An efficient new iterative method for finding exact solutions of nonlinear time-fractional partial differential equations

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Abstract. In this paper, a new iterative method (NIM) is used to obtain the exact solutions of some nonlinear time-fractional partial differential equations. The fractional derivatives are described in the Caputo sense. The method provides a convergent series with easily computable components in comparison with other existing methods.

Keywords: convergent series, fractional calculus, Caputo derivative, exact solution.

1 Introduction

In recent years, notable contributions have been made to both the theory and applications of the fractional differential equations. These equations are increasingly used to model problems in research areas as diverse as population dynamics, mechanical systems, fiber optics, control, chaos, fluid mechanics, continuous-time random walks, anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomenon, dynamical systems and others. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus [1–4] has become more and more popular.

In general, there exists no method that yields an exact solution for a fractional differential equation. Approximation and numerical solutions are used extensively [5–10]. In the present paper, we use new iterative method (NIM) to construct an exact solution to the generalized nonlinear biological population equation [11–14] with time-fractional derivatives of the form [15–17]:

$$D_t^\alpha u = \left(u^2\right)_{xx} + \left(u^2\right)_{yy} + f(u), \quad t > 0,\ \ x, y \in \Re,\ \ 0 < \alpha \le 1, \tag{1}$$

$$f(u) = hu^a\left(1 - ru^b\right), \tag{2}$$

$$u(x, y, 0) = g(x, y), \tag{3}$$

where u denotes population density, f represents the population supply due to births and deaths, h, a, r, b are real numbers, g is given initial condition and D^α denotes the differential operator in the sense of Caputo.

Also, we use new iterative method to obtain an exact solution of following system of three nonlinear time-fractional partial differential equations [18, 19]:

$$\begin{aligned} D_t^\alpha u &= -v_x w_y + v_y w_x - u, \\ D_t^\alpha v &= -w_x u_y - w_y u_x + v, \quad t > 0,\ \ 0 < \alpha \le 1, \\ D_t^\alpha w &= -u_x v_y - u_y v_x + w, \end{aligned} \tag{4}$$

with initial conditions

$$u(x, y, 0) = h_1(x, y), \quad v(x, y, 0) = h_2(x, y), \quad w(x, y, 0) = h_3(x, y), \tag{5}$$

where D^α denotes the differential operator in the sense of Caputo.

The structure of this article is as follows:

We begin by reviewing the procedure of the new iterative method [6, 20–24]. In Section 3, we introduce some necessary definitions and mathematical preliminaries of the fractional calculus theory in the Caputo sense which are required for establishing our results. In Section 4, we extend the application of the new iterative method to construct our exact solutions for the time-fractional biological population model given by Eqs. (1)–(3) and a system of three nonlinear time-fractional partial differential equations given by Eqs. (4)–(5). We present three examples to show the efficiency and simplicity of the new iterative method. The method is used in a direct way [6, 20–24] without using any linearization, perturbation, polynomials or restrictive assumptions in comparison with other existing methods [14–19].

2 The new iterative method (NIM)

Daftardar-Gejji and Jafari [20] have considered the following functional equation:

$$u = f + L(u) + N(u), \tag{6}$$

where L is a linear operator, N is a nonlinear operator and f is a known function.

We are looking for a solution u of Eq. (6) having the series form:

$$u = \sum_{i=0}^{\infty} u_i. \tag{7}$$

Since L is a linear operator,

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L(u_i). \tag{8}$$

The nonlinear operator N is decomposed as [20–22]

$$\begin{aligned} N\left(\sum_{i=0}^{\infty} u_i\right) &= N(u_0) + \bigl\{N(u_0+u_1) - N(u_0)\bigr\} + \bigl\{N(u_0+u_1+u_2) - N(u_0+u_1)\bigr\} \\ &\quad + \bigl\{N(u_0+u_1+u_2+u_3) - N(u_0+u_1+u_2)\bigr\} + \cdots \\ &= N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \end{aligned} \tag{9}$$

From Eqs. (7)–(9), Eq. (6) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + L\left(\sum_{i=0}^{\infty} u_i\right) + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}. \tag{10}$$

