A seventh order numerical method for singular perturbed differential-difference equations with negative shift

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Abstract. In this paper, a seventh order numerical method is presented for solving singularly perturbed differential-difference equations with negative shift. In recent papers the term negative shift has been used for delay. Such problems are associated with expected first exit time problem of the membrane, potential in models for neuron and in variational problems in control theory. In the numerical treatment for such type of boundary value problems, we first use Taylor approximation to tackle terms containing small shifts which converts into a singularly perturbed boundary value problem. This two point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a seventh order compact difference scheme is employed for the first order system and solved by using the boundary conditions. Several numerical examples are solved and compared with exact solution. We also present least square errors, maximum errors and observed that the present method approximates the exact solution very well.

Keywords: singular perturbation problems, differential-difference equations, negative shift, boundary layer, seventh order numerical method.

1 Introduction

The boundary value problems for singularly perturbed differential-difference equations arise in various practical problems in biomechanics and physics such as in variation problems in control theory and depolarization in Stein's model. For analytical discussion on delay differential equation one can refer Driver [1]. The depolarization in Stein's model [2] is continuous time,continuous state space, Markov process whose sample paths have discontinuities of first kind. Lange and Miura [3, 4] gave an asymptotic approach in study of class of boundary-value problems for linear second- order differential-difference equations in which the highest order derivative is multiplied by small parameter. In [5] M.K. Kadalbajoo and K.K. Sharma presented a numerical method to solve boundary value problems for singularly perturbed differential difference equations of mixed type, i.e., containing both terms having a negative shift and terms having a positive shift. In [6] they presented a numerical method to solve boundary value problems for singularly perturbed differential–difference equations with negative shift. In [7], they described a numerical approach based on finite difference method to solve a mathematical model arising from a model of neuronal variability. The mathematical modeling of the determination of the expected time for generation of action potentials in nerve cells by random synaptic inputs in dendrites includes a general boundary value problem for singularly perturbed differential difference equation with small shifts. In [8] Kailash C. Patidar and Kapil K. Sharma presented non-standard finite difference methods for second order, linear, singularly perturbed differential-difference equations. The non-standard finite difference methods are ε-uniformly convergent.

The objective of this paper is to describe the seventh order numerical method to the boundary value problems for singularly perturbed differential difference equations with negative shift. In this method, we approximate the shifted term by Taylor series and apply a difference scheme, provided shifts are of $o(\varepsilon)$. The two point boundary value problem is transformed into general first order ordinary differential equation system. A discrete approximation of a seventh order compact difference scheme is employed for the first order system and is solved using the boundary conditions. Several numerical examples are solved and compared with exact solution. It is observed that the present method approximates the exact solution very well.

2 Seventh order numerical method

We consider the boundary value problem for a singularly perturbed differential-difference equation, which contains only negative shift in the differentiated term

$$\varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad x \in (0,1), \tag{1}$$

under the boundary conditions

$$\begin{aligned} &y(x) = \phi(x) \quad \text{on } -\delta \le x \le 0,\\ &y(1) = \gamma, \end{aligned} \tag{2}$$

where ε is small parameter, $0 < \varepsilon \ll 1$ and δ is also a small shifting parameter, $0 < \delta \ll 1$, $a(x)$, $b(x)$, $f(x)$, $\delta(\varepsilon)$ and $\phi(x)$ are smooth functions and γ is a constant. Now there are two cases according to the sign of $a(x)$. If $a(x) \ge M > 0$ throughout the interval $[0,1]$, where M is a positive constant, then boundary layer will be in the neighborhood of 0, i.e., on the left side of the interval $[0,1]$. If $a(x) \le M < 0$ throughout the interval $[0,1]$, then the boundary layer will be in the neighborhood of 1, i.e., on the right side of the interval $[0,1]$.

Since the solution $y(x)$ of boundary value problem (1) and (2) is sufficiently differentiable, so we expand the retarded term $y'(x-\delta)$ by Taylor series, we obtain

$$\begin{aligned} y'(x-\delta) &\approx y'(x) - \delta y''(x),\\ y(0) = \phi_0&, \quad y(1) = \gamma. \end{aligned} \tag{3}$$

Using (3) in (1), we obtain

$$\big(\varepsilon - \delta a(x)\big)y''(x) + a(x)y'(x) + b(x)y(x) = f(x). \tag{4}$$

Equation (4) is a second order singular perturbation problem.

