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A semi-analytical solution of micro polar flow in a porous channel with mass injection by using differential transform method

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Abstract. In this letter, the differential transform method (DTM) was applied to the micro-polar flow in a porous channel with mass injection. Approximate solutions of the governing system of nonlinear ordinary differential equations were calculated in the form of DTM series with easily computable terms. The validity of the series solutions were verified by comparison with numerical results obtained using a fourth order Runge–Kutta method. The computed DTM velocity profiles are shown and the influence of Reynolds number on the velocity component in x-direction is discussed.

Keywords: differential transform method, micro-polar flow, porous channel, system of nonlinear ordinary differential equations.

1 Introduction

Most phenomena in our world are essentially nonlinear and were described by nonlinear equations. Some of them were solved using numerical methods and some were solved using the analytic methods of perturbation [1, 2]. The numerical methods such as Runge–Kutta method are based on discretization techniques, and they only permit us to calculate the approximate solutions for some values of time and space variables, which cause us to overlook some important phenomena, in addition to the intensive computer time required to solve the problem. Thus it is often costly and time consuming to get a complete curve of results and so in these methods, stability and convergence should be considered so as to avoid divergence or inappropriate results. Numerical difficulties additionally appear if a nonlinear problem contains singularities or has multiple solutions. In the analytic perturbation methods, we should exert the small parameter in the equation.

Therefore, finding the small parameter and exerting it into the equation are deficiencies of the perturbation methods. Recently, much attention has been devoted to the newly developed methods to construct approximate analytic solutions of nonlinear equations without mentioned deficiencies.

One of the semi-exact methods which do not require small parameters is the DTM. The concept of this method was first introduced by Zhou in 1986 [3] who solved linear and nonlinear problems in electrical circuit problems. Chen and Ho [4] developed this method for partial differential equations and Ayaz [5] applied it to the system of differential equations, this method is very powerful [6]. This method constructs a semi-analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders. The differential transform method is an alternative procedure for obtaining analytic Taylor series solution of the differential equations. In recent years, the DTM has been successfully employed to solve many types of nonlinear problems [7–10].

In this study, the DTM was applied to find an approximate solution for the micropolar flow in a porous channel with mass injection. This problem was studied first by Kelson et al. [11] and Desseaux and Kelson [12] using a perturbation approach. More recently, Ziabakhsh and Domairry [13] also studied this flow using the homotopy analysis method (HAM). In this work, the flow analysis and mathematical formulation are presented in Section 2, and the DTM is applied in Section 3 to construct the approximate solutions for the governing equations. Section 4 contains the results and discussion, and conclusions are given in Section 5.

2 Flow analysis and mathematical formulation

We consider steady, incompressible, laminar flow of a micropolar fluid along a two-dimensional channel with porous walls through which fluid is uniformly injected or removed with speed q. Using Cartesian coordinates, the channel walls are parallel to the x-axis and located at $y = \pm h$, where $2h$ is the channel width. The relevant equations governing the flow are [11]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \left(\nu + \frac{\kappa}{\rho}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\kappa}{\rho}\frac{\partial N}{\partial y}, \tag{2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \left(\nu + \frac{\kappa}{\rho}\right)\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - \frac{\kappa}{\rho}\frac{\partial N}{\partial x}, \tag{3}$$

$$u\frac{\partial N}{\partial x} + v\frac{\partial N}{\partial y} = -\frac{\kappa}{\rho j}\left(2N + \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + \frac{\nu_s}{\rho j}\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right). \tag{4}$$