We define the recurrence relation

$$\begin{aligned} u_0 &= f, \\ u_1 &= L(u_0) + N(u_0), \\ u_{n+1} &= L(u_n) + \bigl\{N(u_0 + \cdots + u_n) - N(u_0 + \cdots + u_{n-1})\bigr\} \\ &= L(u_n) + \left\{N\left(\sum_{j=0}^{n} u_j\right) - N\left(\sum_{j=0}^{n-1} u_j\right)\right\}, \quad n = 1, 2, \ldots. \end{aligned} \tag{11}$$

Then,

$$\sum_{i=1}^{n+1} u_i = L\left(\sum_{i=0}^{n} u_i\right) + N\left(\sum_{i=0}^{n} u_i\right) \tag{12}$$

and

$$\sum_{i=0}^{\infty} u_i = f + L\left(\sum_{i=0}^{\infty} u_i\right) + N\left(\sum_{i=0}^{\infty} u_i\right). \tag{13}$$

It is clear from Eq. (13) that $\sum_{i=0}^{\infty} u_i$ is solution of Eq. (6). Where u_i, $i = 0, 1, 2, \ldots$, are given by algorithm (11). Also, The k-term approximate solution of Eq. (6) can be given by $\sum_{i=0}^{k-1} u_i$. We refer to [20–22] for details of the convergence.

3 Fractional calculus

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 1. A real function $f(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in \Re$ if there exists a real number p $(> \mu)$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,\infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$, $m \in N$.

Definition 2. The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \quad \alpha > 0, \ \ x > 0,$$
$$J^0 f(x) = f(x).$$

Properties of the operator J^α can be found in [1–4], we mention only the following. For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$;
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$;
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1) x^{\alpha+\gamma}}$.

The Riemann–Liouville derivative has certain disadvantages when trying to model real world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D^α proposed by Caputo in his work on the theory of viscoelasticity [25].

Definition 3. The fractional derivative $f(x)$ in the Caputo sense is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)\,\mathrm{d}t \tag{14}$$

for $m - 1 < \alpha \leq m$, $m \in N$, $x > 0$, $f \in C_{-1}^m$.

Also, we need here two of its basic properties.

Lemma 1. *If $m - 1 < \alpha \leq m$, $m \in N$ and $f \in C_\mu^m$ $\mu \geq -1$, then*

$$D^\alpha J^\alpha f(x) = f(x)$$

and

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}\bigl(0^+\bigr) \frac{x^k}{k!}, \quad x > 0.$$

The Caputo fractional derivatives are considered here because it allows traditional initial conditions to be included in the formulation of the problem. In this paper, we consider the biological population equation and a system of three partial differential equations with time-fractional derivatives and the fractional derivatives are taken in Caputo sense as follows.

Definition 4. For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,y,t) = \frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,y,\tau)}{\partial t^m}\, \mathrm{d}\tau & \text{for } m-1 < \alpha < m, \\ \frac{\partial^m u(x,y,t)}{\partial t^m} & \text{for } \alpha = m \in N. \end{cases} \tag{15}$$

For more information on the mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

4 Applications

In this section, we present three examples with analytical solution to show the efficiency of methods described in the Section 2

Example 1. We consider the Eqs. (1)–(3) with $a = 1$, $r = 0$ (*Malthusian law* [11]) and $g(x,y) = \sqrt{xy}$, we have the following time-fractional biological population equation with initial condition:

$$D_t^\alpha u = (u^2)_{xx} + (u^2)_{yy} + hu, \quad t > 0, \ 0 < \alpha \le 1, \tag{16}$$

$$u(x,y,0) = \sqrt{xy}. \tag{17}$$

Firstly, for the solution, applying operator J^α, the inverse of the operator D_t^α, on both side of Eq. (16), using the initial condition and Lemma 1 yields

$$u = \sqrt{xy} + J^\alpha\big((u^2)_{xx} + (u^2)_{yy}\big) + J^\alpha(hu). \tag{18}$$

