Alternating-direction method for a mildly nonlinear elliptic equation with nonlocal integral conditions

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Received: 23 December 2010 / **Revised:** 23 March 2011 / **Published online:** 30 May 2011

Abstract. The present paper deals with a generalization of the alternating-direction implicit (ADI) method for the two-dimensional nonlinear Poisson equation in a rectangular domain with integral boundary condition in one coordinate direction. The analysis of results of computational experiments is presented.

Keywords: elliptic equation, nonlocal integral conditions, finite-difference method, alternating-direction method, convergence of iterative method.

1 Introduction

In this paper, we consider finite-difference approximations for the following nonlinear elliptic equation

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y, u), \quad (x, y) \in \Omega, \tag{1}$$

with Dirichlet boundary conditions in one coordinate direction

$$u(x, 0) = \mu^l(x), \quad u(x, L_y) = \mu^r(x), \quad x \in [0, L_x], \tag{2}$$

and with nonlocal integral conditions in another coordinate direction:

$$u(0, y) = \gamma_0 \int\limits_0^{L_x} u(x, y)\,\mathrm{d}x + \nu^l(y), \quad y \in [0, L_y], \tag{3}$$

$$u(L_x, y) = \gamma_1 \int\limits_0^{L_x} u(x, y)\,\mathrm{d}x + \nu^r(y), \quad y \in [0, L_y], \tag{4}$$

where $\Omega = (0, L_x) \times (0, L_y)$ is a rectangular domain, γ_0 and γ_1 are given constants.

The function $f(x, y, u)$ satisfies the condition

$$\frac{\partial f}{\partial u} \leqslant 0. \tag{5}$$

The investigation of methods for solving an equation, commonly referred to as mildly nonlinear, began long ago. One of the first articles where the finite difference method has been studied is [1].

The finite difference scheme of high-order accuracy for the stationary problem with a Dirichlet boundary condition was investigated in [2].

The purpose of this paper is to find numerical solution to this equation with special-type nonlocal conditions. Nonlocal boundary conditions (3)–(4), which can be called nonlocal conditions according to one variable, are that of the typical nonlocal conditions. Currently, they are intensively researched. Theoretical investigation of problems with different types of nonlocal boundary conditions is an actual problem, and recently much attention has been paid to them in the scientific literature.

Elliptic equation with integral conditions of another type that of (3)–(4) is the object of study in the works [3–12]. Various statements of different problems with nonlocal conditions and research methods can be found in [11, 13–27].

The remaining part of this paper is organized as follows. In Section 2, we formulate a difference problem and write the alternating-direction implicit method. In Sections 3 and 4, we present the analysis of convergence and the results of numerical experiments. Section 5 contains some brief conclusions and comments.

2 Statement of a difference problem. ADI method

In the domain $\overline{\Omega}$ we consider the grids:

$$\begin{aligned}
&\overline{\omega}_x^h := \{x_0 = 0,\ x_1,\ \ldots,\ x_n = L_x\}, \quad h_x = x_i - x_{i-1} = L_x/n,\\
&\overline{\omega}_y^h := \{y_0 = 0,\ y_1,\ \ldots,\ y_m = L_y\}, \quad h_y = y_j - y_{j-1} = L_y/m,\\
&\omega_x^h := \{x_1, \ldots, x_{n-1}\}, \quad \omega_y^h := \{y_1, \ldots, y_{m-1}\}.
\end{aligned}$$

In the closed domain $\overline{\Omega}$ we consider the rectangular grids $\overline{\omega}^h := \overline{\omega}_x^h \times \omega_y^h$, $\omega^h := \omega_x^h \times \omega_y^h$ and $\partial\omega^h := \overline{\omega}^h \setminus \omega^h$.

If ω is one of these grids, we define the space of grid functions $\mathcal{F}(\omega)$.

We introduce second order central difference operators δ_x^2 and δ_y^2:

$$\delta_x^2 u_{i\,j} := \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h_x^2}, \quad \delta_y^2 u_{i\,j} := \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h_y^2}.$$

The function f is approximated by grid function f_{ij} on the grid $\overline{\omega}^h$, functions ν^l, ν^r by ν_j^l, ν_j^r on the grid $\overline{\omega}_y^h$ and functions μ^l, μ^r by μ_i^l, μ_i^r on the grid $\overline{\omega}_x^h$.

