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Time-delayed impulsive control for discrete-time nonlinear systems with actuator saturation*

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Abstract. This paper focuses on the problem of time-delayed impulsive control with actuator saturation for discrete-time dynamical systems. By establishing a delayed impulsive difference inequality, combining with convex analysis and inequality techniques, some sufficient conditions are obtained to ensure exponential stability for discrete-time dynamical systems via time-delayed impulsive controller with actuator saturation. The designed controller admits the existence of some transmission delays in impulsive feedback law, and the control input variables are required to stay within an availability zone. Several numerical simulations are also given to demonstrate the effectiveness of the proposed results.

Keywords: discrete-time dynamical systems, delayed impulsive control, exponential stability, actuator saturation.

1 Introduction

For the engineering applications, it is essential to formulate discrete-time system that are an analogue of continuous ones, while the discretization may not preserve the dynamics of the continuous-time counter part even for a small sampling period [10, 28, 30], which promotes the investigation direct for the discrete-time dynamic system. Further, in many

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evolutionary processes, the state of the dynamical systems may abruptly change at some point. Consequently, it is natural to assume that these perturbations act instantaneously [17,18,20–22,33,39,40], that is, those processes can be modelled by impulsive dynamical systems. However, the theory of impulsive difference equations have developed a little slowly [24, 32, 38], and not better than the theory of impulsive differential equation. Generally speaking, the stability analysis of impulsive differential equation is not applicable to the impulsive difference equations. Therefore, further study for impulsive discrete-time equations is necessary.

Recently, increasing attention has been paid to the study of dynamics and control of impulsive systems in which the impulses involve time delays [3,4,11,20,21]. Such kind of impulses that are sometimes called delayed impulses is regarded as a better way to model many practical problems. For example, in communication security systems based on impulsive synchronization [11], due to the limit speed of signal sampling, a type of delays called sampling delays, which depend on the historical states at the sampling points, will occur in impulsive transients. Another example is found in population dynamics such as fishing industry, where effective impulsive control such as harvesting and releasing can keep the balance of fishing, and the quantities of every impulsive harvesting or releasing are not only measured by the current numbers of fish but also depend on the numbers in recent history due to the fact that the immature fish need some time to grow. While in some other models arising from digital communication, neural networks and ecological models, delayed impulses also have potential applications [4]. There are some important results on analysis of delayed impulsive systems in the existing literatures [11,20,21,30,37,39].

Moreover, in practice, the system frequency is required to stay within an ideal region close to its nominal value [2, 5, 7–9, 12, 26, 29, 31]. Otherwise, the system needs to take relevant control actions immediately to keep the frequency in an acceptable region, and then to restore the frequency back to its nominal value [5, 9, 26, 31]. In [27], Michel et al. presented a qualitative analysis for a class of synchronous discrete-time neural networks defined on hypercubes in the state space. A complete stability analysis of discrete-time linear systems under saturation was considered by Hu et al. [6]. The impulsive controller with partial input saturation is proposed to synchronize the discrete delayed coupled systems by Razumikhin-type technique [23]. Also, recently, we consider the state constraint impulsive control for continuous nonlinear dynamic systems combining with convex analysis technique [14, 15]. Based on actual demand and theoretical challenges, the rigorous handling of constraints in a control design has become an important research topic. In recent years, there are various techniques to solve the constraint control problem based on set invariance [7, 8], model predictive control, adaptive variable structure control [26, 31] and so on. This key point motivates us to solve the problems of the state constraints and delayed impulsive control of nonlinear systems.

In conclusion, some researchers have attempted to solve the control problem of constraint systems because the actuator saturation appear in most physical systems. However, the control problem for difference dynamic system via delayed impulsive control with actuator saturation has not been fully investigated yet, and many challenging questions remain. So far, there have been many studies on impulsive control, but the impulses are basically unconstrained [1, 13, 16, 32, 35, 36, 38]. Based on this, this paper proposes a special type of delayed-impulse controller with actuator saturation, the controller of which engages the feedback control laws and analyzes the exponentially stability of the closed loop. And the designed controller admits the existence of some transmission delays in impulsive feedback law. More specifically, the main contributions of the present paper are as follows. Firstly, by the schematic of delayed impulsive control system, it gives a brief introduction to the system about its working principle and structure. Secondly, a delayed impulsive difference inequality is established, and some global exponential stability criteria are provided. Finally, two instances with numerical simulations are offered to demonstrate the availability of our results.

This paper is organized as follows. In the following section, some preliminaries are described. The exponential stabilization criteria for discrete-time dynamical systems via delayed-impulsive controller with actuator saturation are established in Section 3. The examples are carried out to demonstrate the effectiveness of the obtained results in Section 4, and the conclusion is drawn in Section 5.

