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Unsteady Oscillatory Flow and Heat Transfer in a Horizontal Composite Porous Medium Channel

J. C. Umavathi[1], **A. J. Chamkha**[2], **A. Mateen**[1], **A. Al-Mudhaf**[2]

[1]Department of Mathematics Gulbarga University
Gulbarga, 585 106, India

[2]Manufacturing Engineering Department
The Public Authority for Applied Education and Training
Shuweikh, 70654, Kuwait
achamkha@yahoo.com

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Abstract. The problem of unsteady oscillatory flow and heat transfer in a horizontal composite porous medium is performed. The flow is modeled using the Darcy-Brinkman equation. The viscous and Darcian dissipation terms are also included in the energy equation. The partial differential equations governing the flow and heat transfer are solved analytically using two-term harmonic and non-harmonic functions in both regions of the channel. Effect of the physical parameters such as the porous medium parameter, ratio of viscosity, oscillation amplitude, conductivity ratio, Prandtl number and the Eckert number on the velocity and/or temperature fields are shown graphically. It is observed that both the velocity and temperature fields in the channel decrease as either of the porous medium parameter or the viscosity ratio increases while they increase with increases in the oscillation amplitude. Also, increasing the thermal conductivity ratio is found to suppress the temperature in both regions of the channel. The effects of the Prandtl and Eckert numbers are found to decrease the thermal state in the channel as well.

Keywords: unsteady, composite porous medium, horizontal channel.

Nomenclature

A	real positive constant
C_p	specific heat at constant pressure
Ec	Eckert number
h	channel half width
K	thermal conductivity
k	ratio of thermal conductivities
m	ratio of viscosities
P	non-dimensional pressure gradient
p	pressure
Pr	Prandtl number
s	permeability of porous matrix
T	temperature
T_w	wall temperature
t	time
u, v	velocity components of velocity along and perpendicular to the plates, resp.
$\bar{u}_l$	average velocity
v_0	scale of suction
x, y	Cartesian coordinates

Greek letters

ε	coefficient of periodic parameter	ωt	periodic frequency parameter
δ	Kronecker delta	ν	kinematic viscosity
ρ_0	fluid density	σ	porous medium parameter
μ	fluid dynamic viscosity	θ	non-dimensional temperature
ω	frequency parameter		

Subscripts

$1, 2$ quantities for region I and region II, respectively
eff effective value for porous matrix

Superscripts

$*$ dimensionless quantity

1 Introduction

In recent years considerable interest has been evidenced in the study of flow past a porous medium because of its natural occurrence and importance in both geophysical and engineering environments. Research on thermal interaction between heat generating porous bed and overlying fluid layer was largely motivated by the nuclear reactor severe accident problems. Another area of nuclear engineering applications is the design of pebble bed reactor which requires a proper understanding of forced convection through packed beds under normal operating conditions and of free convection either in the case of loss of coolant or during cold shut down. The applications also include problems involving porous bearings [1–6], porous rollers [7], porous layer insulation consisting of solid and porous media [8] and, in biomathematics. Composite fluid and porous layers also find its application in porous journal bearings. Under high pressure conditions, the lubricant is squeezed into the porous matrix which releases the fluid as soon as the pressure decreases thereby maintaining a liquid-film between the shaft and in human body hip and knee, etc., acts on a similar principle. The surfaces of the joint are articular cartilage, a smooth rubbery material which is attached to the solid bone. The surface of the articular cartilage is rough and porous, and hence, can trap the synovial fluid. It has been suggested that because of the porous nature of the articular cartilage, other lubricating material is squeezed into the joint when it is under stress. One theory is that pressure causes lubricating “threads” to squeeze out of the cartilage into the joint; one end of each lubricating threads remains in the cartilage and as the pressure reduces, the threads are pulled back into the holes. Zaturaska et al. [9] reported on the flow of viscous fluid driven along a channel by suction at porous walls. More recently, King and Cox [10] performed an asymptotic analysis of the steady-state and time-dependent laminar porous channel flows.

Problems involving multiphase flow and heat transfer and multi-component mass transfer arise in a number of scientific and engineering disciplines and are important in the petroleum extraction and transport. For example, the reservoir rock of an oil field always contains several immiscible fluids in its pores. Part of the pore volume is occupied either by oil or gas or both. Crude oils often contain dissolved gases which may be released into the reservoir rock when the pressure decreases. There has been some theoretical and experimental work on stratified laminar flow of two immiscible fluids in a horizontal pipe [11–14]. Chamkha [15] reported analytical solutions for flow of two-immiscible fluids in porous and non-porous parallel-plates. Later on, MHD two-fluid convective flow and heat transfer in composite porous medium was analyzed by Malashetty et al. [16–18].

All of the above studies pertain to steady flow. However, a significant number of problems of practical interest are unsteady. The flow unsteadiness may be caused by a change in the free stream velocity or in the surface temperature or in both. When there is an impulsive change in the velocity field, the inviscid flow is developed instantaneously, but the flow in the viscous layer near the wall is developed slowly which becomes fully developed steady flow after sometime. Raptis and Kafousias [19] studied the influence of a magnetic field on the steady free convection flow through a porous medium bounded by an infinite vertical isothermal plate with a constant suction velocity. Umavathi [20] studied oscillatory flow of unsteady convective fluid in a infinite vertical stratum.

The objective of this paper is to consider unsteady flow and heat transfer in a horizontal composite channel consisting of two parallel permeable plates with half of the distance between them filled by a fluid-saturated porous layer and the other half by a clear viscous fluid.

2 Mathematical formulation

Consider unsteady, fully developed, laminar flow of an incompressible viscous fluid through a an infinitely-long composite channel (see Fig. 1).

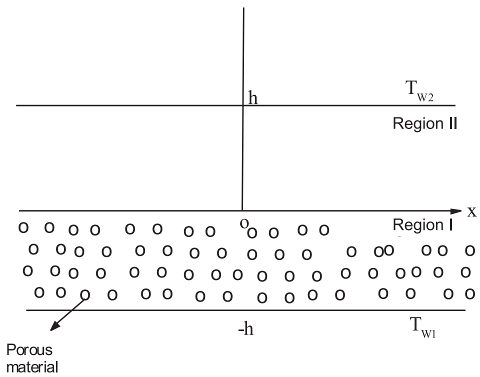

Fig. 1. Physical configuration.

The region $-h < y < 0$ (region I) is filled with a porous matrix and the region $0 < y < h$ (region II) is occupied by a clear viscous fluid. The two walls of the channel are held at constant different temperatures T_{w_1} and T_{w_2} with $T_{w_1} < T_{w_2}$ and the infinite plates are placed horizontally. It should be noted here that since the plates of the channel are assumed to be infinite, all of the physical dependent variables except pressure will only depend on y and t. The thermo-physical properties of the fluid and the effective properties of the porous medium are assumed constant. In region I, both the fluid and the porous matrix are assumed to be in local thermal equilibrium. Further, the flow in both regions of the channel is assumed to be driven by a common pressure gradient $(-\frac{\partial P}{\partial x})$ and temperature gradient $\Delta T = T_{w_1} - T_{w_2}$.