Compared with Newtonian fluids, the governing equations include the micro rotation or angular velocity N whose direction of rotation is in the xy-plane, and the material parameters j, κ and ν_s [11]. For consistency with other micropolar studies, all material parameters are taken as independent and constant. When these constants are zero, the governing equations reduce to those given by Berman [14]. The appropriate physical boundary conditions are

$$u(x, \pm h) = 0, \quad v(x, \pm h) = \pm q, \quad N(x, \pm h) = -s \left.\frac{\partial u}{\partial y}\right|_{(x,\pm h)}, \tag{5}$$

for symmetric flow about

$$\frac{\partial u}{\partial y}(x, 0) = v(x, 0) = 0, \tag{6}$$

where $q > 0$ corresponds to suction, $q < 0$ to injection, and s is a boundary parameter that is used to model the extent to which microelements are free to rotate in the vicinity of the channel walls. For example, the value $s = 0$ corresponds to the case where microelements close to a wall are unable to rotate, whereas the value $s = 1/2$ corresponds to the case where the microrotation is equal to the fluid vorticity at the boundary (see Lukaszewicz [15]).

To simplify the governing equations, Kelson et al. [11] introduced the following similarity transforms

$$\psi = -qxf(\eta), \quad N = \frac{qx}{h^2} g(\eta), \tag{7}$$

where

$$\eta = \frac{y}{h}, \quad u = \frac{\partial \psi}{\partial y} = -\frac{q\,x}{h} f'(\eta), \quad v = -\frac{\partial \psi}{\partial x} = qf(\eta). \tag{8}$$

The Navier–Stokes equations (1)–(4) are reduced by using (7) and (8)

$$(1 + N_1) f^{IV} - N_1 g'' - Re(ff''' - f'f'') = 0, \tag{9}$$

$$N_2 g'' + N_1(f'' - 2g) - N_3 Re(fg' - f'g) = 0, \tag{10}$$

where primes denote differentiation with respect to η. N_1, N_2, N_3 and Re are dimensionless parameters and introduced as follow

$$N_1 = \frac{\kappa}{\rho\nu}, \quad N_2 = \frac{\nu_s}{\rho\nu h^2}, \quad N_3 = \frac{j}{h^2} \quad \text{and} \quad Re = \frac{qh}{\nu}. \tag{11}$$

Where $Re > 0$ corresponds to suction, and $Re < 0$ to injection. The boundary conditions are

$$f(\pm 1) = 1, \quad f'(\pm 1) = 0, \quad g(\pm 1) = sf''(\pm 1). \tag{12}$$

For the symmetric flow

$$f(0) = f''(0) = f'(1) = 0, \quad f(1) = 1, \quad g(1) = sf''(1). \tag{13}$$

In this work, following Kelson et al. [11] we set $s = 0$, $N_1 = N_2 = 1$, $N_3 = 0.1$,

$$2f^{IV} - g'' - Re(ff''' - f'f'') = 0, \tag{14}$$

$$g'' + f'' - 2g - 0.1Re(fg' - f'g) = 0, \tag{15}$$

the boundary conditions are

$$f(0) = f''(0) = f'(1) = 0, \quad f(1) = 1, \quad g(0) = 0, \quad g(1) = 0, \tag{16}$$

and investigate solution behavior of the symmetric flow with mass injection $(q < 0)$ as Re is varied.

3 The differential transform method

Differential transformation of the function $f(\eta)$ is defined as follows

$$F(k) = \frac{1}{k!}\left[\frac{d^k f(\eta)}{d\eta^k}\right]_{\eta=\eta_0}, \tag{17}$$

in equation (17) $f(\eta)$ is the original function and $F(k)$ is transformed function which is called the T-function (it is also called the spectrum of the $f(\eta)$ at $\eta = \eta_0$, in the K domain). The differential inverse transformation of $F(k)$ is defined as

$$f(\eta) = \sum_{k=0}^{\infty} F(k)(\eta - \eta_0)^k, \tag{18}$$

combining equation (17) and (18), we obtain

$$f(\eta) = \sum_{k=0}^{\infty}\left[\frac{d^k f(\eta)}{d\eta^k}\right]_{\eta=\eta_0}\frac{(\eta - \eta_0)^k}{k!}. \tag{19}$$

Equation (19) shows the concept of the differential transformation that is derived from Taylor's series expansion, but the method does not evaluate the derivatives symbolically. However, relative derivative are calculated by iterative procedure that are described by the transformed equations of the original functions.

