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Existence results for a class of (p, q) Laplacian systems

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Abstract. We establish the existence of a nontrivial solution for inhomogeneous quasilinear elliptic systems:

$$\begin{cases} -\Delta_p u = \lambda a(x)u|u|^{\gamma-2} + \frac{\alpha}{\alpha+\beta} b(x)u|u|^{\alpha-2}|v|^{\beta} + f & \text{in } \Omega, \\ -\Delta_q v = \mu d(x)v|v|^{\gamma-2} + \frac{\beta}{\alpha+\beta} b(x)|u|^{\alpha}v|v|^{\beta-2} + g & \text{in } \Omega, \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega). \end{cases}$$

Our result depending on the local minimization.

Keywords: elliptic systems, Nehari manifold, Ekeland variational principle, local minimization.

1 Introduction

In this paper we deal with the nonlinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda a(x)u|u|^{\gamma-2} + \frac{\alpha}{\alpha+\beta} b(x)u|u|^{\alpha-2}|v|^{\beta} + f & \text{in } \Omega, \\ -\Delta_q v = \mu d(x)v|v|^{\gamma-2} + \frac{\beta}{\alpha+\beta} b(x)|u|^{\alpha}v|v|^{\beta-2} + g & \text{in } \Omega, \\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega), \end{cases} \tag{1}$$

where $1 < p,\ q < N$ and Ω is a regular set of R^N, $N \geq 3$, $\alpha > 0$, $\beta > 0$, λ and μ are positive parameters, functions $a(x)$, $b(x)$ and $d(x) \in C(\overline{\Omega})$ are smooth functions with change sign on $\overline{\Omega}$, we assume here that $1 < \gamma < \min(p,q)$, $\gamma < \alpha+\beta$, $\alpha+\beta > \max(p,q)$ and $\alpha/p + \beta/q = 1$. For $p \geq 1$ $\Delta_p u$ is the p-Laplacian defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and $W_0^{1,p}(\Omega)$ is the closer of $C_0^{\infty}(\Omega)$ equipped by the norm $\|u\|_{1,p} := \|\nabla u\|_p$, where $\|.\|_p$ represent the norm of Lebesgue space $L^p(\Omega)$. The Lebesgue integral in Ω will be denote by the symbol $\int$ whenever the integration is carried out over all Ω.

Let p' be the conjugate to p, $W_0^{-1,p'}(\Omega)$ is the dual space to $W_0^{1,p}(\Omega)$ and we denote by $\|.\|_{-1,p'}$ its norm. We denote by $\langle x^*, x\rangle_{X^*,X}$ the natural duality paring between X and

X^*. The problem

$$\begin{cases} -\Delta_p u = u|u|^{\alpha-1}|v|^{\beta+1} + f & \text{in } \Omega, \\ -\Delta_q v = |u|^{\alpha+1}v|v|^{\beta-1} + g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain, $f \in D_0^{-1,p'}(\Omega)$, $g \in D_0^{-1,q'}(\Omega)$ has been studied in [1] for $p \neq q$ and in a recent paper [2] for $p \neq q$ on arbitrary domains with lack of compactness.

Let us define $X = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ equipped with the norm $\|(u,v)\|_X = \|u\|_{1,p} + \|v\|_{1,q}$ and $(X, \|.\|)$ is a reflexive and separable Banach space.

Definition 1 (**Weak solution**). We say that $(u,v) \in X$ is a weak solution of (1) if:

$$\begin{aligned} &\int |\nabla u|^{p-2}\nabla u.\nabla w_1 \,\mathrm{d}x \\ &\quad = \lambda \int a(x)u|u|^{\gamma-2}w_1 \,\mathrm{d}x + \frac{\alpha}{\alpha+\beta}\int b(x)u|u|^{\alpha-2}|v|^{\beta}w_1 \,\mathrm{d}x + \int f w_1 \,\mathrm{d}x, \\ &\int |\nabla v|^{q-2}\nabla v.\nabla w_2 \,\mathrm{d}x \\ &\quad = \mu \int \mathrm{d}(x)v|v|^{\gamma-2}w_2 \,\mathrm{d}x + \frac{\beta}{\alpha+\beta}\int b(x)|u|^{\alpha}v|v|^{\beta-2}w_2 \,\mathrm{d}x + \int g w_2 \,\mathrm{d}x. \end{aligned}$$

for all $(w_1, w_2) \in X$.

It is clear that problem (1) has a variational structure.

It is well known if the Euler function ϕ is bounded below and ϕ has a minimizer on X, then this minimizer is a critical point of ϕ. However, the Euler function $\phi(u,v)$, associated with the problem (1), is not bounded below on the whole space X, but is bounded on an appropriate subset, and has a minimizer on this set (if it exists) which gives rise to solution to (1). Clearly, the critical points of ϕ are the weak solutions of problem (1).