With the assumptions mentioned above, the governing equations of motion and energy are:

$$\frac{\partial v_i}{\partial y} = 0, \tag{1}$$

$$\rho_0\left(\frac{\partial u_i}{\partial t} + v_i\frac{\partial u_i}{\partial y}\right) = \chi_\mu\frac{\partial^2 u_i}{\partial y^2} - \frac{\partial p}{\partial x} - \chi\frac{\mu}{s}u_i, \tag{2}$$

$$\rho_0 C_p\left(\frac{\partial T_i}{\partial t} + v_i\frac{\partial Y_i}{\partial y}\right) = \chi_K\frac{\partial^2 T_i}{\partial y^2} - \chi_\mu\left(\frac{\partial u_i}{\partial y}\right)^2 + \chi\frac{\mu}{s}u_i^2, \tag{3}$$

where $i = 1, 2$ gives equations for regions I and II, respectively, u is the x-component of fluid velocity, v is the y-component of fluid velocity and T is temperature of the fluid. ρ_0, μ and C_p are the fluid density, dynamic viscosity and specific heat at constant pressure, respectively. The parameter s is the porous medium permeability. The other coefficients appearing in equations (2) and (3) are such that

$\chi_\mu = \mu_{eff}$ for porous matrix region,
$\chi_\mu = \mu$ for clear fluid region,

$\chi = 1$ for porous matrix region,
$\chi = 0$ for clear fluid region,

$\chi_K = K_{eff}$ for porous matrix region,
$\chi_K = K$ for clear fluid region.

The boundary conditions on velocity are the no-slip boundary conditions which require that the x-component of velocity must vanish at the wall. The boundary conditions on temperature are the isothermal conditions. It is also assumed that the velocity, shear stress, temperature and heat flux at the interfaces are continuous.

The boundary and interface conditions on velocity for the two fluids can then be written as

$$\begin{aligned} &u_1(-h) = 0, \quad u_2(h) = 0, \\ &u_1(0) = u_2(0), \\ &\mu_{eff}\frac{\partial u_1}{\partial y} = \mu\frac{\partial u_2}{\partial y} \quad \text{at } y = 0. \end{aligned} \tag{4}$$

The thermal boundary and interface conditions for both fluids are given by

$$
\begin{aligned}
&T_1(-h) = T_{w_1}, \quad T_2(h) = T_{w_2},\\
&T_1(0) = T_2(0),\\
&K_{eff}\frac{\partial T_1}{\partial y} = K\frac{\partial T_2}{\partial y} \quad \text{at } y = 0.
\end{aligned} \tag{5}
$$

The continuity equations of both fluids (equation (1)) imply that, v_1 and v_2 are independent of y, they can be utmost a function of time alone. Hence, we can write (assuming $v_1 = v_2 = v$)

$$
v = v_0\big(1 + \varepsilon A e^{i\omega t}\big), \tag{6}
$$

where A is real positive constant, ω is the frequency parameter and ε is small such that $\varepsilon A \leq 1$. Here, it is assumed that the transpiration velocity varies periodically with time about a non-zero constant mean v_0. When $\varepsilon A = 0$, the case of constant transpiration velocity is recovered. By use of the following non-dimensional quantities

$$
\begin{aligned}
&u_i = \bar{u}_1 u_i^*, \quad y = \frac{\nu}{v_0} y^*, \quad t = \frac{\nu}{v_0^2} t^*, \quad v = v_0 v^*, \quad \sigma^2 = \frac{\nu^2}{s v_0^2},\\
&P = \frac{\nu^2}{\mu v_0^2 U_0}\left(\frac{\partial P}{\partial x}\right), \quad \theta = \frac{T - T_{w_2}}{T_{w_1} - T_{w_2}}, \quad Ec = \frac{U_0^2}{C_p \Delta T}.
\end{aligned} \tag{7}
$$

Equations (2) and (3) are placed in dimensionless form as

$$
\frac{\partial u_i}{\partial t} + v\frac{\partial u_i}{\partial y} = A_i \frac{\partial^2 u_i}{\partial y^2} - \chi\sigma^2 u_i - P, \tag{8}
$$

$$
\frac{\partial \theta_i}{\partial t} + v\frac{\partial \theta_i}{\partial y} = B_i \frac{\partial^2 \theta_i}{\partial y^2} + A_i Ec\left(\frac{\partial u_i}{\partial y}\right)^2 + \chi\sigma^2 Ec\, u_i^2, \tag{9}
$$

where $i = 1, 2$ gives equations for regions I and II and

$$
A_1 = m, \quad A_2 = 1, \quad B_1 = \frac{k}{Pr}, \quad B_2 = \frac{1}{Pr}.
$$

$Pr = \frac{\rho_0 \nu C_p}{K}$, $m = \frac{\mu_{eff}}{\mu}$ is the ratio of viscosities and $k = \frac{K_{eff}}{K}$ is the ratio of thermal conductivities.

The non-dimensional form of the hydrodynamic and thermal boundary and interface conditions reduce to

$$
\begin{aligned}
&u_1(-1) = 0, \quad u_2(1) = 0,\\
&u_1(0) = u_2(0),\\
&m\frac{\partial u_1}{\partial y} = \frac{\partial u_2}{\partial y} \quad \text{at } y = 0;
\end{aligned} \tag{10}
$$

$$\begin{aligned}
&\theta_1(-1) = 1, \quad \theta_2(1) = 0,\\
&\theta_1(0) = \theta_2(0),\\
&k\frac{\partial \theta_1}{\partial y} = \frac{\partial \theta_2}{\partial y} \quad \text{at } y = 0.
\end{aligned} \tag{11}$$

(The asterisks have been dropped for simplicity.)

3 Closed-form solutions

The governing momentum and energy equations (8) and (9) are solved subject to the boundary and interface conditions (10) and (11) for the velocity and temperature distributions in both regions. These equations are coupled partial differential equations that can not be solved in closed form. However, they can be reduced to set of ordinary differential equations that can be solved analytically. This can be done by representing the velocity and temperature as

$$u_i(y,t) = u_{i0}(y) + \varepsilon e^{i\omega t} u_{i1}(y) + O\big(\varepsilon^2\big) + \ldots, \quad i = 1, 2, \tag{12}$$

$$\theta_i(y,t) = \theta_{i0}(y) + \varepsilon e^{i\omega t} \theta_{i1}(y) + O\big(\varepsilon^2\big) + \ldots, \quad i = 1, 2. \tag{13}$$

This is a valid assumption because of the choice of v as defined in equation (6) that the amplitude $\varepsilon A \ll 1$.

By substituting equations (12) and (13) into equations (8) to (9), equating the harmonic and non-harmonic terms and neglecting the higher order terms of $O(\varepsilon^2)$, one obtains the following pairs of equations for (u_{i0}, θ_{i0}) and (u_{i1}, θ_{i1}).

Non-periodic coefficients $O(\varepsilon^0)$

$$A_i \frac{\mathrm{d}^2 u_{i0}}{\mathrm{d}y^2} + \frac{\mathrm{d}u_{i0}}{\mathrm{d}y} - \chi\sigma^2 u_{i0} = P, \tag{14}$$

$$B_i \frac{\mathrm{d}^2 \theta_{i0}}{\mathrm{d}y^2} + \frac{\mathrm{d}\theta_{i0}}{\mathrm{d}y} = -A_i Ec\left(\frac{\mathrm{d}u_{i0}}{\mathrm{d}y}\right)^2 - \chi\sigma^2 Ec\, u_{i0}^2. \tag{15}$$

Periodic coefficients $O(\varepsilon^1)$

$$A_i \frac{\mathrm{d}^2 u_{i1}}{\mathrm{d}y^2} + \frac{\mathrm{d}u_{i1}}{\mathrm{d}y} - \chi\big(\sigma^2 + i\omega\big) u_{i1} = -A\frac{\mathrm{d}u_{i0}}{\mathrm{d}y}, \tag{16}$$

$$B_i \frac{\mathrm{d}^2 \theta_{i1}}{\mathrm{d}y^2} + \frac{\mathrm{d}\theta_{i1}}{\mathrm{d}y} - i\omega\theta_{i1} = -A\frac{\mathrm{d}u_{i0}}{\mathrm{d}y} - 2Ec\, A_i \frac{\mathrm{d}u_{i0}}{\mathrm{d}y}\frac{\mathrm{d}u_{i1}}{\mathrm{d}y} - 2\chi\sigma^2 Ec\, u_{i0} u_{i1}. \tag{17}$$

Using equations (12) and (13), the boundary and interface conditions may be written as:

$$\begin{aligned}
&u_{1i}(-1) = 0, \quad u_{2i}(1) = 0,\\
&u_{1i}(0) = u_{2i}(0),\\
&m\frac{\partial u_{1i}}{\partial y} = \frac{\partial u_{2i}}{\partial y} \quad \text{at } y = 0;
\end{aligned} \tag{18}$$

$$
\begin{aligned}
&\theta_{1i}(-1)=1-\delta_{1i}, \quad \theta_{2i}(1)=0,\\
&\theta_{1i}(0)=\theta_{2i}(0),\\
&k\frac{\partial\theta_{1i}}{\partial y}=\frac{\partial\theta_{2i}}{\partial y} \quad \text{at } \ y=0,
\end{aligned} \tag{19}
$$

where $i=0,1$ gives the boundary and interface conditions for non-periodic $(O(\varepsilon^0))$ and periodic $(O(\varepsilon^1))$ coefficients, respectively and δ_{1i} is the Kronecker delta.