From the definitions of equations (17) and (18), it is easily proven that the transformed functions comply with the basic mathematical operations shown in below. In real applications, the function $f(\eta)$ in equation (18) is expressed by a finite series and can be written as

$$f(\eta) \cong \sum_{k=0}^{N} F(k)(\eta - \eta_0)^k. \tag{20}$$

Equation (20) implies that $\sum_{k=N+1}^{\infty} F(k)(\eta - \eta_0)^k$ is negligibly small, where N is series size.

Theorems to be used in the transformation procedure, which can be evaluated from equations (17) and (18), are given below

Theorem 1. *If* $f(\eta) = g(\eta) \pm h(\eta)$*, then* $F(k) = G(k) \pm H(k)$.

Theorem 2. *If* $f(\eta) = cg(\eta)$*, then* $F(k) = cG(k)$*, where* c *is a constant.*

Theorem 3. *If* $f(\eta) = \frac{d^n g(\eta)}{d\eta^n}$*, then* $F(k) = \frac{(k+n)!}{k!} G(k+n)$.

Theorem 4. *If* $f(\eta) = g(\eta)h(\eta)$*, then* $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.

Taking differential transform of equations (14) and (15), can be obtained

$$\begin{aligned} &2(k+4)(k+3)(k+2)(k+1)F(k+4) - (k+2)(k+1)G(k+2) \\ &\quad - Re \sum_{r=0}^{k} \big[(k+3-r)(k+2-r)(k+1-r)F(r)F(k+3-r) \\ &\qquad - (k+1-r)(k+2-r)(r+1)F(r+1)F(k+2-r)\big] = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} &(k+2)(k+1)G(k+2) + (k+2)(k+1)F(k+2) - 2G(k) \\ &\quad - 0.1Re \sum_{r=0}^{k} \big[(k+1-r)F(r)G(k+1-r) \\ &\qquad - (r+1)F(r+1)G(k-r)\big] = 0, \end{aligned} \tag{22}$$

where $F(k)$ and $G(k)$ are the differential transforms of $f(t)$ and $g(t)$. The transform of the boundary conditions are

$$\begin{aligned} &F(0) = 0, \quad F(1) = \alpha, \quad F(2) = 0, \quad F(3) = \beta, \\ &G(0) = 0, \quad G(1) = \gamma, \end{aligned} \tag{23}$$

where α, β and γ are constants. For these constants, the problem was solved with (23) and then the boundary conditions (16) were applied

$$f(1) = 1, \quad f'(1) = 0, \quad g(1) = 0. \tag{24}$$

For $Re = -10$ and $N = 20$, we have

$$\begin{aligned} &\alpha = 1.5322204297572861, \quad \beta = -0.5553448285583471, \\ &\gamma = -0.3999429995789404, \end{aligned} \tag{25}$$

and the solutions of above equations (using the DTM) are as follows

$$\begin{aligned} f(\eta) \cong\ & 1.53222\eta - 0.555345\eta^3 + 0.0105508\eta^5 + 0.0180515\eta^7 \\ & - 0.0080363\eta^9 + 0.00370628\eta^{11} - 0.00177786\eta^{13} \\ & + 0.000846618\eta^{15} - 0.000392725\eta^{17} + 0.000176144\eta^{19}, \end{aligned} \tag{26}$$

$$\begin{aligned} g(\eta) \cong\ & -0.399943\eta + 0.42203\eta^3 + 0.0108015\eta^5 - 0.0173915\eta^7 \\ & + 0.00912927\eta^9 - 0.00439804\eta^1 1 + 0.00212283\eta^1 3 \\ & - 0.00100527\eta^1 5 + 0.000462899\eta^{17} - 0.000206222\eta^{19}, \end{aligned} \tag{27}$$

4 Results and discussion

Table 1 shows results obtained from the DTM for the functions $f(\eta)$ and $g(\eta)$ at $Re = -10$ and $N = 20$, compared with a numerical solution using a fourth-order Runge–Kutta method. Excellent agreement can be seen. From this table and Fig. 1, it can also be seen that the maximum error for the DTM occurs near the middle of the interval ($\eta = 0.5$).

