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Infinite point and Riemann–Stieltjes integral conditions for an integro-differential equation

Ahmed El-Sayed[a], **Reda Gamal**[b],

[a]Faculty of Science, Alexandria University,
Alexandria, Egypt
amasayed@alexu.edu.eg

[b]Faculty of Science, Al-Azhar University,
Cairo, Egypt
redagamal@azhar.edu.eg

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Abstract. In this paper, we study the existence of solutions for two nonlocal problems of integro-differential equation with nonlocal infinite-point and Riemann–Stieltjes integral boundary conditions. The continuous dependence of the solution will be studied.

Keywords: existence of solutions, continuous dependence, nonlocal condition, Riemann–Stieltjes condition, infinite point condition.

1 Introduction

In the last few years, some investigators have established a lot of useful and interesting functional differential equation with the nonlocal condition in order to achieve various goals; see [1–9, 11, 12, 14–21] and the references cited therein.

In this paper, we are concerned with the nonlocal problem for the integro-differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f\left(t, x(t), \int_0^t g\bigl(s, x(s)\bigr)\,\mathrm{d}s\right), \quad \text{a.e. } t \in (0,1), \tag{1}$$

with the nonlocal condition

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0, \quad a_k \geqslant 0,\ \tau_k \in (0,1). \tag{2}$$

The existence of solution, under certain conditions, will be proved. The continuous dependence of the solution on the nonlocal parameter a_k and on the function g will be studied.

As applications, the nonlocal problem of equation (1) with the Riemann–Stieltjes integral condition

$$\int_0^1 x(s)\,\mathrm{d}g(s) = x_0 \tag{3}$$

will be studied. Also, the nonlocal problem of equation (1) with infinite-point boundary condition

$$\sum_{k=1}^{\infty} a_k x(\tau_k) = x_0 \tag{4}$$

will be studied.

2 Main results

2.1 Integral representation

Lemma 1. *Let* $B = \sum_{k=1}^{m} a_k \neq 0$*, the solution of the nonlocal problem* (1)–(2)*, if it exist, then it can be represented by the integral equation*

$$\begin{aligned} x(t) = B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] \\ + \int_0^t f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s. \end{aligned} \tag{5}$$

Proof. Let x be a solution of the nonlocal problem (1)–(2). Integrating both sides of (1), we get

$$x(t) = x(0) + \int_0^t f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s. \tag{6}$$

Using the nonlocal condition (2), we get

$$\sum_{k=1}^{m} a_k x(\tau_k) = x(0)\sum_{k=1}^{m} a_k + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s,$$

then

$$x(0) = \frac{1}{\sum_{k=1}^{m} a_k}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr]. \tag{7}$$

Using (6) and (7), we obtain

$$x(t) = B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s\right] + \int_0^t f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s. \qquad \square$$

2.2 Existence of solution

2.2.1 Functional equation approach

Consider the nonlocal problem (1)–(2) with the assumptions:

(i) $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Caratheodory condition, i.e., f is measurable in t for any $x, y \in \mathbb{R}$ and continuous in x, y for almost all $t \in [0, 1]$. There exist a function $c_1 \in L^1[0, 1]$ and a positive constant $b_1 > 0$ such that

$$\big|f(t, x, y)\big| \leqslant c_1(t) + b_1|x| + b_1|y|.$$

(ii) $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition, i.e., g is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0, 1]$. There exist a function $c_2 \in L^1[0, 1]$ and a positive constant $b_2 > 0$ such that

$$\big|g(t, x)\big| \leqslant c_2(t) + b_2|x|.$$

(iii)
$$\sup_{t\in[0,1]} \int_0^t c_1(s)\,\mathrm{d}s \leqslant M_1, \qquad \sup_{t\in[0,1]} \int_0^t \int_0^s c_2(\theta)\,\mathrm{d}\theta\,\mathrm{d}s \leqslant M_2.$$

(iv)
$$2b_1 + b_1 b_2 < 1.$$

Definition 1. By a solution of the nonlocal problem (1)–(2) we mean a function $x \in C[0, 1]$ that satisfies (1)–(2).

Theorem 1. *Let assumptions* (i)–(iv) *be satisfied, then the nonlocal problem* (1)–(2) *has at least one solution.*

Proof. Define the operator A associated with the integral equation (5) by

$$Ax(t) = B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s\right] + \int_0^t f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s.$$

Let $Q_r = \{x \in \mathbb{R}\colon \|x\| \leqslant r\}$, where $r = B^{-1}(|x_0| + 2M_1 + 2b_1 M_2)/(1 - (2b_1 + b_1 b_2))$. Then we have, for $x \in Q_r$,