Equations (1)–(4) are replaced with finite-difference equations:

$$-\left(\delta_x^2 + \delta_y^2\right)u_{ij} = f_{ij}(u_{ij}), \quad (x_i, y_j) \in \omega^h, \tag{6}$$

$$u_{0j} = \gamma_0 h_x \left(\frac{u_{0j} + u_{nj}}{2} + \sum_{i=1}^{n-1} u_{ij}\right) + \nu_j^l, \tag{7}$$

$$u_{nj} = \gamma_1 h_x \left(\frac{u_{0j} + u_{nj}}{2} + \sum_{i=1}^{n-1} u_{ij}\right) + \nu_j^r, \tag{8}$$

$$u_{i0} = \mu_i^l, \quad u_{im} = \mu_i^r, \quad x_i \in \overline{\omega}_x^h. \tag{9}$$

Now we write the Peaceman–Rachford alternating-direction implicit method [28] for the system (6)–(9) as follows:

$$\frac{u_{ij}^{k+1/2} - u_{ij}^k}{\tau_{k+1}} = \delta_x^2 u_{ij}^{k+1/2} + \delta_y^2 u_{ij}^k + f_{ij}\left(u_{ij}^k\right), \quad i = 1, \dots, n-1, \tag{10}$$

$$\frac{u_{ij}^{k+1} - u_{ij}^{k+1/2}}{\tau_{k+1}} = \delta_x^2 u_{ij}^{k+1/2} + \delta_y^2 u_{ij}^{k+1} + f_{ij}\left(u_{ij}^{k+1/2}\right), \quad j = 1, \dots, m-1, \tag{11}$$

where τ_{k+1} are parameters.

For each fixed value $i = 1, \dots, n-1$, we solve equation (11) with boundary conditions

$$u_{i0}^{k+1/2} = \mu_i^l, \quad u_{im}^{k+1/2} = \mu_i^r. \tag{12}$$

For each fixed value $j = 1, \dots, m-1$, we solve equation (10) with nonlocal boundary conditions

$$u_{0j}^{k+1} = \gamma_0 h_x \left(\frac{u_{0j}^{k+1} + u_{nj}^{k+1}}{2} + \sum_{i=1}^{n-1} u_{ij}^{k+1}\right) + \nu_j^l, \tag{13}$$

$$u_{nj}^{k+1} = \gamma_1 h_x \left(\frac{u_{0j}^{k+1} + u_{nj}^{k+1}}{2} + \sum_{i=1}^{n-1} u_{ij}^{k+1}\right) + \nu_j^r. \tag{14}$$

3 Analysis of the convergence

We investigate the convergence of the ADI method. Let us write the system of difference equations (6)–(9) in the matrix form. We consider two one-dimensional difference problems with nonlocal or homogeneous Dirichlet conditions

$$\frac{v_{i-1} - 2v_i + v_{i+1}}{h_x^2} = p_i, \quad i = 1, \dots, n-1, \tag{15}$$

$$v_0 = \gamma_0 h_x \left(\frac{v_0 + v_n}{2} + \sum_{i=1}^{n-1} v_i\right), \tag{16}$$

$$v_n = \gamma_1 h_x \left(\frac{v_0 + v_n}{2} + \sum_{i=1}^{n-1} v_i\right) \tag{17}$$

and

$$\frac{w_{j-1} - 2w_j + w_{j+1}}{h_x^2} = q_j, \quad j = 1, \dots, m-1, \tag{18}$$

$$w_0 = 0, \quad w_m = 0, \tag{19}$$

where p_i, $i = 1, \dots, n-1$ and q_j, $j = 1, \dots, m-1$ are given values.

Let us interpret Eqs. (16)–(17) for each fixed value $j = 1, \dots, m-1$ as a system of two equations with the unknown variables v_0, v_n. We express these variables in other unknown variables:

$$v_0 = a \sum_{i=1}^{n-1} v_i, \quad v_n = b \sum_{i=1}^{n-1} v_i, \tag{20}$$

$$a = \frac{\gamma_0 h_x}{D}, \quad b = \frac{\gamma_1 h_x}{D}, \quad D = 1 - \frac{(\gamma_0 + \gamma_1) h_x}{2}. \tag{21}$$

If h_x is small enough $h_x < 2/(\gamma_0 + \gamma_1)$, the determinant $D \neq 0$ and v_0, v_n are expressed by formulas (20)–(21) uniquely.

