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Analytic approximate solutions for unsteady boundary-layer flow and heat transfer due to a stretching sheet by homotopy analysis method

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Abstract. In this work, the homotopy analysis method is applied to study the unsteady boundary-layer flow and heat transfer due to a stretching sheet. The analytic solutions of the system of nonlinear ordinary differential equations are constructed in the series form. The convergence of the obtained series solutions is carefully analyzed. The velocity and temperature profiles are shown and the influence of non-dimensional parameter on the heat transfer is discussed in detail. The validity of our solutions is verified by the numerical results.

Keywords: homotopy analysis method, system of nonlinear ordinary differential equations, convergence, stretching sheet.

1 Introduction

Nonlinear differential equations are usually arising from mathematical modeling of many physical systems. Some of them are solved using numerical methods and some are solved using the analytic methods such as perturbation [1, 2]. The numerical methods such as Rung-Kutta method are based on discretization techniques, and they only permit us to calculate the approximate solutions for some values of time and space variables, which cause us to overlook some important phenomena, in addition to the intensive computer time required to solve the problem. Thus it is often costly and time consuming to get a complete curve of results and so in these methods, stability and convergence should be considered so as to avoid divergence or inappropriate results. Numerical difficulties additionally appear if a nonlinear problem contains singularities or has multiple solutions. Perturbation techniques are based on the existence of small/large parameters, the so-called perturbation quantity. Unfortunately, many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Some nonperturbative techniques, such as the artificial small parameter method [3], the δ-expansion method [4] and the Adomian's decomposition method [5], have been developed. Different from perturbation techniques, these nonperturbative methods are independent upon small parameters. However, both of the perturbation techniques and the nonperturbative methods themselves cannot provide us with a simple way to adjust or control the convergence region and rate of given approximate series.

In 1992, Liao [6] employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM), [7–11]. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation. Therefore, the HAM can overcome the foregoing restrictions and limitations of perturbation methods [12]. The HAM also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation and avoidance of physically unrealistic assumptions. Furthermore, the HAM always provides us with a family of solution expressions in the auxiliary parameter $\hbar$ the convergence region and rate of each solution might be determined conveniently by the auxiliary parameter $\hbar$. Besides, the HAM is rather general and contains the homotopy perturbation method (HPM) [11, 12], the Adomian decomposition method (ADM) [13] and δ-expansion method.

In recent years, the homotopy analysis method has been successfully employed to solve many types of nonlinear problems such as the nonlinear equations arising in heat transfer [14], the nonlinear model of diffusion and reaction in porous catalysts [15], the chaotic dynamical systems [16], the non-homogeneous Blasius problem [17], the generalized three-dimensional MHD flow over a porous stretching sheet [18], the wire coating analysis using MHD Oldroyd 8-constant fluid [19], the axisymmetric flow and heat transfer of a second grade fluid past a stretching sheet [20], the MHD flow of a second grade fluid in a porous channel [21], the generalized Couette flow [22], the squeezing flow between two infinite plates [23], the Glauert-jet problem [24], the Burger and regularized long wave equations [25], the laminar viscous flow in a semi-porous channel in the presence of a uniform magnetic field [26], and other problems. All of these successful applications verified the validity, effectiveness and flexibility of the HAM.

The flow and heat transfer of a viscous and incompressible fluid induced by a continuously moving or stretching surface in a resting fluid is relevant to many manufacturing processes such as polymers involves the cooling of continuous strips or filaments by drawing them through a quiescent fluid [33]. Further, glass blowing, continuous casting of metals and spinning of fibers involve the flow due to a stretching surface. Crane [27] was first to study the boundary-layer flow due to a stretching surface in an ambient fluid and applied a similarity transformation for the steady boundary-layer flow by stretching of a sheet when its velocity varying linearly with the distance from a fixed point. Carragher and Crane [28] considered the influence of heat transfer in the flow over a stretching surface in the case when the temperature difference between the surface and the ambient fluid is proportional to a power of distance from the fixed point [33]. Dutta [29], Grubka and Bobba [30] studied the temperature field in the flow over a stretching surface when a uniform heat flux is exerted to the surface. Elbashbeshy [31] considered the case of a stretching surface with a variable surface heat flux. The unsteady flow filed and heat transfer occur when a flat plat stretches suddenly or a step change of the temperature or heat flux of the sheet [33]. Elbashbeshy and Bazid [32] studied the unsteady flow and heat transfer over a stretching sheet.