We solve (4) subject to the boundary conditions (3) by using seventh order numerical method described as follows:

We consider the second order linear differential equation

$$y''(x) + \tilde{a}(x)y'(x) + \tilde{b}(x)y(x) = \tilde{f}(x) \tag{5}$$

with the boundary conditions

$$y(0) = \alpha \quad \text{and} \quad y(1) = \beta.$$

The first order linear system corresponding to the above BVP is

$$Y' = A(x)Y + R(x), \quad x \in [a, b], \tag{6}$$

with the boundary conditions $B_1Y(0) + B_2Y(1) = D$, where A, B_1 and B_2 are 2×2 matrices. Y, R, D are two dimensional vectors.

Now we divide the interval $[0, 1]$ into N equal parts with constant mesh length H. Let $0 = x_0,\ x_1,\ x_2, \ldots, x_N = 1$ be the mesh points. Again we divide each subinterval $[x_i, x_{i+1}]$ into six equal smaller sub intervals. Let $t_1,\ t_2, \ldots, t_7$ are the grids in the subinterval $[x_i, x_{i+1}]$ and corresponding values of the variables and its derivatives are $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7$ and $Y_1', Y_2', Y_3', Y_4', Y_5', Y_6', Y_7'$.

By considering Taylor's expansions of $Y_1, Y_2, Y_3, Y_5, Y_6, Y_7$ at the fractional grid t_4, (Dianyun Peng [9]) we have,

$$\frac{h^{n+1}}{(n+1)!}Y_4^{(n+1)} = \sum_{j=1}^{7} a_j^n Y_j + a_8^n Y_4' + O\big(h^8 Y_4^{(6)}\big), \quad n = 1, 2, 3, 4, 5, 6, \tag{7}$$

where $h = \frac{x_{i+1} - x_i}{6}$ and the coefficients a_j^n are given by:

$$\begin{aligned}
&a_3^1 = a_5^1 = \frac{3}{4}, \quad a_4^1 = -\frac{49}{36}, \quad a_8^1 = 0, \quad a_1^1 = a_7^1 = \frac{1}{180}, \quad a_2^1 = a_6^1 = -\frac{3}{40},\\
&a_1^2 = -a_7^2 = -\frac{1}{540}, \quad a_2^2 = -a_6^2 = \frac{3}{80}, \quad a_3^2 = -a_5^2 = -\frac{3}{4}, \quad a_4^2 = 0, \quad a_8^2 = -\frac{49}{36},\\
&a_1^3 = a_7^3 = -\frac{1}{144}, \quad a_2^3 = a_6^3 = \frac{1}{12}, \quad a_3^3 = a_5^3 = -\frac{13}{48}, \quad a_4^3 = \frac{7}{18}, \quad a_8^1 = 0,\\
&a_1^4 = a_7^4 = -\frac{1}{432}, \quad a_2^4 = -a_6^4 = -\frac{1}{24}, \quad a_3^4 = -a_5^4 = \frac{13}{48}, \quad a_4^4 = 0, \quad a_8^4 = \frac{7}{18},\\
&a_1^5 = a_7^5 = \frac{1}{720}, \quad a_2^5 = a_6^5 = -\frac{1}{120}, \quad a_3^5 = a_5^5 = \frac{1}{48}, \quad a_4^5 = -\frac{1}{36}, \quad a_8^5 = 0,\\
&a_1^6 = -a_7^6 = \frac{1}{2160}, \quad a_2^6 = -a_6^6 = \frac{1}{240}, \quad a_3^6 = -a_5^6 = -\frac{1}{48}, \quad a_4^6 = 0, \quad a_8^6 = -\frac{1}{36}.
\end{aligned} \tag{8}$$

By taking the Taylor's series expansions of Y_1', Y_2', Y_3', Y_5', Y_6', Y_7' at the grid point t_4 and substituting (7), we get

$$Y_k' = \frac{1}{h}\sum_{j=1}^{7} b_j^k Y_j + b_8^k Y_4' + O\big(h^7 Y_4^{(6)}\big) \quad \text{for } k = 1,2,3,5,6,7. \tag{9}$$