Notations. Throughout this paper, $\mathbb{Z}$ be the integers, $\mathbb{Z}^+$ be the nonnegative integers, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times m}$ denote the m-dimensional Euclidean space and the set of $m \times m$ matrices, respectively. $\mathbb{N} = \{1, 2, \dots\}$. The notations B^{T} and B^{-1} mean the transpose of B and the inverse of a square matrix B, respectively. $Y > 0$ ($Y \geqslant 0$) means that Y is a real symmetric and positive definite (positive semi-definite) matrix. I is the $n \times n$ identity matrix. Denote $\|c\|$ as a vector norm of the vector $c \in \mathbb{R}^m$. Matrix set $\varPi = \{H\colon \|Hx\|_\infty \leqslant 1,\ H \in \mathbb{R}^{m\times m},\ x \in \mathbb{R}^m\}$. $\|B\|$ is the matrix norm of $\|B\| = \sqrt{\lambda_{\max}(B^{\mathrm{T}}B)}$, where $\lambda_{\max}(B)$ denotes the maximum eigenvalue of the symmetric matrix B. $\operatorname{sign} s$ is sign function. $\operatorname{co}\{\cdot\}$ denotes the convex hull of group of vectors. Let $\mathcal{G}$ be the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. The each elements of $\mathcal{G}$ is labeled as Σ_i, $i \in \mathbb{M}_m \triangleq \{1, 2, 3, \dots, 2^m\}$.

2 Preliminaries

We consider a class of discrete-time nonlinear systems described by

$$\begin{aligned} &x(n+1) = Bx(n) + Af\bigl(n, x(n)\bigr) \triangleq \varPhi\bigl(n, \bar{x}(n)\bigr),\\ &y(n) = \varphi\bigl(x(n)\bigr),\\ &x(0) = x_0. \end{aligned} \tag{1}$$

Here $x(n) \in \mathbb{R}^m$ is the discrete-time state vector, $f(n, x(n)) : \mathbb{Z}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous function in their respective domains of definition, and $f(n, 0) = 0$. $A, B \in \mathbb{R}^{m\times m}$ are feedback matrices. $y(n) \in \mathbb{R}^l$ is the measured output vector, where $\varphi(x(n)) : \mathbb{R}^m \to \mathbb{R}^l$. $x(0) = x_0 \in \mathbb{R}^m$ denote the initial condition.

A delayed impulsive control law of system (1) is given by a sequence $\{n_k, U_k(y(\cdot))\}$, where $0 = n_0 < n_1 < \cdots < n_k < \cdots$, $\lim_{k\to\infty} n_k = \infty$. $U_k(\cdot)$ denotes the control input at each impulsive instant n_k, $k \in \mathbb{N}$. It works as in Fig. 1. Module "Delayed" can be transmission delays or other perturbed delays. Here $\bar{x}(n)$ denotes the state x at time n.

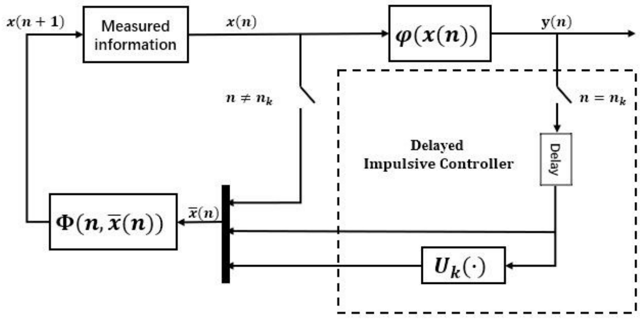

Figure 1. Schematic of delayed impulsive control systems.

According to the schematic of delayed impulsive control system, the state of the impulsive control system satisfies the following equations:

$$\begin{aligned} &x(n+1) = \Phi\bigl(n, \bar{x}(n)\bigr), \\ &\bar{x}(n) = \begin{cases} x(n), & n \neq n_k, \\ y(\cdot) + U_k(y(\cdot)), & n = n_k, \end{cases} \\ &y(n) = \varphi\bigl(x(n)\bigr), \\ &x(0) = x_0. \end{aligned} \tag{2}$$

In this paper, let $\varphi(x(n)) = x(n)$, and the controller is designed as $U_k(\cdot) = U_k(x(n_k - \tau(k)))$, where $U_k(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is impulsive disturbed function with $U_k(0) = 0$. $\tau(k) \in \mathbb{Z}^+$ is the impulsive delay with $\inf_{k\in\mathbb{N}}\{\tau(k)\} = \tau$ and $\tau(k) < n_k - n_{k-1}$. Then systems (2) can be rewritten as the following impulsive system:

$$\begin{aligned} &x(n+1) = Bx(n) + Af\bigl(n, x(n)\bigr), \quad n \neq n_k - 1, \\ &x(n_k) = x\bigl(n_k - \tau(k)\bigr) + U_k\bigl(x\bigl(n_k - \tau(k)\bigr)\bigr), \quad n = n_k, \\ &y(n) = x(n), \\ &x(0) = x_0. \end{aligned} \tag{3}$$

Design the controller with actuator saturation as follows:

$$U_k(\cdot) = \mathrm{Sat}\bigl(u_k(\cdot)\bigr). \tag{4}$$

In this paper, assume feedback control law $u_k(\cdot) = Fx(n_k - \tau(k))$ at impulsive time $n = n_k$, where $F = \mathrm{diag}(F_1, F_2, \dots, F_m)$, and $\mathrm{Sat}(q) = \mathrm{sign}(q)\min\{1, |q|\}$ for $q \in \mathbb{R}$.

