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On the Practical Output Feedback Stabilization for Nonlinear Uncertain Systems

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Abstract. In this paper, we treat the problem of output feedback stabilization of nonlinear uncertain systems. We propose an output feedback controller that guarantees global uniform practical stability of the closed loop system.

Keywords: output feedback, uncertain system, practical stability.

1 Introduction

The problem of stabilization for uncertain systems has been widely investigated for many years [1–11]. In these studies, the origin was not supposed to be an equilibrium point of the uncertain system. So we can no longer expect to design a controller that guarantee the stability of the origin as an equilibrium point. In [1], a class of state feedback controls is proposed in order to guarantee uniform ultimate boundedness of every system response within an arbitrarily small neighborhood of the zero state. [5], [9] and [6] presented controllers that guarantee exponential stability of a ball containing the origin of the state space and the radius of this ball can be made arbitrary small. In order to study uncertain dynamical systems, the authors in [12] introduced the notion of input to state practical stability. In [13], the concept of input to state practical stability is extended to stochastic case and an output feedback controller is proposed for a class of stochastic nonlinear systems with uncertain nonlinear functions.

Most of the recent nonlinear controllers are designed for an uncertain system that has a nominal linear part and the controller is designed based on the knowledge of the upper bound, possibly time varying and state dependent, of the uncertainties vector norm. Another class of uncertain systems which has also received considerable attention, namely systems with nominal part which is affine in the control. Such a class of systems is important because it may represent many physical systems. In [2, 8, 10] authors investigated the state feedback stabilization problem for these systems. In this paper, we will synthesis an output feedback controller for this class of systems. It should be noted that output feedback stabilization problem for uncertain system with linear nominal part has been discussed in ([3, 4, 7, 11]). Under the assumption that the uncertain part is bounded by a known function that depends only on the output, they construct an output feedback controller that guarantees global exponential stability of the closed loop system. Here, we will suppose that the unknown part is bounded by a function that depends on the input and the output. We will design an output feedback controller that guarantees global uniform practical stability of the closed loop system. In Section 2, we recall the definition of global uniform practical stability and we give a sufficient condition to assure it. In Section 3, we state the main result. Throughout this paper $\|.\|$ denotes the Euclidean norm of $\mathbb{R}^n$.

2 Practical stability

Consider a system described by

$$\dot{x} = F(t, x) \tag{1}$$

with $t \in \mathbb{R}_+$ is the time and $x \in \mathbb{R}^n$ is the state. As a first step, we need to recall what is meant by global uniform practical stability of (1).

For $r \geq 0$, denote $B_r = \{x \in \mathbb{R}^n / \|x\| \leq r\}$.

Definition 1. The system (1) is said globally uniformly practically stable if there exists $r \geq 0$ such that:

(i) for all $\varepsilon > r$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for all $t_0 \geq 0$,

$$\|x_0\| < \delta \ \Rightarrow \ \|x(t)\| < \varepsilon, \quad \forall t \geq t_0;$$

(ii) for all $\varepsilon > r$ and $c > 0$, there exists $T(\varepsilon, c) > 0$ such that, for all $t_0 \geq 0$,

$$\|x(t)\| < \varepsilon, \quad \forall t \geq t_0 + T(\varepsilon, c), \quad \|x_0\| < c.$$

The origin may not be an equilibrium point of the system (1). But, since the B_r ball is an attractor and $0 \in B_r$, it follows that at least zero cannot be a globally unstable equilibrium provided that zero is an equilibrium but it could be a locally unstable equilibrium, even when the ball is an attractor for $r > 0$. If system (1) satisfies the requirements of Definition 1 with $r = 0$, then it is globally uniformly asymptotically stable.

The following theorem gives sufficient conditions to assure global practical stability. Its proof can be deduced from [1]. It uses the following comparison function definitions. A function $\alpha \colon [0, +\infty[\to [0, +\infty[$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded.