The main goal of the present study is to find the totally analytic solution for unsteady boundary-layer flow and heat transfer due to a stretching sheet by homotopy analysis method. This problem studied first by Sharidan [33] in 2006 and exerted the similarity solution. Liao and Pop [34] applied the HAM to solve a steady boundary-layer flow due to a stretching sheet. In this way, the Letter has been organized as follows. In Section 2, the flow analysis and mathematical formulation are presented. In Section 3, we extend the application of the HAM to construct the approximate solutions for the governing equations. The convergence of the obtained series solutions is carefully analyzed in Section 4. Section 5 contains the results and discussion. The conclusions are summarized in Section 6.

2 Flow analysis and mathematical formulation

Fig. 1 shows the unsteady flow and heat transfer of a viscous and incompressible fluid past a semi-infinite stretching sheet in the region $y > 0$. Keeping the origin fixed, two equal and opposite forces are suddenly applied along the x-axis. These forces stretch the sheet and the flow is generated. The wall temperature $T_w(x,t)$ of the sheet is suddenly raised from T_∞ to $T_w(t,x) > T_\infty$ or there is suddenly imposed a heat flux $q_w(t,x)$ at the wall [34].

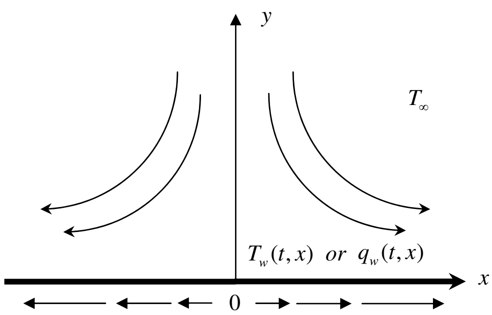

Fig. 1. Geometry of the problem.

With these assumptions, the governing equations for the unsteady boundary-layer flow due to the stretching sheet are given as follow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

and the boundary conditions are

$$
\begin{aligned}
&u = u_w(t,x), \quad v = 0, && \text{at } \ y = 0,\\
&T = T_w(t,x) \ \text{(VWT)} \quad \text{or} \quad \frac{\partial T}{\partial y} = -\frac{q_w(t,x)}{k} \ \text{(VHF)} && \\
&u \to 0, \quad T \to T_\infty && \text{at } \ y \to \infty,
\end{aligned} \tag{4}
$$

where t is the time, u and v are the velocity components along the x- and y-axes respectively, T is the temperature, α is the thermal diffusivity, ν is the kinematic viscosity and k is the thermal conductivity.

The velocity of the sheet $u_w(t,x)$ the sheet temperature $T_w(t,x)$ and the heat flux $q_w(t,x)$ are defined

$$
\begin{aligned}
&u_w(t,x) = \frac{cx}{1-\gamma t}, \quad T_w(t,x) = T_\infty + \frac{c}{2\nu x^2(1-\gamma t)^{3/2}},\\
&q_w(t,x) = \frac{q_{w_0}}{2x^2}\left(\frac{c}{\nu}\right)^{3/2}\frac{1}{(1-\gamma t)^2},
\end{aligned} \tag{5}
$$

where c is the stretching rate being a positive constant, γ is a positive constant, which measures the unsteadiness and q_{w_0} is a characteristic heat transfer quantity [33]. Sharidan [33] introduced the following similarity transforms

$$
\begin{aligned}
&\eta = \sqrt{\frac{c}{\nu(1-\gamma t)}}\, y, \quad \psi = \sqrt{\frac{c\nu}{1-\gamma t}}\, x f(\eta),\\
&T = T_\infty + \frac{c}{2\nu x^2(1-\gamma t)^{3/2}}\,\theta(\eta) \qquad \text{(VWT)},\\
&T = T_\infty + \frac{q_{w_0}}{k}\frac{c}{2\nu x^2(1-\gamma t)^{3/2}}\,\theta(\eta) \qquad \text{(VHF)},
\end{aligned} \tag{6}
$$

