Perturbation solutions of fifth order oscillatory nonlinear systems

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Abstract. Oscillatory systems play an important role in the nature. Many engineering problems and physical systems of fifth degrees of freedom are oscillatory and their governing equations are fifth order nonlinear differential equations. To investigate the solution of fifth order weakly nonlinear oscillatory systems, in this article the Krylov–Bogoliubov–Mitropolskii (KBM) method has been extended and desired solution is found. An example is solved to illustrate the method. The results obtain by the extended KBM method show good agreement with those obtained by numerical method.

Keywords: nonlinearity, oscillatory systems, perturbation, eigenvalues.

1 Introduction

In oscillatory problems, the method of Krylov–Bogoliubov–Mitropolskii (KBM) [1, 2] is particularly convenient, and is the widely used technique to obtain analytical approximate solution of nonlinear systems with a small non-linearity. In fact, the method developed by Krylov and Bogoliubov [2] for obtaining periodic solutions was amplified and justified by Bogoliubov and Mitropolskii [1], and later extended by Popov [3] and Meldelson [4], for damped nonlinear oscillations. Murty [5] has developed a unified KBM method for solving second-order nonlinear systems. Sattar [6] has studied a third-order over-damped nonlinear system. Bojadziev [7] studied the damped oscillations modeled by a three-dimensional nonlinear system. Shamsul and Sattar [8] developed a method for third-order critically-damped nonlinear equations. Islam and Akbar [9] investigated a new solution of third order more critically damped nonlinear systems. Shamsul and Sattar [10] presented a unified KBM method for solving third-order nonlinear systems. Murty et al. [11] extended the method to the fourth-order over-damped nonlinear systems in a way which we think too much laborious and cumbersome. Akbar et al. [12] has presented a method for solving the fourth-order over-damped nonlinear systems which is easier than that of Murty et al. [11]. Later, Akbar et al. [13] extended the method presented in [12] to the damped oscillatory systems. Islam et al. [14] investigated the solutions of fourth order more critically damped nonlinear systems. Akbar [15] examined a different type solution of fourth order more critically damped differential systems. Rahman et al. [16] studied fourth order nonlinear oscillatory systems when two of the eigenvalues of the corresponding linear systems are real and negative and the other two are complex numbers.

The aim of this article is to obtain the analytical approximate solutions of fifth-order weakly nonlinear oscillatory systems by extending the KBM method. The results obtained by the perturbation solution are compared with those obtained by the fourth-order Runge–Kutta method.

2 The method

Consider a fifth-order weakly-nonlinear oscillatory system, governed by the differential equation:

$$\frac{\mathrm{d}^5x}{\mathrm{d}t^5} + k_1\frac{\mathrm{d}^4x}{\mathrm{d}t^4} + k_2\frac{\mathrm{d}^3x}{\mathrm{d}t^3} + k_3\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + k_4\frac{\mathrm{d}x}{\mathrm{d}t} + k_5x = -\varepsilon f(x), \tag{1}$$

where ε is a small parameter, $f(x)$ is the nonlinear function, k_1, k_2, k_3, k_4, k_5, are the characteristic parameters of the system defined by

$$k_1 = \sum_{i=1}^{5}\lambda_i, \quad k_2 = \sum_{\substack{i=1,\,j=1\\ i\neq j}}^{5}\lambda_i\lambda_j, \quad k_3 = \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{5}\lambda_i\lambda_j\lambda_k,$$
$$k_4 = \sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l}}^{5}\lambda_i\lambda_j\lambda_k\lambda_l \quad \text{and} \quad k_5 = \prod_{i=1}^{5}\lambda_i,$$

and where $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4$ and $-\lambda_5$ are the eigenvalues of the unperturbed equation of (1). Suppose four of the eigenvalues are imaginary and the other is real and negative (as the system is oscillatory). The oscillatory system is represented by these imaginary eigenvalues. Therefore, the unperturbed solution is:

$$x(t,0) = \sum_{j=1}^{5} a_{j,0}e^{-\lambda_j t}, \tag{2}$$

where $a_{j,0}$, $j = 1,2,3,4,5$, are arbitrary constants.