Theorem 1. *Consider system* (1) *with* $F(., .)$ *continuous. Suppose that there exist a* C^1 *function* $V(., .) \colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$*,* $\mathcal{K}_\infty$ *functions* $\alpha_1(.)$*,* $\alpha_2(.)$*, a class* $\mathcal{K}$ $\alpha_3(.)$ *and a small positive real number* ϱ *such that the following inequalities hold for all* $t \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^n$.

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|),$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\big(F(t,x)\big) \leq -\alpha_3(\|x\|) + \varrho.$$

Then, system (1) *is globally uniformly practically stable with* $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)$ *and*

$$B_r = \big\{x \in \mathbb{R}^n / \|x\| \leq \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varrho)\big\},$$

where "$\circ$*" denotes the composition.*

Remark 1. Consider a control system having x as state and u as input: $(\Sigma)\, \dot{x} = f(x,u)$, $x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$, where f is a smooth function. This system is said to be input to state practical stable (see [12]) if there exist a function β of class $\mathcal{KL}$ (ie $\beta\colon [0,+\infty[\times[0,+\infty[\to [0,+\infty[$, such that for each fixed t, the function $\beta(.,t)$ is of class $\mathcal{K}$ and for each fixed s, the function $\beta(s,.)$ is non increasing and tends to zero at infty), a function γ of class $\mathcal{K}$ and a nonnegative constant d such that for any initial condition $x(0)$ and each measurable essentially bounded control u the associated solution $x(t)$ exists and satisfies $\|x(t)\| \leq \beta(\|x(0)\|,t) + \gamma(\|u\|_\infty) + d$, where $\|u\|_\infty$ denotes $\sup\{\|u(t)\|,\ t \geq 0\}$. It is clear that if system (Σ) is input to state practical stable then it is globally uniformly practically stable in the sense of Definition 1 when $u = 0$.

3 Output feedback stabilization

Throughout this paper, we deal with uncertain dynamical systems described by

$$\begin{cases} \dot{x} = f(t,x) + g(t,x)u + g(t,x)\xi(t,x,u), \\ y = h(t,x), \end{cases} \tag{2}$$

where $t \in \mathbb{R}_+$ is the time, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $y \in \mathbb{R}^p$ is the output. $f\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n,\ g\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^p$ are known functions satisfying $f(t,0) = 0$ and $h(t,0) = 0$ for all $t \in \mathbb{R}_+$. f and g are supposed to be locally Lipschitz in x and continuous in t. The function $\xi\colon \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ represents uncertainties in the plant. The nominal system corresponding to system (2) is given by

$$\begin{cases} \dot{x} = f(t,x) + g(t,x)u, \\ y = h(t,x). \end{cases} \tag{3}$$

Our aim is to design an output feedback controller such that system (2) is globally practically stable. We consider the following assumptions pertaining to system (2).

(A1) There exist nonnegative real scalar functions $\rho_1(.,.),\ \rho_2(.,.)$, with

$$\rho_1(t,y) < 1$$

such that

$$\|\xi(t,x,u)\| \leq \rho_1(t,y)\|u\| + \rho_2(t,y)$$

for all $t \in \mathbb{R}_+$, $y \in \mathbb{R}^p$, $u \in \mathbb{R}^m$.

In order to design an output feedback variable structure control, the author in [14] consider uncertain systems with linear nominal part and such that the unknown function satisfies assumption (A1).

We will consider the problem of choosing an output feedback $u(t,y)$ such that, for all uncertainties satisfying the assumption (A1), system (2) is globally uniformly practically stable. We will assume that the nominal system (3) is globally asymptotically stabilizable. Indeed, we suppose that the assumptions below are fulfilled.

(A2) There exist an output feedback $u_0(t,y)$, a C^1 function $V(.,.)$, $\mathcal{K}_\infty$ functions $\alpha_1(.)$, $\alpha_2(.)$, a class $\mathcal{K}$ $\alpha_3(.)$ satisfying

$$\alpha_1(\|x\|) \le V(t,x) \le \alpha_2(\|x\|), \tag{4}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\big(f(t,x) + g(t,x)u_0(t,y)\big) \le -\alpha_3(\|x\|) \tag{5}$$

for all $x \in \mathbb{R}^n$.