Table 1. The results of DTM and Runge–Kutta numerical method for $f(\eta)$ and $g(\eta)$ when $Re = -10$ and $N = 20$.

	$f(\eta)$			$g(\eta)$		
η	DTM	Numerical	Error	DTM	Numerical	Error
0.0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.1	0.15266681	0.15261477	0.00005203	−0.03957238	−0.03953400	0.00003838
0.2	0.30200493	0.30190458	0.00010035	−0.07661603	−0.07654220	0.00007383
0.3	0.44470125	0.44456001	0.00014124	−0.10861796	−0.10851455	0.00010340
0.4	0.57748176	0.57731069	0.00017107	−0.13310413	−0.13297987	0.00012425
0.5	0.69714877	0.69696245	0.00018632	−0.14767519	−0.14754153	0.00013366
0.6	0.80063401	0.80045027	0.00018374	−0.15005557	−0.14992643	0.00012914
0.7	0.88506598	0.88490551	0.00016047	−0.13815323	−0.13804462	0.00010861
0.8	0.94784823	0.94773361	0.00011462	−0.11012516	−0.11005419	0.00007097
0.9	0.98674644	0.98669724	0.00004920	−0.06444625	−0.06442561	0.00002064
1.0	1.00000000	1.00000001	−0.00000001	0.00000000	−0.00000001	−0.00000001

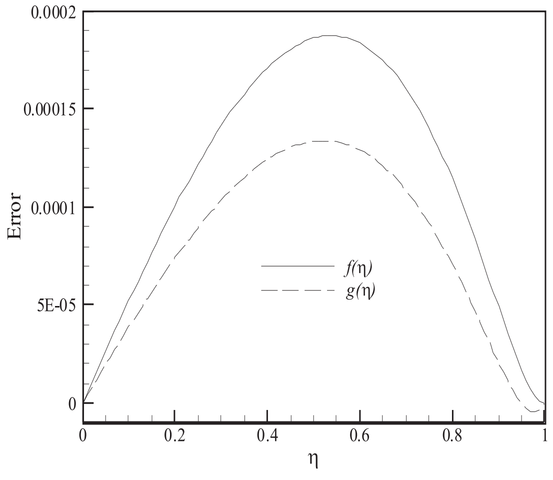

Fig. 1. The errors of the DTM solutions for $f(\eta)$ and $g(\eta)$, when $Re = -10$ and $N = 20$.

The rate of convergence for $f(\eta)$ and $g(\eta)$ for middle point of interval at some values of Re is shown in Table 2. The tabulated data indicates that to achieve comparible accuracy, larger N values are needed for higher mass injection rates. Fig. 2 shows the profiles of $f(\eta), f'(\eta)$ and $g(\eta)$ obtained using the DTM, along with the numerical solution using the fourth-order Runge–Kutta method. As noted earlier, we can see a very good agreement between the DTM and Runge–Kutta numerical results.

The effect of Reynolds number on the dimensionless velocity component in x-direction $f'(t)$ is considered in Figs. 3 and 4. These figures elucidate that increasing the magnitude of Reynolds number (i.e., increasing mass injection) increases the maximum value of this component.

Table 2. The rate of convergence for $f(\eta)$ and $g(\eta)$ for middle point of interval.