When $\varepsilon \neq 0$, following Shamsul [17], we seek the solution of the nonlinear differential equation (1) of the form:

$$x(t,\varepsilon) = \sum_{j=1}^{5} a_j(t)e^{-\lambda_j t} + \varepsilon u_1(a_1,a_2,a_3,a_4,a_5,t) + \cdots, \tag{3}$$

where each $a_j(t)$, $j = 1,2,3,4,5$, satisfies the differential equation

$$\dot{a}_j(t) = \varepsilon A_j(a_1,a_2,a_3,a_4,a_5,t) + \cdots. \tag{4}$$

Confining only to the first few terms $1, 2, 3, \ldots, m$ in the series expansions of equation (3) and (4), we evaluate the functions u_1 and A_j, $j = 1, 2, 3, 4, 5$, such that $a_j(t)$, appearing in equation (3) and (4), satisfy the given differential equation (1) with an accuracy of ε^{m+1}. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly-growing algebraic complexity for the derivation of the formulae, the solution is, in general, confined to a lower order, usually the first [5]. In order to determine these functions, it is assumed that the functions u_1 do not contain the fundamental terms which are included in the series expansion (3) at order ε^0.

Differentiating equation (3) five times with respect to t, substituting x and the derivatives $\frac{\mathrm{d}^5x}{\mathrm{d}t^5}, \frac{\mathrm{d}^4x}{\mathrm{d}t^4}, \frac{\mathrm{d}^3x}{\mathrm{d}t^3}, \frac{\mathrm{d}^2x}{\mathrm{d}t^2}, \frac{\mathrm{d}x}{\mathrm{d}t}$ in the original equation (1), utilizing the relations in equation (4) and equating the coefficients of ε, we obtain:

$$\prod_{j=1}^{5}\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_j\right)u_1+\sum_{j=1}^{5}e^{-\lambda_j t}\left(\prod_{k=1,\,j\neq k}^{5}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_j+\lambda_k\right)\right)A_j = -f^{(0)}(a_1,a_2,a_3,a_4,a_5,t), \tag{5}$$

where $f^{(0)} = f(x_0)$ and $x_0 = \sum_{j=1}^{5} a_j(t)e^{-\lambda_j t}$.

In general, the functional $f^{(0)}$ can be expanded in a Taylor series (see Murty and Deekshatulu [18] for details) as:

$$f^{(0)}=\sum_{\substack{m_1=-\infty\\ \cdots\\ m_5=-\infty}}^{\infty\ldots\infty} F_{m_1,m_2,m_3,m_4,m_5}a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5}e^{-(m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4+m_5\lambda_5)t}.$$

According to KBM [1,2], Sattar [19] and Shamsul [17,20], u_1 does not contain the fundamental terms. Therefore equation (5) can be separated into six equations for unknown functions u_1 and A_1, A_2, A_3, A_4, A_5 (see [17] for details). Substituting the functional values of and equating the coefficients of $e^{-\lambda_j t}$, $j = 1, 2, 3, 4, 5$, we obtain:

$$\begin{aligned} &e^{-\lambda_1 t}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_1+\lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_1+\lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_1+\lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_1+\lambda_5\right)A_1\\ &\quad=-\sum_{\substack{m_1=-\infty\\ \cdots\\ m_5=-\infty}}^{\infty\ldots\infty} F_{m_1,m_2,\ldots,m_5}a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5}e^{-(m_1\lambda_1+m_2\lambda_2+\ldots+m_5\lambda_5)t},\\ &m_3=m_4,\ \ m_1=m_2+1, \end{aligned} \tag{6}$$