(A3) There exists a function $\varphi(.,.)\colon \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ such that

$$\frac{\partial V}{\partial x} g(t,x) = \varphi^T(t,y). \tag{6}$$

For any $\varepsilon_1,\ \varepsilon_2 > 0$, the proposed controller is given by

$$u(t,y) = u_0(t,y) + u_1(t,y) + u_2(t,y), \tag{7}$$

where u_0 is given by assumption (A2) and

$$u_1(t,y) = -\frac{\theta_1(t,y)^2\|u_0(t,y)\|^2}{\theta_1(t,y)\|\varphi(t,y)\|\|u_0(t,y)\| + \varepsilon_1}\varphi(t,y), \tag{8}$$

$$u_2(t,y) = -\frac{\theta_2(t,y)^2}{\theta_2(t,y)\|\varphi(t,y)\| + \varepsilon_2}\varphi(t,y), \tag{9}$$

where $\theta_1(t,y) = \rho_1(t,y)(1-\rho_1(t,y))^{-1}$, $\theta_2(t,y) = \rho_2(t,y)(1-\rho_1(t,y))^{-1}$ and V is the Lyapunov function satisfying assumption (A2).

We have the following result.

Theorem 2. *Consider an uncertain system described by* (2) *satisfying assumptions* (A1), (A2) *and* (A3) *and subject to the control given by* (7)*. Then the resulting closed-loop system is globally uniformly practically stable.*

Proof. First, note that controller (7) satisfies

$$\begin{aligned}\|u(t,y)\| &\le \|u_0(t,y)\| + \|u_1(t,y)\| + \|u_2(t,y)\| \\ &\le \big(1 + \theta_1(t,y)\big)\|u_0(t,y)\| + \theta_2(t,y).\end{aligned} \tag{10}$$

We will use the function V as a Lyapunov function candidate for the closed-loop system. Its derivative along the trajectories of (2) is given by

$$\dot{V}(t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\big(f(t,x) + g(t,x)u + g(t,x)\xi(t,x,u)\big).$$

Taking into account assumption (A1) we have

$$\begin{aligned}\dot{V}(t) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\big(f(t,x) + g(t,x)u\big) + \frac{\partial V}{\partial x}g(t,x)\xi(t,x,u)\\ &\le \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\big(f(t,x) + g(t,x)u\big) + \left\|\frac{\partial V}{\partial x}g(t,x)\right\|\|\xi(t,x,u)\|\\ &\le \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\big(f(t,x) + g(t,x)u\big) + \left\|\frac{\partial V}{\partial x}g(t,x)\right\|\big(\rho_1(t,x)\|u\| + \rho_2(t,x)\big).\end{aligned}$$

By assumption (A3) and (10) we have

$$\begin{aligned}\dot{V}(t) \le{}& \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\big(f(t,x) + g(t,x)u_0(t,y)\big) + \varphi^T(t,y)u_1(t,y)\\ &+ \varphi^T(t,y)u_2(t,y) + \big(\rho_2(t,y) + \rho_1(t,y)\theta_2(t,y)\big)\|\varphi(t,y)\|\\ &+ \rho_1(t,y)\big(1 + \theta_1(t,y)\big)\|u_0(t,y)\|\|\varphi(t,y)\|.\end{aligned}$$

On the one hand, using (8) and the fact that

$$\rho_1(t,y)\big(1 + \theta_1(t,y)\big) = \theta_1(t,y)$$

we get

$$\varphi^T(t,y)u_1(t,y) + \rho_1(t,y)\big(1 + \theta_1(t,y)\big)\|u_0(t,y)\|\|\varphi(t,y)\| \le \varepsilon_1. \tag{11}$$