Remark 1. With respect to traditional impulsive control, which suggest that the impulsive control input is unconstrained. That is, the control input variables can also be arranged arbitrarily, or it is just satisfied the linear conditions. In fact, there are artificial constraints or physical constraints inherent in the actual engineering system. Until now, the research in discrete-time impulsive difference equations with constraints is poor. So, in the following, we shall consider the exponential stability for discrete-time dynamical networks via time-delayed impulsive controller with actuator saturation.

In this paper, we consider a class of activation function of Lipschitz type, which may be unbounded and defined by

$$\mathcal{L} = \big\{ f(\cdot) \bigm| f_i \in C(\mathbb{R}, \mathbb{R})\colon \exists L_i > 0,\ \big|f_i(x_i) - f_i(y_i)\big| \leqslant L_i|x_i - y_i| \\ \forall x_i, y_i \in \mathbb{R},\ i = 1, 2, \dots, n\big\}.$$

Remark 2. Based on the above, obviously, $x(n) = 0$ is a unique solution of (3), which we call the zero solution. And one can easily confirm that $|f_i(x_i)| \leqslant L_i|x_i|$. We denote $L = \max\{L_1, L_2, \dots, L_m\}$ in the sequel.

In this paper, we also need the following definition and some lemmas.

Definition 1. (See [39].) The zero solution of (3) is said to be globally exponentially stable if there exist scalars $\lambda > 0$, $M(x_0) \geqslant 0$ such that

$$\big\|x(n, x_0)\big\| \leqslant M\mathrm{e}^{-\lambda n}, \quad n \geqslant 0.$$

Lemma 1. (*See* [15].) *For any* $\varepsilon > 0$, $r, s \in \mathbb{R}^n$, *the inequality* $2r^{\mathrm{T}}s \leqslant \varepsilon^{-1}r^{\mathrm{T}}r + \varepsilon s^{\mathrm{T}}s$ *holds.*

Lemma 2. (*See* [7].) *Let* $a, b \in \mathbb{R}^m$, $a = (a_1, a_2, \dots, a_m)^{\mathrm{T}}$, $b = (b_1, b_2, \dots, b_m)^{\mathrm{T}}$. *Suppose that* $|b_i| \leqslant 1$, *then* $\mathrm{Sat}(a) \in \mathrm{co}\{\Sigma_i a + \Sigma_i^- b\colon i \in \{1, \dots, 2^m\}\}$.

Remark 3. Denote $\Sigma_i^- = I - \Sigma_i$. Given two vectors $a, b \in \mathbb{R}^m$, $\{\Sigma_i a + \Sigma_i^- b\colon i \in \mathbb{M}_m\}$ is the set of vectors formed by choosing some elements from a and the rest from b. Given two matrices $E, F \in \mathbb{R}^{m\times n}$, $\{\Sigma_i E + \Sigma_i^- F\colon i \in \mathbb{M}_m\}$ is the set of matrices formed by choosing some rows from E and the rest from F. Suppose $\|Fx\|_\infty \leqslant 1$, we have $\mathrm{Sat}(Ex) \in \mathrm{co}\{\Sigma_i Ex + \Sigma_i^- Fx\colon i \in \mathbb{M}_m\}$, i.e. there exist $\vartheta_i \in [0, 1]$ satisfying $\sum_{i=1}^{2^m} \vartheta_i = 1$ such that $\mathrm{Sat}(Ex) = \sum_{i=1}^{2^m} \vartheta_i(\Sigma_i E + \Sigma_i^- F)x$.

Lemma 3. *Assume that there exist a function* $V(n, x(n)) : \mathbb{Z} \times \mathbb{R}^m \to \mathbb{R}^+$ *and positive constants* p, c_1, c_2, d_k *and* $\mu > 1$ *such that for any* $k \in \mathbb{N}$, *the following conditions hold*:

(i) $c_1\|x(n)\|^p \leqslant V(n, x(n)) \leqslant c_2\|x(n)\|^p$;
(ii) *When* $n \neq n_k - 1$, $V(n+1, x(n+1)) - \mu V(n, x(n)) < 0$;
(iii) $V(n_k, x(n_k)) \leqslant d_k V(n_k - \tau(k), x(n_k - \tau(k)))$, *where* $\sup_{k\in\mathbb{N}}\{d_k\} = d < 1$;
(iv) $\max\{n_k - n_{k-1}\} \leqslant \xi < (\tau \ln\mu - \ln d)/\ln\mu$.

