The Laplace transform of Dirichlet L-functions

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Abstract. In the paper, a formula for the Laplace transform of the square of Dirichlet L-functions on the critical line is obtained.

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1 Introduction

Let $s = \sigma + it$ be a complex variable. The Laplace transform $\mathfrak{L}(s)$ of the function $f(x)$ is defined by

$$\mathfrak{L}(s) = \int_0^\infty f(x)\mathrm{e}^{-sx}\,\mathrm{d}x$$

provided the integral exists for $\sigma > \sigma_0$ with some σ_0. The function $\mathfrak{L}(s)$ can be applied for the investigation of the asymptotics for the mean value

$$M_T(f) \stackrel{\mathrm{def}}{=} \int_0^T f(x)\,\mathrm{d}x.$$

Suppose that $f(x) \geq 0$ for all $x \in [0, \infty)$ and, for a given $m > 0$,

$$\mathfrak{L}(\delta) \sim \frac{1}{\delta}\log^m\frac{1}{\delta}$$

as $\delta \to 0$. Then [1, Chap. VII],

$$M_T(f) \sim T\log^m T$$

as $T \to \infty$. The latter example shows that Laplace transforms can be used in the moment problem of zeta and L-functions. Let χ be a Dirichlet character mod q, and let $L(s,\chi)$ denote the corresponding L-function which is defined, for $\sigma > 1$, by

$$L(s,\chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s}.$$

Denote by χ_0 the principal character mod q. Then it is well known that the function $L(s,\chi_0)$ is analytically continued to the whole complex plane, except for a simple pole at $s = 1$ with residue

$$\prod_{p|q} \left(1 - \frac{1}{p}\right),$$

where p denotes a prime number. If $\chi \neq \chi_0$, then $L(s,\chi)$ is analytically continued to an entire function.

In the theory of Dirichlet L-functions, usually the moments

$$\sum_{\chi = \chi \,(\mathrm{mod}\, q)} \int_0^T \left|L(\sigma + \mathrm{i}t, \chi)\right|^{2k} \mathrm{d}t, \quad k \geq 0, \ \sigma \geq \frac{1}{2},$$

are considered, see, for example, [2]. This corresponds to the Laplace transform

$$\sum_{\chi = \chi \,(\mathrm{mod}\, q)} \int_0^\infty \left|L(\sigma + \mathrm{i}t, \chi)\right|^{2k} \mathrm{e}^{-sx} \, \mathrm{d}x.$$

The aim of this note is an explicit formula for the individual Laplace transform $\mathfrak{L}(s,\chi)$ of $|L(1/2 + \mathrm{i}x, \chi)|^2$, i.e., for the function

$$\mathfrak{L}(s,\chi) = \int_0^\infty \left|L\left(\frac{1}{2} + \mathrm{i}x, \chi\right)\right|^2 \mathrm{e}^{-sx} \, \mathrm{d}x.$$

In forthcoming papers, this formula will be applied for the investigation of the Mellin transform of $|L(1/2 + \mathrm{i}x, \chi)|^2$, for the case of $|\zeta(\rho + \mathrm{i}x)|^2$ with $1/2 < \rho < 1$, see [3]. In a series of papers, A. Ivič, M. Jutila and Y. Motohashi developed the method of Mellin transforms for the moment problem in the theory of zeta-functions, see, for example, [4]. These remarks show the motivation of the paper. For the statement of the formula, we need some notation.

Let $G(\chi)$ denote the Gauss sum, i.e.,

$$G(\chi) = \sum_{l=1}^{q} \chi(l) \mathrm{e}^{2\pi \mathrm{i} l/q}.$$

Moreover, let

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1, \end{cases}$$

$$E(\chi) = \begin{cases} \epsilon(\chi) & \text{if } a = 0, \\ \epsilon_1(\chi) & \text{if } a = 1, \end{cases}$$

where

$$\epsilon(\chi) = \frac{G(\chi)}{\sqrt{q}}, \quad \epsilon_1(\chi) = -\frac{G(\chi)}{\sqrt{q}}.$$

