag{48}$$

Next, we perform integrals par parts and use notation (38)–(41), (44)–(46) to obtain the following variational formulation of the electro-viscoelastic Problem P.

Problem P_V**.** *Find a displacement field* $u\colon[0,T]\to V$*, an electric potential field* $\varphi\colon[0,T]\to W$ *and a bonding field* $\beta\colon[0,T]\to L^2(\Gamma_3)$ *such that, for all* $t\in[0,T]$*,*

$$\begin{aligned}&a_\theta\big(\dot u(t),w\big)+a_\mu\big(u(t),w\big)+a_e^*\big(\varphi(t),w\big)+j\big(\beta(t),u(t),w\big)\\&\qquad=\big(f(t),w\big)_V\quad\forall w\in V,\end{aligned} \tag{49}$$

$$a_\alpha\big(\varphi(t),\psi\big)-a_e\big(u(t),\psi\big)=\big(q(t),\psi\big)_W\quad\forall\psi\in W, \tag{50}$$

$$\dot\beta(t)=-\big(\gamma\beta(t)R\big(u(t)\big)^2-\epsilon_a\big)_+, \tag{51}$$

and

$$u(0)=u_0,\quad \beta(0)=\beta_0. \tag{52}$$

Similar arguments lead to the following variational formulation of the electro-elastic Problem Q.

Problem Q_V**.** *Find a displacement field* $u\colon[0,T]\to V$*, an electric potential field* $\varphi\colon[0,T]\to W$ *and a bonding field* $\beta\colon[0,T]\to L^2(\Gamma_3)$ *such that, for all* $t\in[0,T]$*,*

$$a_\mu\big(u(t),w\big)+a_e^*\big(\varphi(t),w\big)+j\big(\beta(t),u(t),w\big)=\big(f(t),w\big)_V\quad\forall w\in V, \tag{53}$$

$$a_\alpha\big(\varphi(t),\psi\big)-a_e\big(u(t),\psi\big)=\big(q(t),\psi\big)_W\quad\forall\psi\in W, \tag{54}$$

$$\dot\beta(t)=-\big(\gamma\beta(t)R\big(u(t)\big)^2-\epsilon_a\big)_+, \tag{55}$$

and

$$\beta(0)=\beta_0. \tag{56}$$

Well-posedness of the variational Problems P_V and Q_V will be proved in Theorems 1 and 2 below. We conclude by these theorems the existence of a unique *weak solution* to Problems P and Q, respectively.

3 Study of the electro-viscoelastic problem

Our main existence and uniqueness result in the study Problem P_V is the following.

Theorem 1. *Assume that* (25)–(37) *hold. Then, there exists a unique solution of Problem* (49)–(52)*. Moreover, the solution satisfies*

$$u \in W^{2,\infty}(0,T;V), \tag{57}$$

$$\varphi \in W^{1,\infty}(0,T;W), \tag{58}$$

$$\beta \in W^{1,\infty}\big(0,T;L^2(\Gamma_3)\big) \cap \mathcal{Z}. \tag{59}$$

The proof of Theorem 1 will be carried out in several steps and is based on arguments similar to those used in [14]. The modifications arise mainly in the treatment of the variational equation (49) since, unlike [14], here the process is assumed to be mechanically quasistatic. The treatment of the variational equation (50) as well as that of the differential equation (51) is similar to that in [14] and, for this reason, we omit the corresponding details. We assume in what follows that (25)–(37) hold and below in this section we denote by c a generic positive constant which may depend on Ω, Γ_1, Γ_2, Γ_3, Γ_a, Γ_b, θ, μ, e, α p, L and T, but does not depend on the time, nor on the rest of the input data, and whose value may change from place to place.

Let $\eta \in C([0,T];V)$ be given. The first step of the proof is given by the following existence and uniqueness result for the displacement field.

Lemma 1. *There exists a unique function $u_\eta \in C^1([0,T];V)$ such that*

$$a_\theta\big(\dot{u}_\eta(t),w\big) + \big(\eta(t),w\big)_V = \big(f(t),w\big)_V \quad \forall w \in V,\ t \in [0,T], \tag{60}$$

$$u_\eta(0) = u_0. \tag{61}$$

Proof. We use the properties of the bilinear form a_θ and the Lax-Milgram lemma to see that, for all $t \in [0,T]$, there exists a unique element $v_\eta(t) \in V$ such that

$$a_\theta\big(v_\eta(t),w\big) + \big(\eta(t),w\big)_V = \big(f(t),w\big)_V \quad \forall w \in V. \tag{62}$$

Consider now t_1, $t_2 \in [0,T]$; using (62) and (42) we find that

$$\theta^* \, \|v_\eta(t_1) - v_\eta(t_2)\|_V \le \|\eta(t_1) - \eta(t_2)\|_V + \|f(t_1) - f(t_2)\|_V. \tag{63}$$