$$\begin{aligned}
b_j^1 &= -6a_j^1 + 27a_j^2 - 108a_j^3 + 405a_j^4 - 1458a_j^5 + 5103a_j^6 + \mathrm{Sgn}(j-8),\\
b_j^2 &= -4a_j^1 + 12a_j^2 - 32a_j^3 + 80a_j^4 - 192a_j^5 + 448a_j^6 + \mathrm{Sgn}(j-8),\\
b_j^3 &= -2a_j^1 + 3a_j^2 - 4a_j^3 + 5a_j^4 - 6a_j^5 + 7a_j^6 + \mathrm{Sgn}(j-8),\\
b_j^5 &= 2a_j^1 + 3a_j^2 + 4a_j^3 + 5a_j^4 + 6a_j^5 + 7a_j^6 + \mathrm{Sgn}(j-8),\\
b_j^6 &= 4a_j^1 + 12a_j^2 + 32a_j^3 + 80a_j^4 + 192a_j^5 + 448a_j^6 + \mathrm{Sgn}(j-8),\\
b_j^7 &= 6a_j^1 + 27a_j^2 + 108a_j^3 + 405a_j^4 + 1458a_j^5 + 5103a_j^6 + \mathrm{Sgn}(j-8),
\end{aligned}$$

$$\mathrm{Sgn}(x) = \begin{cases} 1, & x \ge 0,\\ 0, & x < 0. \end{cases}$$

The variable Y and its derivative Y'at grids $t_1, t_2, \ldots, t_7$ subject to equations

$$Y_j' = A_j Y_j + R_j, \quad j = 1,2,3,4,5,6,7, \tag{10}$$

where A_j and R_j are values of A and R at grids t_j. Substituting (10) in (9), we get six linear algebraic equations with respect to seven unknown variables Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7.

By eliminating Y_2, Y_3, Y_4, Y_5, Y_6 from the above equations a relation between Y_1 and Y_7 can be obtained as follows:

$$\frac{1}{h}S_i Y_i + \frac{1}{h}T_i Y_{i+1} = F_i \quad \text{for } i = 1,2,\ldots,N-1, \tag{11}$$

where S_i and T_i are 2×2 matrices and F_i is a two dimensional vector.

By assuming

$$\begin{aligned}
c_1 &= b_5^7 b_3^1 - b_5^1 b_3^7,\\
c_2 &= \big(b_5^7 b_3^1 - b_5^1 b_3^7\big)/c_1, \quad c_3 = \big(b_2^7 b_5^1 - b_2^1 b_5^7\big)/c_1,\\
c_4 &= \big(b_6^1 b_3^7 - b_6^7 b_3^1\big)/c_1, \quad c_5 = \big(b_2^1 b_3^7 - b_2^7 b_3^1\big)/c_1,\\
W_1 &= \big(\big(b_7^7 b_5^1 - b_7^1 b_5^7\big)I - h b_5^1 A_1\big)/c_1,\\
W_2 &= \big(\big(b_1^7 b_5^1 - b_1^1 b_5^7\big)I + h b_5^7 A_1\big)/c_1,\\
W_2 &= \big(\big(b_3^5 b_4^1 - b_3^1 b_4^5\big)I + h\big(b_6^5 b_4^1 - b_6^1 b_4^5\big)A_3\big)/c_1,\\
W_3 &= \big(\big(b_4^7 b_5^1 - b_4^1 b_5^7\big)I + h\big(b_8^7 b_5^1 - b_8^1 b_5^7\big)A_4\big)/c_1,\\
G_1 &= \big(b_5^7 R_1 - b_5^1 R_7 + \big(b_8^7 b_5^1 - b_8^1 b_5^7\big)R_4\big)/c_1,\\
W_4 &= \big(\big(b_7^1 b_3^7 - b_7^7 b_3^1\big)I + h b_3^1 A_7\big)/c_1,\\
W_5 &= \big(\big(b_1^1 b_3^7 - b_1^7 b_3^1\big)I - h b_3^7 A_1\big)/c_1,\\
W_6 &= \big(\big(b_4^1 b_3^7 - b_4^7 b_3^1\big)I + h\big(b_8^1 b_3^7 - b_8^7 b_3^1\big)A_4\big)/c_1,
\end{aligned}$$