$$\begin{aligned} &e^{-\lambda_2 t}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_2+\lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_2+\lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_2+\lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_2+\lambda_5\right)A_2\\ &\quad=-\sum_{\substack{m_1=-\infty\\ \cdots\\ m_5=-\infty}}^{\infty\ldots\infty} F_{m_1,m_2,\ldots,m_5}a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5}e^{-(m_1\lambda_1+m_2\lambda_2+\ldots+m_5\lambda_5)t},\\ &m_3=m_4,\ \ m_1=m_2-1, \end{aligned} \tag{7}$$

$$e^{-\lambda_3 t}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_3+\lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_3+\lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_3+\lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_3+\lambda_5\right)A_3$$
$$=-\sum_{\substack{m_1=-\infty\\ \cdots\\ m_5=-\infty}}^{\infty\ldots\infty} F_{m_1,m_2,\ldots,m_5} a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5} e^{-(m_1\lambda_1+m_2\lambda_2+\ldots+m_5\lambda_5)t},$$
$$m_1=m_2,\;\; m_3=m_4+1, \tag{8}$$

$$e^{-\lambda_4 t}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_4+\lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_4+\lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_4+\lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_4+\lambda_5\right)A_4$$
$$=-\sum_{\substack{m_1=-\infty\\ \cdots\\ m_5=-\infty}}^{\infty\ldots\infty} F_{m_1,m_2,\ldots,m_5} a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5} e^{-(m_1\lambda_1+m_2\lambda_2+\ldots+m_5\lambda_5)t},$$
$$m_1=m_2,\;\; m_3=m_4-1, \tag{9}$$

$$e^{-\lambda_5 t}\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_5+\lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_5+\lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_5+\lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}-\lambda_5+\lambda_4\right)A_5$$
$$=-\sum_{\substack{m_1=-\infty\\ \cdots\\ m_5=-\infty}}^{\infty\ldots\infty} F_{m_1,m_2,\ldots,m_5} a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5} e^{-(m_1\lambda_1+m_2\lambda_2+\ldots+m_5\lambda_5)t},$$
$$m_1=m_2,\;\; m_3=m_4, \tag{10}$$

and

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_5\right)u_1$$
$$=-\sum_{\substack{m_1=-\infty\\ \cdots\\ m_5=-\infty}}^{\infty\ldots\infty} F_{m_1,m_2,\ldots,m_5} a_1^{m_1}a_2^{m_2}a_3^{m_3}a_4^{m_4}a_5^{m_5} e^{-(m_1\lambda_1+m_2\lambda_2+\ldots+m_5\lambda_5)t}, \tag{11}$$

where $\sum'$ excludes those terms for $m_1=m_2\pm1$, $m_3=m_4\pm1$, $m_1=m_2$, $m_3=m_4$.

The particular solutions of the equations (6)–(11) give the functions A_1, A_2, A_3, A_4, A_5 and u_1. Thus, the determination of the first approximate solution is completed.

3 Example

As an example of the method, we consider the Duffing equation type of fifth order differential equation:

$$\frac{\mathrm{d}^5x}{\mathrm{d}t^5}+k_1\frac{\mathrm{d}^4x}{\mathrm{d}t^4}+k_2\frac{\mathrm{d}^3x}{\mathrm{d}t^3}+k_3\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+k_4\frac{\mathrm{d}x}{\mathrm{d}t}+k_5x=-\varepsilon x^3. \tag{12}$$

Here $f(x)=x^3$.