Without going into detail, solution of the equations (14) to (17) using the boundary and interface conditions (18) and (19) can be written as

$$u_{10}=C_1e^{m_1y}+C_2e^{m_2y}-\frac{P}{\sigma^2}, \tag{20}$$

$$u_{20}=C_3+C_4e^{-y}+Py, \tag{21}$$

$$
\begin{aligned}
\theta_{10}=&C_5+C_6e^{m_4y}+k_8y+k_{10}e^{m_1y}+k_{11}e^{m_2y}\\
&+k_{12}e^{2m_1y}+k_{13}e^{2m_2y}+k_{14}e^{m_5y},
\end{aligned} \tag{22}
$$

$$\theta_{20}=C_7+C_8e^{-Pr\,y}+k_{15}e^{-2y}+k_{16}e^{-y}+k_{17}y, \tag{23}$$

$$
\begin{aligned}
u_{11}=&e^{e_1y}(XC_9\cos F_1y+XC_{10}\sin F_1y)+E_2e^{m_1y}+E_3e^{m_2y}\\
&+i\big[e^{e_1y}(YC_9\cos F_1y+YC_{10}\sin F_1y)+F_2e^{m_1y}+F_3e^{m_2y}\big],
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
u_{21}=&e^{e_4y}(XC_{11}\cos F_4y+XC_{12}\sin F_4y)\\
&+i\left[e^{e_4y}(YC_{11}\cos F_4y+YC_{12}\sin F_4y)+\frac{A}{\omega}(C_4e^{-y}+P)\right],
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\theta_{11}=&e^{e_5y}(XC_{13}\cos F_5y+XC_{14}\sin F_5y)+P_{23}e^{2m_1y}+P_{24}e^{2m_2y}+E_7e^{m_4y}\\
&+P_{25}e^{m_5y}+P_{26}e^{m_1y}+P_{27}e^{m_2y}+e^{m_6y}(P_{28}\cos F_1y+P_{29}\sin F_1y)\\
&+e^{m_7y}(P_{30}\cos F_1y+P_{31}\sin F_1y)+e^{e_1y}(P_{32}\cos F_1y+P_{33}\sin F_1y)\\
&+i\big[e^{e_5y}(YC_{13}\cos F_5y+YC_{14}\sin F_5y)+Q_{23}e^{2m_1y}+Q_{24}e^{2m_2y}\\
&\quad+F_7e^{m_4y}+Q_{25}e^{m_5y}+Q_{26}e^{m_1y}+Q_{27}e^{m_2y}\\
&\quad+e^{m_6y}(Q_{28}\cos F_1y+Q_{29}\sin F_1y)\\
&\quad+e^{m_7y}(Q_{30}\cos F_1y+Q_{31}\sin F_1y)\\
&\quad+e^{e_1y}(Q_{32}\cos F_1y+Q_{33}\sin F_1y)+K_{18}\big],
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\theta_{21}=&e^{e_{27}y}(XC_{15}\cos F_{27}y+XC_{16}\sin F_{27}y)+P_{44}e^{-2y}+P_{45}e^{-y}\\
&+e^{m_8y}(P_{46}\cos F_4y+P_{47}\sin F_4y)+e^{E_4y}(P_{48}\cos F_4y+P_{49}\sin F_4y)\\
&+i\big[e^{e_{27}y}(XC_{15}\cos F_{27}y+XC_{16}\sin F_{27}y)+P_{44}e^{-2y}+P_{45}e^{-y}\\
&\quad+k_{28}e^{-Pr\,y}+e^{m_8y}(P_{46}\cos F_4y+P_{47}\sin F_4y)\\
&\quad+e^{e_4y}(P_{48}\cos F_4y+P_{49}\sin F_4y)+k_{29}\big].
\end{aligned} \tag{27}
$$

It should be noted that all of the constants appearing in the above solutions are defined in Appendix section.

4 Results and discussion

The problem of unsteady flow and heat transfer in a horizontal composite porous medium channel is investigated analytically. The closed-form solutions are reported for small such that oscillation amplitude $\varepsilon A \leq 1$. The solution of non-periodic and periodic coefficients of $e^{i\omega t}$ is evaluated for various parametric conditions. The results are depicted graphically in Figs. 2 to 10.

Figs. 2 and 3 display the effect of the porous medium parameter σ on the velocity and temperature profiles, respectively. As the porous medium parameter σ increases, the velocity and temperature decrease in both regions of the channel. This is expected since the porous matrix represents an obstacle to flow and therefore, reduces its velocity and temperature. This result is also similar to the case of fully developed flow through a porous medium as predicted by Rudraiah and Nagraj [21].

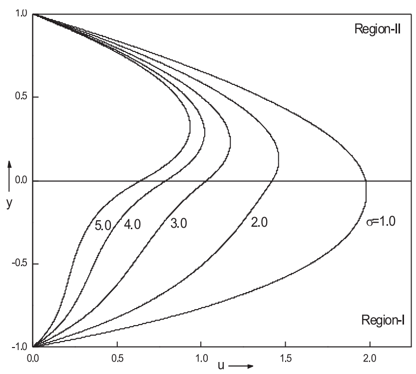

Fig. 2. Velocity profiles for different values of the porous medium parameter σ.

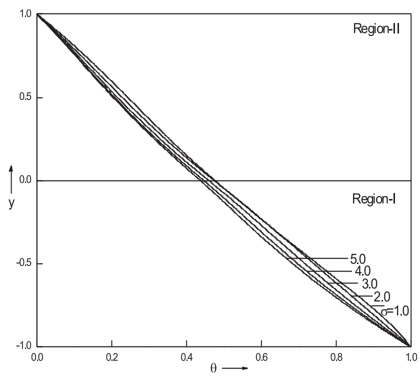

Fig. 3. Temperature profiles for different values of the porous medium parameter σ.

Fig. 4 depicts the effect of Prandtl number on the temperature profiles. The Prandtl number is the ratio of momentum diffusion to heat diffusion. It is a measure of the relative importance of viscosity and heat conduction in a flow field. Thus, as the Prandtl number increases, the viscous forces dominate over heat conduction and hence, the temperature decreases. This is obvious from Fig. 4.

Fig. 5 presents the effect of the Eckert number on the temperature profiles. Physically speaking, the Eckert number represents the effects of the viscous and porous medium dissipations. As the Eckert number increases, the temperature field in the channel decreases. The magnitude of reduction in the temperature field in region II is larger compared to that in region I.

The effect of the viscosity ratio m on the velocity and temperature profiles is shown in Figs. 6 and 7, respectively. As the viscosity ratio increases, both the velocity and temperature profiles are decreased. This is due to the fact that as the fluid viscosity increases, the fluid in both regions of the channel becomes thicker and hence the flow velocity is reduced causing the temperature distribution to reduce as well.