As usual, denote by $d(m)$ the divisor function

$$d(m) = \sum_{d|m} 1.$$

Theorem 1. *Suppose that χ is a primitive character* mod q*, $q > 1$. Then*

$$\mathfrak{L}(s,\chi) = \frac{2\pi \mathrm{i}^a \mathrm{e}^{-\mathrm{i}s/2}}{\sqrt{q}E(\chi)} \sum_{m=1}^{\infty} d(m)\chi(m) \exp\left\{-\frac{2\pi \mathrm{i} m}{q}\mathrm{e}^{-\mathrm{i}s}\right\} + \lambda(s,\chi),$$

where the function $\lambda(s,\chi)$ is analytic in the strip $\{s \in \mathbb{C}\colon |\sigma| < \pi\}$, and, for $|\sigma| \le \theta$, $0 < \theta < \pi$, the estimate

$$\lambda(s,\chi) = O\bigl((1+|s|)^{-1}\bigr)$$

is valid.

Let $\mu(m)$ be the Möbius function, and let γ_0 denote the Euler constant.

Theorem 2. *Suppose that χ_0 is the principal character* mod q*, $q > 1$. Then*

$$\begin{aligned} &\mathfrak{L}(s,\chi_0) \\ &\quad = \mathrm{i}\mathrm{e}^{\mathrm{i}s/2} \prod_{p|q}\left(1-\frac{1}{p}\right)\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2}-s\right)\mathrm{i} + \sum_{p|q}\frac{\log p}{p-1} + \sum_{m|q}\mu(m)\log m\right) \\ &\qquad + 2\pi \mathrm{e}^{-\mathrm{i}s/2} \sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^{\infty} d(k)\exp\left\{-\frac{2\pi \mathrm{i} k n}{m}\mathrm{e}^{-\mathrm{i}s}\right\} + \lambda(s,\chi_0), \end{aligned}$$

where the function $\lambda(s,\chi_0)$ has the same properties as $\lambda(s,\chi)$ in Theorem 1.

The case $q = 1$ corresponds the Riemann zeta-function. Let $\mathfrak{L}_\zeta(s)$ be the Laplace transform of $|\zeta(1/2+\mathrm{i}x)|^2$.

Corollary 1. *We have*

$$\begin{aligned} \mathfrak{L}_\zeta(s) = \mathrm{i}\mathrm{e}^{\mathrm{i}s/2}&\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2}-s\right)\mathrm{i}\right) \\ &+ 2\pi \mathrm{e}^{-\mathrm{i}s/2}\sum_{m=1}^{\infty} d(m)\exp\bigl\{-2\pi \mathrm{i} m \mathrm{e}^{-\mathrm{i}s}\bigr\} + \lambda(s), \end{aligned}$$

where function $\lambda(s)$ has the same properties as $\lambda(s,\chi)$.

2 Lemmas

First we remind the functional equation for $L(s,\chi)$. We preserve the notation used in introduction. Let

$$l(s,\chi)=\left(\frac{\pi}{q}\right)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi).$$

Lemma 1. *Suppose that χ is a primitive character* $\bmod q$, $q>1$. *Then*

$$l(s,\chi)=E(\chi)l(1-s,\bar{\chi}).$$

Proof of the lemma is given, for example, in [5].

Lemma 2. *For $\sigma>1$,*

$$L^2(s,\chi)=\sum_{m=1}^{\infty}\frac{d(m)\chi(m)}{m^s}.$$

Proof. For $\sigma>1$, we have that

$$L^2(s,\chi)=\sum_{m=1}^{\infty}\frac{\chi(m)}{m^s}\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}=\sum_{m=1}^{\infty}\frac{a(m)}{m^s},$$

where

$$a(m)=\sum_{d|m}\chi(d)\chi\left(\frac{m}{d}\right)=\chi(m)d(m). \qquad \square$$

Denote by $\varphi(q)$ the Euler totient function.