		$N=5$	$N=10$	$N=15$	$N=20$	Numerical
$Re=-0.1$	$f(\eta)$	0.69018775	0.69051036	0.69051065	0.69051065	0.69051065
	$g(\eta)$	−0.15661506	−0.15740800	−0.15740851	−0.15740851	−0.15740851
$Re=-1$	$f(\eta)$	0.69019366	0.69132989	0.69134072	0.69134071	0.69134071
	$g(\eta)$	−0.15541294	−0.15668788	−0.156697633	−0.15669762	−0.15669762
$Re=-5$	$f(\eta)$	0.69022096	0.69379912	0.69438107	0.69435187	0.69434988
	$g(\eta)$	−0.14986365	−0.15257933	−0.15300702	−0.15298466	−0.15298314
$Re=-10$	$f(\eta)$	0.69025759	0.69363233	0.69768648	0.69714877	0.69696245
	$g(\eta)$	−0.14241398	−0.14531757	−0.14805139	−0.14767519	−0.14754153

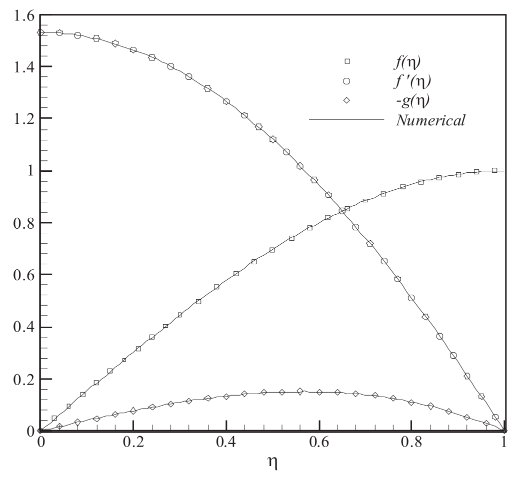

Fig. 2. The profiles $f(\eta), f'(\eta)$ and $g(\eta)$ obtained by the DTM in comparison with the numerical solution, when $Re=-10$.

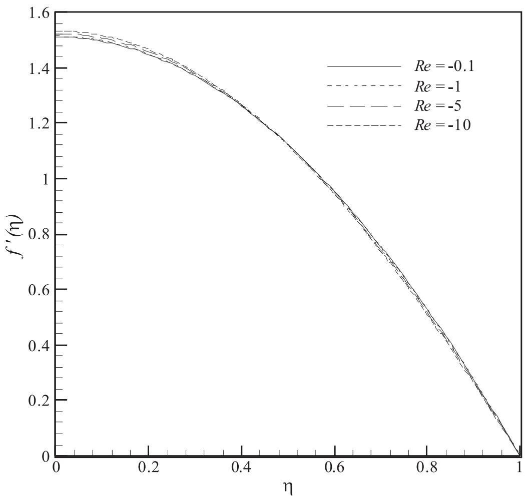

Fig. 3. The effect of Reynolds number on velocity component in x-direction.

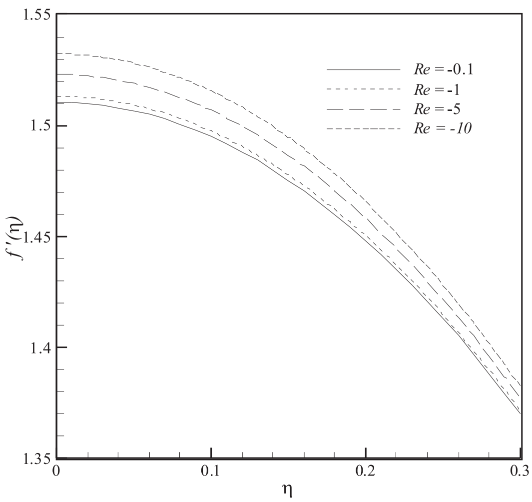

Fig. 4. The effect of Reynolds number on velocity component in x-direction (zoomed).

5 Conclusions

In this paper, the DTM was applied successfully to find the semi-analytical solution of the micro-polar flow in a porous channel with mass injection. Excellent agreement was noted between the DTM solutions and those obtained using a fourth order Runge–Kutta method. The results also show that the differential transform method does not require small parameters in the mathematical formulation, so one limitation of the traditional perturbation methods can be eliminated.

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