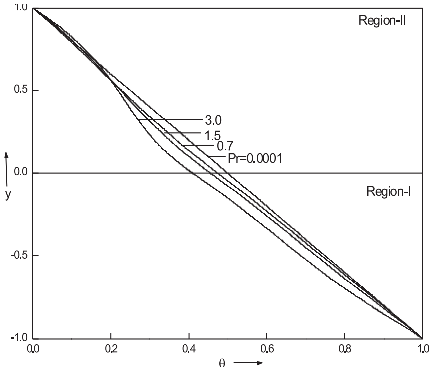

Fig. 4. Temperature profiles for different values of the Prandtl number Pr.

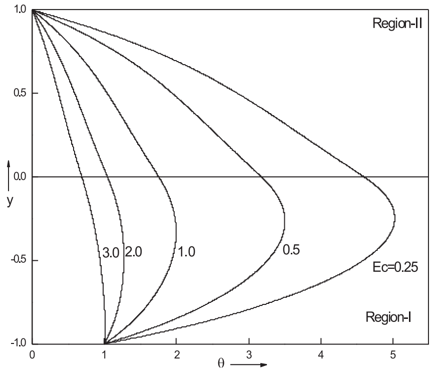

Fig. 5. Temperature profiles for different values of the Eckert number Ec.

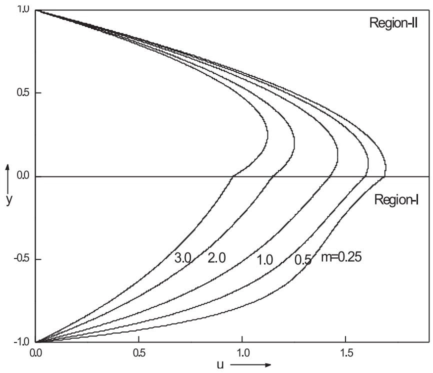

Fig. 6. Velocity profiles for different values of the ratio of viscosities m.

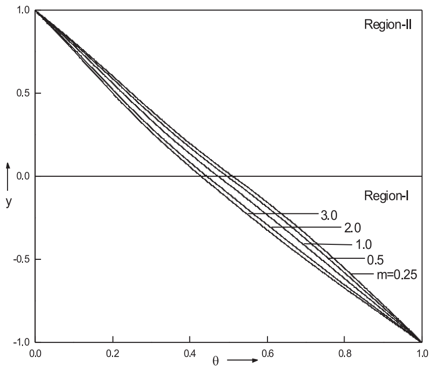

Fig. 7. Temperature profiles for different values of the ratio of viscosities m.

Fig. 8 displays the influence of the thermal conductivity ratio k on the temperature profiles. Increases in the thermal conductivity ratio have the tendency to cool down the thermal state in the channel. This is depicted in the reduction in the fluid temperatures as k increases as shown in Fig. 8.

Figs. 9 and 10 illustrate the effect of the oscillation amplitude εA on the velocity and temperature fields, respectively. It should be reminded that the oscillation amplitude was assumed to be small in evaluating the analytical solutions i.e. $\varepsilon A \leq 1$. The condition of $\varepsilon A = 0$ is for steady state solutions. It is clear from Figs. 9 and 10 that as the oscillation amplitude increases, both the velocity and temperature profiles increase in both regions of the channel.

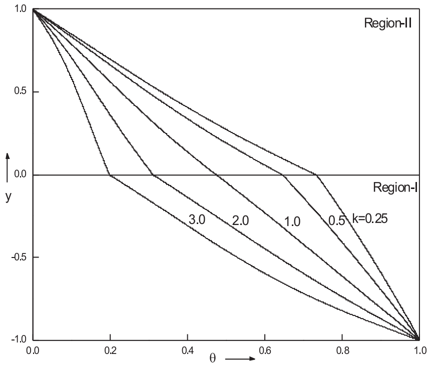

Fig. 8. Temperature profiles for different values of the ratio of conductivities k.

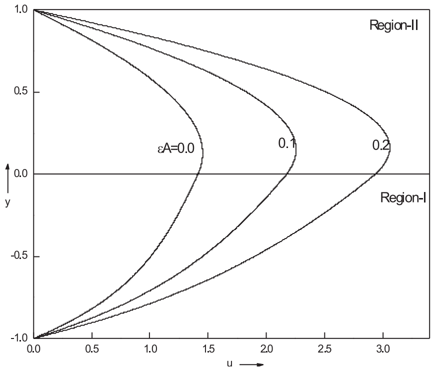

Fig. 9. Velocity profiles for different values of the oscillation amplitude εA.

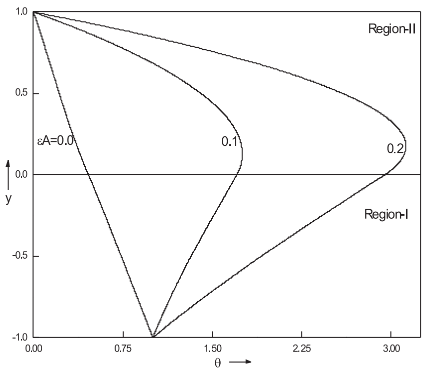

Fig. 10. Temperature profiles for different values of the oscillation amplitude εA.

5 Conclusions

The problem of unsteady flow of a viscous fluid through a horizontal composite channel whose half width is filled with a uniform layer of porous media in the presence of time-dependent oscillatory wall transpiration velocity was investigated analytically. Both the fluid and the porous matrix were assumed to have constant physical properties. Separate closed-form solutions for each region of the channel were obtained taking into consideration suitable interface matching conditions. The closed-form results were numerically evaluated and presented graphically for various values of the porous medium parameter, viscosity and thermal conductivity ratios, oscillation amplitude and the Prandtl and Eckert numbers.

It was predicted that both the velocity and temperature profiles decreased as either of the porous medium parameter or the viscosity ratio was increased. Furthermore, it was concluded that the temperature field decreased as either of the Prandtl number, the Eckert number or the thermal conductivity ratio increased. However, both the velocity and temperature fields in the channel increased as the oscillation amplitude was increased. It can be concluded that the flow and heat transfer aspects in a horizontal composite channel with permeable walls can be controlled by considering different combinations of fluids and porous media having different viscosities, conductivities and also by varying the amplitude of the transpiration velocity at the boundary.

Appendix

$$\begin{aligned}
&B_1 = 2k_2m_6+1, && B_2 = k_2m_6^2+m_6-k_2F_1^2, && B_3 = B_2^2+\omega^2-B_1^2F_1^2,\\
&B_4 = 2k_2m_7+1, && B_5 = k_2m_7^2+m_7-k_2F_1^2, && B_6 = B_5^2+\omega^2-B_4^2F_1^2,\\
&B_7 = 2k_2e_1+1, && B_8 = k_2e_1^2+e_1-k_2F_1^2, && B_9 = B_8^2+\omega^2-B_7^2F_1^2,\\
&B_{10} = 2m_8+Pr, && B_{11} = m_8^2+Pr\,m_8-F_4^2, && B_{12} = B_{11}^2+\omega^2Pr^2-B_{10}^2F_4^2,\\
&B_{13} = 2e_4+Pr, && B_{14} = e_4^2+Pr\,e_4-F_4^2, && B_{15} = B_{14}^2+\omega^2Pr^2-B_{13}^2F_4^2;
\end{aligned}$$

$$\begin{aligned}
&C_1 = \frac{-(l_3e^{-m_2}\sigma^2+l_2P)}{\sigma^2(l_1e^{-m_2}-l_2e^{-m_1})}, && C_2 = \frac{-(P-C_1e^{-m_1}\sigma^2)}{\sigma^2e^{-m_2}},\\
&C_3 = C_1+C_2-C_4-\frac{P}{\sigma^2}, && C_4 = P-m(m_1C_1-m_2C_2),\\
&C_5 = -C_6e^{-m_4}-l_4, && C_6 = \frac{l_9-l_4}{e^{-m_4}-l_8},\\
&C_7 = C_5+C_6-C_8-l_6, && C_8 = \frac{-(km_4C_6+l_7)}{Pr};
\end{aligned}$$