Lemma 3. *Let σ_0 be arbitrary real number. Then, for $\sigma\geq\sigma_0$,*

$$L(s,\chi)=\frac{E_0\varphi(q)}{q(s-1)}+O_{\sigma_0}\left(q^c\left(|t|+2\right)^c\right),$$

where $c=c(\sigma_0)>0$, and

$$E_0=\begin{cases}1 & \textit{if } \chi=\chi_0,\\ 0 & \textit{if } \chi\neq\chi_0.\end{cases}$$

The lemma is Theorem 7.3.2 from [6].

Lemma 4. *The estimate*

$$L'\left(\frac{1}{2}+\mathrm{i}t,\chi\right)=O\left(q^{c_1}\left(|t|+2\right)^{c_1}\right)$$

holds with some $c_1>0$.

Proof. The lemma is a corollary of Lemma 3 and the integral Cauchy formula. $\square$

Lemma 5. *Suppose that* $a > 0$. *Then*

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \Gamma(s) b^{-s} \, \mathrm{d}s = \mathrm{e}^{-b}.$$

The lemma is the well-known Mellin formula, see, for example, [7].

3 Proof of Theorems

Define the function $\lambda(s,\chi)$ by

$$\lambda(s,\chi) = \int\limits_0^\infty \left| L\left(\frac{1}{2} + \mathrm{i}x, \chi\right)\right|^2 \mathrm{e}^{-sx} \, \mathrm{d}x - \frac{\mathrm{e}^{-\mathrm{i}s/2}}{2\mathrm{i}^{1-a}} \int\limits_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \frac{L(z,\chi)L(1-z,\bar{\chi})\mathrm{e}^{-\mathrm{i}z(\pi/2-s)}}{\cos(\frac{\pi a}{2} - \frac{\pi z}{2})} \, \mathrm{d}z. \tag{1}$$

First suppose that $a = 0$. Then the formulae

$$\cos s = \frac{\mathrm{e}^{\mathrm{i}s} + \mathrm{e}^{-\mathrm{i}s}}{2} \quad \text{and} \quad \overline{L(s,\chi)} = L(\bar{s},\bar{\chi})$$

give

$$\begin{aligned}
&\frac{\mathrm{e}^{-\mathrm{i}s/2}}{2\mathrm{i}} \int\limits_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \frac{L(z,\chi)L(1-z,\bar{\chi})\mathrm{e}^{-\mathrm{i}z(\pi/2-s)}}{\cos\frac{\pi z}{2}} \, \mathrm{d}z \\
&= \mathrm{e}^{-\mathrm{i}s/2} \int\limits_{-\infty}^{\infty} \frac{L(\frac{1}{2}+\mathrm{i}x,\chi)L(\frac{1}{2}-\mathrm{i}x,\bar{\chi}) \exp\{-\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2} + \frac{\mathrm{i}s}{2} - xs\}}{\exp\{\frac{\pi\mathrm{i}}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2}\}} \, \mathrm{d}x \\
&= \int\limits_0^\infty \frac{|L(\frac{1}{2}+\mathrm{i}x,\chi)|^2 \exp\{-\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2} - xs\}}{\exp\{\frac{\pi\mathrm{i}}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2}\}} \, \mathrm{d}x \\
&\quad + \int\limits_0^\infty \frac{|L(\frac{1}{2}+\mathrm{i}x,\bar{\chi})|^2 \exp\{-\frac{\pi\mathrm{i}}{4} - \frac{\pi x}{2} + xs\}}{\exp\{\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2}\} + \exp\{-\frac{\pi\mathrm{i}}{4} - \frac{\pi x}{2}\}} \, \mathrm{d}x.
\end{aligned} \tag{2}$$

Since

$$1 - \frac{\exp\{-\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2}\}} = \frac{\exp\{\frac{\pi\mathrm{i}}{4} - \frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4} - \frac{\pi x}{2}\} + \exp\{-\frac{\pi\mathrm{i}}{4} + \frac{\pi x}{2}\}},$$

we find from (1) and (2) that

$$\lambda(s,\chi)=\int_0^\infty \frac{|L(\frac12+\mathrm{i}x,\chi)|^2\mathrm{e}^{-xs}\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}+\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}}\,\mathrm{d}x - \int_0^\infty \frac{|L(\frac12+\mathrm{i}x,\bar\chi)|^2\mathrm{e}^{xs}\exp\{-\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}+\exp\{-\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}\,\mathrm{d}x. \tag{3}$$