$$
\begin{aligned}
G_2 &= \left(b_3^1 R_7 - b_3^7 R_1 + \left(b_3^7 b_8^1 - b_3^1 b_8^7\right) R_4\right)/c_1,\\
c_6 &= b_2^6 + b_3^6 c_3 + b_5^6 c_5,\\
W_7 &= b_7^6 I + b_3^6 W_1 + b_5^6 W_4,\\
W_8 &= b_1^6 I + b_3^6 W_2 + b_5^6 W_5,\\
W_9 &= b_4^6 I + b_3^6 W_3 + b_5^6 W_6 + h b_8^6 A_4,\\
W_{10} &= \left(b_3^6 c_2 + b_5^6 c_4 + b_6^6\right) I - h A_6,\\
G_3 &= R_6 - b_8^6 R_4 - b_3^6 G_1 - b_5^6 G_2,\\
c_7 &= b_6^2 + b_3^2 c_2 + b_5^6 c_4,\\
W_{11} &= b_3^2 W_1 + b_5^2 W_4 + b_7^2 I,\\
W_{12} &= b_1^2 I + b_3^2 W_2 + b_5^2 W_5,\\
W_{13} &= b_4^2 I + b_3^2 W_3 + b_5^2 W_6 + h b_8^2 A_4,\\
W_{14} &= \left(b_2^2 + b_3^2 c_3 + b_5^2 c_5\right) I - h A_2,\\
G_4 &= R_6 - b_8^2 R_4 - b_3^2 + b_7^5 I - h W_4 A_5,\\
W_{16} &= b_3^5 W_2 + b_5^5 W_5 + b_5^1 I - h W_5 A_5,\\
W_{17} &= b_3^5 W_3 + b_5^5 W_6 + b_4^5 I + h\left(b_8^5 A_4 - W_6 A_5\right),\\
W_{18} &= b_3^5 c_2 + b_5^5 c_4 + b_6^5 I - h c_4 A_5,\\
W_{19} &= b_3^5 c_3 + b_5^5 c_5 + b_2^5 I - h c_5 A_5,\\
W_{20} &= b_3^3 W_1 + b_5^3 W_4 + b_7^3 I - h W_1 A_3,\\
G_5 &= R_5 - b_8^5 R_4 - b_3^5 G_1 - b_5^5 G_2 + h A_5 G_2,\\
W_{21} &= b_3^3 W_2 + b_5^3 W_5 + b_7^3 I - h W_2 A_3,\\
W_{22} &= b_3^3 W_3 + b_5^3 W_6 + b_4^3 I + h\left(b_8^3 A_4 - W_3 A_3\right),\\
W_{23} &= \left(b_3^3 c_2 + b_5^3 c_4 + b_6^3\right) I - h c_2 A_3,\\
W_{24} &= \left(b_3^3 c_3 + b_5^3 c_5 + b_2^3\right) I - h c_3 A_3,\\
G_6 &= R_3 - b_8^3 R_4 - \left(b_3^3 - h A_3\right) G_1 - b_5^3 G_2,\\
W_{28} &= W_{10} W_{14} - c_6 c_7 I,\\
W_{25} &= W_{28}^{-1}(c_6 W_{11} - W_7 W_{14}),\\
W_{26} &= W_{28}^{-1}(c_6 W_{12} - W_8 W_{14}),\\
W_{27} &= W_{28}^{-1}(c_6 W_{13} - W_9 W_{14}),\\
G_7 &= W_{28}^{-1}(c_6 G_4 - G_3 W_{14}),\\
W_{29} &= -(W_{10} W_{25} + W_7)/c_6,\\
W_{30} &= -(W_{10} W_{26} + W_8)/c_6,\\
W_{31} &= -(W_{10} W_{27} + W_9)/c_6,\\
G_8 &= -(G_3 + W_{10} G_7)/c_6,
\end{aligned}
$$

$$\begin{aligned}
W_{32} &= W_{19}W_{29} + W_{18}W_{25} + W_{15},\\
W_{33} &= W_{19}W_{30} + W_{18}W_{26} + W_{16},\\
W_{34} &= W_{19}W_{31} + W_{18}W_{27} + W_{17},\\
G_9 &= G_5 + W_{19}G_8 + W_{18}G_7,\\
W_{35} &= W_{24}W_{29} + W_{23}W_{25} + W_{20},\\
W_{36} &= W_{24}W_{30} + W_{23}W_{26} + W_{21},\\
W_{37} &= W_{24}W_{31} + W_{23}W_{27} + W_{22},\\
G_{10} &= G_6 + W_{24}G_8 + W_{23}G_7.
\end{aligned}$$

We get

$$S_i = W_{36} - W_{37}W_{33}W_{34}^{-1}, \quad T_i = W_{35} - W_{37}W_{32}W_{34}^{-1},$$
$$F_i = G_{10} - W_{37}G_9W_{34}^{-1}.$$

The equation (11) is the seventh order compact difference scheme of equation (6) in the ith subinterval. The structure of the seventh order scheme is profiled here as the following matrix:

$$\begin{bmatrix}
S_1 & T_1 & \dots & \dots & \dots & \dots & \dots & F_1\\
\dots & S_2 & T_2 & \dots & \dots & \dots & \dots & F_2\\
\dots & \dots & S_3 & T_3 & \dots & \dots & \dots & F_3\\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots\\
\dots & \dots & \dots & \dots & \dots & S_{N-1} & T_{N-1} & F_{N-1}
\end{bmatrix}.$$

Solving the above system together with the boundary conditions $y(0) = \alpha$ and $y(1) = \beta$ we get the solution.