According to the NIM, in view of the algorithm (11), we construct the following recurrence relation:

$$\begin{aligned} u_0 &= \sqrt{xy}, \\ u_1 &= J^\alpha\big((u_0^2)_{xx} + (u_0^2)_{yy}\big) + J^\alpha\big(hu_0\big), = \sqrt{xy}\frac{ht^\alpha}{\Gamma(\alpha+1)}, \\ u_2 &= J^\alpha\big(((u_0+u_1)^2)_{xx} + ((u_0+u_1)^2)_{yy}\big) - J^\alpha\big((u_0^2)_{xx} + (u_0^2)_{yy}\big) + J^\alpha(hu_1) \\ &= \sqrt{xy}\frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ &\dots \\ u_{n+1} &= J^\alpha\big(((u_0+\dots+u_n)^2)_{xx} + ((u_0+\dots+u_n)^2)_{yy}\big) \\ &\quad - J^\alpha\big(((u_0+\dots+u_{n-1})^2)_{xx} + ((u_0+\dots+u_{n-1})^2)_{yy}\big) + J^\alpha(hu_n) \\ &= \sqrt{xy}\frac{(ht^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)}, \quad n = 2,3,\dots. \end{aligned}$$

Thus, the solution of Eqs. (16)–(17) is

$$\sum_{i=0}^{\infty} u_i = u(x,y,t) = \sqrt{xy}\sum_{k=0}^{\infty}\frac{(ht^\alpha)^k}{\Gamma(k\alpha+1)} = \sqrt{xy}E_\alpha\big(ht^\alpha\big), \tag{19}$$

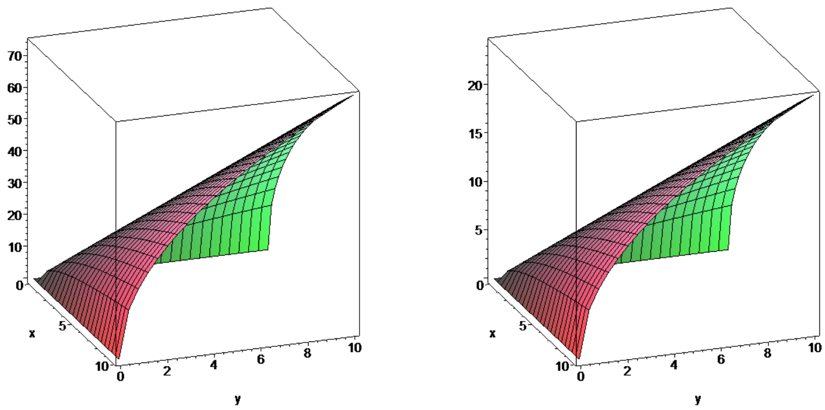
where $E_\alpha(ht^\alpha)$ is the Mittag–Leffler function defined as $E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+k\alpha)}$ [4].

For $\alpha \to 1$, we have

$$u(x,y,t) = \sqrt{xy}\sum_{k=0}^{\infty}\frac{(ht)^k}{k!} = \sqrt{xy}e^{ht} \tag{20}$$

which is an exact solution of the standard form biological population model [15].

Figure 1 shows the exact solutions of Eqs. (16)–(17) for $\alpha = 1$ and $\alpha = 0.5$ when $h = 0.2$ and $t = 10$.



(a) (b)

Fig. 1. The exact solution $u(x,y,t)$ of Eqs. (16)–(17) when $h = 0.2$ and $t = 10$: (a) $\alpha = 1$, (b) $\alpha = 0.5$.

Example 2. Consider the Eqs. (1)–(3) with $a = 1, b = 1$ (*Verhulst law* [11]) and $g(x,y) = e^{\sqrt{hr/8}(x+y)}$, we have the following time-fractional biological population equation with initial condition:

$$D_t^\alpha u = (u^2)_{xx} + (u^2)_{yy} + hu(1-ru), \quad t > 0, \ \ 0 < \alpha \le 1, \tag{21}$$

$$u(x,y,0) = e^{\sqrt{hr/8}(x+y)}. \tag{22}$$

Similar to previous example, for the solution, applying operator J^α on both side of Eq. (21), using the initial condition and Lemma 1 yields

$$u = e^{\sqrt{hr/8}(x+y)} + J^\alpha\big(\big(u^2\big)_{xx} + \big(u^2\big)_{yy} - hru^2\big) + J^\alpha(hu). \tag{23}$$