We note that regularity of f and η combined with (63) imply that $v_\eta \in C([0,T];V)$. Let $u_\eta \colon [0,T] \to V$ be the function defined by

$$u_\eta(t) = \int_0^t v_\eta(s)\, ds + u_0 \quad \forall t \in [0,T]. \tag{64}$$

It follows from (62) and (64) that u_η is a solution of the problem (60)–(61) and it satisfies $u_\eta \in C^1([0,T];V)$. This concludes the existence part of Lemma 1. The uniqueness part follows from the uniqueness of the solution of the variational equation (62), at any $t \in [0,T]$. $\square$

In the next two steps we use the displacement field u_η obtained in Lemma 1 to obtain the following existence and uniqueness result for the electric potential field and the bonding field, respectively.

Lemma 2. *There exists a unique function $\varphi_\eta \in W^{1,\infty}(0,T;W)$ such that*

$$a_\alpha\big(\varphi_\eta(t),\psi\big) - a_e\big(u_\eta(t),\psi\big) = \big(q(t),\psi\big)_W \quad \forall \psi \in W,\ t \in [0,T]. \tag{65}$$

Lemma 3. *There exists a unique function $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ such that*

$$\dot\beta_\eta(t) = -\big(\gamma\beta_\eta(t)R\big(u_\eta(t)\big)^2 - \epsilon_a\big)_+ \quad \forall t \in [0,T], \tag{66}$$

$$\beta_\eta(0) = \beta_0. \tag{67}$$

The proof of Lemma 2 is based on arguments similar to those used in the proof of Lemma 1, see also [14]. The proof of Lemma 3 can be found in [14]; it is based on a version of the Cauchy-Lipschitz theorem, see for instance [23, p. 48] .

Now, for $\eta \in C([0,T];V)$ we denote by u_η, φ_η and β_η the functions obtained in Lemmas 1, 2 and 3, respectively. We use Riesz's representation theorem to define the function $\Lambda\eta\colon [0,T] \to V$ by

$$\big(\Lambda\eta(t),w\big)_V = a_\mu\big(u_\eta(t),w\big) + a_e^*\big(\varphi_\eta(t),w\big) + j\big(\beta_\eta(t),u_\eta(t),w\big) \tag{68}$$

for all $w \in V$ and $t \in [0,T]$. We have the following result.

Lemma 4. *For all $\eta \in C([0,T];V)$ the function $\Lambda\eta$ belongs to $W^{1,\infty}(0,T;V)$. Moreover, there exists a unique element $\eta^* \in W^{1,\infty}(0,T;V)$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let $\eta \in C([0,T];V)$ and let $t_1,\ t_2 \in [0,T]$. Using (68), the continuity of the bilinear forms a_μ and a_e^* and (46), we obtain

$$\begin{aligned}\|\Lambda\eta(t_1)-\Lambda\eta(t_2)\|_V \le c\,\big(&\|u_\eta(t_1)-u_\eta(t_2)\|_V+\|\varphi_\eta(t_1)-\varphi_\eta(t_2)\|_W\\ &+\big\|p\big(\beta_\eta(t_1)\big)R\big(u_\eta(t_1)\big)-p\big(\beta_\eta(t_2)\big)R\big(u_\eta(t_2)\big)\big\|_{L^2(\Gamma_3)}\big).\end{aligned}$$

Now, keeping in mind (24), assumptions on the function p, the inequality $0 \le \beta_\eta \le 1$ and the properties of the operator R we find that

$$\begin{aligned}\|\Lambda\eta(t_1)-\Lambda\eta(t_2)\|_V \le c\,\big(&\|u_\eta(t_1)-u_\eta(t_2)\|_V\\ &+\|\varphi_\eta(t_1)-\varphi_\eta(t_2)\|_W+\|\beta_\eta(t_1)-\beta_\eta(t_2)\|_{L^2(\Gamma_3)}\big).\end{aligned} \tag{69}$$

Since $u_\eta \in C^1([0,T];V)$, $\varphi_\eta \in W^{1,\infty}(0,T;W)$ and $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$, we deduce from inequality (69) that $\Lambda\eta \in W^{1,\infty}(0,T;V)$.