$$\begin{aligned}
&D_1 = 2B_1^2B_2, && D_2 = 2B_1B_1^2-B_1B_3, && D_3 = -B_3B_2,\\
&D_4 = 2\omega B_1B_2, && D_5 = 2B_4^2B_5, && D_6 = 2B_4B_5^2-B_4B_6,\\
&D_7 = -B_6B_5, && D_8 = 2\omega B_4B_5, && D_9 = 2B_7^2B_8,\\
&D_{10} = 2B_7B_8^2-B_7B_9, && D_{11} = -B_9B_8, && D_{12} = 2\omega B_7B_8,\\
&D_{13} = 2B_{10}^2B_{11}, && D_{14} = 2B_{10}B_{11}^2-B_{10}B_{12}, && D_{15} = -B_{11}B_{12},\\
&D_{16} = 2\omega PrB_{10}B_{11}, && D_{17} = 2B_{13}^2B_{14}, && D_{18} = 2B_{13}B_{14}^2-B_{13}B_{15},\\
&D_{19} = -B_{14}B_{15}, && D_{20} = 2\omega PrB_{13}B_{14};
\end{aligned}$$

$$\begin{aligned}
&e_1 = \frac{-1+\sqrt{r_1}\cos(\theta_1/2)}{2m}, && e_2 = \frac{-2Am_1C_1(mm_1^2+m_1-\sigma^2)}{(mm_1^2+m_1-\sigma^2)^2+\omega^2},\\
&e_3 = \frac{-Am_2C_2(mm_2^2+m_2-\sigma^2)}{(mm_2^2+m_2-\sigma^2)^2+\omega^2}, && e_4 = \frac{-1+\sqrt{r_2}\cos(\theta_2/2)}{2},
\end{aligned}$$

$$e_5 = \frac{-1+\sqrt{r_3}\cos(\theta_3/2)}{2k_2}, \qquad e_{5a} = \frac{-2Am_1k_{12}(4k_2m_1^2+2m_1)}{(4km_1^2+2m_1)^2+\omega^2},$$
$$e_6 = \frac{-2Am_2k_{13}(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2}, \qquad e_7 = \frac{-Am_4C_6(k_2m_4^2+m_4)}{(k_2m_4^2+m_4)^2+\omega^2},$$
$$e_8 = \frac{-Am_5k_{14}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2}, \qquad e_9 = \frac{-Am_1k_{10}(k_2m_1^2+m_1)}{(k_2m_1^2+m_1)^2+\omega^2},$$
$$e_{10} = \frac{-2Am_2k_{11}(k_2m_2^2+m_2)}{(k_2m_2^2+m_2)^2+\omega^2}, \qquad e_{11} = \frac{-2Ec\,mm_1^2C_1e_2(4k_2m_1^2+2m_1)}{(4k_2m_1^2+2m_1)^2+\omega^2},$$
$$e_{12} = \frac{-2Ec\,mm_2^2C_2e_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2}, \qquad e_{13} = \frac{-2Ec\,mk_{21}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2},$$
$$e_{14} = \frac{-2Ec\,mm_1^2C_1F_2(4k_2m_1^2+2m_1)}{(4k_2m_1^2+2m_1)^2+\omega^2}, \qquad e_{15} = \frac{-2Ec\,mm_2^2C_2F_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2},$$
$$e_{16} = \frac{-2Ec\,mk_{22}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2}, \qquad e_{17} = \frac{-2Ec\,\sigma^2C_1e_2(4k_2m_1^2+2m_1)}{(4k_2m_1^2+2m_1)^2+\omega^2},$$
$$e_{18} = \frac{-2Ec\,\sigma^2C_2e_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2}, \qquad e_{19} = \frac{-2Ec\,\sigma^2k_{26}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2},$$
$$e_{20} = \frac{-2Ec\,Pe_2(k_2m_1^2+m_1)}{(k_2m_1^2+m_1)^2+\omega^2}, \qquad e_{21} = \frac{-2Ec\,Pe_3(k_2m_2^2+m_2)}{(k_2m_2^2+m_2)^2+\omega^2},$$
$$e_{22} = \frac{-2Ec\,\sigma^2C_1F_2(4k_2m_1^2+2m_1)}{(4k_2m_1^2+2m_1)^2+\omega^2}, \qquad e_{23} = \frac{-2Ec\,\sigma^2C_2F_3(4k_2m_2^2+2m_2)}{(4k_2m_2^2+2m_2)^2+\omega^2},$$
$$e_{24} = \frac{-2Ec\,\sigma^2k_{27}(k_2m_5^2+m_5)}{(k_2m_5^2+m_5)^2+\omega^2}, \qquad e_{25} = \frac{-2Ec\,PF_2(k_2m_1^2+m_1)}{(k_2m_1^2+m_1)^2+\omega^2},$$
$$e_{26} = \frac{-2Ec\,PF_3(k_2m_2^2+m_2)}{(k_2m_2^2+m_2)^2+\omega^2}, \qquad e_{27} = \frac{-Pr+\sqrt{r_4}\cos(\theta_4/2)}{2},$$
$$e_{28} = \frac{2PrAk_{15}(4-2Pr)}{(4-2Pr)^2+\omega^2Pr^2}, \qquad e_{29} = \frac{PrAPk_{16}(1-Pr)}{(1-Pr)^2+\omega^2Pr^2},$$
$$e_{30} = \frac{-2Ec\,PrAC_4^2(4-2Pr)}{\omega((4-2Pr)^2+\omega^2Pr^2)}, \qquad e_{31} = \frac{-2Ec\,PrAC_4(1-Pr)}{\omega((1-Pr)^2+\omega^2Pr^2)};$$

$$F_1 = \frac{\sqrt{r_1}\sin(\theta_1/2)}{2m}, \qquad F_2 = \frac{-Am_1C_1\omega}{(mm_1^2+m_1-\sigma^2)^2+\omega^2},$$
$$F_3 = \frac{-Am_2C_2\omega}{(mm_2^2+m_2-\sigma^2)^2+\omega^2}, \qquad F_4 = \frac{\sqrt{r_2}\sin(\theta_2/2)}{2},$$
$$F_5 = \frac{\sqrt{r_3}\sin(\theta_3/2)}{2k_2}, \qquad F_{5a} = \frac{-2Am_1k_{12}\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},$$