The associated Euler–Lagrange functional to system (1) $\phi\colon X \to R$ is defined by

$$\begin{aligned} \phi(u,v) = {}& \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q - \frac{1}{\gamma}\left[\lambda\int a(x)|u|^{\gamma} + \mu\int d(x)|v|^{\gamma}\right] \\ & - \frac{1}{\alpha+\beta}\int b(x)|u|^{\alpha}|v|^{\beta} - \langle f,u\rangle - \langle g,v\rangle. \end{aligned} \tag{2}$$

Consider the Nehari manifold associated to problem (1) given by

$$\Lambda = \big\{(u,v) \in X \setminus \{(0,0)\};\ \phi'(u,v)(u,v) = 0\big\}, \quad m_1 = \inf_{(u,v)\in\Lambda} J(u,v).$$

Consequently, for every $(u,v) \in \Lambda$, (2) becomes

$$\begin{aligned} \phi_{|\Lambda}(u,v) = {}& A(p)\|u\|_{1,p}^p + A(q)\|v\|_{1,q}^q - A(\gamma)\left[\lambda\int a(x)|u|^{\gamma} + \mu\int d(x)|v|^{\gamma}\right] \\ & - A(1)\langle f,u\rangle - A(1)\langle g,v\rangle, \end{aligned}$$

where for all $t > 0$, $A(t) = 1/t - 1/(\alpha + \beta)$.

We introduce the operators $J_1, J_2, D_1, D_2, B_1, B_2 \colon X \to X^*$ in the following way

$$\begin{aligned}
\langle J_1(u,v),(w,z)\rangle_X &:= \int |\nabla u|^{p-2}\nabla u \nabla w,\\
\langle J_2(u,v),(w,z)\rangle_X &:= \int |\nabla v|^{q-2}\nabla v \nabla z,\\
\langle D_1(u,v),(w,z)\rangle_X &:= \int a(x)|u|^{\gamma-2}uw,\\
\langle D_2(u,v),(w,z)\rangle_X &:= \int d(x)|v|^{\gamma-2}vz,\\
\langle B_1(u,v),(w,z)\rangle_X &:= \int b(x)|u|^{\alpha-2}|v|^{\beta}uw,\\
\langle B_2(u,v),(w,z)\rangle_X &:= \int b(x)|u|^{\alpha}|v|^{\beta-2}vz.
\end{aligned}$$

2 Main results

Our main result is the following:

Theorem 1. *Suppose that $(f, g) \in W_0^{-1,p'}(\Omega) \times W_0^{-1,q'}(\Omega)$, non of the functions f and g is identically to zero on Ω and*:

$$\text{(a) } 1 < \gamma < \min(p,q), \quad \text{(b) } \gamma < \alpha + \beta, \quad \text{(c) } \alpha + \beta > \max(p,q).$$

Then, there exists a pair $(u^*, v^*) \in \Lambda$ such that the sequence (u_n, v_n) converges strongly to (u^*, v^*) in X, Moreover, (u^*, v^*) is a solution of system (1) satisfies the property $\phi(u^*, v^*) < 0$.

Definition 2. We say that ϕ satisfies the Palais–Smale condition $(PS)_c$ if every sequence $(u_m, v_m) \subset X$ such that $\phi(u_m, v_m)$ is bounded and $\phi'(u_m, v_m) \to 0$ in X^* as $m \to \infty$, is relatively compact in X.

Lemma 1. *The operators J_i, D_i, B_i, $i = 1, 2$, are well defined. Also J_i, $i = 1, 2$, are continuous and the operators D_i, B_i, $i = 1, 2$, are compact.*

Proof. This lemma is proved in [3]. □

Lemma 2. *Let (u_n, v_n) be a bounded sequence in X such that $\phi(u_n, v_n)$ is bounded and $\phi'(u_n, v_n) \to 0$ as $n \to \infty$. Then (u_n, v_n) has a convergent subsequence.*

Proof. Since the sequence (u_n, v_n) is bounded in X, we may consider that there is a subsequence (denote again by (u_n, v_n)), which is weakly convergent in X.