For example (12), we have

$$f^{(0)} = \left(a_1 e^{-\lambda_1 t} + a_2 e^{-\lambda_2 t} + a_3 e^{-\lambda_3 t} + a_4 e^{-\lambda_4 t} + a_5 e^{-\lambda_5 t}\right)^3$$

or

$$\begin{aligned} f^{(0)} &= a_1^3 e^{-3\lambda_1 t} + 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + a_2^3 e^{-3\lambda_2 t} \\ &+ 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t} + 3a_1^2 a_5 e^{-(2\lambda_1+\lambda_5)t} \\ &+ 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} + 6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} \\ &+ 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2 a_5 e^{-(2\lambda_2+\lambda_5)t} \\ &+ 3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} + 3a_1 a_4^2 e^{-(\lambda_1+2\lambda_4)t} + 3a_1 a_5^2 e^{-(\lambda_1+2\lambda_5)t} \\ &+ 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 6a_1 a_3 a_5 e^{-(\lambda_1+\lambda_3+\lambda_5)t} + 6a_1 a_4 a_5 e^{-(\lambda_1+\lambda_4+\lambda_5)t} \\ &+ 3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + 3a_2 a_4^2 e^{-(\lambda_2+2\lambda_4)t} + 3a_2 a_5^2 e^{-(\lambda_2+2\lambda_5)t} \\ &+ 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + 6a_2 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 6a_2 a_3 a_5 e^{-(\lambda_2+\lambda_3+\lambda_5)t} \\ &+ a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} \\ &+ 3a_3^2 a_5 e^{-(2\lambda_3+\lambda_5)t} + 6a_3 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 3a_4^2 a_5 e^{-(2\lambda_4+\lambda_5)t} \\ &+ 3a_3 a_5^2 e^{-(\lambda_3+2\lambda_5)t} + 3a_4 a_5^2 e^{-(\lambda_4+2\lambda_5)t} + a_5^3 e^{-3\lambda_5 t}. \end{aligned} \tag{13}$$

Therefore equations (6)–(11) become:

$$\begin{aligned} &e^{-\lambda_1 t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1 + \lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1 + \lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1 + \lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_1 + \lambda_5\right)A_1 \\ &\quad = -3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} - 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t}, \end{aligned} \tag{14}$$

$$\begin{aligned} &e^{-\lambda_2 t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2 + \lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2 + \lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2 + \lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_2 + \lambda_5\right)A_2 \\ &\quad = -3a_1 a_2^2 e^{-(2\lambda_1+\lambda_2)t} - 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t}, \end{aligned} \tag{15}$$

$$\begin{aligned} &e^{-\lambda_3 t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_3 + \lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_3 + \lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_3 + \lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_3 + \lambda_5\right)A_3 \\ &\quad = -3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} - 6a_1 a_2 a_3 e^{-(\lambda_2+\lambda_3+\lambda_4)t}, \end{aligned} \tag{16}$$

$$\begin{aligned} &e^{-\lambda_4 t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_4 + \lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_4 + \lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_4 + \lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_4 + \lambda_5\right)A_4 \\ &\quad = -3a_3 a_4^2 e^{-(2\lambda_3+\lambda_4)t} - 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t}, \end{aligned} \tag{17}$$

$$\begin{aligned} &e^{-\lambda_5 t}\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_5 + \lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_5 + \lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_5 + \lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - \lambda_5 + \lambda_4\right)A_5 \\ &\quad = -6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} - 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t}, \end{aligned} \tag{18}$$

$$\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_3\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_4\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}+\lambda_5\right)u_1\\
&\quad= -\big(a_1^3e^{-3\lambda_1 t}+a_2^3e^{-3\lambda_2 t}+3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t}\\
&\qquad+3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t}+3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t}\\
&\qquad+3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t}+3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t}+3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t}\\
&\qquad+3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t}+6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t}+6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t}\\
&\qquad+3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t}+3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t}+3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t}\\
&\qquad+6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t}+6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t}+a_3^3e^{-3\lambda_3 t}+a_4^3e^{-3\lambda_4 t}\\
&\qquad+3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t}+3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t}+3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t}\\
&\qquad+3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t}+a_5^3e^{-3\lambda_5 t}\big). \qquad (19)
\end{aligned}$$