Let now $\eta_1, \eta_2 \in C([0,T];V)$ and let $t \in [0,T]$. In what follows we use the notation $u_i = u_{\eta_i}$, $v_i = v_{\eta_i} = \dot u_{\eta_i}$, $\varphi_i = \varphi_{\eta_i}$ and $\beta_i = \beta_{\eta_i}$ for $i = 1,2$. Using arguments similar to those in the proof of (69) we find that

$$\begin{aligned}\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_V \le c\,\big(&\|u_1(t)-u_2(t)\|_V\\ &+\|\varphi_1(t)-\varphi_2(t)\|_W+\|\beta_1(t)-\beta_2(t)\|_{L^2(\Gamma_3)}\big).\end{aligned} \tag{70}$$

On the other hand, from (65), (66) and (67), it was proved in [14] that

$$\|\varphi_1(t) - \varphi_2(t)\|_W \leq c\,\|u_1(t) - u_2(t)\|_V, \tag{71}$$

$$\|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)} \leq c \int_0^t \|u_1(s) - u_2(s)\|_V \,\mathrm{d}s. \tag{72}$$

We combine now the inequalities (70), (71) and (72) to obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c\,\|u_1(t) - u_2(t)\|_V + c \int_0^t \|u_1(s) - u_2(s)\|_V \,\mathrm{d}s.$$

Also, since u_1 and u_2 have the same initial value it follows that

$$\|u_1(t) - u_2(t)\|_V \leq \int_0^t \|v_1(s) - v_2(s)\|_V \,\mathrm{d}s.$$

We use now the last two inequalities to obtain

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c \int_0^t \|v_1(s) - v_2(s)\|_V \,\mathrm{d}s. \tag{73}$$

Next, (62) and the properties of the form a_θ yield

$$\|v_1(s) - v_2(s)\|_V \leq c\,\|\eta_1(s) - \eta_2(s)\|_V \quad \forall s \in [0, T]$$

and, using this inequality in (73), we deduce that

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_V \leq c \int_0^t \|\eta_1(s) - \eta_2(s)\|_V \,\mathrm{d}s. \tag{74}$$

Reiterating this inequality m times yields

$$\|\Lambda^m\eta_1 - \Lambda^m\eta_2\|_{C([0,T];V)} \leq \frac{c^m T^m}{m!}\|\eta_1 - \eta_2\|_{C([0,T];V)},$$

which implies that for m sufficiently large a power Λ^m of Λ is a contraction in the Banach space $C([0,T];V)$. Therefore, there exists a unique element $\eta^* \in C([0,T];V)$ such that and $\Lambda\eta^* = \eta^*$. The regularity $\eta^* \in W^{1,\infty}(0,T;V)$ follows from the regularity $\Lambda\eta^* \in W^{1,\infty}(0,T;V)$, which concludes the proof. $\square$

Now, we have all the ingredients necessary to prove Theorem 1.

Proof of Theorem 1. Existence. Let $\eta^* \in W^{1,\infty}(0,T;V)$ be the fixed point of the operator Λ and let u, φ, β be the functions defined in Lemmas 1, 2 and 3, respectively, for $\eta = \eta^*$, i.e. $u = u_{\eta^*}$, $\varphi = \varphi_{\eta^*}$, $\beta = \beta_{\eta^*}$. Clearly, equalities (50)–(52) hold from Lemmas 1–3. Moreover, since $\eta^* = \Lambda\eta^*$ it follows from (60) and (68) that (49) holds, too. The regularity of the solution expressed in (58) and (59) follows from Lemmas 2 and 3, respectively. Also, it follows form (63), (47) and (64) that $\dot{u} \in W^{1,\infty}(0,T;V)$, i.e. u satisfies (57). We conclude that (u,φ,β) is a solution of Problem P_V and it satisfies (57)–(59).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of Λ and the uniqueness part in Lemmas 1–3. $\square$

4 Study of the electro-elastic problem

The proof of the unique solvability of the electro-elastic Problem Q_V could be obtained by using arguments similar to those used in the proof of Theorem 1. However, since the viscosity term is missing, in this case the corresponding inequality (74) would not contain an integral term. As a consequence, the use of the Banach fixed point arguments would require a smallness assumption on the problem's data and therefore would restrict the solvability of the problem. To avoid this restriction, in the study of electro-elastic Problem Q_V we shall use a method which is different from that used in the study of the electro-viscoelastic Problem P_V. We start by reinforcing assumption (27) as follows:

$$\mu \in L^\infty(\Omega) \text{ and there exists } \mu^* > 0 \text{ such that } \mu(\boldsymbol{x}) \geq \mu^* \text{ a.e. } \boldsymbol{x} \in \Omega. \tag{75}$$

We note that in this case the bilinear form a_μ is V-elliptic, since it safisfies

$$a_\mu(v,v) \geq \mu^* \|v\|_V^2 \quad \forall v \in V. \tag{76}$$

Our main result concerning the unique solvability of Problem Q_V is the following.