Then the discrete-time dynamical system (3) *with delayed impulses is globally exponentially stable, and the convergence rate is* $(-\ln\mu + (\tau\ln\mu - \ln d)/\xi)/p$.

Proof. Since $\mu \geqslant 1$, $0 < d < 1$, then $\ln d - \tau\ln\mu < 0$. For $n \in [n_{k-1}, n_k)$, from condition (iv) we have $n < n_k = (n_k - n_{k-1}) + (n_{k-1} - n_{k-2}) + \dots + (n_1 - n_0) \leqslant k\xi$ and $n/\xi < k$. For convenience sake, let $V(n) = V(n, x(n))$.

For $n \in [n_0 + 1,\, n_1)$, from (ii) we have

$$V(n) < \mu V(n-1) < \mu^2 V(n-2) < \cdots < \mu^{n-n_0} V(n_0) = \mu^n V(0).$$

When $n = n_1$, from condition (iii) and the above inequality we have

$$V(n_1) \leqslant d_1 V\bigl(n_1 - \tau(1)\bigr) \leqslant \cdots \leqslant d_1 \mu^{n_1 - \tau(1)} V(0).$$

For $n \in [n_1 + 1,\, n_2)$, from (ii) we have

$$\begin{aligned} V(n) &< \mu V(n-1) < \cdots < \mu^{n-n_1} V(n_1) = \mu^{n-n_1} d_1 \mu^{n_1 - \tau(1)} V(0) \\ &= d_1 \mu^{n-\tau(1)} V(0). \end{aligned}$$

From above we have

$$V(n) \leqslant d_1 \mu^{n-\tau(1)} V(0), \quad n \in [n_1, n_2).$$

When $n = n_2$, from condition (iii) and the above inequality we have

$$V(n_2) \leqslant d_1 V\bigl(n_2 - \tau(2)\bigr) \leqslant \cdots \leqslant d_1 d_2 \mu^{n_2 - \tau(1) - \tau(2)} V(0).$$

For $n \in [n_2 + 1,\, n_3)$, from (ii) we have

$$\begin{aligned} V(n) &< \mu^{n-n_2} V(n_2) = \mu^{n-n_2} d_1 d_2 \mu^{n_2 - \tau(1) - \tau(2)} V(0) \\ &= d_1 d_2 \mu^{n - \tau(1) - \tau(2)} V(0). \end{aligned}$$

From above we have

$$V(n) \leqslant d_1 d_2 \mu^{n - \tau(1) - \tau(2)} V(0), \quad n \in [n_2, n_3).$$

By the mathematical induction procedures, we can prove that for $n \in [n_{k-1}, n_k)$, $k \in \mathbb{N}$, we have

$$V(n) \leqslant d_1 d_2 \cdots d_{k-1} \mu^{n - \tau(1) - \tau(2) - \cdots - \tau(k-1)} V(0).$$

Combining with (iv) and $\tau = \inf_{k \in \mathbb{N}}\{\tau(k)\}$, $d = \sup_{k \in \mathbb{N}}\{d_k\}$, note that $\ln d - \tau \ln \mu < 0$. Then for $n \in [n_{k-1}, n_k)$, we have

$$\begin{aligned} V(n) &\leqslant d^{k-1} \mu^{\{n-(k-1)\tau\}} V(0) \\ &= \exp\bigl\{\ln \mu^{n-(k-1)\tau} + \ln d^{k-1}\bigr\} V(0) \\ &= \exp\bigl\{n \ln \mu + (k-1)[\ln d - \tau \ln \mu]\bigr\} V(0) \\ &= \exp\bigl\{n \ln \mu + k(\ln d - \tau \ln \mu) - \ln d + \tau \ln \mu\bigr\} V(0) \\ &\leqslant \exp\biggl\{n \ln \mu + \frac{n}{\xi}(\ln d - \tau \ln \mu)\biggr\} \frac{\mu^\tau}{d} V(0) \\ &\leqslant \frac{\mu^\tau}{d} V(0) \exp\biggl\{-n\biggl[-\ln \mu + \frac{\tau \ln \mu - \ln d}{\xi}\biggr]\biggr\}. \end{aligned} \tag{5}$$

This, together with condition (i), gives

$$\|x(n)\| \leqslant \left(\frac{c_2\mu^\tau}{c_1 d}\right)^{1/p} \|x_0\| \exp\left\{-n\frac{-\ln\mu + \frac{\tau\ln\mu - \ln d}{\xi}}{p}\right\}. \tag{6}$$

Combining with Definition 1, this implies that the zero solution of system is globally exponentially stable. The proof is completed. □

Remark 4. The continuous nonlinear differential systems with delayed impulses or delay-dependent impulsive control method has been studied in great detail [20, 21]. However, those method cannot be used directly for discrete-time dynamical systems. In [39], Zhang investigated the exponential stability of delayed impulses in difference equations by Lyapunov functions together with Razumikhin technique. And the conditions related to the state were not easy to verify in computation. Based on this, in our model, we simplifies the system model, assume that there is no impulses in the section of continuous system. In addition, for the delayed problem in impulsive systems, although the exponential stability of delay-impulsive difference equations can be investigated by Lyapunov functions together with Razumikhin technique, the condition was not easy to verify if the method used in concrete model. The easier rule of verification method is one of the key research directions in our future work.