3 Numerical examples

Layer on the left side. If $a(x) \geq M > 0$ throughout the interval $[0, 1]$, where M is some positive constant, then the boundary layer will be in the neighborhood of 0, i.e., on the left side of the interval $[0, 1]$. To demonstrate the efficiency of the method, we consider some numerical experiments.

Example 1. Consider equation (1) with $a(x) = 1$, $b(x) = -1$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = 1$. The singular perturbed delay differential equation is

$$\varepsilon y''(x) + y'(x - \delta) - y(x) = 0, \quad x \in [0, 1],$$

with

$$y(0) = 1 \quad \text{and} \quad y(1) = 1.$$

The exact solution is given by

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{(e^{m_2} - e^{m_1})},$$

where $m_1 = (-1 - \sqrt{1 + 4(\varepsilon - \delta)})/2(\varepsilon - \delta)$ and $m_2 = (-1 + \sqrt{1 + (4\varepsilon - \delta)})/2(\varepsilon - \delta)$.

The numerical results are given in Table 1 for different choices ε and δ.

Table 1. Numerical results of Example 1.

(a) $\varepsilon = 0.1$, $\delta = 0.01$, $N = 100$

x	Numerical solution	Exact solution
0.00	1.000000000000000	1.000000000000000
0.01	0.935337607335685	0.935337607335687
0.02	0.878458854537734	0.878458854537735
0.03	0.828485389097690	0.828485389097691
0.04	0.784638488976589	0.784638488976591
0.05	0.746227766411489	0.746227766411492
0.10	0.616565591274272	0.616565591274277
0.20	0.532078010100237	0.532078010100244
0.30	0.540282525159504	0.540282525159513
0.40	0.579556005422677	0.579556005422686
0.50	0.631715657547513	0.631715657547521
0.60	0.691648119603336	0.691648119603343
0.70	0.758196262968007	0.758196262968013
0.80	0.831426995345893	0.831426995345899
0.90	0.911814697709268	0.911814697709270
1.00	1.000000000000000	1.000000000000000

Least square error $6.604865977274242\,\mathrm{e}-014$
Maximum error $9.658940314238862\,\mathrm{e}-015$

(b) $\varepsilon = 0.1$, $\delta = 0.08$, $N = 100$

x	Numerical solution	Exact solution
0.00	1.000000000000000	1.000000000000000
0.01	0.754088201945571	0.754088201941058
0.02	0.607903494090242	0.607903494084819
0.03	0.521630893329375	0.521630893324486
0.04	0.471357225298980	0.471357225295063
0.05	0.442720120123168	0.442720120120222
0.10	0.417487990544566	0.417487990544094
0.20	0.456321084303574	0.456321084303555
0.30	0.503317952401640	0.503317952401627
0.40	0.555183123973670	0.555183123973659
0.50	0.612392993229185	0.612392993229173
0.60	0.675498160054451	0.675498160054440
0.70	0.745106115330908	0.745106115330899
0.80	0.821886950896788	0.821886950896781

x	Numerical solution	Exact solution
0.90	0.906579809446906	0.906579809446902
1.00	1.000000000000000	1.000000000000000

Least square error $1.031630343641800\,\mathrm{e}-011$
Maximum error $5.423106408386502\,\mathrm{e}-012$

(c) $\varepsilon = 0.001$, $\delta = 0.001$, $N = 100$

x	Numerical solution	Exact solution
0.00	1.000000000000000	1.000000000000000
0.01	0.579815895827152	0.579815892739947
0.02	0.445387209366442	0.445387207353913
0.03	0.404124628442542	0.404124627458576
0.04	0.393254633082114	0.393254632654488
0.05	0.392316681040498	0.392316680866267
0.10	0.409826314975035	0.409826314973749
0.20	0.452518444521641	0.452518444521638
0.30	0.499668244835440	0.499668244835438
0.40	0.551730781326308	0.551730781326305
0.50	0.609217932516787	0.609217932516784
0.60	0.672694911833319	0.672694911833316
0.70	0.742785824667049	0.742785824667047
0.80	0.820179804575386	0.820179804575385
0.90	0.905637788840211	0.905637788840210
1.00	1.000000000000000	1.000000000000000