Solving the equations (14)–(18) and substituting, $\lambda_1 = -i\omega_1$, $\lambda_2 = i\omega_1$, $\lambda_3 = -i\omega_2$, $\lambda_4 = i\omega_2$ and $\lambda_5 = \xi$, we get:

$$\begin{aligned}
A_1 &= \frac{-3a_1^2a_2-6a_1a_3a_4}{2\omega_1(\omega_1^2-\omega_2^2)(\omega_1-i\xi)}, \quad A_2 = \frac{-3a_1a_2^2-6a_2a_3a_4}{2\omega_1(\omega_1^2-\omega_2^2)(\omega_1+i\xi)},\\
A_3 &= \frac{-3a_3^2a_4-6a_1a_2a_3}{2\omega_2(\omega_2^2-\omega_1^2)(\omega_2-i\xi)}, \quad A_4 = \frac{-3a_3a_4^2-6a_1a_2a_4}{2\omega_2(\omega_2^2-\omega_1^2)(\omega_2+i\xi)},\\
A_5 &= \frac{-6a_1a_2a_5-6a_3a_4a_5}{(\xi^2+\omega_1^2)(\xi^2+\omega_2^2)}.
\end{aligned} \qquad (20)$$

Substituting A_j, $j = 1,\dots,5$, in equation (4), we get:

$$\begin{aligned}
\dot a_1 &= \frac{-\varepsilon(3a_1^2a_2+6a_1a_3a_4)}{2\omega_1(\omega_1^2-\omega_2^2)(\omega_1-i\xi)}, \quad \dot a_2 = \frac{-\varepsilon(3a_1a_2^2+6a_2a_3a_4)}{2\omega_1(\omega_1^2-\omega_2^2)(\omega_1+i\xi)},\\
\dot a_3 &= \frac{-\varepsilon(3a_3^2a_4+6a_1a_2a_3)}{2\omega_2(\omega_2^2-\omega_1^2)(\omega_2-i\xi)}, \quad \dot a_4 = \frac{-\varepsilon(3a_3a_4^2+6a_1a_2a_4)}{2\omega_2(\omega_2^2-\omega_1^2)(\omega_2+i\xi)},\\
\dot a_5 &= \frac{-\varepsilon(6a_1a_2a_5-6a_3a_4a_5)}{(\xi^2+\omega_1^2)(\xi^2+\omega_2^2)}.
\end{aligned} \qquad (21)$$

Now substituting $a_1 = \frac{1}{2}ae^{i\varphi_1}$, $a_2 = \frac{1}{2}ae^{-i\varphi_1}$, $a_3 = \frac{1}{2}be^{i\varphi_2}$, $a_4 = \frac{1}{2}be^{-i\varphi_2}$ and $a_5 = c$ in equation (21) and simplifying, we obtain:

$$\begin{aligned}
\dot a &= \varepsilon l_1 a\big(a^2+2b^2\big), \qquad \dot b = \varepsilon l_3 b\big(b^2+2a^2\big),\\
\dot c &= \varepsilon l_5 c\big(a^2+b^2\big), \qquad \dot\varphi_1 = \varepsilon l_2\xi\big(a^2+2b^2\big),\\
\dot\varphi_2(t) &= \varepsilon l_4\xi\big(b^2+2a^2\big),
\end{aligned} \qquad (22)$$

where:

$$l_1 = \frac{-3}{8(\omega_1^2 - \omega_2^2)(\omega_1^2 + \xi^2)}, \quad l_3 = \frac{-3}{8(\omega_2^2 + \xi^2)(\omega_2^2 - \omega_1^2)},$$
$$l_5 = \frac{-3}{2(\xi^2 + \omega_1^2)(\xi^2 + \omega_2^2)}, \quad l_2 = \frac{-3}{8\omega_1(\omega_1^2 - \omega_2^2)(\omega_1^2 + \xi^2)}, \tag{23}$$
$$l_4 = \frac{-3}{8\omega_2(\omega_2^2 + \xi)(\omega_2^2 - \omega_1^2)}.$$