Now let $a=1$. In this case, in view of the formula

$$\sin s=\frac{\mathrm{e}^{\mathrm{i}s}-\mathrm{e}^{-\mathrm{i}s}}{2\mathrm{i}},$$

we find that

$$\begin{aligned}
&\frac{\mathrm{e}^{-\mathrm{i}s/2}}{2}\int_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\frac{L(z,\chi)L(1-z,\bar\chi)\mathrm{e}^{-\mathrm{i}z(\pi/2-s)}}{\sin\frac{\pi z}{2}}\,\mathrm{d}z\\
&\quad=-\mathrm{e}^{-\mathrm{i}s/2}\int_{-\infty}^{\infty}\frac{L(\frac12+\mathrm{i}x,\chi)L(\frac12-\mathrm{i}x,\bar\chi)\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}+\frac{\mathrm{i}s}{2}-xs\}}{\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}}\,\mathrm{d}x\\
&\quad=-\int_0^\infty\frac{|L(\frac12+\mathrm{i}x,\chi)|^2\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}-xs\}}{\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}}\,\mathrm{d}x\\
&\qquad-\int_0^\infty\frac{|L(\frac12+\mathrm{i}x,\bar\chi)|^2\exp\{-\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}+xs\}}{\exp\{\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}-\exp\{-\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}\,\mathrm{d}x.
\end{aligned}\tag{4}$$

Clearly, we have

$$1+\frac{\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}}=\frac{\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}},$$

thus, from (1) and (4), it follows that

$$\lambda(s,\chi)=\int_0^\infty \frac{|L(\frac12+\mathrm{i}x,\chi)|^2\mathrm{e}^{-xs}\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}-\exp\{-\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}}\,\mathrm{d}x + \int_0^\infty \frac{|L(\frac12+\mathrm{i}x,\bar\chi)|^2\mathrm{e}^{xs}\exp\{-\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}{\exp\{\frac{\pi\mathrm{i}}{4}+\frac{\pi x}{2}\}-\exp\{-\frac{\pi\mathrm{i}}{4}-\frac{\pi x}{2}\}}\,\mathrm{d}x. \tag{5}$$

By estimates for $L(1/2+\mathrm{i}x,\chi)$ of Lemma 3, we have that the integrals in (3) and (5) converge uniformly in compact subsets of the strip $\{s\in\mathbb{C}\colon |\sigma|<\pi\}$, thus, the function $\lambda(s,\chi)$ is analytic in that strip.

It remains to estimate the function $\lambda(s,\chi)$. Suppose that $|\sigma| \le \theta$, where $0 < \theta < \pi$. First let $|s|$ is small. Then the integrals in (3) and (5) are bounded by a constant. If $|s|$ is large, then integrating by parts with respect to $\mathrm{e}^{\pm sx}$ and using the estimate

$$\begin{aligned}
&\left(\left|L\left(\frac{1}{2}+\mathrm{i}x,\chi\right)\right|^2\right)' \\
&= \left(L\left(\frac{1}{2}+\mathrm{i}x,\chi\right)L\left(\frac{1}{2}-\mathrm{i}x,\bar{\chi}\right)\right)' \\
&= \mathrm{i}L'\left(\frac{1}{2}+\mathrm{i}x,\chi\right)L\left(\frac{1}{2}-\mathrm{i}x,\bar{\chi}\right) - \mathrm{i}L\left(\frac{1}{2}+\mathrm{i}x,\chi\right)L'\left(\frac{1}{2}-\mathrm{i}x,\bar{\chi}\right) \\
&= O\big(q\big(|x|+2\big)\big)^{c_2}
\end{aligned}$$