So we can rewrite (15)–(17) in the matrix form

$$\Lambda_x v = p, \tag{22}$$

where Λ_x is the $(n-1)$ order matrix

$$\Lambda_x = \frac{1}{h_x^2} \begin{pmatrix} -2+a & 1+a & a & \dots & a & a \\ 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -2 & 1 \\ b & b & b & \dots & 1+b & -2+b \end{pmatrix}. \tag{23}$$

Rewriting the system (18)–(19) in the form

$$\Lambda_y w = q, \tag{24}$$

we define Λ_y as an $(m-1)$-order tridiagonal matrix

$$\Lambda_y = \frac{1}{h_y^2} \begin{pmatrix} -2 & 1 & 0 & \dots & 0 & 0 \\ 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 1 & -2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -2 & 1 \\ 0 & 0 & 0 & \dots & 1 & -2 \end{pmatrix}. \tag{25}$$

Now we can define matrices A_1, A_2 and I of order $(n-1)(m-1)$ using the Kronecker (tensor) product of matrices:

$$A_1 = -I_{n-1} \otimes \Lambda_x, \quad A_2 = -\Lambda_y \otimes I_{m-1}, \quad I = I_{m-1} \otimes I_{n-1},$$

where I_k is the identity matrix of order k.

Now the iteration method (10)–(11) with boundary conditions (12)–(14) is equivalent to the following iteration method:

$$\big(I + \tau_{k+1} A_1\big) u^{k+1/2} = \big(I - \tau_{k+1} A_2\big) u^k + \tau_{k+1} f(u^k), \tag{26}$$

$$\big(I + \tau_{k+1} A_2\big) u^{k+1} = \big(I - \tau_{k+1} A_1\big) u^{k+1/2} + \tau_{k+1} f(u^{k+1/2}). \tag{27}$$

Let us define $u^* = \{u^*_{ij}\}$ as the exact solution of system (6)–(9) and

$$z^k = u^* - u^k. \tag{28}$$

According to (26)–(27) the following system of equations is true for the error z^k

$$\big(I + \tau_{k+1} A_1\big) z^{k+1/2} = \big(I - \tau_{k+1} A_2\big) z^k - \tau_{k+1} D_1 z^k, \tag{29}$$

$$\big(I + \tau_{k+1} A_2\big) z^{k+1} = \big(I - \tau_{k+1} A_1\big) z^{k+1/2} - \tau_{k+1} D_2 z^{k+1/2}, \tag{30}$$

where D_l, $l = 1, 2$ are diagonal matrices with diagonal elements

$$d_l = \big\{d^l_{ij}\big\} = -\partial f\big(\tilde{u}^l_{ij}\big)/\partial u, \quad l = 1, 2,$$

and $\tilde{u}^l$ is an intermediate point.

Let us indicate the basic properties of matrices Λ_x, Λ_y and A_1, A_2:

1. Λ_y is a symmetric matrix. All the eigenvalues of Λ_y are positive and distinct [29]. The eigenvalues of the matrix Λ_y are given by (see, [29]):

$$\lambda_j = \frac{4}{h_y^2} \sin^2 \frac{\pi j h_y}{2}, \quad j = 1, \dots, m-1. \tag{31}$$

2. Λ_x is a nonsymmetric matrix (it becomes symmetric iff $\gamma_0 = \gamma_1 = 0$, namely, if there are no nonlocal conditions). Its eigenvalues are given in [30].

   - If $\gamma_0 + \gamma_1 = 0$, then there exists one single eigenvalue $\lambda = 0$ and all the other remaining eigenvalues are positive.
   - If $\gamma_0 + \gamma_1 > 2$ and $h_x < 2/(\gamma_0 + \gamma_1)$, then there exists one single eigenvalue $\lambda < 0$

   $$\lambda = -\frac{4}{h_x^2} \sinh^2 \frac{\beta h_x}{2},$$

   where β is the unique positive root of the equation

   $$\tanh \frac{\beta}{2} - \frac{2}{h_x(\gamma_0 + \gamma_1)} \tanh \frac{\beta h_x}{2} = 0, \tag{32}$$

   and all the other eigenvalues are positive.