Equations in (22) are nonlinear and they have no exact solutions. But since ε is a small quantity, $\dot{a}, \dot{b}, \dot{c}, \dot{\varphi}_1$ and $\dot{\varphi}_2$ are slowly varying functions of time t. Therefore, we can solve (22) by assuming that a, b, c, φ_1 and φ_2 are constants in the right-hand sides of (22). This assumption was first made by Murty et al. [11, 18] to solve similar type of nonlinear equations. Thus the solutions of the equations of (22) are

$$a(t) = a_0 + \varepsilon l_1 a_0(a_0^2 + 2b_0^2), \qquad b(t) = b_0 + \varepsilon l_3 b_0(b_0^2 + 2a_0^2),$$
$$c(t) = c_0 + \varepsilon l_5 c_0(a_0^2 + b_0^2), \qquad \varphi_1(t) = \varphi_{1,0} + \varepsilon l_2 \xi(a_0^2 + 2b_0^2), \tag{24}$$
$$\varphi_2(t) = \varphi_{2,0} + \varepsilon l_4 \xi(b_0^2 + 2a_0^2).$$

Therefore, we obtain the solution of equation (12) in the following form

$$x = a\cos(\omega_1 t + \varphi_1) + b\cos(\omega_2 t + \varphi_2) + ce^{-\xi t} + \varepsilon u_1. \tag{25}$$

Here equation (25) is the first order approximate solution of equation (12), where a, b, c, φ_1 and φ_2 are given by the equations of (24).

4 Results and discussion

In order to test the accuracy of an analytical approximate solution obtained by a certain perturbation method, we compare the approximate solution to the numerical solution (considered to be exact). With regard to such a comparison concerning the presented technique of this article, we refer the work of Murty et al. [11]. In the present article, for different sets of initial conditions as well as different sets of eigenvalues we have compared the results obtained by perturbation solution (25) to those obtained by the fourth order Runge–Kutta method. Beside this, we have computed the Pearson correlation between the perturbation results and the corresponding numerical results. From figures, we observed that the perturbation results from equation (25) show good coincidence with the numerical results.

First of all, for $\omega_1 = 2.00$, $\omega_2 = 1.00$, $\xi = 0.003$ and $\varepsilon = 0.1$, $x(t, \varepsilon)$ has been computed (25), in which $a, b, c, \varphi_1, \varphi_2$ by the equations (24) with initial conditions

$$a_0 = 0.005, \quad b_0 = 0.005, \quad c_0 = 0.0075, \quad \varphi_{1,0} = 2\pi/3 \quad \text{and} \quad \varphi_{2,0} = \pi/2,$$

i.e.,

$$x(0) = 0.005000, \quad \frac{\mathrm{d}x(0)}{\mathrm{d}t} = -0.013683, \quad \frac{\mathrm{d}^2x(0)}{\mathrm{d}t^2} = 0.010000,$$
$$\frac{\mathrm{d}^3(0)}{\mathrm{d}t^3} = 0.039641 \quad \text{and} \quad \frac{\mathrm{d}^4x(0)}{\mathrm{d}t^4} = -0.040000.$$

In these cases, the perturbation results obtained by the solution (25) and the corresponding numerical results computed by a fourth order Runge–Kutta method with a small time increment $\Delta t = 0.05$, are plotted in the Fig. 1. The correlation between these two results has also been calculated: which is 0.999999956.