$$\begin{aligned}
\bigl|Ax(t)\bigr| &\leqslant B^{-1}\left[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left| f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\, \mathrm{d}\theta\right)\right| \mathrm{d}s\right] \\
&\quad + \int_0^t \left| f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\, \mathrm{d}\theta\right)\right| \mathrm{d}s \\
&\leqslant B^{-1}\left[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left(c_1(s) + b_1\bigl|x(s)\bigr| + b_1 \int_0^s \bigl|g\bigl(\theta, x(\theta)\bigr)\bigr|\, \mathrm{d}\theta\right) \mathrm{d}s\right] \\
&\quad + \int_0^t \left(c_1(s) + b_1\bigl|x(s)\bigr| + b_1 \int_0^s g\bigl(\theta, x(\theta)\bigr)\, \mathrm{d}\theta\right) \mathrm{d}s \\
&\leqslant B^{-1}\left[|x_0| + \sum_{k=1}^{m} a_k \left(M_1 + b_1 r + b_1 \int_0^{\tau_k}\int_0^s c_2(\theta) + b_2\bigl|x(\theta)\bigr|\, \mathrm{d}\theta\, \mathrm{d}s\right)\right] \\
&\quad + M_1 + b_1 r + b_1 \int_0^t \int_0^s \bigl(c_2(\theta) + b_2\bigl|x(\theta)\bigr|\bigr)\, \mathrm{d}\theta\, \mathrm{d}s \\
&\leqslant B^{-1}|x_0| + M_1 + b_1 r + b_1 M_2 + \frac{1}{2} b_1 b_2 r + M_1 + b_1 r + b_1 M_2 + \frac{1}{2} b_1 b_2 r \\
&= B^{-1}|x_0| + 2M_1 + 2b_1 r + 2b_1 M_2 + b_1 b_2 r = r.
\end{aligned}$$

This prove that $A : Q_r \to Q_r$ and the class of functions $\{Ax\}$ is uniformly bounded in Q_r.

Now, let $t_1, t_2 \in (0,1)$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
&\bigl|Ax(t_2) - Ax(t_1)\bigr| \\
&\quad = \left| \int_0^{t_2} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\, \mathrm{d}\theta\right) \mathrm{d}s - \int_0^{t_1} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\, \mathrm{d}\theta\right) \mathrm{d}s \right| \\
&\quad \leqslant \int_{t_1}^{t_2} \left| f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\, \mathrm{d}\theta\right)\right| \mathrm{d}s \\
&\quad \leqslant \int_{t_1}^{t_2} \left(c_1(s) + b_1\bigl|x(s)\bigr| + b_1 \int_0^s \bigl|g\bigl(\theta, x(\theta)\bigr)\bigr|\, \mathrm{d}\theta\right) \mathrm{d}s
\end{aligned}$$

$$\leqslant \int_{t_1}^{t_2} c_1(s)\,\mathrm{d}s + (t_2 - t_1)b_1 r + b_1 \int_{t_1}^{t_2}\int_0^s c_2(\theta)\,\mathrm{d}\theta\,\mathrm{d}s + \frac{1}{2}b_1 b_2 r\bigl(t_2^2 - t_1^2\bigr).$$

This mean that the class of functions $\{Ax\}$ is equicontinuous in Q_r.

Let $x_n \in Q_r$, $x_n \to x$ ($n \to \infty$), then from continuity of the functions f and g we obtain $f(t, x_n(t), y_n(t)) \to f(t, x(t), y(t))$ and $g(t, x_n(t)) \to g(t, x(t))$ as $n \to \infty$. Also

$$\lim_{n\to\infty} Ax_n(t) = \lim_{n\to\infty}\Biggl[B^{-1}\Biggl[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\Biggl(s, x_n(s), \int_0^s g\bigl(s, x_n(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] + \int_0^t f\Biggl(s, x_n(s), \int_0^s g\bigl(\theta, x_n(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr]. \tag{8}$$

Using assumptions (i)–(ii) and Lebesgue dominated convergence theorem [13], from (8) we obtain

$$\lim_{n\to\infty} Ax_n(t) = \Biggl[B^{-1}\Biggl[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} \lim_{n\to\infty} f\Biggl(s, x_n(s), \int_0^s g\bigl(\theta, x_n(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] + \int_0^t \lim_{n\to\infty} f\Biggl(s, x_n(s), \int_0^s g\bigl(\theta, x_n(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] = Ax(t).$$

Then $Ax_n \to Ax$ as $n \to \infty$. This mean that the operator A is continuous.

$$\lim_{t\to 1} x(t) = \Biggl\{B^{-1}\Biggl[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] + \int_0^1 f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr\} \in C[0,1],$$

and

$$\lim_{t\to 0} x(t) = B^{-1}\Biggl[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] \in C[0,1].$$

Then by Schauder fixed point theorem [10] there exist at least one solution $x \in C[0,1]$ of the integral equation (5).

To complete the proof, differentiating (5) we obtain

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Biggl\{B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] \\ &\quad + \int_0^t f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr\} \\ &= 0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s \\ &= f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr).\end{aligned}$$

Also, from the integral equation (5), we obtain

$$\begin{aligned}x(\tau_k) &= B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] \\ &\quad + \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{m} a_k x(\tau_k) &= \sum_{k=1}^{m} a_k B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s\Biggr] \\ &\quad + \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Biggl(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Biggr)\,\mathrm{d}s.\end{aligned}$$

Then

$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0.$$

Then there exist at least one solution $x \in C[0,1]$ of the nonlocal problem of functional differential equation (1)–(2). □

2.2.2 *Coupled system approach*

Let the function f and g satisfies the conditions:

(i*) $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Caratheodory condition, i.e., f is measurable in t for any $x, y \in \mathbb{R}$ and continuous in x, y for almost all $t \in [0,1]$. There exist a function $m_1 \in L^1[0,1]$ such that

$$\bigl|f(t,x,y)\bigr| \leqslant m_1(t).$$

(ii*) $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition, i.e., g is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in [0,1]$. There exist a function $m_2 \in L^1[0,1]$ such that

$$\left|g(t,x)\right| \leqslant m_2(t).$$

(iii*)
$$\sup_{t\in[0,1]} \int_0^t m_1(s)\,\mathrm{d}s \leqslant M_1, \qquad \sup_{t\in[0,1]} \int_0^t m_2(s)\,\mathrm{d}s \leqslant M_2.$$

Now, let

$$y(t) = \int_0^t g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta, \tag{9}$$

then

$$x(t) = B^{-1}\left[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s\right] + \int_0^t f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s. \tag{10}$$

Let X be the Banach space of all order pairs (x,y) with the norm

$$\bigl\|(x,y)\bigr\|_X = \|x\|_C + \|y\|_C = \sup_{t\in[0,1]} \bigl|x(t)\bigr| + \sup_{t\in[0,1]} \bigl|y(t)\bigr|.$$

Definition 2. By a solution of the nonlocal problem (1)–(2) we mean a function $x \in C[0,1]$ that satisfies (1)–(2).

Theorem 2. *Let assumptions* (i*)–(iii*) *be satisfied, then the nonlocal problem* (1)–(2) *has at least one solution.*

Proof. Define the operator A associated with the integral equation (9)–(10) by

$$A\bigl(x(t), y(t)\bigr) = \left(B^{-1}\left[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\bigl(s, x(s), y(s)\bigr)\right] + \int_0^t f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s, \int_0^t g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right).$$

Let $Q_r = \{(x,y) \in \mathbb{R}^2\colon \|x\| \leqslant r_1, \|y\| \leqslant r_2, \|(x,y)\| \leqslant r_1 + r_2 = r\}$, where $r = M_1 + M_2$.

Then we have, for $(x,y) \in Q_r$

$$A\bigl(x(t), y(t)\bigr) = \left(B^{-1}\left[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f\bigl(s, x(s), y(s)\bigr)\right] + \int_0^t f\bigl(s, x(s), y(s)\bigr)\,\mathrm{d}s, \int_0^t g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right),$$

but

$$\left|B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\right] + \int_0^t f\big(s, x(s), y(s)\big)\,\mathrm{d}s\right|$$
$$\leqslant B^{-1}\left[|x_0| + \sum_{k=1}^{m} a_k \int_0^{\tau_k} m_1(s)\,\mathrm{d}s\right] + \int_0^t m_1(s)\,\mathrm{d}s$$
$$\leqslant B^{-1}|x_0| + 2M_1 \tag{11}$$

and

$$\left|\int_0^t g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right| \leqslant \int_0^t m_2(\theta)\,\mathrm{d}\theta \leqslant M_2. \tag{12}$$

From (11) and (12) we get

$$\big\|A(x,y)\big\|_X \leqslant B^{-1}|x_0| + 2M_1 + M_2.$$

This prove that $A : Q_r \to Q_r$ and the class of functions $\{A(x,y)\}$ is uniformly bounded in Q_r.

Now, let $t_1, t_2 \in (0,1)$ such that $|t_2 - t_1| < \delta$, then

$$\big|A\big(x(t_2), y(t_2)\big) - A\big(x(t_1), y(t_1)\big)\big|$$
$$= \Bigg|\Bigg(B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\right] + \int_0^{t_2} f\big(s, x(s), y(s)\big)\,\mathrm{d}s,$$
$$\int_0^{t_2} g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)$$
$$- \Bigg(B^{-1}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\right] + \int_0^{t_1} f\big(s, x(s), y(s)\big)\,\mathrm{d}s,$$
$$\int_0^{t_1} g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)\Bigg|$$
$$= \left|\left(\int_{t_1}^{t_2} f\big(s, x(s), y(s)\big)\,\mathrm{d}s, \int_{t_1}^{t_2} g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\right|,$$

but

$$\left|\int_{t_1}^{t_2} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\right| \leqslant \int_{t_1}^{t_2} m_1(s)\,\mathrm{d}s, \qquad \left|\int_{t_1}^{t_2} g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right| \leqslant \int_{t_1}^{t_2} m_2(s)\,\mathrm{d}s. \tag{13}$$

From (13) we get

$$\bigl|A\bigl(x(t_2), y(t_2)\bigr) - A\bigl(x(t_1), y(t_1)\bigr)\bigr| \leqslant \int_{t_1}^{t_2} \bigl(m_1(s) + m_2(s)\bigr)\, \mathrm{d}s.$$

This mean that the class of functions $\{A(x, y)\}$ is equicontinuous in Q_r.