- If $\gamma_0 + \gamma_1 < 2$, then all eigenvalues are positive

$$\lambda_i = \frac{4}{h_x^2} \sinh^2 \frac{\alpha_i h_x}{2}, \quad i = 1, \dots, n-1,$$

where some of α_i doesn't depend on γ_0 and γ_1, i.e.,

$$\alpha_i = 2i\pi, \quad i = 1, \dots, \left[\frac{n-1}{2}\right], \tag{33}$$

and the other α_i are the roots of the equation

$$\tan \frac{\alpha}{2} - \frac{2}{h_x(\gamma_0 + \gamma_1)} \tan \frac{\alpha h_x}{2} = 0 \tag{34}$$

in the interval $(0, n\pi)$.

3. With all γ_0, γ_1 values the matrices A_1 and A_2 are commutative [9]

$$A_1 A_2 = A_2 A_1 = -\Lambda_y \otimes \Lambda_x. \tag{35}$$

4. With all γ_0, γ_1 values the matrices Λ_x, Λ_y are of simple structure. Therefore the matrices $A_1, A_2, A_1 + A_2, A_1 A_2, A_2 A_1$ have the same system of eigenvectors [9].

Let us now write the iteration method (29)–(30) as a matrix equation:

$$z^{k+1} = S z^k, \tag{36}$$

where

$$S = (I + \tau_{k+1} A_2)^{-1} \big(I - \tau_{k+1}(A_1 + D_2)\big)(I + \tau_{k+1} A_1)^{-1} \big(I - \tau_{k+1}(A_2 + D_1)\big). \tag{37}$$

Theorem 1. *If $\gamma_0 + \gamma_1 < 2$ and $\tau_{k+1} > 0$ are small enough numbers, then the iterative method* (10)–(11) *is convergent.*

Proof. In order to prove the convergence of the iterative method (10)–(11), it suffices to prove that $\|z^k\| \to 0$ as $k \to \infty$.

Firstly, we consider the case $f(x, y, u) = -Cu$, where $C \geqslant 0$ is constant. Then D_1 and D_2 are diagonal matrices with element C on the diagonal. So we see that all the four factors in the expression of matrix S (37) have the same system of eigenvectors. Thus,

$$\lambda(S) = \frac{(1 - \tau_{k+1}(\lambda(A_1) + C))(1 - \tau_{k+1}(\lambda(A_2) + C))}{(1 + \tau_{k+1}\lambda(A_1))(1 + \tau_{k+1}\lambda(A_2))}. \tag{38}$$

If $\gamma_0 + \gamma_1 < 2$, then $\lambda(A_1) > 0$, $\lambda(A_2) > 0$. Therefore

$$\big|\lambda(S)\big| < 1 \tag{39}$$

with $\tau_{k+1} > 0$ sufficiently small, namely,

$$\tau_{k+1} < \frac{2}{C}. \tag{40}$$

Now we consider $f(x, y, u) \neq -Cu$, but $\partial f/\partial u \leqslant 0$. Since the eigenvalues of any matrix are continuous functions of elements of the matrices, the inequalities $\lambda(A_1) > 0$, $\lambda(A_2) > 0$, $|\lambda(S)| < 1$ are true for $D_1 = D_2 = 0$, hence there exists such a number $\tau_0 > 0$ that inequality (39) is true for all $\tau_{k+1} \in (0, \tau_s]$. The theorem is proved. □

In practice, it is important to know what value τ_0 takes and how fast the iterative method (10)–(11) converges.

These questions are still uninvestigated theoretically. In the next section, we partially answer these questions using computer simulation methods.

Let us denote the smallest and the largest eigenvalues of the matrices A_1, A_2 by $\delta_1, \Delta_1, \delta_2, \Delta_2$. From the expressions of the eigenvalues of A_1, A_2 we obtain

$$\delta_1 = \frac{4}{h_x^2} \min_k \sin^2 \frac{\alpha_k h_x}{2}, \quad \Delta_1 = \frac{4}{h_x^2} \max_k \sin \frac{\alpha_k h_x}{2},$$
$$\delta_2 = \frac{4}{h_y^2} \sin^2 \frac{\pi h_y}{2}, \qquad \Delta_2 = \frac{4}{h_y^2} \cos^2 \frac{\pi h_y}{2}.$$

4 Numerical experiment

We consider a model problem (1)–(4) [31] in a unit square domain $[0, 1] \times [0, 1]$.