Least square error $3.842907993345486\,\mathrm{e}-009$
Maximum error $3.087205602270160\,\mathrm{e}-009$

(d) $\varepsilon = 0.01$, $\delta = 0.008$, $N = 100$

x	Numerical solution	Exact solution
0.00	1.000000000000000	1.000000000000000
0.01	0.376635238403165	0.376522213185011
0.02	0.376074123097157	0.376072594873774
0.03	0.379816443230839	0.379816427732523
0.04	0.383625816298184	0.383625816158480
0.05	0.387473599750042	0.387473599748877
0.10	0.407299226182605	0.407299226182620
0.20	0.450045599928066	0.450045599928081
0.30	0.497278239177914	0.497278239177929
0.40	0.549467981021062	0.549467981021076
0.50	0.607135077268772	0.607135077268785
0.60	0.670854380568590	0.670854380568601
0.70	0.741261074804999	0.741261074805008
0.80	0.819057006910136	0.819057006910142
0.90	0.905017683203006	0.905017683203009
1.00	1.000000000000000	1.000000000000000

Least square error $1.130355503620759\,\mathrm{e}-004$
Maximum error $1.130252181538727\,\mathrm{e}-004$

(e) $\varepsilon = 0.01$, $\delta = 0.008$, $N = 1000$

x	Numerical solution	Exact solution
0.000	1.000000000000000	1.000000000000000
0.001	0.751554634741489	0.751554634737560
0.002	0.601160629424041	0.601160629419279
0.003	0.510178602252454	0.510178602248124
0.004	0.455195961257210	0.455195961253711
0.005	0.422026416827804	0.422026416825151
0.100	0.407299226182634	0.407299226182620
0.200	0.450045599928094	0.450045599928081
0.300	0.497278239177942	0.497278239177929
0.400	0.549467981021088	0.549467981021076
0.500	0.607135077268796	0.607135077268785
0.600	0.670854380568611	0.670854380568601
0.700	0.741261074805015	0.741261074805008
0.800	0.819057006910147	0.819057006910142
0.900	0.905017683203012	0.905017683203009
1.000	1.000000000000000	1.000000000000000

Least square error $9.138929540480388\,\mathrm{e}-012$
Maximum error $4.761413485709909\,\mathrm{e}-012$

Example 2. Now consider an example of the BVP with variable coefficients. Consider equation (1) with $a(x) = e^{-0.5x}$, $b(x) = -1$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = 1$.

The variable coefficient singular perturbed delay differential equation is

$$\varepsilon y''(x) + e^{-0.5x} y'(x - \delta) - y(x) = 0, \quad x \in [0, 1],$$

with

$$y(0) = 1 \quad \text{and} \quad y(1) = 1.$$

for which exact solution is not known.

The numerical results are given in Table 2 for different values of ε and δ.

Table 2. Numerical results of Example 2.

(a) $\varepsilon = 0.1$, $N = 100$

	Numerical solution		
x	$\delta = 0$	$\delta = 0.03$	$\delta = 0.08$
0.00	1.000000000000000	1.000000000000000	1.000000000000000
0.01	0.937049836668530	0.911611554988328	0.737093174971516
0.02	0.880954090331866	0.836124596721541	0.578622171448609
0.03	0.830985134520517	0.771664742588349	0.482444400443664
0.04	0.786495676530814	0.716640187786036	0.423876031496795
0.05	0.746909630068549	0.669697773325889	0.388282190950921
0.10	0.606411315051172	0.521373798278011	0.343081265295241
0.20	0.494666529585409	0.437478401087115	0.370685251542266
0.30	0.482627934088460	0.448542239746484	0.412663855580776
0.40	0.510370525412707	0.488445457966694	0.461750404306160

	Numerical solution		
x	$\delta = 0$	$\delta = 0.03$	$\delta = 0.08$
0.50	0.557898125063073	0.542065721409428	0.519046038680500
0.60	0.618939522578240	0.606560582841149	0.586162125456289
0.70	0.692274835293883	0.682474795871793	0.665082859516572
0.80	0.778782175810307	0.771580318389072	0.758244735478542
0.90	0.880434893486833	0.876369722353556	0.868648225944063
1.00	1.000000000000000	1.000000000000000	1.000000000000000