Moreover, we have that

$$
\begin{aligned}
&\big\langle \phi'(u_n,v_n)-\phi'(u_m,v_m),(u_n-u_m,v_n-v_m)\big\rangle\\
&\quad=\int\big(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_m|^{p-2}\nabla u_m\big)(\nabla u_n-\nabla u_m)\\
&\qquad+\int\big(|\nabla v_n|^{q-2}\nabla v_n-|\nabla v_m|^{q-2}\nabla v_m\big)(\nabla v_n-\nabla v_m)\\
&\qquad-\lambda\int a(x)\big(|u_n|^{\gamma-2}u_n-|u_m|^{\gamma-2}u_m\big)(u_n-u_m)\\
&\qquad-\mu\int d(x)\big(|v_n|^{\gamma-2}v_n-|v_m|^{\gamma-2}v_m\big)(v_n-v_m)\\
&\qquad-\frac{\alpha}{\alpha+\beta}\int b(x)\big(|u_n|^{\alpha-2}|v_n|^{\beta}u_n-|u_m|^{\alpha-2}|v_m|^{\beta}u_m\big)(u_n-u_m)\\
&\qquad-\frac{\beta}{\alpha+\beta}\int b(x)\big(|u_n|^{\alpha}|v_n|^{\beta-2}v_n-|u_m|^{\alpha}|v_m|^{\beta-2}v_m\big)(v_n-v_m)\\
&\qquad-\int\big(f(x_n)-f(x_m)\big)(u_n-u_m)-\int\big(g(x_n)-g(x_m)\big)(v_n-v_m).
\end{aligned}
$$

Since (u_n,v_n) converges strongly in $L^p(\Omega)\times L^q(\Omega)$, it is a Cauchy sequence in $L^p(\Omega)\times L^q(\Omega)$. Using Holder inequality (since $\alpha/p+\beta/q=1$ and $(\alpha-1)/\alpha+1/\alpha=1$) we have

$$
\begin{aligned}
&\int b(x)|u_n|^{\alpha-2}|v_n|^{\beta}u_n(u_n-u_m)\\
&\quad\le\|b\|_\infty\int|u_n|^{\alpha-1}|v_n|^{\beta}|u_n-u_m|\\
&\quad\le\|b\|_\infty\bigg[\int\big(|u_n|^{\alpha-1}|u_n-u_m|\big)^{\frac{p}{\alpha}}\bigg]^{\frac{\alpha}{p}}\bigg[\int\big(|v_n|^{\beta}\big)^{\frac{q}{\beta}}\bigg]^{\frac{\beta}{q}}\\
&\quad\le\|b\|_\infty\bigg[|u_n|^{\frac{(\alpha-1)p}{\alpha}}|u_n-u_m|^{\frac{p}{\alpha}}\bigg]^{\frac{\alpha}{p}}\|v_n\|_q^{\beta}\\
&\quad\le\|b\|_\infty\bigg[\int\big(|u_n|^{\frac{(\alpha-1)p}{\alpha}}\big)^{\frac{\alpha}{\alpha-1}}\bigg]^{\frac{\alpha}{p}\times\frac{\alpha-1}{\alpha}}\bigg[\int|u_n-u_m|^{\frac{p}{\alpha}\times\alpha}\bigg]^{\frac{\alpha}{p}\times\frac{1}{\alpha}}\|v_n\|_q^{\beta}\\
&\quad=\|b\|_\infty\|u_n\|_p^{\alpha-1}\|u_n-u_m\|_p\|v_n\|_q^{\beta}\to 0.
\end{aligned}
$$

Similarly

$$
\int b(x)\big(|u_n|^{\alpha}|v_n|^{\beta-2}v_n-|u_m|^{\alpha}|v_m|^{\beta-2}v_m\big)(v_n-v_m)\to 0.
$$

From the compactness of the operators B_i, D_i $(i=1,2)$, [4], continuity of f and g, we obtain (passing to a subsequence, if necessary) that

$$\int \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\big)(\nabla u_n - \nabla u_m) + \int \big(|\nabla v_n|^{q-2}\nabla v_n - |\nabla v_m|^{q-2}\nabla v_m\big)(\nabla v_n - \nabla v_m) \to 0$$

which implies (see [5]) that (u_n, v_n) converges strongly in X. □

Lemma 3. *Let $c \in R$. Then the functional $\phi(u, v)$ satisfies the $(PS)_c$ condition.*

Proof. According to Lemma 2, it sufficient to prove that the sequence (u_n, v_n) is bounded in X. We have

Let (u_n, v_n) be such a sequence, that is

$$\phi(u_n, v_n) = c + o_n(1) \quad \text{and} \quad \phi'(u_n, v_n) = o_n\big(\|(u_n, v_n)\|_X\big),$$

then

$$\begin{aligned}
&\phi(u_n, v_n) - \frac{1}{\alpha+\beta}\big\langle \phi'(u_n, v_n), (u_n, v_n)\big\rangle \\
&\quad = A(p)\|u_n\|_{1,p}^p + A(q)\|v_n\|_{1,q}^q - A(\gamma)\left[\lambda \int a(x)|u_n|^\gamma + \mu \int b(x)|v_n|^\gamma\right] \\
&\qquad - A(1)\langle f, u_n\rangle - A(1)\langle g, v_n\rangle \\
&\quad = c + o_n\big(\|(u_n, v_n)\|_X\big) + o_n(1).
\end{aligned}$$