Let $x_n \in Q_r$, $x_n \to x$ ($n \to \infty$), then from continuity of the functions f and g we obtain $f(t, x_n(t), y_n(t)) \to f(t, x(t), y(t))$ and $g(t, x_n(t)) \to g(t, x(t))$ as $n \to \infty$. Also

$$\begin{aligned}
&\lim_{n\to\infty} A\bigl(x_n(t), y_n(t)\bigr) \\
&\quad = \lim_{n\to\infty} \Biggl(B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\bigl(s, x_n(s), y_n(s)\bigr)\, \mathrm{d}s\Biggr] + \int_0^t f\bigl(s, x_n(s), y_n(s)\bigr)\, \mathrm{d}s, \\
&\qquad\qquad \int_0^t g\bigl(s, x_n(\theta)\bigr)\, \mathrm{d}\theta\Biggr) \\
&\quad = \Biggl(\lim_{n\to\infty} B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\bigl(s, x_n(s), y_n(s)\bigr)\, \mathrm{d}s\Biggr] + \int_0^t f\bigl(s, x_n(s), y_n(s)\bigr)\, \mathrm{d}s, \\
&\qquad\quad \lim_{n\to\infty} \int_0^t g\bigl(s, x_n(\theta)\bigr)\, \mathrm{d}\theta\Biggr).
\end{aligned} \tag{14}$$

Using assumptions (i)–(ii) and Lebesgue dominated convergence theorem [13], from (14) we obtain

$$\begin{aligned}
&\lim_{n\to\infty} A\bigl(x_n(t), y_n(t)\bigr) \\
&\quad = \Biggl(B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\bigl(s, x(s), y(s)\bigr)\, \mathrm{d}s\Biggr] + \int_0^t f\bigl(s, x(s), y(s)\bigr)\, \mathrm{d}s, \\
&\qquad \int_0^t g\bigl(s, x(\theta)\bigr)\, \mathrm{d}\theta\Biggr) \\
&\quad = A\bigl(x(t), y(t)\bigr).
\end{aligned}$$

Then $Ax_n \to Ax$ as $n \to \infty$. This mean that the operator A is continuous.

$$\begin{aligned}
\lim_{t\to 1} x(t) = {}& \Biggl\{B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\bigl(s, x(s), y(s)\bigr)\, \mathrm{d}s\Biggr] + \int_0^1 f\bigl(s, x(s), y(s)\bigr)\, \mathrm{d}s\Biggr\} \\
&\in C[0, 1],
\end{aligned}$$

and

$$\lim_{t\to 1} y(t) = \int_0^1 g\big(s, x(\theta)\big)\,\mathrm{d}\theta \in C[0,1],$$
$$\lim_{t\to 0} x(t) = B^{-1}\Bigg[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\Bigg] \in C[0,1],$$
$$\lim_{t\to 0} y(t) = 0 \in C[0,1],$$

Then by Schauder fixed point theorem [10] there exist at least one solution $x \in C[0,1]$ of the integral equation (9)–(10).

To complete the proof, differentiating (10), we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{B^{-1}\Bigg[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\Bigg] + \int_0^t f\big(s, x(s), y(s)\big)\,\mathrm{d}s\Bigg\}$$
$$= 0 + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t f\big(s, x(s), y(s)\big)\,\mathrm{d}s = f\big(s, x(s), y(s)\big),$$
$$y(t) = \int_0^t g\big(s, x(\theta)\big)\,\mathrm{d}\theta.$$

Also, from the integral equation (9)–(10) we obtain

$$x(\tau_k) = B^{-1}\Bigg[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\Bigg] + \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s,$$
$$y(t) = \int_0^t g\big(s, x(\theta)\big)\,\mathrm{d}\theta,$$

and

$$\sum_{k=1}^{m} a_k x(\tau_k) = \sum_{k=1}^{m} a_k B^{-1}\Bigg[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s\Bigg]$$
$$+ \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\big(s, x(s), y(s)\big)\,\mathrm{d}s$$
$$y(t) = \int_0^t g\big(s, x(\theta)\big)\,\mathrm{d}\theta.$$

Then
$$\sum_{k=1}^{m} a_k x(\tau_k) = x_0.$$
Hence, the nonlocal problem (1)–(2) is equivalent to integral equation (9)–(10). □

2.3 Uniqueness of the solution

Let f and g satisfy the following assumptions:

(v) $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is measurable in t for any $x, y \in \mathbb{R}$ and satisfies the Lipschitz condition
$$\bigl|f(t,x,y) - f(t,u,v)\bigr| \leqslant b_1|x-u| + b_1|y-v|.$$

(vi) $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ is measurable in t for any $x \in \mathbb{R}$ and satisfies the Lipschitz condition
$$\bigl|g(t,x) - g(t,u)\bigr| \leqslant b_2|x-u|.$$

(vii)
$$\sup_{t\in[0,1]} \int_0^t \bigl|f(s,0,0)\bigr|\,\mathrm{d}s \leqslant L_1, \qquad \sup_{t\in[0,1]} \int_0^t \int_0^s \bigl|g(\theta,0)\bigr|\,\mathrm{d}\theta\,\mathrm{d}s \leqslant L_2.$$

Theorem 3. *Let assumptions* (v)–(vii) *be satisfied, then the solution of the nonlocal problem* (1)–(2) *is unique.*

Proof. From assumption (v) we have that f is measurable in t for any $x, y \in \mathbb{R}$ and satisfies the Lipschitz condition, then it is continuous in $x, y \in \mathbb{R}$ for all $t \in [0,1]$, and
$$\bigl|f(t,x,y)\bigr| \leqslant b_1|x| + b_1|y| + \bigl|f(t,0,0)\bigr|.$$
Condition (i) is satisfied. Also by the same way we can show that assumption (ii) satisfied by assumption (vi). Now, from Theorem 1 the solution of the nonlocal problem (1)–(2) exists.