where ψ is the stream function and is defined as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. The governing equations are reduced by using (6) as follow

$$
f''' + f\,f'' - f'^2 - A\left(f' + \frac{1}{2}\eta f''\right) = 0, \tag{7}
$$

$$
\frac{1}{Pr}\theta'' + f\theta' + 2f'\theta - \frac{1}{2}A(3\theta + \eta\theta') = 0, \tag{8}
$$

with boundary conditions

$$
\begin{aligned}
&f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0,\\
&\theta(0) = 1 \ \text{(VWT)} \quad \text{or} \quad \theta'(0) = 1 \ \text{(VHF)}, \quad \theta(\infty) = 0,
\end{aligned} \tag{9}
$$

where Pr is the Prandtl number, $A = \gamma/c$ is a non-dimensional constant which measures the flow and heat transfer unsteadiness and primes denote the differentiation with respect to the similarity variable η [33].

The skin friction coefficient C_f and the local Nusselt number Nu_x are the important physical quantities in this problem and are defined as

$$C_f = \frac{\tau_w}{\rho u_w^2}, \quad Nu_x = \frac{x q_w}{k(T - T_w)}, \tag{10}$$

where the skin friction τ_w and the heat transfer from the sheet q_w are given by

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad q_w = -k \left(\frac{\partial T}{\partial y}\right)_{y=0}, \tag{11}$$

and μ is the dynamic viscosity. By using equations (5) and (6), it is obvious to get

$$C_f Re_x^{1/2} = f''(0), \quad \frac{Nu_x}{Re_x^{1/2}} = -\theta'(0) \text{ (VWT)}, \quad \frac{Nu_x}{Re_x^{1/2}} = \frac{1}{\theta(0)} \text{ (VHF)}, \tag{12}$$

where $Re_x = u_w x/\nu$ is the local Reynolds number.

3 HAM solution

To investigate the explicit and totally analytic solutions of equations (7) and (8) by using HAM, we choose

$$f_0(\eta) = 1 - e^{-\eta}, \tag{13}$$
$$\theta_0(\eta) = e^{-\eta}, \tag{14}$$

as initial approximations of $f(\eta)$ and $\theta(\eta)$ which satisfy the boundary conditions (9). Besides, we select the auxiliary linear operators $\mathrm{L}_1(f)$ and $\mathrm{L}_2(\theta)$ as

$$\mathrm{L}_1(f) = f''' + f'', \tag{15}$$
$$\mathrm{L}_2(\theta) = \theta'' + \theta', \tag{16}$$

satisfying the following properties

$$\mathrm{L}_1(c_1 e^{-\eta} + c_2 \eta + c_3) = 0, \tag{17}$$
$$\mathrm{L}_2(c_4 e^{-\eta} + c_5) = 0, \tag{18}$$

where $c_i,\ i = 1$–5 are arbitrary constants. If $p \in [0, 1]$ is an embedding parameter, $\hbar_f$ and $\hbar_\theta$ are auxiliary nonzero parameters and $\mathrm{H}_f(\eta)$ and $\mathrm{H}_\theta(\eta)$ are auxiliary functions, then the zeroth-order deformation equations are of the following form

$$(1 - p)\mathrm{L}_1\big[\widehat{f}(\eta; p) - f_0(\eta)\big] = p\hbar_f \mathrm{H}_f(\eta)\mathrm{N}_1\big[\widehat{f}(\eta; p)\big], \tag{19}$$
$$(1 - p)\mathrm{L}_2\big[\widehat{\theta}(\eta; p) - \theta_0(\eta)\big] = p\hbar_\theta \mathrm{H}_\theta(\eta)\mathrm{N}_2\big[\widehat{f}(\eta; p), \widehat{\theta}(\eta; p)\big], \tag{20}$$