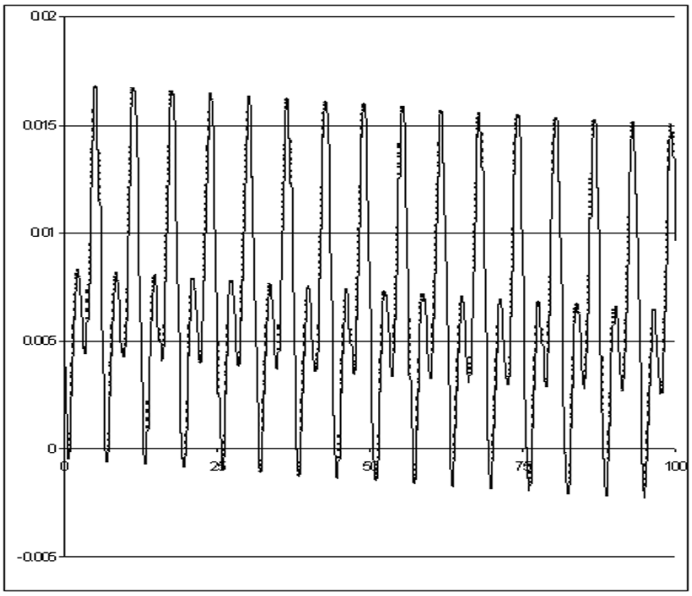

Fig. 1. Perturbation solution plotted by dot line and numerical solution plotted by continuous line.

Secondly, we have considered $\omega_1 = 1.5$, $\omega_2 = 2.0$, $\xi = 0.25$ and $\varepsilon = 0.1$ with initial conditions

$$a_0 = 0.20, \quad b_0 = 0.20, \quad c_0 = 0.40, \quad \varphi_{1,0} = \pi/2 \quad \text{and} \quad \varphi_{2,0} = \pi/3,$$

i.e.,

$$x(0) = 0.541421, \quad \frac{\mathrm{d}x(0)}{\mathrm{d}t} = -0.683469, \quad \frac{\mathrm{d}^2x(0)}{\mathrm{d}t^2} = -0.540695,$$
$$\frac{\mathrm{d}^3x(0)}{\mathrm{d}t^3} = 1.801215 \quad \text{and} \quad \frac{\mathrm{d}^4x(0)}{\mathrm{d}t^4} = 2.264117.$$

In these cases, the perturbation results obtained by the solution (25), and the corresponding numerical results computed by a fourth-order Runge–Kutta method with a small time increment$\Delta t = 0.05$, are plotted in the Fig. 2. The correlation between these two results have also been calculated: which is 0.99992479.

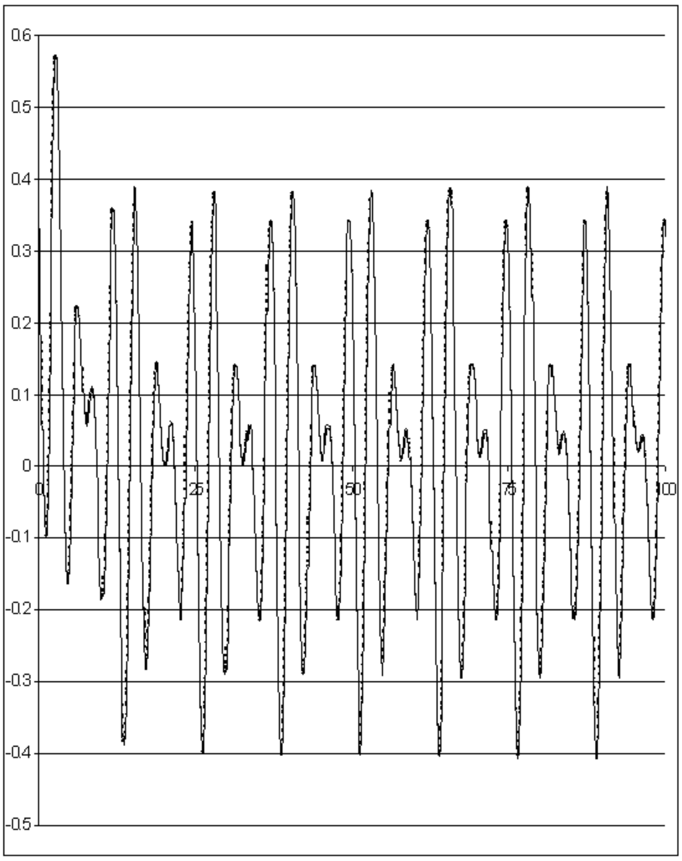

Fig. 2. Perturbation results are plotted by dotted line and numerical results are plotted by continuous line.