On the other hand, using (9) and the fact that

$$\rho_2(t,y) + \rho_1(t,y)\theta_2(t,y) = \theta_2(t,y)$$

we get

$$\varphi^T(t,y)u_2(t,y) + \big(\rho_2(t,y) + \rho_1(t,y)\theta_2(t,y)\big)\|\varphi(t,y)\| \le \varepsilon_2. \tag{12}$$

So, by assumption (A2), (11) and (12) we obtain the following upper bound on $\dot{V}$

$$\dot{V} \le -\alpha_3(\|x\|) + \varepsilon_1 + \varepsilon_2.$$

Letting $r = \alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(\varepsilon_1 + \varepsilon_2)$ we deduce by Theorem 1 that the closed loop system is globally practically stable. □

Remark 2. Since $\alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}$ is continuous and $\alpha_1^{-1} \circ \alpha_2 \circ \alpha_3^{-1}(0) = 0$, we deduce that, if $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ then $r \to 0$. That is the controller (7) insures that solutions of the closed loop system converge towards an arbitrarily small neighborhood of the origin.

Remark 3. It should be remarked that when the nominal system is linear, i.e. $f(t,x) = Ax$, $g(t,x) = B$ and $h(t,x) = Cx$, the result of Theorem 2 will recover that of [7].

In the rest of this section, we give a class nominal systems satisfying assumptions (A2) and (A3). We consider systems (2) with nominal autonomous part described by

$$\begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x), \end{cases} \tag{13}$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, f and the m columns of g are smooth, $f(0) = 0$ and $h(0) = 0$. We will suppose that the uncertainties satisfy assumption (A1). For the nominal system (13) we will make the following assumption.

(A4) There exist a C^1 function $V(.)\colon \mathbb{R}^n \to \mathbb{R}$, $\mathcal{K}_\infty$ functions $\alpha_1(.)$, $\alpha_2(.)$, a class $\mathcal{K}$ $\alpha_3(.)$ satisfying

$$\begin{aligned} &\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \\ &L_f V(x) \le -\alpha_3(\|x\|), \\ &L_g V(x) = h(x)^T \end{aligned}$$

for all $x \in \mathbb{R}^n$.

Systems (13) satisfying assumption (A4) are said to be C^1 strictly passive. Letting $\phi(.)\colon \mathbb{R}^m \to \mathbb{R}^m$ be any smooth function such that

$$\phi(0) = 0, \;\; y^T\phi(y) > 0 \quad \forall y \neq 0, \tag{14}$$

then $u(y) = -\phi(y)$ globally asymptotically stabilizes the equilibrium $x = 0$ of system (13) (see [15]).

Here, we consider the output feedback stabilization problem for uncertain system (2) with nominal autonomous part satisfying assumptions (A1) and (A4). We propose the following controller

$$u(y) = -\phi(y) + u_1(t,y) + u_2(t,y), \tag{15}$$

where

$$\begin{aligned} u_1(t,y) &= -\frac{\theta_1(t,y)^2\|\phi(y)\|^2}{\theta_1(t,y)\|y\|\|\phi(y)\| + \varepsilon_1} y, \\ u_2(t,y) &= -\frac{\theta_2(t,y)^2}{\theta_2(t,y)\|y\| + \varepsilon_2} y, \end{aligned}$$

where $\theta_1(t,y) = \rho_1(t,y)(1-\rho_1(t,y))^{-1}$, $\theta_2(t,y) = \rho_2(t,y)(1-\rho_1(t,y))^{-1}$, $\varepsilon_1,\ \varepsilon_2 > 0$, V is the Lyapunov function given by assumption (A4) and ϕ is any nonlinearity satisfying condition (14).

We have the following corollary of Theorem 2.