Theorem 2. *Assume that* (26)–(37) *and* (75) *hold. Then, there exists a unique solution of Problem* (53)–(56)*. Moreover, the solution satisfies*

$$u \in W^{1,\infty}(0,T;V), \tag{77}$$

$$\varphi \in W^{1,\infty}(0,T;W), \tag{78}$$

$$\beta \in W^{1,\infty}\big(0,T;L^2(\Gamma_3)\big) \cap \mathcal{Z}. \tag{79}$$

The proof of Theorem 2 will be carried out in several steps. We assume in what follows that (26)–(37) and (75) hold; below in this section c will denote a generic positive constant which may depend on Ω, Γ_1, Γ_2, Γ_3, Γ_a, Γ_b, μ, e, α p, L and T, but does not depend on t, nor on the rest of the input data, and whose value may change from place to place.

Let $\beta \in \mathcal{Z}$ be given. The first step of the proof is given by the following existence and uniqueness result.

Lemma 5. *There exists a unique couple* $(u_\beta, \varphi_\beta) \in C([0,T]; V \times W)$ *such that, for all* $t \in [0,T]$,

$$a_\mu\big(u_\beta(t), w\big) + a_e^*\big(\varphi_\beta(t), w\big) + j\big(\beta(t), u(t), w\big) = \big(f(t), w\big)_V \quad \forall w \in V, \tag{80}$$

$$a_\alpha\big(\varphi_\beta(t), \psi\big) - a_e\big(u_\beta(t), \psi\big) = \big(q(t), \psi\big)_W \quad \forall \psi \in W. \tag{81}$$

Proof. We consider the product space $X = V \times W$ together with the inner product

$$(x, y)_X = (u, w)_V + (\varphi, \psi)_W \quad \forall x = (u, \varphi) \in X,\ \forall y = (w, \psi) \in X \tag{82}$$

and the associated norm $\|\cdot\|_X$. Let $t \in [0,T]$ be given; we define the operator $A_\beta(t)\colon X \to X$ and the element $h(t) \in X$ by

$$\big(A_\beta(t)\, x, y\big)_X = a_\mu(u, w) + a_\alpha(\varphi, \psi) + a_e^*(\varphi, w) - a_e(u, \psi) + j\big(\beta(t), u, w\big)$$
$$\forall x = (u, \varphi),\ \ \forall y = (w, \psi) \in X, \tag{83}$$

$$\big(h(t), y\big)_X = \big(f(t), w\big)_V + \big(q(t), \psi\big)_W \quad \forall y = (w, \psi) \in X. \tag{84}$$

It is easy to see that equalities (80) and (81) hold if and only if the element $x_\beta(t) = (u_\beta(t), \varphi_\beta(t)) \in X$ satisfies the following equation in X:

$$A_\beta(t)\, x_\beta(t) = h(t). \tag{85}$$

In order to solve (85), we investigate the properties of the operator $A_\beta(t)$. First, we use (46), (29) and (12) to see that

$$j\big(\beta(t), u_1, u_2 - u_1\big) + j\big(\beta(t), u_2, u_1 - u_2\big) \le 0,$$
$$\big|j\big(\beta(t), u_1, v\big) - j\big(\beta(t), u_2, v\big)\big| \le c\, \|u_1 - u_2\|_V \|v\|_V,$$

for all $u_1,\ u_2,\ v \in V$. Next, we use the previous two inequalities, (83), (76), (43) and (82) to find that $A_\beta(t)$ satisfies

$$\big(A_\beta(t)\, x_1 - A_\beta(t)\, x_2, x_1 - x_2\big)_X \ge c\, \|x_1 - x_2\|_X^2,$$
$$\|A_\beta(t)\, x_1 - A_\beta(t)\, x_2\|_X \le c\, \|x_1 - x_2\|_X,$$

for all $x_1,\ x_2 \in X$. We conclude that $A_\beta(t)$ is a strongly monotone Lipschitz continuous operator and therefore, using a standard existence and uniqueness result, we obtain the existence of a unique element $x_\beta(t) \in X$ which solves (85). We conclude from above that there exists a unique couple of functions $(u_\beta(t), \varphi_\beta(t))$ which solve (80) and (81), at any $t \in [0,T]$.

Next, we let $t_1,\ t_2 \in [0,T]$ and use the notation $u_\beta(t_i) = u_i$, $\varphi_\beta(t_i) = \varphi_i$, $\beta(t_i) = \beta_i$, $f(t_i) = f_i$, $q(t_i) = q_i$ for $i = 1,\ 2$. We use standard arguments in (80) and (81) to find

$$a_\mu(u_1 - u_2, u_1 - u_2) + a_e^*(\varphi_1 - \varphi_2, u_1 - u_2)$$
$$+\, j(\beta_1, u_1, u_1 - u_2) - j(\beta_2, u_2, u_1 - u_2) = (f_1 - f_2, u_1 - u_2)_V,$$
$$a_\alpha(\varphi_1 - \varphi_2, \varphi_1 - \varphi_2) - a_e(u_1 - u_2, \varphi_1 - \varphi_2) = (q_1 - q_2, \varphi_1 - \varphi_2)_W.$$