Using successively the Holder's inequality and the Young inequality on the terms $\langle f, u_n\rangle$ and $\langle g, v_n\rangle$, we can write

$$\begin{aligned}
&\left[A(p)\|u_n\|_{1,p}^p - \frac{A(1)}{p}\theta^p\|u_n\|_{1,p}^p - \lambda A(\gamma)\|u_n\|_{1,p}^\gamma\right] \\
&+ \left[A(q)\|v_n\|_{1,q}^q - \frac{A(1)}{q}\nu^q\|v_n\|_{1,q}^q - \mu A(\gamma)\|v_n\|_{1,q}^\gamma\right] \\
&\quad \le \frac{A(1)}{p'}\theta^{-p'}\|f\|_{-1,p'}^{p'} + \frac{A(1)}{q'}\nu^{-q'}\|g\|_{-1,q'}^{q'} + c + o_n\big(\|(u_n, v_n\|\big) + o_n(1).
\end{aligned}$$

Since the real numbers θ and ν being arbitrary, a suitable choose of θ and ν assure the boundedness of the sequence (u_n, v_n). □

Lemma 4. *The critical value of ϕ on Λ, $m_1 = \inf_{(u,v)\in\Lambda}\phi(u, v)$, has the following property*:

$$m_1 < \min\left[-\frac{\|f\|_{-1,p'}^{p'}}{p'}, -\frac{\|g\|_{-1,q'}^{q'}}{q'}\right].$$

Proof. Let u_f be the unique solution of the Dirichlet problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and let v_g be the unique solution of the problem

$$\begin{cases} -\Delta_q v = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

It is clear that $(u_f, 0), (0, v_g)$ are two elements of Λ and we have

$$m_1 \le \phi(u_f, 0) = \left[\frac{1}{p}\|\nabla u_f\|_p^p - \langle f, u_f\rangle\right] = -\left(1 - \frac{1}{p}\right)\|\nabla u_f\|_p^p = -\frac{1}{p'}\|\nabla u_f\|_p^p,$$
$$m_1 \le \phi(0, v_g) = \left[\frac{1}{q}\|\nabla v_g\|_q^q - \langle g, v_g\rangle\right] = -\left(1 - \frac{1}{q}\right)\|\nabla v_g\|_q^q = -\frac{1}{q'}\|\nabla v_g\|_q^q.$$

Similarly to proof of J. Velin [13, 4.2], we can show that

$$\|f\|_{-1,p'}^{p'} = \|\nabla u_f\|_p^p,$$
$$\|g\|_{-1,q'}^{q'} = \|\nabla v_g\|_q^q.$$

Then

$$m_1 \le \min\left[-\frac{1}{p'}\|f\|_{-1,p'}^{p'}, -\frac{1}{q'}\|g\|_{-1,q'}^{q'}\right].$$

Thus, the Lemma is proved. □

3 Proof of the Theorem 1

We show that ϕ is bounded below on Λ. Let (u, v) be an arbitrary element in Λ. We have

$$\phi_{|\Lambda}(u, v) \ge \left[A(p)\|u\|_{1,p}^p - \frac{A(1)}{p}\theta^p\|u\|_{1,p}^p\right] + \left[A(q)\|v\|_{1,q}^q - \frac{A(1)}{q}\nu^q\|v\|_{1,q}^q\right]$$
$$- \frac{A(1)}{p'}\theta^{-p'}\|f\|_{-1,p'}^{p'} + \frac{A(1)}{q'}\nu^{-q'}\|g\|_{-1,q'}^{q'}.$$

This inequality follows from $a(x), d(x)$ are sign chaining functions and we can choose $(u, v) \in X$ with these properties that $\sup u \subseteq \Omega_1 = \{x \in \Omega;\ a(x) < 0\}$ and $\sup v \subseteq \Omega_2 = \{x \in \Omega;\ d(x) < 0\}$.

We choose $\theta = \{pA(p)/A(1)\}^{1/p}$ and $\nu = \{qA(q)/A(1)\}^{1/q}$. Consequently, we have, for every $(u, v) \in \Lambda$

$$\phi(u, v) \ge -\frac{A(1)}{p'}\theta^{-p'}\|f\|_{-1,p'}^{p'} - \frac{A(1)}{q'}\nu^{-q'}\|g\|_{-1,q'}^{q'}.$$

Hence, we have shown that ϕ is bounded blow on Λ. Then Ekeland variational principle [6] imply the existence of a solution of (1), such that $\phi(u^*, v^*) < 0$.

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