$$F_6 = \frac{-2Am_2k_{13}\omega}{(4k_2m_2^2+2m_2)^2+\omega^2}, \quad F_7 = \frac{-Am_4C_6\omega}{(k_2m_4^2+m_4)^2+\omega^2},$$
$$F_8 = \frac{-Am_5k_{14}\omega}{(k_2m_5^2+m_2)^2+\omega^2}, \quad F_9 = \frac{-Am_1k_{10}\omega}{(k_2m_1^2+m_1)^2+\omega^2},$$
$$F_{10} = \frac{-Am_2k_{11}\omega}{(k_2m_2^2+m_2)^2+\omega^2}, \quad F_{11} = \frac{-2Ec\,mm_1^2C_1e_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},$$
$$F_{12} = \frac{-2Ec\,mm_2^2C_2e_3\omega}{(4k_2m_2^2+2m_2)^2+\omega^2}, \quad F_{13} = \frac{-2Ec\,mk_{21}\omega}{(k_2m_5^2+m_5)^2+\omega^2},$$
$$F_{14} = \frac{-2Ec\,mm_1^2C_1F_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2}, \quad F_{15} = \frac{-2Ec\,mm_2^2C_2F_3\omega}{(4k_2m_2^2+2m_2)^2+\omega^2},$$
$$F_{16} = \frac{-2Ec\,mk_{22}\omega}{(k_2m_5^2+m_5)^2+\omega^2}, \quad F_{17} = \frac{-2Ec\,\sigma^2C_1e_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2},$$
$$F_{18} = \frac{-2Ec\,\sigma^2C_2e_3\omega}{(4k_2m_2^2+2m_2)^2+\omega^2}, \quad F_{19} = \frac{-2Ec\,\sigma^2k_{26}\omega}{(k_2m_5^2+m_5)^2+\omega^2},$$
$$F_{20} = \frac{-2Ec\,Pe_2\omega}{(k_2m_1^2+m_1)^2+\omega^2}, \quad F_{21} = \frac{-2Ec\,Pe_2\omega}{(k_2m_2^2+m_2)^2+\omega^2},$$
$$F_{22} = \frac{-2Ec\,\sigma^2C_1F_2\omega}{(4k_2m_1^2+2m_1)^2+\omega^2}, \quad F_{23} = \frac{-2Ec\,\sigma^2C_2F_3\omega}{(4k_2m_2^2+2m_2)^2+\omega^2},$$
$$F_{24} = \frac{-2Ec\,\sigma^2k_{27}\omega}{(k_2m_5^2+m_5)^2+\omega^2}, \quad F_{25} = \frac{-2Ec\,PF_2\omega}{(k_2m_1^2+m_1)^2+\omega^2},$$
$$F_{26} = \frac{-2Ec\,PF_3\omega}{(k_2m_2^2+m_2)^2+\omega^2}, \quad F_{27} = \frac{\sqrt{r_4}\sin(\theta_4/2)}{2},$$
$$F_{28} = \frac{-2Pr^2Ak_{15}\omega}{(4-2Pr)^2+\omega^2Pr^2}, \quad F_{29} = \frac{Pr^2Ak_{16}\omega}{(1-Pr)^2+\omega^2Pr^2},$$
$$F_{30} = \frac{-2Ec\,Pr^2AC_4^2}{(4-2Pr)^2+\omega^2Pr^2}, \quad F_{29} = \frac{-2Ec\,Pr^2AC_4}{(1-Pr)^2+\omega^2Pr^2};$$

$$k_2 = \frac{k}{Pr}, \quad k_3 = \frac{-mEc\,m_1^2C_1^2}{4m_1^2k_2+2m_1}, \quad k_4 = \frac{-mEc\,m_2^2C_2^2}{4m_2^2k_2+2m_2},$$
$$k_5 = \frac{-2mEc\,m_1m_2C_1C_2}{m_5^2k_2+m_5}, \quad k_6 = \frac{-\sigma^2Ec\,C_1^2}{4m_1^2k_2+2m_1}, \quad k_7 = \frac{-\sigma^2Ec\,C_2^2}{4m_2^2k_2+2m_2},$$
$$k_8 = \frac{-Ec\,P^2}{\sigma^2}, \quad k_9 = \frac{-2\sigma^2Ec\,C_1C_2}{m_5^2k_2+m_5}, \quad k_{10} = \frac{2Ec\,C_1P}{m_1^2k_2+m_1},$$
$$k_{11} = \frac{2Ec\,C_2P}{m_2^2k_2+m_2}, \quad k_{12} = k_3+k_6, \quad k_{13} = k_4+k_7,$$
$$k_{14} = k_5+k_9, \quad k_{15} = \frac{-Ec\,PrC_4^2}{4-2Pr}, \quad k_{16} = \frac{2Ec\,Pr\,PC_4}{1-Pr},$$

$$k_{17} = -Ec\,P^2, \qquad k_{18} = \frac{-Ak_8}{\omega}, \qquad k_{19} = \frac{2Ec\,mm_1C_1}{4B_1^2B_2^2F_1^2 + B_3^2},$$
$$k_{20} = \frac{2Ec\,mm_2C_2}{4B_4^2B_5^2F_1^2 + B_6^2}, \qquad k_{21} = m_1m_2(C_1e_3 + C_2e_2), \qquad k_{22} = m_1m_2(C_1F_3 + C_2F_2),$$
$$k_{23} = \frac{2Ec\,\sigma^2C_1}{4B_1^2B_2^2F_1^2 + B_3^2}, \qquad k_{24} = \frac{2Ec\,\sigma^2C_2}{4B_4^2B_5^2F_1^2 + B_6^2}, \qquad k_{25} = \frac{2Ec\,P}{\sigma^2(4B_7^2B_8^2F_1^2 + B_9^2)},$$
$$k_{26} = C_1e_3 + C_2e_2, \qquad k_{27} = C_1F_3 + C_2F_2, \qquad k_{28} = \frac{PrAC_8}{\omega},$$
$$k_{29} = \frac{-Ak_{17}}{\omega}, \qquad k_{30} = \frac{-2Ec\,PrC_4}{4B_{10}^2B_{11}^2F_4^2 + B_{12}^2}, \qquad k_{31} = \frac{-2Ec\,Pr\,P}{4B_{13}^2B_{14}^2F_4^2 + B_{15}^2)};$$

$$l_1 = 1 - (e^{-1} - 1)mm_1, \qquad l_2 = 1 - (e^{-1} - 1)mm_2, \qquad l_3 = Pe^{-1} - \frac{P}{\sigma^2},$$
$$l_4 = -k_8 + k_{10}e^{-m_1} + k_{11}e^{-m_2} + k_{12}e^{-2m_1} + k_{13}e^{-2m_2} + k_{14}e^{-m_5} - 1,$$
$$l_5 = k_{15}e^{-2} + k_{16}e^{-1} + k_{17}, \qquad l_6 = k_{10} + k_{11} + k_{12} + k_{13} + k_{14} - k_{15} - k_{16},$$
$$l_7 = k(k_8 + m_1k_{10} + m_2k_{11} + 2m_1k_{12} + 2m_2k_{13} + m_5k_{14}) + 2k_{15} + k_{16} - k_{17},$$

$$l_8 = 1 - \frac{km_4(e^{-Pr} - 1)}{Pr}, \qquad l_9 = l_5 + l_6 - \frac{l_7(e^{-Pr} - 1)}{Pr},$$
$$l_{10} = e^{-e_1}\cos F_1, \qquad l_{11} = -e^{-e_1}\sin F_1,$$
$$l_{12} = e_2e^{-m_1} + e_3e^{-m_2}, \qquad l_{13} = F_2e^{-m_1} + F_3e^{-m_2},$$
$$l_{14} = e^{e_4}\cos F_4, \qquad l_{15} = e^{e_4}\sin F_4,$$
$$l_{16} = \frac{A}{\omega}\big(C_4e^{-1} + P\big), \qquad l_{17} = e_2 + e_3,$$
$$l_{18} = F_2 + F_3 - \frac{A}{\omega}(C_4 + P), \qquad l_{19} = m(m_1e_2 + m_2e_3),$$
$$l_{20} = m(m_1F_2 + m_2F_3) + \frac{AC_4}{\omega}, \qquad l_{21} = \frac{F_4l_{14}}{l_{15}}e_4;$$
$$l_{22} = \frac{F_4l_{16}}{l_{15}} + L_{20}, \qquad l_{23} = me_1 + l_{21}, \qquad l_{24} = l_{21}l_{17} + l_{19},$$
$$l_{25} = l_{21}l_{18} + l_{22}, \qquad l_{26} = e^{-e_5}\cos F_5, \qquad l_{27} = -e^{-e_5}\sin F_5,$$