3 Main results

In the following, we will discuss the exponential stability of delayed-impulsive discrete-time dynamical networks (3) with actuator saturation.

Theorem 1. *Assume that functions $f_i(\cdot) \in \mathcal{L}$, and there exist two matrixes $H \in \varPi$, $0 < P \in \mathbb{R}^{m\times m}$ and some constants $\mu > 0$, $d < 1$, $\xi \in \mathbb{Z}^+$ such that the following conditions hold*:

$$\begin{aligned} &B^{\mathrm{T}}PB + \varepsilon^{-1}B^{\mathrm{T}}PAA^{\mathrm{T}}PB + \varepsilon L^2 + L^2A^{\mathrm{T}}PA - P < \mu P, \\ &(I + \Sigma_i F + \Sigma_i^- H)^{\mathrm{T}}P(I + \Sigma_j F + \Sigma_j^- H) \leqslant dP, \quad i, j \in \mathbb{M}_m, \\ &\max\{n_k - n_{k-1}\} \leqslant \xi < \frac{\tau\ln(\mu+1) - \ln d}{\ln(\mu+1)}. \end{aligned} \tag{7}$$

Then the discrete-time dynamical system (3) *with delayed impulses and actuator saturation is globally exponentially stable, and the convergence rate is* $(-\ln(\mu+1) + (\tau\ln(\mu+1) - \ln d)/\xi)/2$.

Proof. Choose a Lyapunov functional candidate for system (3) as

$$V(n) = x^{\mathrm{T}}(n)Px(n).$$

When $n \neq n_k - 1$, combining with Lemma 1, we calculate the difference $\Delta V(n) = V(n+1) - V(n)$:

$$\begin{aligned}\Delta V(n) &= x^{\mathrm T}(n+1)Px(n+1) - x^{\mathrm T}(n)Px(n)\\ &= x^{\mathrm T}(n)B^{\mathrm T}PBx(n) + 2x^{\mathrm T}(n)B^{\mathrm T}PAf\bigl(x(n)\bigr)\\ &\quad + f^{\mathrm T}\bigl(x(n)\bigr)A^{\mathrm T}PAf\bigl(x(n)\bigr) - x^{\mathrm T}(n)Px(n)\\ &\leqslant x^{\mathrm T}(n)B^{\mathrm T}PBx(n) + \varepsilon^{-1}x^{\mathrm T}(n)B^{\mathrm T}PAA^{\mathrm T}PBx(n)\\ &\quad + \varepsilon f^{\mathrm T}\bigl(x(n)\bigr)f\bigl(x(n)\bigr) + f^{\mathrm T}(x(n))A^{\mathrm T}PAf\bigl(x(n)\bigr)\\ &\quad - x^{\mathrm T}(n)Px(n)\\ &\leqslant x^{\mathrm T}(n)\bigl(B^{\mathrm T}PB + \varepsilon^{-1}B^{\mathrm T}PAA^{\mathrm T}PB + \varepsilon L^2\\ &\quad + L^2A^{\mathrm T}PA - P\bigr)x(n)\\ &< \mu V(n),\end{aligned} \tag{8}$$

that is,

$$V(n+1) \leqslant (1+\mu)V(n).$$

When $n = n_k$, combining (7) and Lemma 2, we have

$$\begin{aligned}V(n_k) &= x^{\mathrm T}(n_k)Px(n_k)\\ &= \bigl(x\bigl(n_k - \tau(k)\bigr) + \mathrm{Sat}\bigl(Fx\bigl(n_k - \tau(k)\bigr)\bigr)\bigr)^{\mathrm T}\\ &\quad P\bigl(x\bigl(n_k - \tau(k)\bigr) + \mathrm{Sat}\bigl(Fx(n_k - \tau(k))\bigr)\bigr)\\ &\leqslant \left[\left(I + \sum_{i=1}^{2^m}\vartheta_i\bigl(\Sigma_i F + \Sigma_i^- H\bigr)\right)x\bigl(n_k - \tau(k)\bigr)\right]^{\mathrm T}\\ &\quad \times P\left[\left(I + \sum_{i=1}^{2^m}\vartheta_i\bigl(\Sigma_i F + \Sigma_i^- H\bigr)\right)x\bigl(n_k - \tau(k)\bigr)\right]\\ &\leqslant dV\bigl(n_k - \tau(k)\bigr).\end{aligned} \tag{9}$$

From (8), (9) we have

$$\begin{aligned}&V(k+1) < (1+\mu)V(k), \quad n \neq n_k - 1,\\ &V(n_k) \leqslant dV(n_k - \tau(k)), \quad n = n_k.\end{aligned} \tag{10}$$