Then, we add these equalities and use (76) and (43) to obtain

$$\begin{aligned}\mu^* \|u_1 - u_2\|_V^2 + \alpha^* \|\varphi_1 - \varphi_2\|_W^2 \\ \le (f_1 - f_2, u_1 - u_2)_V + (q_1 - q_2, \varphi_1 - \varphi_2)_W \\ + j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2).\end{aligned} \tag{86}$$

We use again (46), (29) and (12) to see that

$$j(\beta_1, u_1, u_2 - u_1) + j(\beta_2, u_2, u_1 - u_2) \le c\, \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)} \|u_1 - u_2\|_V \tag{87}$$

and therefore, combining (86) and (87), after some algebra we find

$$\begin{aligned}\|u_1 - u_2\|_V + \|\varphi_1 - \varphi_2\|_W \\ \le c\left(\|f_1 - f_2\|_V + \|q_1 - q_2\|_W + \|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}\right).\end{aligned} \tag{88}$$

This inequality and the regularity of the functions f, q and β show that $u_\beta \in C([0,T];V)$ and $\varphi_\beta \in C([0,T];W)$. Thus, we conclude the existence part in Lemma 5 and we note that the uniqueness of the solution follows from of the unique solvability of (80) and (81), at any $t \in [0,T]$. □

In the next step we use the displacement field u_β obtained in Lemma 5 and the arguments used in the proof of Lemma 3 to obtain the following existence and uniqueness result for the bonding field.

Lemma 6. *There exists a unique function* $\xi_\beta \in W^{1,\infty}(0,T;L^2(\Gamma_3)) \cap \mathcal{Z}$ *such that*

$$\dot{\xi}_\beta(t) = -\left(\gamma \xi_\beta(t) R\big(u_\beta(t)\big)^2 - \epsilon_a\right)_+ \quad \forall t \in [0,T], \tag{89}$$

$$\xi_\beta(0) = \beta_0. \tag{90}$$

It follows from Lemma 6 that for all $\beta \in \mathcal{Z}$ the solution ξ_β of problem (89)–(90) belongs to $\mathcal{Z}$. Therefore, we may define the operator $\Lambda\colon \mathcal{Z} \to \mathcal{Z}$ by

$$\Lambda\beta = \xi_\beta. \tag{91}$$

Moreover, we have the following result.

Lemma 7. *There exists a unique element* $\beta^* \in \mathcal{Z}$ *such that* $\Lambda\beta^* = \beta^*$.

Proof. Suppose that β_1 and β_2 are two functions in $\mathcal{Z}$ and denote by u_i, φ_i and ξ_i the functions obtained in Lemmas 5 and 6 for $\beta = \beta_i$, $i = 1,2$. Let $t \in [0,T]$. We use arguments similar to those used in the proof of (88) to deduce that

$$\|u_1(t) - u_2(t)\|_V + \|\varphi_1(t) - \varphi_2(t)\|_V \le c\, \|\beta_1(t) - \beta_2(t)\|_{L^2(\Gamma_3)}. \tag{92}$$

On the other hand, (91) and the estimate (72) obtained for the Cauchy problem (89)–(90) leads to

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \le c \int_0^t \|u_1(s) - u_2(s)\|_V \,\mathrm{d}s. \tag{93}$$

We now combine (93) and (92) to see that

$$\|\Lambda\beta_1(t) - \Lambda\beta_2(t)\|_{L^2(\Gamma_3)} \le c \int_0^t \|\beta_1(s) - \beta_2(s)\|_{L^2(\Gamma_3)}\, \mathrm{d}s$$

and, by reiterating this last inequality m times, we obtain

$$\|\Lambda^m\beta_1 - \Lambda^m\beta_2\|_{C([0,T];L^2(\Gamma_3))} \le \frac{c^m T^m}{m!}\,\|\beta_1 - \beta_2\|_{C([0,T];L^2(\Gamma_3))}. \tag{94}$$

Recall that $\mathcal{Z}$ is a nonempty closed set in the Banach space $C([0,T];L^2(\Gamma_3))$ and note that (94) shows that for m sufficiently large the operator $\Lambda^m : \mathcal{Z} \to \mathcal{Z}$ is a contraction. Then we use the Banach fixed point theorem to conclude the proof.

Now, we have all the ingredients needed to prove Theorem 2.