Let x, y be two the solution of (1)–(2), then
$$\begin{aligned}
\bigl|x(t) - y(t)\bigr| = \Bigg| & B^{-1}\Bigg[-\sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Bigg(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg] \\
&+ \int_0^t f\Bigg(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&- B^{-1}\Bigg[-\sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Bigg(s, y(s), \int_0^s g\bigl(\theta, y(\theta)\bigr)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg] \\
&- \int_0^t f\Bigg(s, y(s), \int_0^s g\bigl(\theta, y(\theta)\bigr)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg|
\end{aligned}$$

$$\begin{aligned}
&\leqslant B^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left| f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\mathrm{d}s \right.\\
&\quad \left. - f\left(s, y(s), \int_0^s g\big(\theta, y(\theta)\big)\,\mathrm{d}\theta\right)\right| \mathrm{d}s\\
&\quad + \int_0^t \left| f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\mathrm{d}s - f\left(s, y(s), \int_0^s g\big(\theta, y(\theta)\big)\,\mathrm{d}\theta\right)\right| \mathrm{d}s,\\
&\leqslant B^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \left(b_1\|x-y\| + b_1 \int_0^s \big|g\big(\theta, x(\theta)\big) - g\big(\theta, y(\theta)\big)\big|\,\mathrm{d}\theta\right)\mathrm{d}s\\
&\quad + \int_0^t \left(b_1\|x-y\| + b_1 \int_0^s \big|g\big(\theta, x(\theta)\big) - g\big(\theta, y(\theta)\big)\big|\,\mathrm{d}\theta\right)\mathrm{d}s\\
&\leqslant b_1\|x-y\| + \frac{1}{2}b_1 b_2\|x-y\| + b_1\|x-y\| + \frac{1}{2}b_1 b_2\|x-y\|\\
&= (2b_1 + b_1 b_2)\|x-y\|.
\end{aligned}$$

Hence,

$$(1 - 2b_1 + b_1 b_2)\|x-y\| \leqslant 0.$$

Since $(2b_1 + b_1 b_2) < 1$, then $x(t) = y(t)$, and the solution of the nonlocal problem (1)–(2) is unique. □

2.4 Continuous dependence

2.4.1 *Continuous dependence on* x_0

Definition 3. The solution $x \in C[0,1]$ of the nonlocal problem (1)–(2) depends continuously on x_0 if

$$\forall \epsilon > 0,\ \exists \delta(\epsilon)\colon\quad |x_0 - x_0^*| < \delta \quad\Longrightarrow\quad \|x - x^*\| < \epsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{\mathrm{d}x^*}{\mathrm{d}t} = f\left(t, x^*(t), \int_0^t g\big(s, x^*(s)\big)\,\mathrm{d}s\right), \quad \text{a.e. } t \in (0,1), \tag{15}$$

with the nonlocal condition

$$\sum_{k=1}^{n} a_k x^*(\tau_k) = x_0^*, \quad a_k \geqslant 0,\ \tau_k \in (0,1). \tag{16}$$

Theorem 4. *Let the assumptions of Theorem* 3 *be satisfied, then the solution of the nonlocal problem* (1)–(2) *depends continuously on* x_0.

Proof. Let x, x^* be two solutions of the nonlocal problems (1)–(2) and (15)–(16), respectively. Then

$$\begin{aligned}
&\big|x(t) - x^*(t)\big| \\
&\quad = \Bigg|B^{-1}\Bigg[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg] \\
&\qquad + \int_0^t f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&\qquad - B^{-1}\Bigg[x_0^* - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Bigg(s, x^*(s), \int_0^s g\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg] \\
&\qquad + \int_0^t f\Bigg(s, x^*(s), \int_0^s g\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg| \\
&\quad \leqslant B^{-1}|x_0 - x_0^*| \\
&\qquad + B^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \Bigg|f\Bigg(s, x^*(s), \int_0^s g\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg) - f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)\Bigg|\,\mathrm{d}s \\
&\qquad + \int_0^t \Bigg|f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg) - f\Bigg(s, x^*(s), \int_0^s g\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg)\Bigg|\,\mathrm{d}s, \\
&\quad \leqslant B^{-1}|x_0 - x_0^*| \\
&\qquad + B^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \Bigg(b_1\|x - x^*\| + b_1 \int_0^s \big|g\big(\theta, x^*(\theta)\big) - g\big(\theta, x(\theta)\big)\big|\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&\qquad + \int_0^t \Bigg(b_1\|x - x^*\| + b_1 \int_0^s \big|g\big(\theta, x(\theta)\big) - g\big(\theta, x^*(\theta)\big)\big|\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&\quad \leqslant B^{-1}|x_0 - x_0^*| + b_1\|x - y\| + \frac{1}{2}b_1 b_2\|x - x^*\| + b_1\|x - x^*\| \\
&\qquad + \frac{1}{2}b_1 b_2\|x - x^*\| \\
&\quad \leqslant B^{-1}\delta + (2b_1 + b_1 b_2)\|x - x^*\|.
\end{aligned}$$

Hence,

$$\|x - x^*\| \leqslant \frac{B^{-1}\delta}{[1 - (2b_1 + b_1 b_2)]} = \epsilon.$$

This mean that the solution of the nonlocal problem (1)–(2) depends continuously on x_0. The proof is completed. □