$$\begin{aligned} l_{28} = {} & P_{23}e^{-2m_1} + P_{24}e^{-2m_2} + e_7e^{-m_4} + P_{25}e^{-m_5} + P_{26}e^{-m_1} + P_{27}e^{-m_2} \\ & + e^{-m_6}(P_{28}\cos F_1 - P_{29}\sin F_1) + e^{-m_7}(P_{30}\cos F_1 - P_{31}\sin F_1) \\ & + e^{-e_1}(P_{32}\cos F_1 - P_{33}\sin F_1), \end{aligned}$$
$$\begin{aligned} l_{29} = {} & Q_{23}e^{-2m_1} + Q_{24}e^{-2m_2} + F_7e^{-m_4} + Q_{25}e^{-m_5} + Q_{26}e^{-m_1} + Q_{27}e^{-m_2} \\ & + e^{-m_6}(Q_{28}\cos F_1 - Q_{29}\sin F_1) + e^{-m_7}(Q_{30}\cos F_1 - Q_{31}\sin F_1) \\ & + e^{-e_1}(Q_{32}\cos F_1 - Q_{33}\sin F_1) + k_{18}, \end{aligned}$$
$$l_{30} = e^{e_{27}}\cos F_{27}, \qquad l_{31} = e^{e_{27}}\sin F_{27},$$

$$
\begin{aligned}
l_{32} &= P_{44}e^{-2} + P_{45}e^{-1} + e^{m_8}(P_{46}\cos F_4 - P_{47}\sin F_4)\\
&\quad + e^{e_4}(P_{48}\cos F_4 + P_{49}\sin F_4),\\
l_{33} &= Q_{44}e^{-2} + Q_{45}e^{-1} + e^{m_8}(Q_{46}\cos F_4 + Q_{47}\sin F_4)\\
&\quad + e^{e_4}(Q_{48}\cos F_4 + Q_{49}\sin F_4) + k_{28}e^{-Pr} + k_{29},\\
l_{34} &= P_{23} + P_{24} + e_7 + P_{25} + P_{26} + P_{27} + P_{28} + P_{30} + P_{32}\\
&\quad - P_{44} - P_{45} - P_{46} - P_{48},\\
l_{35} &= Q_{23} + Q_{24} + F_7 + Q_{25} + Q_{26} + Q_{27} + Q_{28} + Q_{30} + Q_{32}\\
&\quad - Q_{44} - Q_{45} - Q_{46} - Q_{48} - k_{28} - k_{29},\\
l_{36} &= k(2m_1P_{23} + 2m_2P_{24} + m_4e_7 + m_5P_{25} + m_1P_{26} + m_2P_{27} + m_6P_{28}\\
&\qquad + F_1P_{29} + m_7P_{30} + F_1P_{31} + e_1P_{32} + F_1P_{33})\\
&\quad + 2P_{44} + P_{45} - m_8P_{46} - F_4P_{47} - e_4P_{48} - F_4P_{49},\\
l_{37} &= k(2m_1Q_{23} + 2m_2Q_{24} + m_4F_7 + m_5Q_{25} + m_1Q_{26} + m_2Q_{27} + m_6Q_{28}\\
&\qquad + F_1Q_{29} + m_7Q_{30} + F_1Q_{31} + e_1Q_{32} + F_1Q_{33})\\
&\quad + 2Q_{44} + Q_{45} - m_8Q_{46} - F_4Q_{47} - e_4Q_{48} - F_4Q_{49} + Prk_{28},\\
l_{38} &= \frac{F_{27}l_{30}}{l_{31}} - e_{27}, \quad l_{39} = \frac{F_{27}l_{32}}{l_{31}} + l_{36}, \quad l_{40} = \frac{F_{27}l_{33}}{l_{31}} + l_{37},\\
l_{41} &= ke_5 + l_{38}, \qquad l_{42} = l_{38}l_{34} + l_{39}, \qquad l_{43} = l_{38}l_{35} + l_{40};
\end{aligned}
$$

$$
\begin{aligned}
m_1 &= \frac{-1 + sqrt1 + 4\sigma^m}{2m}, & m_2 &= \frac{-1 - sqrt1 + 4\sigma^m}{2m},\\
m_4 &= \frac{-1}{k_2}, & m_5 &= m_1 + m_2;
\end{aligned}
$$

$$
\begin{aligned}
P_1 &= e_1XC_9 + F_1XC_{10}, & P_2 &= e_1XC_{10} - F_1XC_9,\\
P_3 &= -D_1F_1^2P_1 + D_2F_1P_2 + D_3P_1, & P_4 &= -D_1F_1^2P_2 - D_2F_1P_1 + D_3P_2,\\
P_5 &= -D_5F_1^2P_1 + D_5F_1P_2 + D_7P_1, & P_6 &= -D_5F_1^2P_2 - D_6F_1P_1 + D_7P_2,\\
P_7 &= -D_1F_1^2Q_1 + D_2F_1P_2 + D_3P_1, & P_8 &= -D_1F_1^2Q_2 - D_2F_1Q_1 + D_3Q_2,\\
P_9 &= -D_5F_1^2Q_1 + D_5F_1Q_2 + D_7Q_1, & P_{10} &= -D_5F_1^2Q_2 - D_6F_1Q_1 + D_7Q_2,\\
P_{11} &= -D_1F_1^2XC_9 + D_2F_1XC_{10} + D_3XC_9,\\
P_{12} &= -D_1F_1^2XC_{10} - D_2F_1XC_9 + D_3XC_{10},\\
P_{13} &= -D_5F_1^2XC_9 + D_6F_1XC_{10} + D_7XC_9,\\
P_{14} &= -D_5F_1^2XC_{10} - D_6F_1XC_9 + D_7XC_{10},\\
P_{15} &= -D_9F_1^2XC_9 + D_{10}F_1XC_{10} + D_{11}XC_9,\\
P_{16} &= -D_9F_1^2XC_{10} - D_{10}F_1XC_9 + D_{11}XC_{10},\\
P_{17} &= -D_1F_1^2YC_9 + D_2F_1YC_{10} + D_3YC_9,\\
P_{18} &= -D_1F_1^2YC_{10} - D_2F_1YC_9 + D_3YC_{10},
\end{aligned}
$$

$$P_{19} = -D_5F_1^2YC_9 + D_6F_1YC_{10} + D_7YC_9,$$
$$P_{20} = -D_5F_1^2YC_{10} - D_6F_1YC_9 + D_7YC_{10},$$
$$P_{21} = -D_9F_1^2YC_9 + D_{10}F_1YC_{10} + D_{11}YC_9,$$
$$P_{22} = -D_9F_1^2YC_{10} - D_{10}F_1YC_9 + D_{11}YC_{10},$$
$$P_{23} = e_{5a}+e_{11}-F_{14}+e_{17}-F_{22}, \quad P_{24} = e_6+e_{12}-F_{15}+e_{18}-F_{23},$$
$$P_{25} = e_8+e_{13}-F_{16}+e_{19} - F_{24}, \quad P_{26} = e_9+e_{20} - F_{25},$$
$$P_{27} = e_{10}+e_{21}-F_{26}, \quad P_{28} = k_{19}(P_3-Q_7)+k_{23}(P_{11}-Q_{17}),$$
$$P_{29} = k_{19}(P_4-Q_8)+k_{23}(P_{12}-Q_{18}), \quad P_{30} = k_{20}(P_5-Q_9)+k_{24}(P_{13}-Q_{19}),$$
$$P_{31} = k_{20}(P_6-Q_{10})+k_{24}(P_{14}-Q_{20}), \quad P_{32} = k_{25}(P_{15}-Q_{21}),$$
$$P_{33} = k_{25}(P_{16}-Q_{22}), \quad P_{34} = e_4XC_{11}+F_4XC_{12},$$
$$P_{35} = e_4XC_{12}-F_4XC_{11}, \quad P_{36} = -D_{13}F_4^2P_{34}+D_{14}F_4P_{35}+D_{15}P_{34},$$
$$P_{37} = -D_{13}F_4^2P_{35} - D_{14}F_4P_{34} + D_{15}P_{35},$$
$$P_{38} = -D_{17}F_4^2P_{34} + D_{18}F_4P_{35} + D_{19}P_{34},$$
$$P_{39} = -D_{17}F_4^2P_{35} - D_{18}F_4P_{34} + D_{19}P_{35},$$
$$P_{40} = -D_{13}F_4^2Q_{34} + D_{14}F_4Q_{35} + D_{15}Q_{34},$$
$$P_{41} = -D_{13}F_4^2Q_{35} - D_{14}F_4Q_{34} + D_{15}Q_{35},$$
$$P_{42} = -D_{17}F_4^2Q_{34} + D_{18}F_4Q_{35} + D_{19}Q_{34},$$
$$P_{43} = -D_{17}F_4^2Q_{35} - D_{18}F_4Q_{34} + D_{19}Q_{35},$$
$$P_{44} = e_{28} - F_{30}, \quad P_{45} = e_{29} - F_{31}, \quad P_{46} = k_{30}(P_{36} - Q_{40}),$$
$$P_{47} = k_{30}(P_{37} - Q_{41}), \quad P_{48} = k_{31}(P_{38} - Q_{42}), \quad P_{49} = k_{31}(P_{39} - Q_{43});$$