subject to the boundary conditions

$$\begin{aligned}&\widehat{f}(0;p)=0,\quad \widehat{f}'(0;p)=1,\quad \widehat{f}'(\infty;p)=0,\\&\widehat{\theta}(0;p)=1\ \text{(VWT)}\quad\text{or}\quad \widehat{\theta}'(0;p)=-1\ \text{(VHF)},\quad \widehat{\theta}(\infty;p)=0,\end{aligned}\tag{21}$$

in which we define the nonlinear operators N_1 and N_2 as

$$\begin{aligned}\mathrm{N}_1=&\frac{\partial^3\widehat{f}(\eta;p)}{\partial\eta^3}+\widehat{f}(\eta;p)\,\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}-\left(\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\right)^2\\&-A\left(\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}+\frac{1}{2}\,\eta\,\frac{\partial^2\widehat{f}(\eta;p)}{\partial\eta^2}\right),\end{aligned}\tag{22}$$

$$\begin{aligned}\mathrm{N}_2=&\frac{1}{Pr}\frac{\partial^2\widehat{\theta}(\eta;p)}{\partial\eta^2}+\widehat{f}(\eta;p)\,\frac{\partial\widehat{\theta}(\eta;p)}{\partial\eta}+2\,\frac{\partial\widehat{f}(\eta;p)}{\partial\eta}\,\widehat{\theta}(\eta;p)\\&-\frac{1}{2}\,A\left(3\widehat{\theta}(\eta;p)+\eta\,\frac{\partial\widehat{\theta}(\eta;p)}{\partial\eta}\right).\end{aligned}\tag{23}$$

For $p=0$ and $p=1$, we have

$$\widehat{f}(\eta;0)=f_0(\eta),\quad \widehat{f}(\eta;1)=f(\eta),\quad \widehat{\theta}(\eta;0)=\theta_0(\eta),\quad \widehat{\theta}(\eta;1)=\theta(\eta).\tag{24}$$

As p increases from 0 to 1, $\widehat{f}(\eta;p)$ and $\widehat{\theta}(\eta;p)$ vary from $f_0(\eta)$ to $f(\eta)$ and $\theta(\eta)$. By Taylor's theorem and equations (24) one obtains

$$\widehat{f}(\eta;p)=f_0(\eta)+\sum_{m=1}^{+\infty}f_m(\eta)p^m,\tag{25}$$

$$\widehat{\theta}(\eta;p)=\theta_0(\eta)+\sum_{m=1}^{+\infty}\theta_m(\eta)p^m,\tag{26}$$

where

$$f_m(\eta)=\frac{1}{m!}\frac{\partial^m\widehat{f}(\eta;p)}{\partial p^m}\bigg|_{p=0},\quad \theta_m(\eta)=\frac{1}{m!}\frac{\partial^m\widehat{\theta}(\eta;p)}{\partial p^m}\bigg|_{p=0}.\tag{27}$$

As pointed by Liao [7], the convergence of the series (25) and (26) strongly depend upon auxiliary parameters $\hbar_f$ and $\hbar_\theta$. Assume that $\hbar_f$ and $\hbar_\theta$ are selected such that the series (25) and (26) are convergent at $p=1$ then due to equation (24) we have

$$f(\eta)=f_0(\eta)+\sum_{m=1}^{+\infty}f_m(\eta),\tag{28}$$

$$\theta(\eta)=\theta_0(\eta)+\sum_{m=1}^{+\infty}\theta_m(\eta).\tag{29}$$

For the mth-order deformation equations, we differentiate equations (19) and (20) m times with respect to p divide by $m!$ and then set $p = 0$. The resulting deformation equations at the mth-order are

$$\mathrm{L}_1\big[\widehat{f}_m(\eta) - \chi_m f_{m-1}(\eta)\big] = \hbar_f \mathrm{H}_f(\eta) R_{1,m}(\eta), \tag{30}$$

$$\mathrm{L}_2\big[\widehat{\theta}_m(\eta) - \chi_m \theta_{m-1}(\eta)\big] = \hbar_\theta \mathrm{H}_\theta(\eta) R_{2,m}(\eta), \tag{31}$$