with some $c_2 > 0$, which follows from Lemmas 3 and 4, we obtain that

$$\lambda(s,\chi) = O\big(|s|^{-1}\big).$$

So, in all cases, we have that, for $|\sigma| \le \theta$, $0 < \theta < \pi$,

$$\lambda(s,\chi) = O\big(\big(1+|s|\big)^{-1}\big).$$

Equality (1) shows that

$$\mathfrak{L}(s,\chi) = \frac{\mathrm{e}^{-\mathrm{i}s/2}}{2\mathrm{i}^{1-a}} \int\limits_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty} \frac{L(z,\chi)L(1-z,\bar{\chi})\mathrm{e}^{-\mathrm{i}z(\pi/2-s)}}{\cos(\frac{\pi a}{2}-\frac{\pi z}{2})}\,\mathrm{d}z + \lambda(s,\chi), \tag{6}$$

where the function $\lambda(s,\chi)$ is analytic in the strip $\{s \in \mathbb{C}\colon |\sigma| < \pi\}$, and, for $|\sigma| \le \theta$, $0 < \theta < \pi$, the estimate

$$\lambda(s,\chi) = O\big(\big(1+|s|\big)^{-1}\big)$$

is valid.

It remains to calculate the integral in (6). Using Lemma 1, we find

$$L(1-z,\bar{\chi}) = E^{-1}(\chi)\left(\frac{\pi}{q}\right)^{-z+1/2} \frac{\Gamma(\frac{z}{2}+\frac{a}{2})}{\Gamma(\frac{1}{2}+\frac{a}{2}-\frac{z}{2})} L(z,\chi). \tag{7}$$

Taking into account the formulas, see [8],

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z},$$

and

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2\sqrt{\pi}2^{-2z}\Gamma(2z),$$

we obtain that

$$\frac{\Gamma(\frac{z}{2})}{\Gamma(\frac{1}{2}-\frac{z}{2})}=\frac{\Gamma(\frac{z}{2})\Gamma(\frac{1}{2}+\frac{z}{2})}{\Gamma(\frac{1}{2}-\frac{z}{2})\Gamma(\frac{1}{2}+\frac{z}{2})}=\frac{\Gamma(\frac{z}{2})\Gamma(\frac{1}{2}+\frac{z}{2})\cos\frac{\pi z}{2}}{\pi}$$
$$=\frac{2\sqrt{\pi}2^{-z}\Gamma(z)\cos\frac{\pi z}{2}}{\pi}=2^{1-z}\pi^{-1/2}\Gamma(z)\cos\frac{\pi z}{2},$$

and

$$\frac{\Gamma(\frac{1}{2}+\frac{z}{2})}{\Gamma(1-\frac{z}{2})}=\frac{\Gamma(\frac{1}{2}+\frac{z}{2})\Gamma(\frac{1}{2}-\frac{z}{2})}{\Gamma(1-\frac{z}{2})\Gamma(\frac{1}{2}-\frac{z}{2})}=\frac{\pi}{2\sqrt{\pi}2^{-1+z}\Gamma(1-z)\cos\frac{\pi z}{2}}$$
$$=\frac{\pi^{-1/2}\Gamma(z)\sin\pi z}{2^z\cos\frac{\pi z}{2}}=2^{1-z}\pi^{-1/2}\Gamma(z)\sin\frac{\pi z}{2}.$$

Since $a=0$ or $a=1$, hence we have that

$$\frac{\Gamma(\frac{z}{2}+\frac{a}{2})}{\Gamma(\frac{1}{2}+\frac{a}{2}-\frac{z}{2})}=2^{1-z}\pi^{-1/2}\Gamma(z)\cos\left(\frac{\pi a}{2}-\frac{\pi z}{2}\right),$$

and, in view of (7), this gives

$$L(1-z,\bar{\chi})=E^{-1}(\chi)2^{1-z}\pi^{-z}q^{z-1/2}\Gamma(z)\cos\left(\frac{\pi a}{2}-\frac{\pi z}{2}\right)L(z,\chi).$$

Hence, we find that

$$\frac{\mathrm{e}^{-\mathrm{i}s/2}}{2\mathrm{i}^{1-a}}\int\limits_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\frac{L(z,\chi)L(1-z,\bar{\chi})\mathrm{e}^{-\mathrm{i}z(\pi/2-s)}}{\cos(\frac{\pi a}{2}-\frac{\pi z}{2})}\,\mathrm{d}z$$
$$=\frac{\mathrm{e}^{-\mathrm{i}s/2}}{\mathrm{i}^{1-a}\sqrt{q}E(\chi)}\int\limits_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\Gamma(z)L^2(z,\chi)\left(\frac{2\pi\mathrm{i}}{q}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\mathrm{d}z. \tag{8}$$

Next we prove Theorem 1.