(b) $\varepsilon = 0.01$, $N = 1000$

	Numerical solution	
x	$\delta = 0$	$\delta = 0.08$
0.000	1.000000000000000	1.000000000000000
0.001	0.931601141073168	0.716594937162734
0.002	0.869796883464067	0.544777128853008
0.003	0.813951289736812	0.440523855514915
0.004	0.763490018121925	0.377230958744245
0.005	0.717894330826354	0.338800748936650
0.010	0.547997437919281	0.284704640443383
0.050	0.301007270550615	0.290987069442839
0.100	0.311549076695967	0.306434103779498
0.200	0.346353779820135	0.341154406555376
0.300	0.387133749633707	0.381859933399911
0.400	0.435113235985297	0.429841132883670
0.500	0.491875825441656	0.486719308324638
0.600	0.559418059809898	0.554544342779584
0.700	0.640273202459981	0.635924991709127
0.800	0.737676782764715	0.734203807510623
0.900	0.855789562092193	0.853693671684283
1.000	1.000000000000000	1.000000000000000

Remark. We have considered numerical results for these test examples to show the effect of small shifts on boundary layer solution of the problem. From the numerical experiments presented here, we observe as δ increases, the thickness of the boundary layer decreases and maximum error decreases as the grid size h decreases, which shows the convergence of the computed solution to the exact solution.

Layer on the right side. If $a(x) \leq M < 0$ throughout the interval $[0, 1]$, then the boundary layer will be in the neighborhood of 1, i.e., on the right side of the interval $[0, 1]$. To demonstrate the efficiency of the method, we consider some numerical experiments.

Example 3. Consider equation (1) with $a(x) = -1$, $b(x) = -1$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = -1$. The singular perturbed delay differential equation is

$$\varepsilon y''(x) - y'(x - \delta) - y(x) = 0, \quad x \in [0, 1],$$

with

$$y(0) = 1 \quad \text{and} \quad y(1) = -1.$$

The exact solution is given by

$$y(x) = \frac{[(e^{m_2}+1)e^{m_1 x} - (1+e^{m_1})e^{m_2 x}]}{(e^{m_2} - e^{m_1})},$$

where $m_1 = (1 - \sqrt{1+4(\varepsilon+\delta)})/2(\varepsilon+\delta)$ and $m_2 = (1 + \sqrt{1+(4\varepsilon+\delta})/2(\varepsilon+\delta)$.

The numerical results are given in Table 3 for different choices of ε and δ.

Table 3. Numerical results of Example 3.

(a) $\varepsilon = 0.01$, $\delta = 0.007$, $N = 100$

x	Numerical solution	Exact solution
0.00	1.0000000000000000	1.000000000000000
0.10	0.9063266883481120	0.906326688348113
0.20	0.8214280660120560	0.821428066012058
0.30	0.7444821787849000	0.744482178784903
0.40	0.6747440676323050	0.674744067632309
0.50	0.6115385562995810	0.611538556299585
0.60	0.5542537144722820	0.554253714472286
0.70	0.5023349114206060	0.502334911420610
0.80	0.4552707826982130	0.455270782698227
0.90	0.4091599182326700	0.409159918234437
0.95	0.3237611594601670	0.323761159477694
0.96	0.2633756708964280	0.263375670921927
0.97	0.1567431781617890	0.156743178196566
0.98	−0.034025613920763	−0.034025613878602
0.99	−0.377836138800961	−0.377836138762625
1.00	−1.000000000000000	−1.000000000000000

Least square error 7.508592426852532e − 011
Maximum error 4.216065691009518e − 011

(b) $\varepsilon = 0.01$, $\delta = 0.025$, $N = 100$

x	Numerical solution	Exact solution
0.00	1.000000000000000	1.000000000000000
0.10	0.907805196606813	0.907805196606808
0.20	0.824110274917940	0.824110274917932
0.30	0.748131588773705	0.748131588773694
0.40	0.679157717627285	0.679157717627272
0.50	0.616542398836372	0.616542398836359
0.60	0.559690678081091	0.559690678081075
0.70	0.507903762464284	0.507903762464268
0.80	0.457503603942831	0.457503603942822
0.90	0.346773450048870	0.346773450048917
0.95	0.083827704468978	0.083827704469102
0.96	−0.028306922677932	−0.028306922677800
0.97	−0.177619475485561	−0.177619475485421
0.98	−0.376899059691852	−0.376899059691723

x	Numerical solution	Exact solution
0.99	−0.643329580659146	−0.643329580659061
1.00	−1.000000000000000	−1.000000000000000

Least square error 7.321245876809389 e − 010
Maximum error 4.804800723956504 e − 010