with the following boundary conditions

$$\begin{aligned} &f_m(0) = 0, \quad f'_m(0) = 0, \quad f'_m(\infty) = 0, \\ &\theta_m(0) = 0 \ \text{(VWT)} \quad \text{or} \quad \theta'_m(0) = 0 \ \text{(VHF)}, \quad \theta_m(\infty) = 0, \end{aligned} \tag{32}$$

where

$$\begin{aligned} R_{1,m} = {} & \frac{\partial^3 f_{m-1}(\eta)}{\partial \eta^3} + \sum_{n=0}^{m-1} \left(f_n(\eta) \frac{\partial^2 f_{m-1-n}(\eta)}{\partial \eta^2} + \frac{\partial f_n(\eta)}{\partial \eta} \frac{\partial f_{m-1-n}(\eta)}{\partial \eta} \right) \\ & - A\left(\frac{\partial f_{m-1}(\eta)}{\partial \eta} + \frac{1}{2}\eta \frac{\partial^2 f_{m-1}(\eta)}{\partial \eta^2} \right), \end{aligned} \tag{33}$$

$$\begin{aligned} R_{2,m} = {} & \frac{1}{Pr} \frac{\partial^2 \theta_{m-1}(\eta)}{\partial \eta^2} + \sum_{n=0}^{m-1} \left(f_n(\eta) \frac{\partial \theta_{m-1-n}(\eta)}{\partial \eta} + 2 \frac{\partial f_n(\eta)}{\partial \eta} \theta_{m-1-n}(\eta) \right) \\ & - \frac{1}{2}\left(\theta_{m-1}(\eta) + \eta \frac{\partial \theta_{m-1}(\eta)}{\partial \eta} \right), \end{aligned} \tag{34}$$

and

$$\chi = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{35}$$

According to the rule of solution expression, the rule of coefficient ergodicity and the rule of solution existence as discussed by Liao [7], we choose auxiliary functions as follow

$$\mathrm{H}_f(\eta) = e^{-\eta}, \tag{36}$$

$$\mathrm{H}_\theta(\eta) = e^{-\eta}, \tag{37}$$

and use the symbolic software MATHEMATICA to solve the system of linear equations, equations (30) and (31), with the boundary conditions equation (32), and successively obtain

$$f_1(\eta) = 0.125 A\hbar_f - 0.125 A\hbar_f e^{-\eta} - 0.125 A\hbar_f \eta e^{-2\eta}, \tag{38}$$

$$\begin{aligned} \theta_1(\eta) = {} & 0.5\hbar_\theta e^{-3\eta} - 0.5\hbar_\theta e^{-2\eta} - 0.375 A\hbar_\theta e^{-2\eta} - 1.110223 \cdot 10^{-16} \hbar_\theta e^{-\eta} \\ & + 0.375 A\hbar_\theta e^{-\eta} + \frac{0.5\hbar_\theta e^{-2\eta}}{Pr} - \frac{0.5\hbar_\theta e^{-\eta}}{Pr} + 0.25 A\hbar_\theta e^{-2\eta}. \end{aligned} \tag{39}$$

4 Convergence of HAM solution

Note that the two series (28) and (28) contain the auxiliary parameter $\hbar_f$ and $\hbar_\theta$ which influences the convergent rate and region of the two series. To ensure that these two series converge, we first focus on how to choose proper values of $\hbar_f$ and $\hbar_\theta$. To see the range of admissible values of these parameters, the curves of $\hbar_f$ and $\hbar_\theta$ are plotted in Figs. 2–4 for the 20th-order of approximation.

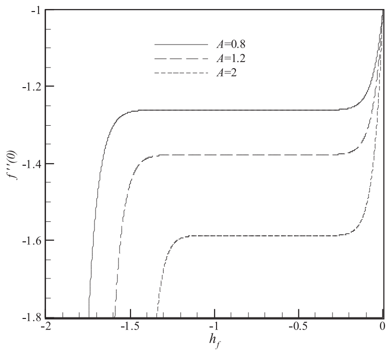

Fig. 2. The $\hbar_f$-curves of $f''(0)$ obtained by the 20th-order approximation of the HAM for different values of A.