Note that $\lambda_{\min}(P)\|x(n)\|^2 \leqslant V(n) \leqslant \lambda_{\max}(P)\|x(n)\|^2$, let $M_1 = ((\mu+1)^{\tau} \times \lambda_{\max}(P)/(d\lambda_{\min}(P)))^{1/2}$. Combining with (10), Lemma 3 and Definition 1, we have that the conclusion holds, that is,

$$\bigl\|x(n)\bigr\| \leqslant M_1\|x_0\| \exp\left\{-n\frac{-\ln(\mu+1) + \frac{\tau\ln(\mu+1) - \ln d}{\xi}}{2}\right\}.$$

The proof is completed. □

Theorem 2. *Assume that functions $f_i(\cdot) \in \mathcal{L}$, and there exist matrix $H \in \varPi$ and some constants $\mu > 1$, $d < 1$, $\xi \in \mathbb{Z}^+$ such that the following conditions hold*:

$$\begin{aligned} &\|B\| + L\|A\| < \mu, \\ &\left\| I + \sum_{i=1}^{2^m} \vartheta_i\bigl(\Sigma_i F + \Sigma_i^- H\bigr) \right\| \leqslant d, \\ &\max\{n_k - n_{k-1}\} \leqslant \xi < \frac{\tau \ln\mu - \ln d}{\ln\mu}. \end{aligned} \tag{11}$$

Then the discrete-time dynamical system (3) *with delayed impulses and actuator saturation is globally exponentially stable, and the convergence rate is* $-\ln\mu + (\tau\ln\mu - \ln d)/\xi$.

Proof. Choose a Lyapunov functional candidate for system (3) as

$$V(k) = \bigl\|x(n)\bigr\| = \sum_{i=1}^{N} \bigl|x_i(n)\bigr|.$$

When $n \neq n_k - 1$, we can get that

$$\begin{aligned} V(n+1) &= \bigl\|x(n+1)\bigr\| = \bigl\|Bx(n) + Af\bigl(x(n)\bigr)\bigr\| \\ &\leqslant \bigl\|Bx(n)\bigr\| + \bigl\|Af\bigl(x(n)\bigr)\bigr\| \\ &\leqslant \bigl(\|B\| + L\|A\|\bigr)\bigl\|x(n)\bigr\| \\ &< \mu V(n). \end{aligned} \tag{12}$$

When $n = n_k$, combining with Lemma 2 yields

$$\begin{aligned} V(n_k) &= \bigl\|x(n_k)\bigr\| = \sum_{i=1}^{N} \bigl|x_i(n_k)\bigr| \\ &= \bigl\|x\bigl(n_k - \tau(k)\bigr) + \mathrm{Sat}\bigl(Fx\bigl(n_k - \tau(k)\bigr)\bigr)\bigr\| \\ &\leqslant \left\| \left[I + \sum_{i=1}^{2^m} \vartheta_i\bigl(\Sigma_i F + \Sigma_i^- H\bigr) \right] x\bigl(n_k - \tau(k)\bigr) \right\| \\ &\leqslant dV\bigl(n_k - \tau(k)\bigr). \end{aligned} \tag{13}$$

From (12), (13) we have

$$\begin{aligned} &V(n+1) < \mu V(n), \quad n \neq n_k - 1, \\ &V(n_k) \leqslant dV\bigl(n_k - \tau(k)\bigr), \quad n = n_k. \end{aligned} \tag{14}$$

Combining (14), Lemma 3 and Definition 1, we have

$$\bigl\|x(n)\bigr\| \leqslant \frac{\mu^\tau}{d}\|x_0\| \exp\left\{ -n\left(-\ln\mu + \frac{\tau\ln\mu - \ln d}{\xi} \right) \right\}.$$

The proof is completed. □

Remark 5. From Theorems 1 and 2 it can be found that the dynamical system (1) without impulsive may be unstable, the delayed-impulses can stabilize the original system, and the time interval between the nearest two impulses must meet certain requirements.

Remark 6. The exponential stability of nonlinear impulsive difference dynamical equations with or without delays have been studied in [19, 24, 25, 32, 34, 38, 39]. But most of them considered the impulsive phenomenon that the impulsive transients just depend on their current states of the system, have not depend on their historical states of the system. And most of existing literature were carried out in the linear assumption for impulses. In this paper, the delayed-impulsive controller with actuator saturation is given, which has combined with the reality.

Remark 7. If there is no delays in controller, we can also get some results. And the method in this paper can be applied in traditional model [10, 24, 28]. Equations (2) can be rewritten as the following impulsive system:

$$\begin{aligned} &x(n+1) = Bx(n) + Af(n, x(n)), \quad n \neq n_k - 1, \\ &x(n_k) = \operatorname{Sat}\bigl(Fx(n_k - 1)\bigr), \quad n = n_k, \\ &x(0) = x_0. \end{aligned} \tag{15}$$

Some useful corollary will be given in the following by the same derivation method.