2.4.2 *Continuous dependence on a_k*

Definition 4. The solution $x \in C[0,1]$ of the nonlocal problem (1)–(2) depends continuously on a_k if

$$\forall \epsilon > 0,\ \exists \delta(\epsilon)\colon \quad |a_k - a_k^*| < \delta \quad \Longrightarrow \quad \|x - x^*\| < \epsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{\mathrm{d}x^*}{\mathrm{d}t} = f\left(t, x^*(t), \int_0^t g\bigl(s, x^*(s)\bigr)\,\mathrm{d}s\right), \quad \text{a.e. } t \in (0,1), \tag{17}$$

with the nonlocal condition

$$\sum_{k=1}^{n} a_k^* x^*(\tau_k) = x_0, \quad a_k \geqslant 0,\ \tau_k \in (0,1). \tag{18}$$

Theorem 5. *Let the assumptions of Theorem* 3 *be satisfied, then the solution of the nonlocal problem* (1)–(2) *depends continuously on a_k.*

Proof. Let $B^* = \sum_{k=1}^{n} a_k^* \neq 0$, and let x, x^* be two solutions of the nonlocal problems (1)–(2) and (17)–(18), respectively. Then

$$\begin{aligned}
&\bigl|x(t) - x^*(t)\bigr| \\
&\quad = \Biggl|B^{-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\Biggr] \\
&\qquad + \int_0^t f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s \\
&\qquad - B^{*-1}\Biggl[x_0 - \sum_{k=1}^{m} a_k^* \int_0^{\tau_k} f\left(s, x^*(s), \int_0^s g\bigl(\theta, x^*(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\Biggr] \\
&\qquad - \int_0^t f\left(s, x^*(s), \int_0^s g\bigl(s, x^*(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\Biggr|
\end{aligned}$$

$$\leqslant B^{-1}B^{*-1}m\delta x_0$$
$$+ B^{*-1}\sum_{k=1}^{m} a_k^* \int_0^{\tau_k} \left| f\left(s, x^*(s), \int_0^s g\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\right) - f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\right| \mathrm{d}s$$
$$+ B^{*-1}\sum_{k=1}^{m} |a_k^* - a_k| \int_0^{\tau_k} \left| f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\right| \mathrm{d}s$$
$$+ B^{-1}B^{*-1}\sum_{k=1}^{m} |a_k - a_k^*| \sum_{k=1}^{m} a_k \int_0^{\tau_k} \left| f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\right| \mathrm{d}s$$
$$+ \int_0^t \left| f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) - f\left(s, x^*(s), \int_0^s g\big(s, x^*(\theta)\big)\,\mathrm{d}\theta\right)\right| \mathrm{d}s$$
$$\leqslant B^{-1}B^{*-1}m\delta x_0 + (2b_1 + b_1b_2)\|x - x^*\|$$
$$+ B^{*-1}m\delta\big(2b_1\|x\| + b_1b_2\|x\| + 2L_1 + 2b_1L_2\big).$$

Hence,

$$\|x - x^*\| \leqslant \frac{m\delta x_0 + m\delta B((2b_1 + b_1b_2)\|x\| + 2L_1 + 2b_1L_2)}{[1 - (2b_1 + b_1b_2)]BB^*} = \epsilon.$$

This mean that the solution of the nonlocal problem (1)–(2) depends continuously on a_k. The proof is completed. □

2.4.3 Continuous dependence on the function g

Definition 5. The solution $x \in C[0, 1]$ of the nonlocal problem (1)–(2) depends continuously on the function g if

$$\forall\epsilon > 0,\ \exists\delta(\epsilon)\colon\quad |g - g^*| < \delta \quad\Longrightarrow\quad \|x - x^*\| < \epsilon,$$

where x^* is the solution of the nonlocal problem

$$\frac{\mathrm{d}x^*}{\mathrm{d}t} = f\left(t, x^*(t), \int_0^t g^*\big(s, x^*(s)\big)\,\mathrm{d}s\big(s, x^*(s)\big)\right), \quad \text{a.e. } t \in (0, 1), \tag{19}$$

with the nonlocal condition

$$\sum_{k=1}^{n} a_k x^*(\tau_k) = x_0, \quad a_k \geqslant 0,\ \tau_k \in (0, 1). \tag{20}$$

Theorem 6. *Let the assumptions of Theorem* 3 *be satisfied, then the solution of the nonlocal problem* (1)–(2) *depends continuously on the function g.*

Proof. Let x, x^* be two solutions of the nonlocal problem (1)–(2) and (19)–(20), respectively. Then