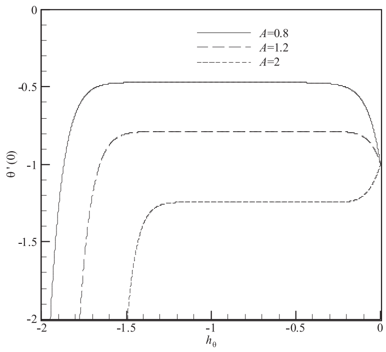

Fig. 3. The $\hbar_\theta$-curves of $\theta'(0)$ (VWT case) obtained by the 20th-order approximation of the HAM for different values of A, when $Pr = 1$.

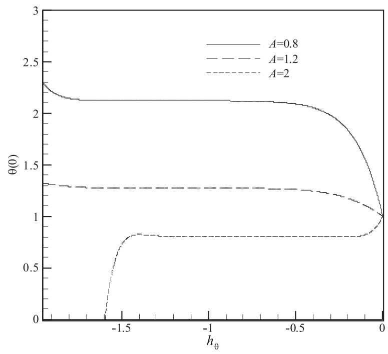

Fig. 4. The $\hbar_\theta$-curves of $\theta(0)$ (VHF case) obtained by the 20th-order approximation of the HAM for different values of A, when $Pr = 1$.

As pointed by Liao [7], the valid region of $\hbar_f$ and $\hbar_\theta$ is a horizontal line segment. For better presentation, we listed these valid regions in Table 1. A wide valid zone is evident in these figures ensuring convergence of the series for both VWT and VHF cases. Table 2 show several best values obtained for the auxiliary parameters $\hbar_f$ and $\hbar_\theta$ for the VWT and VHF cases for different values of the non-dimensional constant. This figure elucidates that the size of the valid region strongly depends on the non-dimensional constant. In fact, the interval for admissible values of $\hbar_f$ and $\hbar_\theta$ shrinks towards zero by increasing the non-dimensional constant.

Table 1. The admissible values of $\hbar_f$ and $\hbar_\theta$ for different values of A when $Pr = 1$

Series solution	A		
	0.8	1.2	2
$f(\eta)$	$-1.6 \le \hbar_f \le -0.2$	$-1.4 \le \hbar_f \le -0.2$	$-1.2 \le \hbar_f \le -0.2$
$\theta(\eta)$ (VWT)	$-1.7 \le \hbar_\theta \le -0.2$	$-1.6 \le \hbar_\theta \le -0.2$	$-1.4 \le \hbar_\theta \le -0.2$
$\theta(\eta)$ (VHF)	$-1.7 \le \hbar_\theta \le -0.5$	$-1.7 \le \hbar_\theta \le -0.5$	$-1.3 \le \hbar_\theta \le -0.2$

Table 2. The best values of $\hbar_f$ and $\hbar_\theta$ for different values of A when $Pr = 1$

A	$\hbar_f$	$\hbar_\theta$ (VWT)	$\hbar_\theta$ (VHF)
0.8	-0.8	-1.3	-1.2
1.2	-0.8	-1.0	-1.0
2	-0.7	-0.7	-0.8

5 Results and discussion

Equations (7) and (8) with the boundary conditions (9) are solved using HAM for some values of the parameter A. The rate of convergence for $f''(0),\ \theta'(0)$ (VWT case) and $\theta(0)$ (VHF case) at some values of A are shown in Tables 3–5, respectively. The results obtained from HAM solution are compared with results of Sharidan [34]. The results show that HAM gives an analytical solution with high order of accuracy with a few iterations.

Table 3. The rate of convergence for $f''(0)$ at some values of A

A	5th-order	10th-order	15th-order	20th-order	Sharidan [34]
0.8	-1.261088	-1.261063	-1.261048	-1.261042	-1.261042
1.2	-1.377594	-1.377647	-1.377710	-1.377721	-1.377722
2.0	-1.587983	-1.587347	-1.587382	-1.587360	-1.587362

Table 4. The rate of convergence for $\theta(0)$ (VWT case) at some values of A

A	5th-order	10th-order	15th-order	20th-order	Sharidan [34]
0.8	-0.472381	-0.471278	-0.471195	-0.471190	-0.471190
1.2	-0.787782	-0.787869	-0.788142	-0.788169	-0.788173
2.0	-1.247413	-1.243933	-1.243792	-1.243739	-1.243741

The profiles $f(\eta)$ and $f'(\eta)$ obtained by the 20th-order approximation of the HAM are shown in Figs. 5 and 6 for different values of A. Figs. 7 and 8 show the effect of the non-dimensional parameter, A, on the temperature profiles for both the VWT and VHF cases.