The right-hand side (RHS) function $f(x, y, u)$ is given by

$$f(x, y, u) = \frac{\pi^2}{4} u(1 - u) + g(x, y), \tag{41}$$

where

$$g(x, y) = 2 \sin\left(\frac{\pi}{2} y\right) + \frac{\pi^2}{4} \left(1 - x^2\right)^2 \sin^2\left(\frac{\pi}{2} y\right). \tag{42}$$

The exact solution to this test problem is given by

$$u(x, y) = \left(1 - x^2\right) \sin\left(\frac{\pi}{2} y\right). \tag{43}$$

The initial and boundary conditions were prescribed to satisfy the exact solution (43).

We consider uniform grids with different mesh sizes $h = h_x = h_y$ and analyze the convergence and accuracy of the computed solution from the present ADI scheme. We compute the maximum norm of the error of the numerical solution with respect to the exact solution, which is defined as

$$\varepsilon_h = \max_{j=1,\cdots,m} \max_{i=1,\cdots,n} \left|u(x_i, y_j) - u_{ij}\right|.$$

We define the number p as

$$p = \frac{\varepsilon_{2h}}{\varepsilon_h},$$

which theoretically must be approximately $p \approx 4$.

The results of the numerical test are listed in Table 1. Note that inequality (5) in the neighborhood of the point $x = 0$, $y = 1$ is not satisfied.

Table 1. The errors for different γ_0, γ_1 in the case of the RHS function (41).

γ_0	γ_1	h	ε_h	p	number of iter.
0.0	0.0	0.25	$1.08749 \cdot 10^{-3}$		10
		0.125	$2.90549 \cdot 10^{-4}$	3.7429	15
		0.06250	$7.35617 \cdot 10^{-5}$	3.9497	19
		0.03125	$1.85111 \cdot 10^{-5}$	3.9736	24
1.0	-1.0	0.25	$1.23522 \cdot 10^{-3}$		10
		0.125	$3.23759 \cdot 10^{-4}$	3.8152	15
		0.06250	$8.25150 \cdot 10^{-5}$	3.9236	19
		0.03125	$2.07282 \cdot 10^{-5}$	3.9808	24
1.0	1.0	0.25	$2.00268 \cdot 10^{-3}$		12
		0.125	$4.95807 \cdot 10^{-4}$	4.0392	17
		0.06250	$1.25950 \cdot 10^{-4}$	3.9365	21
		0.03125	$3.80358 \cdot 10^{-5}$	3.3114	26
10.0	-10.0	0.25	$6.69128 \cdot 10^{-3}$		10
		0.125	$1.72136 \cdot 10^{-3}$	3.8872	15
		0.06250	$4.34652 \cdot 10^{-4}$	3.9603	19
		0.03125	$4.35843 \cdot 10^{-4}$	0.9973	24

In the second test problem, we choose $f(x, y, u)$ as

$$f(x, y, u) = -Cu + \sin\left(\frac{\pi}{2}y\right)\left(2 + \left(1 - x^2\right)\frac{\pi^2}{4}\right) + C\left(1 - x^2\right)\sin\left(\frac{\pi}{2}y\right). \tag{44}$$

The exact solution to this test problem is given by (43).

Table 2 presents the performance of the algorithm for various values of constant C. Note that for large values of $|\gamma_0|$, $|\gamma_1|$ the error increases. The function (41) holds the condition (5) only in the part of the domain.

Table 2. The errors for different γ_0, γ_1 and C in the case of the RHS function (44).