Proof. Let χ be a non-principal character. Then the integrand in (8) is a regular function in the strip $\{z\in\mathbb{C}\colon 1/2<\Re z<2\}$. Therefore,

$$\frac{\mathrm{e}^{-\mathrm{i}s/2}}{\mathrm{i}^{1-a}\sqrt{q}E(\chi)}\int\limits_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\Gamma(z)L^2(z,\chi)\left(\frac{2\pi\mathrm{i}}{q}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\mathrm{d}z$$
$$=\frac{\mathrm{e}^{-\mathrm{i}s/2}}{\mathrm{i}^{1-a}\sqrt{q}E(\chi)}\int\limits_{2-i\infty}^{2+\mathrm{i}\infty}\Gamma(z)L^2(z,\chi)\left(\frac{2\pi\mathrm{i}}{q}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\mathrm{d}z. \tag{9}$$

Moreover, in view of Lemmas 2 and 5,

$$\int\limits_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty} \Gamma(z)L^2(z,\chi)\left(\frac{2\pi\mathrm{i}}{q}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\mathrm{d}z = \sum_{m=1}^{\infty} d(m)\chi(m)\int\limits_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty}\Gamma(z)\left(\frac{2\pi\mathrm{i}m}{q}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\mathrm{d}z$$
$$= 2\pi\mathrm{i}\sum_{m=1}^{\infty} d(m)\chi(m)\exp\left\{-\frac{2\pi\mathrm{i}m}{q}\mathrm{e}^{-\mathrm{i}s}\right\}.$$

This, (8), (9) and (6) prove Theorem 1. □

Next we prove Theorem 2.

Proof. Let χ_0 be the principal character $\bmod q$. Then we have that

$$L(s,\chi_0) = \zeta(s)\prod_{p|q}\left(1-\frac{1}{p^s}\right),$$

where $\zeta(s)$ is the Riemann zeta-function. Using the functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

we find that

$$L(1-z,\chi_0) = \zeta(1-z)\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right)$$
$$= \zeta(z)2^{1-z}\pi^{-z}\Gamma(z)\cos\frac{\pi z}{2}\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right).$$

Therefore,

$$L(z,\chi_0)L(1-z,\chi_0)\mathrm{e}^{-\mathrm{i}z(\pi/2-s)}$$
$$= \zeta^2(z)2^{1-z}\pi^{-z}\Gamma(z)\cos\frac{\pi z}{2}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right),$$

and, similarly to the case of non-principal character, we obtain

$$\mathfrak{L}(s,\chi_0) = \frac{\mathrm{e}^{-\mathrm{i}s/2}}{\mathrm{i}}\int\limits_{1/2-\mathrm{i}\infty}^{1/2+\mathrm{i}\infty}\Gamma(z)\zeta^2(z)\left(2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right)\mathrm{d}z$$
$$+\lambda(s,\chi_0). \tag{10}$$

The product

$$\prod\nolimits_q(z) = \prod_{p|q}\left(1-\frac{1}{p^{1-z}}\right)$$

has zeroes lying on the line $\sigma = 1$. Therefore, we write $\prod_q(z)$ as a sum. Using the Möbius function $\mu(m)$, we obtain that