Table 5. The rate of convergence for (VHF case) at some values of

A	5th-order	10th-order	15th-order	20th-order	Sharidan [34]
0.8	2.099092	2.121981	2.122792	2.122869	2.122870
1.2	1.269349	1.269058	1.268772	1.268760	1.268756
2.0	0.802819	0.803925	0.804006	0.804021	0.804026

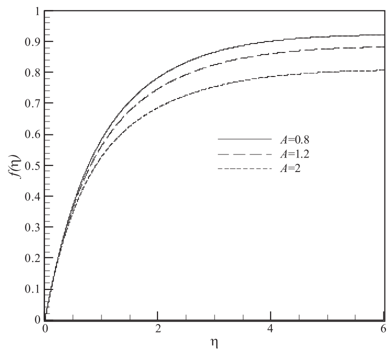

Fig. 5. The profile $f(\eta)$ obtained by the 20th-order approximation of the HAM for different values of A.

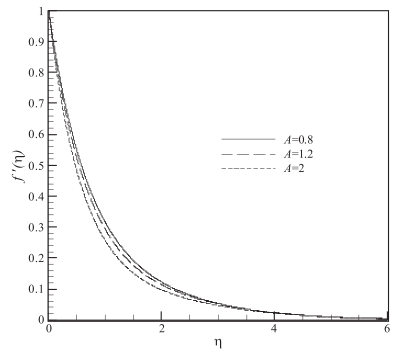

Fig. 6. The profile $f'(\eta)$ obtained by the 20th-order approximation of the HAM for different values of A.

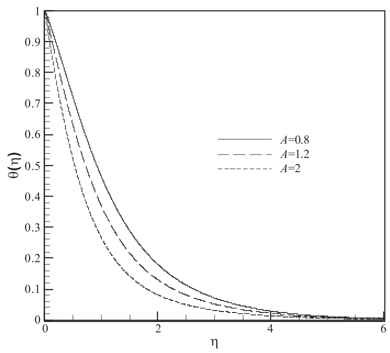

Fig. 7. The profile $\theta(\eta)$ for VWT obtained by the 20th-order approximation of the HAM for different values of A, when $Pr = 1$.

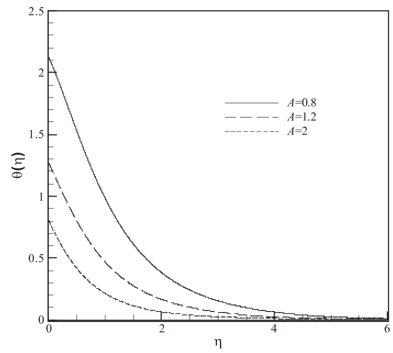

Fig. 8. The profile $\theta(\eta)$ for VHF obtained by the 20th-order approximation of the HAM for different values of A, when $Pr = 1$.

6 Conclusions

In this Letter, the homotopy analysis method (HAM) was used for finding the totally analytic solutions of the system of nonlinear ordinary differential equations derived from similarity transform for unsteady boundary-layer flow and heat transfer due to a stretching sheet. The validity of our solutions is verified by the numerical results. We analyzed the convergence of the obtained series solutions, carefully. Unlike perturbation methods, the HAM does not depend on any small physical parameters. Thus, it is valid for both weakly and strongly nonlinear problems. Besides, different from all other analytic methods, the HAM provides us a simple way to adjust and control the convergence region of the series solution by means of auxiliary parameter $\hbar$. Thus the auxiliary parameter $\hbar$ plays an important role within the frame of the HAM which can be determined by the so-called $\hbar$-curves. The solution obtained by means of the HAM is an infinite power series for appropriate initial approximation, which can be, in turn, expressed in a closed form.

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