In view of the algorithm (11), we construct the recurrence relation:

$$
\begin{aligned}
u_0 &= e^{\sqrt{hr/8}(x+y)},\\
u_1 &= J^\alpha\big(\big(u_0^2\big)_{xx} + \big(u_0^2\big)_{yy} - hru_0^2\big) + J^\alpha(hu_0) = e^{\sqrt{hr/8}(x+y)}\frac{ht^\alpha}{\Gamma(\alpha+1)},\\
u_2 &= J^\alpha\big(\big((u_0+u_1)^2\big)_{xx} + \big((u_0+u_1)^2\big)_{yy} - hr(u_0+u_1)^2\big)\\
&\quad - J^\alpha\big(\big(u_0^2\big)_{xx} + \big(u_0^2\big)_{yy} - hru_0^2\big) + J^\alpha(hu_1) = e^{\sqrt{hr/8}(x+y)}\frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)},\\
&\ \ \dots\\
u_{n+1} &= J^\alpha\big(\big((u_0+\cdots+u_n)^2\big)_{xx} + \big((u_0+\cdots+u_n)^2\big)_{yy} - hr(u_0+\cdots+u_n)^2\big)\\
&\quad - J^\alpha\big(\big((u_0+\cdots+u_{n-1})^2\big)_{xx} + \big((u_0+\cdots+u_{n-1})^2\big)_{yy}\\
&\qquad - hr(u_0+\cdots+u_{n-1})^2\big) + J^\alpha(hu_n) = e^{\sqrt{hr/8}(x+y)}\frac{(ht^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)},\\
n &= 2,3,\dots.
\end{aligned}
$$

Thus, the solution of Eqs. (21)–(22) is

$$
\sum_{i=0}^\infty u_i = u(x,y,t) = e^{\sqrt{hr/8}(x+y)}\sum_{k=0}^\infty \frac{(ht^\alpha)^k}{\Gamma(k\alpha+1)} = e^{\sqrt{hr/8}(x+y)}E_\alpha\big(ht^\alpha\big), \tag{24}
$$

where $E_\alpha(ht^\alpha)$ is the Mittag–Leffler function.

For $\alpha \to 1$, we have

$$
u(x,y,t) = e^{\sqrt{hr/8}(x+y)}\sum_{k=0}^\infty \frac{(ht)^k}{k!} = e^{\sqrt{hr/8}(x+y)+ht} \tag{25}
$$

which is an exact solution of the standard form biological population model [16].

Example 3. Now, we consider the Eqs. (4)–(5) with $h_1(x,y) = e^{(x+y)}$, $h_2(x,y) = e^{(x-y)}$ and $h_3(x,y) = e^{(-x+y)}$.

Similar to previous examples, for the solution of system, applying operator J^α on both sides of Eqs. (4), using the above initial conditions and Lemma 1 yields

$$
\begin{aligned}
u &= e^{(x+y)}J^\alpha(-v_xw_y + v_yw_x) + J^\alpha(-u),\\
v &= e^{(x-y)}J^\alpha(-w_xu_y - w_yu_x) + J^\alpha(v), \quad t>0,\ \ 0<\alpha\le 1,\\
w &= e^{(-x+y)}J^\alpha(-u_xv_y - u_yv_x) + J^\alpha(w).
\end{aligned} \tag{26}
$$

In view of the algorithm (11), we construct the following recurrence relation:

$$
\begin{aligned}
u_0 &= e^{(x+y)},\\
v_0 &= e^{(x-y)},\\
w_0 &= e^{(-x+y)},\\
u_1 &= J^\alpha\big(-(v_0)_x(w_0)_y + (v_0)_y(w_0)_x\big) + J^\alpha(-u_0) = -e^{(x+y)}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
v_1 &= J^\alpha\big(-(w_0)_x(u_0)_y - (w_0)_y(u_0)_x\big) + J^\alpha(v_0) = e^{(x-y)}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
w_1 &= J^\alpha\big(-(u_0)_x(v_0)_y - (u_0)_y(v_0)_x\big) + J^\alpha(w_0) = e^{(-x+y)}\frac{t^\alpha}{\Gamma(\alpha+1)},\\
&\cdots\\
u_{n+1} &= J^\alpha\big(-(v_0+\cdots+v_n)_x(w_0+\cdots+w_n)_y + (v_0+\cdots+v_n)_y(w_0+\cdots+w_n)_x\big)\\
&\quad -J^\alpha\big(-(v_0+\cdots+v_{n-1})_x(w_0+\cdots+w_{n-1})_y\\
&\qquad +(v_0+\cdots+v_{n-1})_y(w_0+\cdots+w_{n-1})_x\big) + J^\alpha(-u_n)\\
&= e^{(x+y)}\frac{(-t^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)}, \quad n=1,2,\dots,\\
v_{n+1} &= J^\alpha\big(-(w_0+\cdots+w_n)_x(u_0+\cdots+u_n)_y - (w_0+\cdots+w_n)_y(u_0+\cdots+u_n)_x\big)\\
&\quad -J^\alpha\big(-(w_0+\cdots+w_{n-1})_x(u_0+\cdots+u_{n-1})_y\\
&\qquad -(w_0+\cdots+w_{n-1})_y(u_0+\cdots+u_{n-1})_x\big) + J^\alpha(v_n)\\
&= e^{(x-y)}\frac{(t^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)}, \quad n=1,2,\dots,\\
w_{n+1} &= J^\alpha\big(-(u_0+\cdots+u_n)_x(v_0+\cdots+v_n)_y - (u_0+\cdots+u_n)_y(v_0+\cdots+v_n)_x\big)\\
&\quad -J^\alpha\big(-(u_0+\cdots+u_{n-1})_x(v_0+\cdots+v_{n-1})_y\\
&\qquad -(u_0+\cdots+u_{n-1})_y(v_0+\cdots+v_{n-1})_x\big) + J^\alpha(w_n)\\
&= e^{(-x+y)}\frac{(t^\alpha)^{n+1}}{\Gamma((n+1)\alpha+1)}, \quad n=1,2,\dots.
\end{aligned}
$$

We get the solutions of Eqs. (4)–(5) as follows:

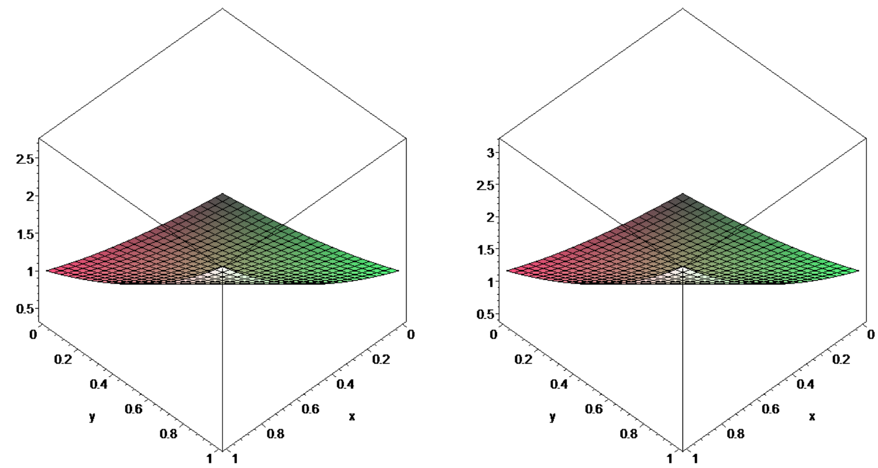
$$
\begin{aligned}
\sum_{i=0}^\infty u_i &= u(x,y,t) = e^{(x+y)}\sum_{k=0}^\infty \frac{(-t^\alpha)^k}{\Gamma(k\alpha+1)} = e^{(x+y)}E_\alpha\big(-t^\alpha\big),\\
\sum_{i=0}^\infty v_i &= v(x,y,t) = e^{(x-y)}\sum_{k=0}^\infty \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} = e^{(x-y)}E_\alpha\big(t^\alpha\big),\\
\sum_{i=0}^\infty w_i &= w(x,y,t) = e^{(-x+y)}\sum_{k=0}^\infty \frac{(t^\alpha)^k}{\Gamma(k\alpha+1)} = e^{(-x+y)}E_\alpha\big(t^\alpha\big).
\end{aligned}
\tag{27}
$$