$$\prod\nolimits_q(z) = \sum_{p|q} \mu(m) m^{z-1}.$$

The function

$$\Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1}$$

has the pole of order 2 at $z = 1$. Moreover, since

$$\zeta(z) = \frac{1}{z-1} + \gamma_0 + \gamma_1(z-1) + \dots,$$

and $\Gamma(1) = 1$, $\Gamma'(1) = -\gamma_0$, we find that

$$\begin{aligned}
&\operatorname*{Res}_{z=1} \Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z} \\
&\quad = \lim_{z\to1} \bigl((z-1)^2\Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z}\bigr)' = -\frac{\mathrm{e}^{\mathrm{i}s}}{2\pi \mathrm{i}}\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)\mathrm{i}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\operatorname*{Res}_{z=1} \Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1} \\
&\quad = \lim_{z\to1} \left((z-1)^2\Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1}\right)' \\
&\quad = \lim_{z\to1} \bigl((z-1)^2\Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z}\bigr)' \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1} \\
&\qquad + \lim_{z\to1} (z-1)^2\Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z} \prod_{p|q} \left(1 - \frac{1}{p^z}\right) \sum_{p|q} \frac{\frac{1}{p^z}\log p}{1 - \frac{1}{p^z}} m^{z-1} \\
&\qquad + \lim_{z\to1} (z-1)^2\Gamma(z)\zeta^2(z)\bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-z} \log m \prod_{p|q} \left(1 - \frac{1}{p^z}\right) m^{z-1} \\
&\quad = \frac{\mathrm{e}^{\mathrm{i}s}}{2\pi \mathrm{i}}\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)\mathrm{i}\right) \prod_{p|q} \left(1 - \frac{1}{p}\right) \\
&\qquad + \bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-1} \prod_{p|q} \left(1 - \frac{1}{p}\right) \sum_{p|q} \frac{\log p}{p-1} + \bigl(2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}s}\bigr)^{-1} \log m \prod_{p|q} \left(1 - \frac{1}{p}\right) \\
&\quad = \frac{\mathrm{e}^{\mathrm{i}s}}{2\pi \mathrm{i}} \prod_{p|q} \left(1 - \frac{1}{p}\right)\left(\gamma_0 - \log 2\pi - \left(\frac{\pi}{2} - s\right)\mathrm{i} + \sum_{p|q} \frac{\log p}{p-1} + \log m\right).
\end{aligned}$$

Thus, in view of (10),

$$\begin{aligned}
&\mathfrak{L}(s,\chi_0)\\
&\quad= \mathrm{i}\mathrm{e}^{\mathrm{i}s/2}\prod_{p|q}\left(1-\frac{1}{p}\right)\left(\gamma_0-\log 2\pi-\left(\frac{\pi}{2}-s\right)\mathrm{i}+\sum_{p|q}\frac{\log p}{p-1}+\sum_{m|q}\mu(m)\log m\right)\\
&\qquad-\mathrm{i}\mathrm{e}^{-\mathrm{i}s/2}\int\limits_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty}\Gamma(z)\zeta^2(z)\left(2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\left(1-\frac{1}{p^{1-z}}\right)\mathrm{d}z\\
&\qquad+\lambda(s,\chi_0). \qquad (11)
\end{aligned}$$

We have that

$$\prod_{p|q}\left(1-\frac{1}{p^z}\right)\left(1-\frac{1}{p^{1-z}}\right)=\sum_{n|q}\sum_{m|q}\frac{\mu(m)}{m}\mu(n)\left(\frac{m}{n}\right)^z.$$

Moreover, for $\sigma>1$,

$$\zeta^2(s)=\sum_{k=1}^{\infty}\frac{d(k)}{k^s}.$$

Therefore, the application of Lemma 5, leads to

$$\begin{aligned}
&\int\limits_{2-\mathrm{i}\infty}^{2+\mathrm{i}\infty}\Gamma(z)\zeta^2(z)\left(2\pi\mathrm{i}\mathrm{e}^{-\mathrm{i}s}\right)^{-z}\prod_{p|q}\left(1-\frac{1}{p^z}\right)\left(1-\frac{1}{p^{1-z}}\right)\mathrm{d}z\\
&\quad=2\pi\mathrm{i}\sum_{n|q}\sum_{m|q}\frac{\mu(m)\mu(n)}{m}\sum_{k=1}^{\infty}d(k)\exp\left\{-\frac{2\pi\mathrm{i}kn}{m}\mathrm{e}^{-\mathrm{i}s}\right\}.
\end{aligned}$$

This together with (11) proves the theorem. □

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