$$
\begin{aligned}
&\big|x(t) - x^*(t)\big| \\
&\quad = \Bigg| B^{-1}\Bigg[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg] \\
&\qquad + \int_0^t f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&\qquad - B^{-1}\Bigg[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\Bigg(s, x^*(s), \int_0^s g^*\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg] \\
&\qquad - \int_0^t f\Bigg(s, x^*(s), \int_0^s g^*\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s\Bigg| \\
&\quad \leqslant B^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \Bigg| f\Bigg(s, x^*(s), \int_0^s g^*\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg) - f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg)\Bigg|\,\mathrm{d}s \\
&\qquad + \int_0^t \Bigg| f\Bigg(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\Bigg) - f\Bigg(s, x^*(s), \int_0^s g^*\big(\theta, x^*(\theta)\big)\,\mathrm{d}\theta\Bigg)\Bigg|\,\mathrm{d}s, \\
&\quad \leqslant B^{-1}\sum_{k=1}^{m} a_k \int_0^{\tau_k} \Bigg(b_1\|x - x^*\| + b_1 \int_0^s \big|g^*\big(\theta, x^*(\theta)\big) - g\big(\theta, x(\theta)\big)\big|\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&\qquad + \int_0^t \Bigg(b_1\|x - x^*\| + b_1 \int_0^s \big|g\big(\theta, x(\theta)\big) - g^*\big(\theta, x^*(\theta)\big)\big|\,\mathrm{d}\theta\Bigg)\,\mathrm{d}s \\
&\quad \leqslant b_1\|x - x^*\| + \frac{1}{2}b_1\delta + \frac{1}{2}b_1 b_2\|x - x^*\| + \frac{1}{2}b_1\delta + b_1\|x - x^*\| \\
&\qquad + \frac{1}{2}b_1 b_2\|x - x^*\| \\
&\quad \leqslant b_1\delta + (2b_1 + b_1 b_2)\|x - x^*\|.
\end{aligned}
$$

Hence,

$$
\|x - x^*\| \leqslant \frac{b_1\delta}{[1 - (2b_1 + b_1 b_2)]\sum_{k=1}^{m} a_k} = \epsilon.
$$

This mean that the solution of the nonlocal problem (1)–(2) depends continuously on the function g. The proof is completed. □

2.5 Nonlocal Riemann–Stieltjes integral condition

Let $x \in C[0,1]$ be the solution of the nonlocal problem (1)–(2). Let $a_k = g(t_k) - g(t_{k-1})$, g is increasing function, $\tau_k \in (t_{k-1}, t_k)$, $0 = t_0 < t_1 < t_2 \cdots < t_m = 1$, then, as $m \to \infty$, the nonlocal condition (2) will be

$$\sum_{k=1}^{m} g(t_k) - g(t_{k-1}) x(\tau_k) = x_0$$

and

$$\lim_{m\to\infty} \sum_{k=1}^{m} g(t_k) - g(t_{k-1}) x(\tau_k) = \int_0^1 x(s)\,\mathrm{d}g(s) = x_0.$$

Theorem 7. *Let assumptions* (i)–(iv) *be satisfied, then the nonlocal problem of* (1)–(3) *has at least one solution.*

Proof. As $m \to \infty$, the solution of the nonlocal problem (1)–(2) will be

$$\begin{aligned}
x(t) &= \lim_{m\to\infty} \frac{1}{\sum_{k=1}^{m} a_k} \left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s \right] \\
&\quad + \int_0^t f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s \\
&= \frac{1}{g(1) - g(0)} \\
&\quad \times \left[x_0 - \lim_{m\to\infty} \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s \,\big(g(t_k) - g(t_{k-1})\big) \right] \\
&\quad + \int_0^t f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s \\
&= \frac{1}{g(1) - g(0)} \left[x_0 - \int_0^1 \int_0^t f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s\,\mathrm{d}g(t) \right] \\
&\quad + \int_0^t f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right) \mathrm{d}s.
\end{aligned}$$

□

2.6 Infinite-point boundary condition

Theorem 8. *Let assumptions* (i)–(iv) *be satisfied, then the nonlocal problem of* (1)–(4) *has at least one solution.*

Proof. Let the assumptions of Theorem 1 be satisfied, and let $\sum_{k=1}^{m} a_k$ be convergent, then

$$x_m(t) = \frac{1}{\sum_{k=1}^{m} a_k}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\right] + \int_0^t f\left(s, x_m(s), \int_0^s g\bigl(\theta, x_m(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s. \tag{21}$$

Taking the limit to (21) as $m \to \infty$, we have

$$\begin{aligned}\lim_{m\to\infty} x_m(t) &= \lim_{n\to\infty}\left[\frac{1}{\sum_{k=1}^{m} a_k}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\right]\right.\\ &\quad \left.+ \int_0^t f\left(s, x_m(s), \int_0^s g\bigl(\theta, x_m(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\right]\\ &= \lim_{n\to\infty}\frac{1}{\sum_{k=1}^{m} a_k}\left[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\right]\\ &\quad + \lim_{n\to\infty}\int_0^t f\left(s, x_m(s), \int_0^s g\bigl(\theta, x_m(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s.\end{aligned} \tag{22}$$

Now, $|a_k x(\tau_k)| \leqslant |a_k|\|x\|$, then by comparison test $\sum_{k=1}^{\infty} a_k x(\tau_k)$ is convergent.

Also

$$\begin{aligned}&\left|\int_0^{\tau_k} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\right|\\ &\quad \leqslant \int_0^{\tau_k}\left(c_1(s) + b_1\bigl|x(s)\bigr| + b_1\int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\\ &\quad \leqslant \int_0^{\tau_k}\left(c_1(s) + b_1\bigl|x(s)\bigr| + b_1\int_0^s \bigl(c_2(s) + b_2\bigl|x(s)\bigr|\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\\ &\quad \leqslant M_1 + b_1\|x\| + b_1 M_2 + \frac{1}{2}b_1 b_2\|x\| \leqslant M,\end{aligned}$$

then

$$\left|a_k\int_0^{\tau_k} f\left(s, x(s), \int_0^s g\bigl(\theta, x(\theta)\bigr)\,\mathrm{d}\theta\right)\mathrm{d}s\right| \leqslant |a_k|M,$$

and by the comparison test $\sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f(s, x(s), \int_0^s g(\theta, x(\theta))\,\mathrm{d}\theta)\,\mathrm{d}s$ is convergent.