Corollary 1. *Consider an uncertain system described by* (2) *satisfying assumptions* (A1) *and* (A4) *and subject to the control given by* (15)*. Then the resulting closed-loop system is globally practically stable.*

4 Example

As an example of systems satisfying assumptions (A1)–(A3), let us consider the following planar system

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 e^{-2t}, \\ \dot{x}_2 = x_1 + u + \cos x_1 + \dfrac{1}{2}\sin u, \\ y = x_2. \end{cases}$$

This system is of the form (2) with

$$f(t,x) = \begin{bmatrix} -x_1 - x_2 e^{-2t} \\ x_1 \end{bmatrix}, \quad g(t,x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad h(t,x) = x_2$$

and

$$\xi(t,x,u) = \cos x_1 + \frac{1}{2}\sin u.$$

We note that assumption (A1) is fulfilled with

$$\rho_1(t,y) = \frac{1}{2} \quad \text{and} \quad \rho_2(t,y) = 1.$$

So $\theta_1(t,y) = 1,\ \theta_2(t,y) = 2$. Using the Lyapunov function

$$V(t,x) = x_1^2 + \big(1 + e^{-2t}\big)x_2^2,$$

it is shown in [16] that the nominal system

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 e^{-2t}, \\ \dot{x}_2 = x_1 + u, \\ y = x_2 \end{cases}$$

in closed loop with the controller $u_0(t,y) = -y(t) = -x_2(t)$ is globally uniformly exponentially stable. It is readily seen that assumption (A2) is satisfied with

$$\alpha_1(s) = s^2, \quad \alpha_2(s) = 2s^2, \quad \alpha_3(s) = s^2.$$

Furthermore,

$$\frac{\partial V}{\partial x} g(t,x) = 2\big(1 + e^{-2t}\big)y.$$

Thus assumption (A3) is also fulfilled with

$$\varphi(t,y) = 2\big(1 + e^{-2t}\big)y.$$

Consequently, for given $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, it follows from (7), (8) and (9) that

$$u(t,y) = -y - \frac{2(1+e^{-2t})y^3}{2(1+e^{-2t})y^2+\varepsilon_1} - \frac{8(1+e^{-2t})y}{4(1+e^{-2t})|y|+\varepsilon_2}.$$

The differential equation of the closed loop system is

$$\begin{cases} \dot{x}_1 = -x_1 - x_2 e^{-2t}, \\ \dot{x}_2 = x_1 - x_2 - \dfrac{2(1+e^{-2t})x_2^3}{2(1+e^{-2t})x_2^2+\varepsilon_1} - \dfrac{8(1+e^{-2t})x_2}{4(1+e^{-2t})|x_2|+\varepsilon_2} + \cos x_1. \end{cases} \tag{16}$$

For simulation we select $x_0 = [1/2, 1/2]^T$, $\varepsilon_1 = 0.1$ and $\varepsilon_2 = 0.1$. The result of simulation is depicted in Fig. 1.

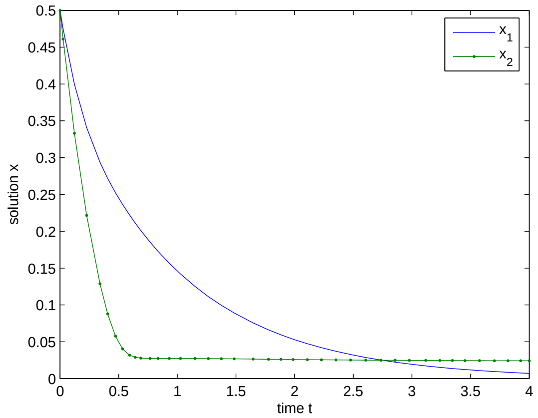

Fig. 1. Trajectories of system (16).

5 Conclusion

In this paper, the problem of output feedback stabilization for nonlinear uncertain systems with nominal part that is affine in the control is investigated. A controller that assures global uniform practical stability of the closed-loop system is proposed, that is, the solutions of the closed-loop system converge towards an arbitrary small neighborhood of the origin. A special case of systems is also considered, namely systems with a strictly passive nominal part.

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