γ_0	γ_1	h	ε_h	p	number of iter.
$C = 1$					
0.0	0.0	0.25	$1.03612 \cdot 10^{-3}$		10
		0.125	$2.84430 \cdot 10^{-4}$	3.6428	15
		0.06250	$7.15912 \cdot 10^{-5}$	3.9730	19
		0.03125	$1.80151 \cdot 10^{-5}$	3.9740	24

γ_0	γ_1	h	ε_h	p	number of iter.
1.0	1.0	0.25	$1.85253 \cdot 10^{-3}$		12
		0.125	$4.63401 \cdot 10^{-4}$	3.9977	17
		0.06250	$1.17050 \cdot 10^{-4}$	3.9590	21
		0.03125	$2.92500 \cdot 10^{-5}$	4.0017	26
1.5	1.5	0.25	$8.88091 \cdot 10^{-3}$		16
		0.125	$1.89226 \cdot 10^{-3}$	4.0392	21
		0.06250	$4.56618 \cdot 10^{-4}$	3.9365	25
		0.03125	$1.13603 \cdot 10^{-4}$	3.3114	29
10.0	-10.0	0.25	$6.99341 \cdot 10^{-3}$		10
		0.125	$1.86917 \cdot 10^{-3}$	3.7415	15
		0.06250	$4.70228 \cdot 10^{-4}$	3.9750	19
		0.03125	$1.17831 \cdot 10^{-4}$	3.9907	24
$C = 20$					
1.0	1.0	0.25	$6.79973 \cdot 10^{-4}$		12
		0.125	$1.85998 \cdot 10^{-4}$	3.6558	17
		0.06250	$4.58929 \cdot 10^{-5}$	4.0529	21
		0.03125	$0.03125 \cdot 10^{-5}$	3.9516	26
$C = 40$					
1.0	1.0	0.25	$3.93329 \cdot 10^{-2}$		12
		0.125	$4.35797 \cdot 10^{-3}$	9.0255	17
		0.06250	$2.37505 \cdot 10^{-2}$	0.18349	21
		0.03125	$7.47679 \cdot 10^{-3}$	3.1766	26

In both cases the set of optimal iterative parameters of the ADI method was chosen according to the monograph [29] where symmetric matrices of an iterative process are used.

5 Conclusions and remarks

The ADI method can be used for a mildly nonlinear Poisson equation. Nonlocal integral conditions with $\gamma_0 + \gamma_1 < 2$ never cause more problems than the classical conditions both in the number of iterations and precision of the solution. But these conditions affect the region of convergence of the method. The convergence domain depends essentially on the coefficients of nonlocality. The values of parameters γ_0 and γ_1 in nonlocal boundary conditions are essential for the stability of the ADI method. The results of the numerical experiment are in good agreement with the existing theoretical results for a two-dimensional Poisson equation in a rectangle domain with an integral boundary condition in one coordinate direction [10].

References

1. L. Bers, On mildly nonlinear partial difference equations of elliptic type, *J. Res. Natl. Bur. Stand.*, **51**, pp. 229–236, 1953.

2. M. Sapagovas, R. Skirmantas. Estimation of the solution error of high order accuracy symmetrized difference schemes, *Liet. Mat. Rink.*, **26**(1), pp. 113–118, 1986 (in Russian).

3. G.K. Berikelashvili, On the convergence rate of the finite-difference solution of a nonlocal boundary value problem for a second-order elliptic equation, *Differ. Equ.*, **39**(7), pp. 945–953, 2003.

4. A.V. Bitsadze, *Some classes of partial differential equations*, Gordon & Breach, New York, 1988.

5. V.A. Il'in, E.I. Moiseev, Two-dimensional nonlocal boundary value problem for Poisson's operator in differential and difference variants, *Math. Model.*, **2**(8), pp. 132–156, 1990 (in Russian).

6. C.V. Pao, Numerical solutions of reaction-diffusion equations with nonlocal boundary conditions, *J. Comput. Appl. Math.*, **136**, pp. 227–243, 2001.

7. M.P. Sapagovas, Difference scheme for two-dimensional elliptic problem with an integral condition, *Lith. Math. J.*, **23**(3), pp. 155–159, 1983.

8. M.P. Sapagovas, Numerical methods for two-dimensional problem with nonlocal conditions, *Differ. Equ.*, **20**(7), pp. 1258–1266, 1984.

9. M.P. Sapagovas, Difference method on increased order of accuracy for the Poisson equation with nonlocal conditions, *Differ. Equ.*, **44**(7), pp. 988–998, 2008.

10. M. Sapagovas, O. Štikonienė, A fourth-order alternating direction method for difference schemes with nonlocal condition, *Lith. Math. J.*, **49**(3), pp. 309–317, 2009.