Finally, it can readily be seen that for $\omega_1 = \sqrt{2}$, $\omega_2 = \sqrt{3}$, $\xi = 0.4$ and $\varepsilon = 0.1$ with initial conditions

$$a_0 = 0.44, \quad b_0 = 0.22, \quad c_0 = 0.33, \quad \varphi_{1,0} = 0 \quad \text{and} \quad \varphi_{2,0} = 3.142,$$

i.e.,

$$x(0) = 0.550000, \quad \frac{\mathrm{d}x(0)}{\mathrm{d}t} = -0.022147, \quad \frac{\mathrm{d}^2x(0)}{\mathrm{d}t^2} = -2.797450,$$
$$\frac{\mathrm{d}^3x(0)}{\mathrm{d}t^3} = 0.000866 \quad \text{and} \quad \frac{\mathrm{d}^4x(0)}{\mathrm{d}t^4} = 21.313951.$$

In these cases, the perturbation results obtained by the solution (25), and the corresponding numerical results computed by a fourth-order Runge–Kutta method with a small time increment $\Delta t = 0.05$ are plotted in Fig. 3. The correlation between these two results have also been calculated: which is 0.999879695.

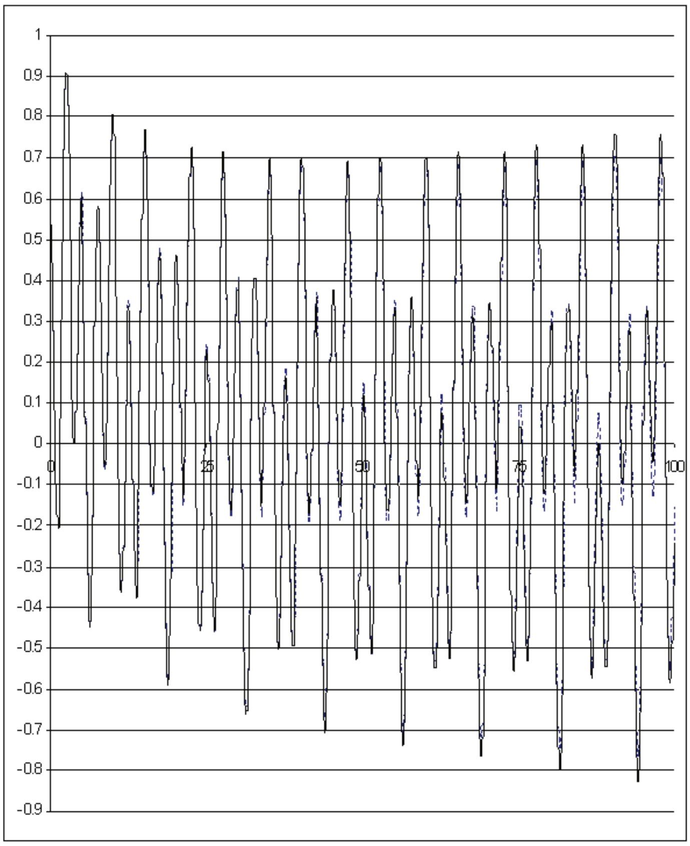

Fig. 3. Perturbation solution plotted by dot line and numerical solution plotted by continuous line.

5 Conclusion

A formula is obtained in this article for obtaining the analytical approximate solution of fifth order nonlinear differential systems, characterized by an oscillatory process, which is based on the KBM [1, 2] method. The correlation between the results of the perturbation solution and the corresponding numerical solution obtain by a fourth-order Runge–Kutta method have been calculated, which shows that these two results are strongly-correlated. The results obtained for different sets of initial conditions, as well as different sets of eigenvalues, show a good coincidence with corresponding numerical results.

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