For $\alpha \to 1$, we have

$$u(x,y,t) = e^{(x+y-t)}, \quad v(x,y,t) = e^{(x-y+t)}, \quad w(x,y,t) = e^{(-x+y+t)} \tag{28}$$

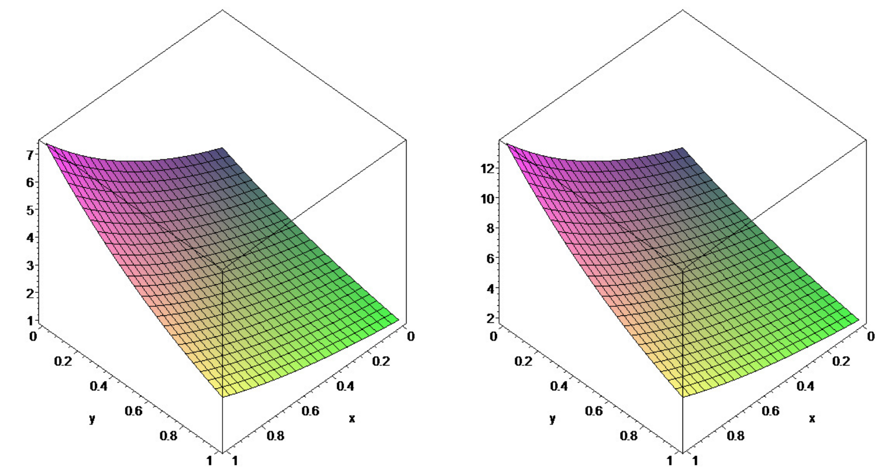
which are the exact solutions of the standard form system [18].

Figures 2–4 show the exact solutions of Eqs. (4)–(5) for $\alpha = 1$ and $\alpha = 0.5$ when $t = 1$.



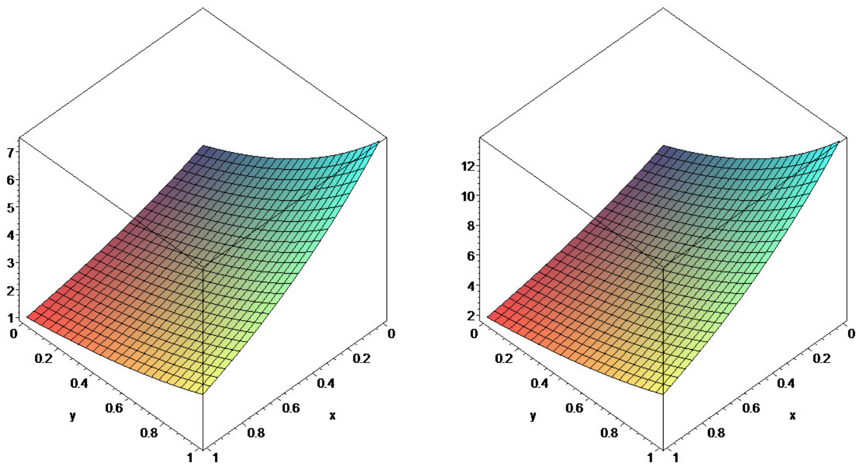
(a) (b)

Fig. 2. The exact solution $u(x,y,t)$ of Eqs. (4)–(5) when $t = 1$: (a) $\alpha = 1$, (b) $\alpha = 0.5$.



(a) (b)

Fig. 3. The exact solution $v(x,y,t)$ of Eqs. (4)–(5) when $t = 1$: (a) $\alpha = 1$, (b) $\alpha = 0.5$.



(a) (b)

Fig. 4. The exact solution $w(x, y, t)$ of Eqs. (4)–(5) when $t = 1$: (a) $\alpha = 1$, (b) $\alpha = 0.5$.

5 Conclusion

In this paper, we have used a new iterative method (NIM) for finding an exact solution of nonlinear time-fractional biological population model and a system of three nonlinear time-fractional partial differential equations. The method is used in a direct way without using any linearization, perturbation, polynomials or restrictive assumptions in comparison with other existing methods. Also, the method gives more realistic series solutions that converge very rapidly in nonlinear fractional problems. Thus, we conclude that new iterative method can be considered as an efficient method for solving linear and nonlinear problems.

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