Proof of Theorem 2. Existence. Let $\beta^* \in \mathcal{Z}$ be the fixed point of Λ and let (u^*, φ^*) be the functions of obtained in Lemma 5 for $\beta = \beta^*$, i.e., $u^* = u_{\beta^*}$ and $\varphi^* = \varphi_{\beta^*}$. It follows from (80) and (81) that the functions u^*, φ^*, β^* satisfy (53) and (54), respectively. Moreover, since $\Lambda\beta^* = \beta^*$ it follows from (89) and (90) that u^* and β^* satisfy (55) and (56), too. Next, since $\beta^* = \Lambda\beta^* = \xi_{\beta^*} \in W^{1,\infty}(0,T,L^2(\Gamma_3))$, using (47), (48) and (88) it follows that the functions u^* and φ^* have the regularity expressed in (77) and (78), respectively, and Lemma 6 shows that β^* has the regularity expressed in (79). We conclude that $(u^*, \varphi^*, \beta^*)$ is a solution of Problem Q_V and it satisfies (77)–(79).

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of Λ and the uniqueness part in Lemmas 5–3. □

5 A convergence result

In this section we investigate the behavior of the weak solution of the electro-viscoelastic Problem P_V as the viscosity converges to zero. In order to outline the dependence on the viscosity coefficient θ, we reformulate Problem P_V as follows.

Problem P_V^θ. *Find a displacement field* $u_\theta\colon [0,T] \to V$, *an electric potential field* $\varphi_\theta\colon [0,T] \to W$ *and a bonding field* $\beta_\theta\colon [0,T] \to L^2(\Gamma_3)$ *such that, for all* $t \in [0,T]$,

$$\begin{aligned} &a_\theta\big(\dot u_\theta(t), w\big) + a_\mu\big(u_\theta(t), w\big) + a_e^*\big(\varphi_\theta(t), w\big) + j\big(\beta_\theta(t), u_\theta(t), w\big) \\ &\qquad = \big(f(t), w\big)_V \quad \forall w \in V, \end{aligned} \tag{95}$$

$$a_\alpha\big(\varphi_\theta(t), \psi\big) - a_e\big(u_\theta(t), \psi\big) = \big(q(t), \psi\big)_W \quad \forall \psi \in W, \tag{96}$$

$$\dot\beta_\theta(t) = -\big(\gamma\beta_\theta(t)R\big(u_\theta(t)\big)^2 - \epsilon_a\big)_+, \tag{97}$$

and

$$u_\theta(0) = u_0, \quad \beta_\theta(0) = \beta_0. \tag{98}$$

Also, we assume in this section that the function p does not depend on the bonding field and therefore we replace (29) by assumption

$$p\colon \Gamma_3 \to \mathbb{R} \quad \text{and} \quad p \in L^\infty(\Gamma_3). \tag{99}$$

Assume in what follows that (25)–(28), (30)–(37), (75) and (99) hold. Then, it follows from Theorem 1 that Problem (95)–(98) has a unique solution $(u_\theta, \varphi_\theta, \beta_\theta)$ with the regularity expressed in (57)–(59). Also, it follows from Theorem 2 that Problem (53)–(56) has a unique solution (u, φ, β) which satisfies (77)–(79). Consider now the additional assumptions

$$\|\theta\|^2_{L^\infty(\Omega)} \to 0, \tag{100}$$

$$\frac{1}{\theta^*}\,\|\theta\|^2_{L^\infty(\Omega)} \to 0, \tag{101}$$

$$u(0) = u_0. \tag{102}$$

It is easy to see that (101) implies (100) but the converse is not true.

The convergence of the solution $(u_\theta, \varphi_\theta, \beta_\theta)$ of Problem P^θ_V to the solution (u, φ, β) of Problem Q_V is given by the following result.

Theorem 3. *Assume that* (25)–(28), (30)–(37), (75) *and* (99) *hold.*

(i) *If* (100) *holds, then*

$$\|u_\theta - u\|_{L^2(0,T;V)} \to 0, \tag{103}$$

$$\|\varphi_\theta - \varphi\|_{L^2(0,T;W)} \to 0, \tag{104}$$

$$\|\beta_\theta - \beta\|_{W^{1,2}(0,T;L^2(\Gamma_3))} \to 0. \tag{105}$$

(ii) *If* (101) *and* (102) *hold, then*

$$\|u_\theta - u\|_{C([0,T];V)} \to 0, \tag{106}$$

$$\|\varphi_\theta - \varphi\|_{C([0,T];W)} \to 0, \tag{107}$$

$$\|\beta_\theta - \beta\|_{W^{1,\infty}(0,T;L^2(\Gamma_3))} \to 0. \tag{108}$$

Proof. (i) Let $t \in [0,T]$. We use (95) and (53) to obtain

$$\begin{aligned} a_\theta\big(\dot u_\theta(t), u_\theta(t) - u(t)\big) &+ a_\mu\big(u_\theta(t) - u(t), u_\theta(t) - u(t)\big) \\ &+ a^*_e\big(\varphi_\theta(t) - \varphi(t), u_\theta(t) - u(t)\big) \\ &+ j\big(\beta_\theta(t), u_\theta(t), u_\theta(t) - u(t)\big) - j\big(\beta(t), u(t), u_\theta(t) - u(t)\big) = 0. \end{aligned} \tag{109}$$