Now, $|f| \leqslant |c_1(s)+b_1\|x\|+b_1M_2+b_1b_2\|x\|$, using assumptions (i)–(ii) and Lebesgue dominated convergence theorem [13], from (22) we obtain

$$x(t) = \frac{1}{\sum_{k=1}^{\infty} a_k}\left[x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\mathrm{d}s\right] + \int_0^t f\left(s, x(s), \int_0^s g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta\right)\mathrm{d}s.$$

The theorem proved. □

3 Examples

In this section, we offer some examples to illustrate our results.

Example 1. Consider the following nonlinear integro-differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^3\mathrm{e}^{-t} + \frac{\ln(1+|x(t)|)}{3+t^2} + \int_0^t \frac{1}{9}\big(\cos(3s+3) + s^5\cos x(s) + \mathrm{e}^{-s}x(s)\big)\mathrm{d}t, \quad \text{a.e. } t \in (0,1), \tag{23}$$

with infinite point boundary condition

$$\sum_{k=1}^{\infty} \frac{1}{k^5} x\left(\frac{k-1}{k}\right) = x_0. \tag{24}$$

Set

$$f\left(t, x(t), \int_0^t g\big(s, x(s)\big)\,\mathrm{d}s\right) = t^3\mathrm{e}^{-t} + \frac{\ln(1+|x(t)|)}{3+t^2} + \frac{1}{9}\int_0^t \big(\cos(3s+3) + s^5\cos x(s) + \mathrm{e}^{-s}x(s)\big)\mathrm{d}t.$$

Then

$$\left|f\left(t, x(t), \int_0^t g\big(s, x(s)\big)\,\mathrm{d}s\right)\right| \leqslant t^3\mathrm{e}^{-t} + \frac{1}{3}\left(|x| + \frac{1}{3}\int_0^t \frac{1}{3}\big|\big(\cos(3s+3) + s^5\cos x(s) + \mathrm{e}^{-s}x(s)\big)\mathrm{d}t\big|\right),$$

and also

$$\big|g\big(s, x(s)\big)\big| \leqslant \frac{1}{3}\big|\cos(3s+3)\big| + \frac{2}{3}\big|x(s)\big|.$$

It is clear that assumptions (i)–(iv) of Theorem 1 are satisfied with $c_1(t) = t^3 \mathrm{e}^{-t} \in L^1[0,1]$, $c_2(t) = |\cos(3t+3)|/2 \in L^1[0,1]$, $b_1 = 1/3$, $b_2 = 2/3, 2b_1 + b_1 b_2 = 2/3 + 2/9 = 8/9 < 1$, and the series $\sum_{k=1}^{\infty} 1/k^5$, is convergent. Therefore, by applying to Theorem 1 the given nonlocal problem (23)–(24) has a continuous solution.

Example 2. Consider the following nonlinear integro-differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = t^3 + t + 1 + \frac{x(t)}{\sqrt{t+3}} + \int_0^t \frac{1}{4}\left(\sin^2(3s+3) + \frac{sx(s)}{2^s(1+x(s))}\right)\mathrm{d}t, \quad \text{a.e. } t \in (0,1), \tag{25}$$

with infinite point boundary condition

$$\sum_{k=1}^{\infty} \frac{1}{k^3} x\left(\frac{k^2+k-1}{k^2+k}\right) = x_0. \tag{26}$$

Set

$$f\left(t, x(t), \int_0^t g\big(s, x(s)\big)\,\mathrm{d}s\right) = t^3 + t + 1 + \frac{x(t)}{\sqrt{2t+4}} + \frac{1}{4}\int_0^t \left(\sin^2(3s+3) + \frac{sx(s)}{2^s(1+x(s))}\right)\mathrm{d}t.$$

Then

$$\left|f\left(t, x(t), \int_0^t g\big(s, x(s)\big)\,\mathrm{d}s\right)\right| \leqslant t^3 + t + 1 + \frac{1}{3}|x| + \frac{1}{3}\int_0^t \frac{3}{4}\left|\sin^2(3s+3) + \frac{sx(s)}{2^s(1+x(s))}\right|\mathrm{d}t,$$

and also

$$\big|g\big(s, x(s)\big)\big| \leqslant \frac{3}{4}\big|\sin^2(3s+3)\big| + \frac{3}{8}\big|x(s)\big|.$$

It is clear that the assumptions (i)–(iv) of Theorem 1 are satisfied with $c_1(t) = t^3 + t + 1 \in L^1[0,1]$, $c_2(t) = (3/4)|(\sin^2(3s+3)| \in L^1[0,1]$, $b_1 = 1/3$, $b_2 = 3/8$, $2b_1 + b_1 b_2 = 2/3 + 1/8 = 19/24 < 1$, and the series $\sum_{k=1}^{\infty} 1/k^3$, is convergent. Therefore, by applying to Theorem 1 the given nonlocal problem (25)–(26) has a continuous solution.

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