Corollary 1. *Assume that functions $f_i(\cdot) \in \mathcal{L}$, and there exist two matrixes $H \in \Pi$, $0 < P \in \mathbb{R}^{m\times m}$ and some constants $\mu > 0$, $d < 1$, $\xi \in \mathbb{Z}^+$ such that the following conditions hold*:

$$\begin{aligned} &B^{\mathrm{T}}PB + \varepsilon^{-1}B^{\mathrm{T}}PAA^{\mathrm{T}}PB + \varepsilon L^2 + L^2A^{\mathrm{T}}PA - P < \mu P, \\ &(I + \Sigma_i F + \Sigma_i^- H)^{\mathrm{T}}P(I + \Sigma_j F + \Sigma_j^- H) \leqslant dP, \quad i, j \in \mathbb{M}_m, \\ &\max\{n_k - n_{k-1}\} \leqslant \xi < \frac{\tau \ln(\mu+1) - \ln d}{\ln(\mu+1)}. \end{aligned} \tag{16}$$

Then the discrete-time dynamical system (15) *with delayed impulses is globally exponentially stable, and the convergence rate is* $(-\ln(\mu+1) + (\tau\ln(\mu+1) - \ln d)/\xi)/2$.

Corollary 2. *Assume that functions $f_i(\cdot) \in \mathcal{L}$, and there exist matrix $H \in \Pi$ and some constants $\mu > 1$, $d < 1$, $\xi \in \mathbb{Z}^+$ such that the following conditions hold*:

$$\begin{aligned} &\|B\| + L\|A\| \leqslant \mu, \\ &\Biggl\| I + \sum_{i=1}^{2^m} \vartheta_i\bigl(\Sigma_i F + \Sigma_i^- H\bigr) \Biggr\| \leqslant d, \\ &\max\{n_k - n_{k-1}\} \leqslant \xi < \frac{\tau\ln\mu - \ln d}{\ln\mu}. \end{aligned} \tag{17}$$

Then the discrete-time dynamical system (15) *with delayed impulses is globally exponentially stable, and the convergence rate is* $-\ln\mu + (\tau\ln\mu - \ln d)/\xi$.

4 Examples

In this section, we provide two numerical examples to illustrate the effectiveness of the proposed criteria in this paper.

Example 1. Consider the following difference system:

$$x(n+1) = \begin{pmatrix} 0.88 & 0.08 \\ 0.69 & 0.93 \end{pmatrix} x(n) + \begin{pmatrix} -0.35 & 0.13 \\ 0.34 & -0.15 \end{pmatrix} f\big(x(n)\big), \tag{18}$$

where $x(n) = (x_1(n), x_2(n))^{\mathrm{T}}$, it is assumed that $f(x) = (f_1(x_1), f_2(x_2))^{\mathrm{T}}$ with $f(s) = 0.5\tanh(s)$. Here the initial conditions of each nodes are chosen as $x(0) = (-1.2, 0.8)^{\mathrm{T}}$. The time responses of the state variables of (18) are depicted in Fig. 2(a). We can find that dynamical system without impulsive controller is unstable.

Now, we design the controller

$$U_k\big(x\big(n_k - \tau(k)\big)\big) = \mathrm{Sat}\left(\begin{pmatrix} -0.74 & 0 \\ 0 & -0.75 \end{pmatrix} x\big(n_k - \tau(k)\big)\right)$$

with $\tau(k) = 3$ and $n_k = 4k$, $k \in \mathbb{N}$. Choose $\varepsilon = 0.28$, $H = \mathrm{diag}(-0.5, -0.5)$, $\xi = 4$, $P = I$. From Theorem 2 we have $L = 0.5$, $\tau = 3$,

$$B^{\mathrm{T}}PB + \varepsilon^{-1}B^{\mathrm{T}}PAA^{\mathrm{T}}PB + \varepsilon L^2 + L^2A^{\mathrm{T}}PA - P = \begin{pmatrix} 4551 & 0.5056 \\ 0.5056 & 0.4531 \end{pmatrix} \leqslant 3P,$$

$$\big(I + \Sigma_i F + \Sigma_i^- H\big)^{\mathrm{T}} P \big(I + \Sigma_j F + \Sigma_j^- H\big) \leqslant 0.25I,$$

$$\max\{n_k - n_{k-1}\} = 4 \leqslant \xi < \frac{\tau\ln(\mu+1) - \ln d}{\ln(\mu+1)} = 4.7582.$$

It is easy to verify that the conditions of Theorem 1 are satisfied, and therefore system (18) has globally exponential stability. The time responses of the state variables are depicted in Fig. 2(b).