(c) $\varepsilon = 0.001$, $\delta = 0.0007$, $N = 1000$

x	Numerical solution	Exact solution
0.000	1.000000000000000	1.000000000000000
0.100	0.904990732600442	0.904990732600459
0.200	0.819008226092685	0.819008226092715
0.300	0.741194854537407	0.741194854537449
0.400	0.670774474407487	0.670774474407536
0.500	0.607044683003707	0.607044683003764
0.600	0.549369812392727	0.549369812392790
0.700	0.497174588985862	0.497174588985927
0.800	0.449938395516639	0.449938395516706
0.900	0.407190078183670	0.407190078183739
0.995	0.298445181010862	0.298445181026878
0.996	0.240366595239761	0.240366595262828
0.997	0.135970726789187	0.135970726820325
0.998	−0.051917266678575	−0.051917266641176
0.999	−0.390308992396563	−0.390308992362868
1.000	−1.000000000000000	−1.000000000000000

Least square error 6.700481309650548 e − 011
Maximum error 3.739840381822290 e − 011

(d) $\varepsilon = 0.001$, $\delta = 0.0025$, $N = 1000$

x	Numerical solution	Exact solution
0.000	1.000000000000000	1.000000000000000
0.100	0.905151968151021	0.905151968151049
0.200	0.819300085447666	0.819300085447718
0.300	0.741591084849256	0.741591084849325
0.400	0.671252630014555	0.671252630014638
0.500	0.607585639184225	0.607585639184318
0.600	0.549957337127898	0.549957337127998
0.700	0.497794966100412	0.497794966100518
0.800	0.450580093301459	0.450580093301568
0.900	0.407843458261002	0.407843458261114
0.995	0.0445144329014424	0.0445144329016372
0.996	−0.064262443948597	−0.064262443948399
0.997	−0.209034413652541	−0.209034413652347
0.998	−0.401753371460064	−0.40175337145993
0.999	−0.658339707193347	−0.658339707193262
1.000	−1.000000000000000	−1.000000000000000

Least square error 2.785076432451545 e − 012
Maximum error 1.979388875028576 e − 013

Example 4. Now we consider an example of the BVP with variable coefficients. Consider equation 1 with $a(x) = -e^x$, $b(x) = -x$, $f(x) = 0$, $\phi(x) = 1$ and $\gamma = 1$.

The variable coefficient singularly perturbed delay differential equation is

$$\varepsilon y''(x) - e^x y'(x-\delta) - xy(x) = 0, \quad x \in [0,1],$$

with

$$y(0) = 0 \quad \text{and} \quad y(1) = 1,$$

for which exact solution is not known.

The numerical results are given in Table 4 for different choices of δ.

Table 4. Numerical results of Example 4.

$\varepsilon = 0.1$, $N = 100$

	Numerical solution	
x	$\delta = 0$	$\delta = 0.008$
0.00	1.000000000000000	1.000000000000000
0.10	0.988882656332727	0.988458430755310
0.20	0.971752068437304	0.971013240385191
0.30	0.950050217434337	0.949096223291772
0.40	0.925046406407414	0.923966076433822
0.50	0.897829489264826	0.896708609975369
0.60	0.869326491959919	0.868286899535815
0.70	0.840432116705029	0.839825533640646
0.80	0.813090302833301	0.814465880418184
0.90	0.802126723824385	0.811784483653044
0.91	0.804664760297485	0.815735007382126
0.92	0.808750205417299	0.821283646787078
0.93	0.814847168246508	0.828822097891890
0.94	0.823566092769773	0.838842703120003
0.95	0.835711964224734	0.851965402152568
0.96	0.852349055080983	0.868972177721747
0.97	0.874888109420313	0.890851156545487
0.98	0.905204047339830	0.918853169683275
0.99	0.945795290704337	0.954564418985984
1.00	1.000000000000000	1.000000000000000

Remark. From the numerical experiments presented here, we observe that as δ increases, the thickness of the right boundary layer increases. As the grid size h decreases, the maximum error decreases, which shows the convergence of the computed solution to the exact solution.

4 Conclusion

We have described a seventh order numerical method for solving boundary value problems for singularly perturbed differential-difference equation with small shifts. Here we have discussed both the cases, when boundary layer is on the left side and when boundary layer is on the right side of the underlying interval. From the numerical experiments considered in the paper, we observe that the small shift affects both the boundary layer solutions in similar fashion but reversely, i.e., as δ increases the thickness of the left boundary layer decreases while that of the right boundary layer increases. This method does not depend on the asymptotic expansion as well as on the matching of coefficients. Thus we have devised an alternative technique of solving singularly perturbed differential-difference equations, which can be easily implemented on computer.

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