Next, it follows from (46), (99) and (12) that

$$\begin{aligned} j\big(\beta_\theta(t), u_\theta(t), u_\theta(t) - u(t)\big) &- j\big(\beta(t), u(t), u_\theta(t) - u(t)\big) \\ &= \int_{\Gamma_3} p\big(R\big(u_\theta(t)\big) - R\big(u(t)\big)\big)\big(u_\theta(t) - u(t)\big)\,\mathrm{d}a \ge 0 \end{aligned}$$

and, using this inequality in (109) yields

$$a_\theta\big(\dot u_\theta(t), u_\theta(t)-u(t)\big) + a_\mu\big(u_\theta(t)-u(t), u_\theta(t)-u(t)\big) + a_e^*\big(\varphi_\theta(t)-\varphi(t), u_\theta(t)-u(t)\big) \le 0. \tag{110}$$

On the other hand, (96) and (54) imply that

$$a_\alpha\big(\varphi_\theta(t)-\varphi(t), \varphi_\theta(t)-\varphi(t)\big) - a_e\big(u_\theta(t)-u(t), \varphi_\theta(t)-\varphi(t)\big) = 0. \tag{111}$$

We add now equality (111) and inequality (110) to see that

$$a_\theta\big(\dot u_\theta(t)-\dot u(t), u_\theta(t)-u(t)\big) + a_\mu\big(u_\theta(t)-u(t), u_\theta(t)-u(t)\big) + a_\alpha\big(\varphi_\theta(t)-\varphi(t), \varphi_\theta(t)-\varphi(t)\big) \le a_\theta\big(\dot u(t), u(t)-u_\theta(t)\big). \tag{112}$$

Let $s \in [0,T]$. We integrate (112) on $[0,s]$ and use (42), (43), (76) and the initial condition $u_\theta(0)=u_0$ to obtain

$$\frac{\theta^*}{2}\,\|u_\theta(s)-u(s)\|_V^2 + \mu^*\int_0^s \|u_\theta(t)-u(t)\|_V^2\,\mathrm{d}t + \alpha^*\int_0^s \|\varphi_\theta(t)-\varphi(t)\|_W^2\,\mathrm{d}t \le \int_0^s a_\theta\big(\dot u(t), u(t)-u_\theta(t)\big)\,\mathrm{d}t + \frac{1}{2}\,\|\theta\|_{L^\infty(\Omega)}\,\|u(0)-u_0\|_V^2. \tag{113}$$

We use now the inequality

$$a_\theta\big(\dot u(t), u(t)-u_\theta(t)\big) \le \|\theta\|_{L^\infty(\Omega)}\|\dot u(t)\|_V\|u_\theta(t)-u(t)\|_V \le \frac{1}{2\mu^*}\,\|\theta\|_{L^\infty(\Omega)}^2\|\dot u(t)\|_V^2 + \frac{\mu^*}{2}\,\|u_\theta(t)-u(t)\|_V^2$$

to see that

$$\int_0^s a_\theta\big(\dot u(t), u(t)-u_\theta(t)\big)\,\mathrm{d}t \le \frac{1}{2\mu^*}\,\|\theta\|_{L^\infty(\Omega)}^2 \int_0^s \|\dot u(t)\|_V^2\,\mathrm{d}t + \frac{\mu^*}{2}\int_0^s \|u_\theta(t)-u(t)\|_V^2\,\mathrm{d}t. \tag{114}$$

Then, we combine (113) and (114) to obtain

$$\frac{\theta^*}{2}\,\|u_\theta(s)-u(s)\|_V^2 + \frac{\mu^*}{2}\int_0^s \|u_\theta(t)-u(t)\|_V^2\,\mathrm{d}t + \alpha^*\int_0^s \|\varphi_\theta(t)-\varphi(t)\|_W^2\,\mathrm{d}t \le \frac{1}{2\mu^*}\,\|\theta\|_{L^\infty(\Omega)}^2\int_0^s \|\dot u(t)\|_V^2\,\mathrm{d}t + \frac{1}{2}\,\|\theta\|_{L^\infty(\Omega)}\,\|u(0)-u_0\|_V^2. \tag{115}$$

Inequality (115) yields

$$\begin{aligned}
&\frac{\mu^*}{2}\int_0^s \|u_\theta(t)-u(t)\|_V^2\,\mathrm{d}t+\alpha^*\int_0^s \|\varphi_\theta(t)-\varphi(t)\|_W^2\,\mathrm{d}t\\
&\qquad\leq \frac{1}{2\mu^*}\,\|\theta\|^2_{L^\infty(\Omega)}\int_0^s \|\dot u(t)\|_V^2\,\mathrm{d}t+\frac{1}{2}\,\|\theta\|_{L^\infty(\Omega)}\,\|u(0)-u_0\|_V^2.
\end{aligned}\tag{116}$$

The convergences (103) and (104) are a direct consequence of (116) and (100).