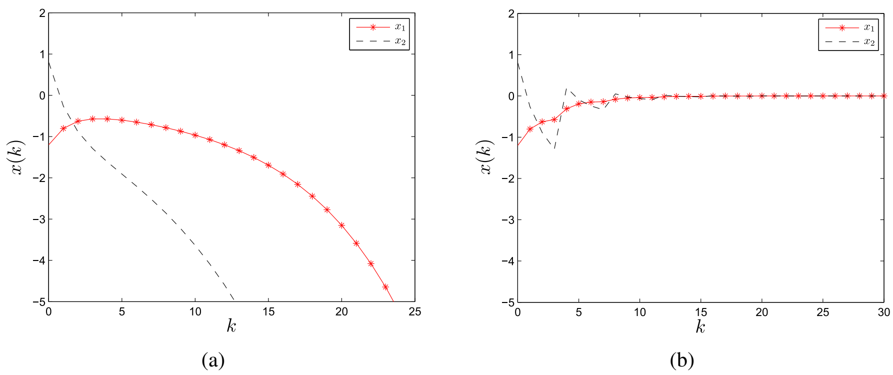

Figure 2. Transient response of state variable $x(t)$ of system (18): (a) without controller; (b) with actuator saturation.

Example 2. Consider the difference system (1) with

$$B = \begin{pmatrix} -0.33 & 0.2 & 0.3 \\ 0 & 0.72 & 0.22 \\ 0.33 & 0.17 & 0.47 \end{pmatrix}, \qquad A = \begin{pmatrix} -0.31 & 0.1 & 0.3 \\ 0 & 0.2 & 0.2 \\ 0.3 & 0.17 & -0.4 \end{pmatrix},$$

where $x(n) = (x_1(n), x_2(n), x_3(n))^{\mathrm{T}}$, it is assumed that $f(x) = (f_1(x_1), f_2(x_2), f_3(x_3))^{\mathrm{T}}$ with $f(s) = \tanh(s)$. Here, the initial conditions of each nodes are chosen as $x(0) = (0.12, -0.4, 0.4)^{\mathrm{T}}$. The time responses of the state variables of (1) are depicted in Fig. 3(a).

Now, we design the controller

$$U_k(\cdot) = \mathrm{Sat}\left(\begin{pmatrix} -0.54 & 0 & 0 \\ 0 & -0.55 & 0 \\ 0 & 0 & -0.57 \end{pmatrix} x\bigl(n_k - \tau(k)\bigr)\right)$$

with $\tau(k) = 4$ and $n_k = 5k$, $k \in \mathbb{N}$. Choose $\vartheta_i = 0.125$, $i = 1, 2, \dots, 8$, $H = \mathrm{diag}(0.2, 0.2, 0.2)$, $\xi = 5$. From Theorem 2 we have $L = 1$, $\tau = 4$, $\|B\| + L\|A\| = 1.5675 \leqslant \mu = 1.5675$, $\|I + \sum_{i=1}^{8} \vartheta_i(\Sigma_i F + \Sigma_i^{-} H)\| \leqslant d = 0.83$, $\max\{n_k - n_{k-1}\} = 5 \leqslant \xi < (\tau \ln \mu - \ln d)/\ln \mu = 5.4145$. It is easy to verify that the conditions of Theorem 2 are satisfied, the dynamical system (1) has globally exponential stability. The time responses of the state variables are depicted in Fig. 3(b).

From above it is easy to see that it is difficult to obtain the results by traditional design methods. In fact, the rate of convergence may not be very fast, efficient, but the controller more corresponds to actual fact and are more direct.

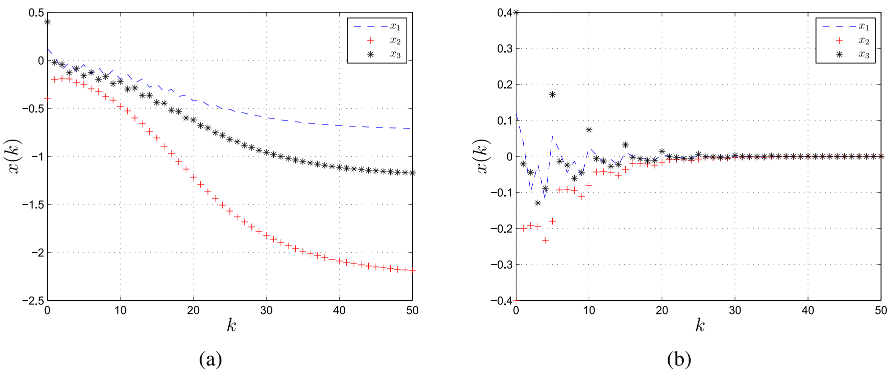

Figure 3. Transient response of state variable $x(t)$ of system (1): (a) without controller; (b) with actuator saturation.

5 Conclusions

Based on a discrete-time delayed-impulsive difference inequality, auxiliary matrix and some other inequality techniques, we obtained some simple sufficient conditions ensuring exponential stability for impulses in discrete-time systems with actuator saturation. The method in this paper can be extended to study other discrete delayed-impulsive systems or delayed-impulsive control problem. Furthermore, studying the delayed impulsive stabilization problem of discrete-time delayed systems will be another interesting topic for future research.

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