$$Q_1 = e_1YC_9 + F_1YC_{10}, \quad Q_2 = e_1YC_{10} - F_1YC_9,$$
$$Q_3 = D_4F_1P_2 - \omega B_3P_1, \quad Q_4 = -D_4F_1P_1 - \omega B_3P_2,$$
$$Q_5 = D_8F_1P_2 - \omega B_6P_1, \quad Q_6 = -D_8F_1P_1 - \omega B_6P_2,$$
$$Q_7 = D_4F_1Q_2 - \omega B_3Q_1, \quad Q_8 = -D_4F_1Q_1 - \omega B_3Q_2,$$
$$Q_9 = D_8F_1Q_2 - \omega B_6Q_1, \quad Q_{10} = -D_8F_1Q_1 - \omega B_6Q_2,$$
$$Q_{11} = D_4F_1XC_{10} - \omega B_3XC_9, \quad Q_{12} = -D_4F_1XC_9 - \omega B_3XC_{10},$$
$$Q_{13} = D_8F_1XC_{10} - \omega B_6XC_9, \quad Q_{14} = -D_8F_1XC_9 - \omega B_6XC_{10},$$
$$Q_{15} = D_{12}F_1XC_{10} - \omega B_9XC_9, \quad Q_{16} = -D_{12}F_1XC_9 - \omega B_9XC_{10},$$
$$Q_{17} = D_4F_1YC_{10} - \omega B_3YC_9, \quad Q_{18} = -D_4F_1YC_9 - \omega B_3YC_{10},$$
$$Q_{19} = D_8F_1YC_{10} - \omega B_6YC_9, \quad Q_{20} = -D_8F_1YC_9 - \omega B_6YC_{10},$$
$$Q_{21} = D_{12}F_1YC_{10} - \omega B_9YC_9, \quad Q_{22} = -D_{12}F_1YC_9 - \omega B_9YC_{10},$$
$$Q_{23} = F_{5a} + F_{11} + e_{14} + F_{17} + e_{22}, \quad Q_{24} = F_6 + F_{12} + e_{15} + F_{18} + e_{23},$$
$$Q_{25} = F_8 + F_{13} + e_{16} + F_{19} + e_{24}, \quad Q_{26} = F_9 + F_{20} + e_{25},$$

$Q_{27} = F_{10} + F_{21} + e_{26}$, $Q_{28} = k_{19}(Q_3+P_7)+k_{23}(Q_{11}+P_{17})$,
$Q_{29} = k_{19}(Q_4+P_8)+k_{23}(Q_{12}+P_{18})$, $Q_{30} = k_{20}(Q_5+P_9)+k_{24}(Q_{13}+P_{19})$,
$Q_{31} = k_{20}(Q_6+P_{10})+k_{24}(Q_{14}+P_{20})$, $Q_{32} = k_{25}(Q_{15} + P_{21})$,
$Q_{33} = k_{25}(Q_{16} + P_{22})$, $Q_{34} = e_4YC_{11} + F_4YC_{12}$,
$Q_{35} = e_4YC_{12} - F_4YC_{11}$, $Q_{36} = D_{16}F_4P_{35} - \omega PrB_{12}Q_{34}$,
$Q_{37} = -D_{16}F_4P_{34} - \omega PrB_{12}P_{35}$, $Q_{38} = D_{20}F_4P_{35} - \omega PrB_{15}Q_{34}$,
$Q_{39} = -D_{20}F_4P_{34} - \omega PrB_{15}P_{35}$, $Q_{40} = D_{16}F_4Q_{35} - \omega PrB_{12}Q_{34}$,
$Q_{41} = -D_{16}F_4Q_{34} - \omega PrB_{12}Q_{35}$, $Q_{42} = D_{20}F_4Q_{35} - \omega PrB_{15}Q_{34}$,
$Q_{43} = -D_{20}F_4Q_{34} - \omega PrB_{15}Q_{35}$, $Q_{44} = F_{28}+e_{30}$, $Q_{45} = F_{29}+e_{31}$,
$Q_{46} = k_{30}(Q_{36} + P_{40})$, $Q_{47} = k_{30}(Q_{37} + P_{41})$,
$Q_{48} = k_{31}(Q_{38} + P_{42})$, $Q_{49} = k_{31}(Q_{39} + P_{43})$;

$$XC_9 = \frac{l_{11}l_{24} - m_1F_1l_{12}}{mF_1l_{10} - l_{11}l_{23}}, \qquad XC_{10} = \frac{-(l_{10}XC_9 + l_{12})}{l_{11}},$$
$$XC_{11} = XC_9 + l_{17}, \qquad XC_{12} = \frac{-l_{14}XC_{11}}{l_{15}},$$
$$XC_{13} = \frac{l_{27}l_{42} - kF_5l_{28}}{kF_5l_{26} - l_{27}l_{41}}, \qquad XC_{14} = \frac{-(l_{26}XC_{13} + l_{28})}{l_{27}},$$
$$XC_{15} = XC_{13} + l_{34}, \qquad XC_{16} = \frac{-(l_{30}XC_{15} + l_{32})}{l_{31}},$$
$$YC_9 = \frac{l_{11}l_{25} - m_1F_1l_{13}}{mF_1l_{10} - l_{11}l_{23}}, \qquad YC_{10} = \frac{-(l_{10}YC_9 + l_{13})}{l_{11}},$$
$$YC_{11} = YC_9 + l_{18}, \qquad YC_{12} = \frac{-l_{14}YC_{11} + l_{16}}{l_{15}},$$
$$YC_{13} = \frac{l_{27}l_{43} - kF_5l_{29}}{kF_5l_{26} - l_{27}l_{41}}, \qquad YC_{14} = \frac{-(l_{26}YC_{13} + l_{29})}{l_{27}},$$
$$YC_{15} = YC_{13} + l_{35}, \qquad YC_{16} = \frac{-(l_{30}YC_{15} + l_{33})}{l_{31}};$$

$$r_1 = \sqrt{(1 + m\sigma^2)^2 + (4\omega m)^2}, \qquad r_2 = \sqrt{1 + 16\omega^2},$$
$$r_3 = \sqrt{1 + 16\omega^2k_2^2}, \qquad r_4 = \sqrt{Pr^2 + (4\omega Pr)^2},$$
$$\theta_1 = \tan^{-1}\frac{4\omega m}{1 + m\sigma^2}, \qquad \theta_2 = \tan^{-1}(4\omega),$$
$$\theta_3 = \tan^{-1}(4\omega k_2), \qquad \theta_4 = \tan^{-1}\frac{4\omega}{Pr}.$$

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