Also, arguments similar to those used to obtain (72), based on (97), (98), (55) and (56), lead to inequalities

$$\|\beta_\theta(s)-\beta(s)\|_{L^2(\Gamma_3)}\leq\int_0^s \|u_\theta(t)-u(t)\|_V\,\mathrm{d}t\quad\forall s\in[0,T],\tag{117}$$

$$\begin{aligned}
\|\dot\beta_\theta(t)-\dot\beta(t)\|_{L^2(\Gamma_3)}\leq c\,\big(&\|\beta_\theta(t)-\beta(t)\|_{L^2(\Gamma_3)}\\
&+\|u_\theta(t)-u(t)\|_V\big)\quad\text{a.e. } t\in(0,T).
\end{aligned}\tag{118}$$

The convergence (105) is now a consequence of inequalities (117), (118) combined with the convergence result (103).

(ii) Assume now that (101) and (102) hold. Then, inequality (115) combined with (102) imply that

$$\|u_\theta(s)-u(s)\|_V^2\leq\frac{1}{\mu^*\theta^*}\,\|\theta\|^2_{L^\infty(\Omega)}\int_0^s \|\dot u(t)\|_V^2\,\mathrm{d}t\quad\forall s\in[0,T]$$

and, using assumption (101), we obtain (106). On the other hand, (96) and (54) yield

$$a_\alpha\big(\varphi_\theta(t)-\varphi(t),\psi\big)-a_e\big(u_\theta(t)-u(t),\psi\big)=0\quad\forall\psi\in W,\ t\in[0,T]$$

which implies that

$$a_\alpha\big(\varphi_\theta(t)-\varphi(t),\varphi_\theta(t)-\varphi(t)\big)=a_e\big(u_\theta(t)-u(t),\varphi_\theta(t)-\varphi(t)\big)\quad\forall t\in[0,T].$$

We use now inequality (43) to see that

$$\alpha^*\|\varphi_\theta(t)-\varphi(t)\|_W\leq\|e\|_{L^\infty(\Omega)}\|u_\theta(t)-u(t)\|_V\quad\forall t\in[0,T]$$

and, combining this last inequality with (106) we obtain (107). Finally, note that (108) is a consequence of (117), (118) and (106), which completes the proof. □

We end this section with some comments on Theorem 3. First, note that the meaning of the convergences (103)–(105) is the following: for every sequence of functions $\{\theta_n\}$ which satisfy (25) for all $n\in\mathbb{N}$, if $\|\theta_n\|_{L^\infty(\Omega)}\to 0$ as $n\to\infty$, then

$\|u_{\theta_n} - u\|_{L^2(0,T;V)} \to 0$, $\|\varphi_{\theta_n} - \varphi\|_{L^2(0,T;W)} \to 0$ and $\|\beta_{\theta_n} - \beta\|_{W^{1,2}(0,T;L^2(\Gamma_3))} \to 0$ as $n \to \infty$. A similar explanation can be made for the convergences (106)–(108).

Next, note that Theorem 3 shows that the convergences (103)–(105) hold under the assumption (100), whatever is the choice of the initial displacement of the electro-visco-elastic Problem P_V^θ. It also shows that, if the initial displacement u_0 is chosen to be the displacement of the corresponding electro-elastic Problem Q at $t = 0$ and (100) is replaced by the stronger assumption (101), then the convergences (103)–(105) can be reinforced by the convergences (106)–(108).

Finally, consider the case of homogeneous viscosity, i.e. the case when assumption (25) is replaced by the assumption

$$\theta(x) = \theta > 0 \quad \text{a.e. } \boldsymbol{x} \in \Omega,$$

where θ is given. In this case $\|\theta\|_{L^\infty(\Omega)} = \theta$, $\theta^* = \theta$ and therefore the convergences (100) and (101) are equivalent to $\theta \to 0$. Therefore, by Theorem 3 we conclude that the solution to the electro-viscoelastic Problem P_V may be approached by the weak solution to the electro-elastic Problem Q_V, as the viscosity is small enough. In addition to the mathematical interest of this result, it is important from the mechanical point of view, since it shows that the electro-elasticity with adhesion can be considered as a limit case of electro-viscoelasticity with adhesion as the viscosity decreases.

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