11. O. Štikonienė, Numerical investigation of fourth-order alternating direction method for Poisson equation with integral conditions, in: *Proceeding of International Conference on Differential Equations and Their Applications, DETA'2009*, pp. 139–146, Kaunas, 2009.

12. Y. Wang, Solutions to nonlinear elliptic equations with a nonlocal boundary condition, *Electron. J. Differ. Equ.*, **2002**(2), pp. 1–16, 2002.

13. B.I. Bandyrskii, I. Lazurchak, V.L. Makarov, M. Sapagovas, Eigenvalue problem for the second order differential equation with nonlocal conditions, *Nonlinear Anal. Model. Control*, **11**(1), pp. 13–32, 2006.

14. A. Bouziani, Solvability of a nonlinear pseudoparabolic equation with nonlocal boundary conditions, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **55**, pp. 883–904, 2003.

15. R. Čiegis, Parallel numerical algorithms for 3D parabolic problem with a non-local boundary condition, *Informatica*, **17**(3), pp. 309–324, 2006.

16. M. Dehghan, A new ADI technique for two-dimensional parabolic equation with an integral condition, *Comput. Math. Appl.*, **43**(12), pp. 1477–1488, 2002.

17. D. Gordeziani, G. Avalishvili, Investigation of the nonlocal initial boundary value problems for some hyperbolic equations, *Hiroshima Math. J.*, **31**(3), pp. 345–366, 2001.

18. A.V. Gulin, N.I. Ionkin, V.A. Morozova, Stability of a nonlocal two-dimensional finite-difference problem, *Diff. Equ.*, **37**(7), pp. 960–978, 2001.
19. F. Ivanauskas, T. Meškauskas, M. Sapagovas, Stability of difference schemes for two-dimensional parabolic equations with non-local boundary conditions, *Appl. Math. Comput.*, **215**(7), pp. 2716–2732, 2009.
20. T. Jankowski, Solvability of three point boundary value problems for second order ordinary differential equations with deviating arguments, *J. Math. Anal. Appl.*, **312**, pp. 620–636, 2005.
21. A.M. Nakhushev, *Equations of Mathematical Biology*, Vysshaya Shkola, Moscow, 1995 (in Russian).
22. S. Pečiulytė, O. Štikonienė, A. Štikonas, Investigation of negative critical points of the characteristic function for problems with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **13**(4), pp. 467–490, 2008.
23. M.P. Sapagovas, G. Kairytė, O. Štikonienė, A. Štikonas, Alternating direction method for a two-dimensional parabolic equation with a nonlocal boundary condition, *Math. Model. Anal.*, **12**(1), pp. 131–142, 2007.
24. M.P. Sapagovas, A.D. Štikonas, On the structure of the spectrum of a differential operator with a nonlocal condition, *Differ. Equ.*, **41**(7), pp. 961–969, 2005.
25. K. Schuegerl, *Bioreaction Engineering. Reactions Involving Microorganisms and Cells*, Vol. 1, John Wiley & Sons, Chister, New York, Brisbane, Toronto, Singapore, 1987.
26. N. Sergejeva, Fucik spectrum for the second order bvp with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **12**(3), pp. 419–429, 2007.
27. J.R.L. Webb, G. Infante, Positive solutions of nonlinear boundary value problems involving integral boundary conditions, *NoDEA, Nonlinear Differ. Equ. Appl.*, **15**, pp. 45–67, 2008.
28. D. Peaceman, J.H.H. Rachford, The numerical solution of parabolic and elliptic differential equations, *J. Soc. Ind. Appl. Math.*, **1**(3), pp. 28–41, 1955.
29. A.A. Samarskii, *The Theory of Difference Schemes*, Moscow, Nauka, 1977 (in Russian); English transl.: pp. Marcel Dekker, Inc., New York, Basel, 2001.
30. M. Sapagovas, On the stability of finite-difference schemes for one-dimensional parabolic equations subject to integral conditions, *Zh. Obchysl. Prykl. Mat.*, **92**(1), pp. 77–90, 2005.
31. H. Y. Hu, J. S. Chen, Radial basis collocation method and quasi-newton iteration for nonlinear elliptic problems, *Numer. Methods Partial Differ. Equations*, **24**, pp. 991–1017, 2008.