Blow-up of the solution of a nonlinear Schrödinger equation system with periodic boundary conditions

Feliksas Ivanauskas, Gintaras Puriuškis

Faculty of Mathematics and Informatics, Vilnius University
Naugarduko str. 24, LT-03225 Vilnius, Lithuania
feliksas.ivanauskas@mif.vu.lt; gintaras.puriuskis@maf.vu.lt

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Abstract. We consider a system of nonlinear Schrödinger equations with periodic boundary conditions of the form

$$\begin{aligned}
&\mathrm{i}\frac{\partial u_j}{\partial t} + D^2u_j = -f_j(u,\overline{u}), \quad t \geqslant 0,\ \ x \in (-2,2),\\
&u_j(0,x) = u_{j0}(x), \quad x \in (-2,2),\\
&D^k u_j(t,-2) = D^k u_j(t,2), \quad t \geqslant 0,\ \ k = 0,1,
\end{aligned}$$

where $D = \partial/\partial x$, $j = 1,\dots,m$, $f_j(u,\overline{u}) = \partial g(u,\overline{u})/\partial\overline{u}$, and $\partial g/\partial u_j = \overline{f}_j$ for some homogenous function $g(u,\overline{u})$ such that $g(\lambda u,\lambda\overline{u}) = \lambda^6 g(u,\overline{u})$. We obtain sufficient conditions for blow-up of solutions of this system in $C^1([0,t_0);H^2(-2,2))$.

Keywords: Schrödinger equations, blow-up, periodic boundary condition.

1 Introduction

In this paper, we consider a following system of nonlinear Schrödinger equations with periodic boundary conditions of the form

$$\mathrm{i}\frac{\partial u_j}{\partial t} + D^2u_j = -f_j(u,\overline{u}), \quad t \geqslant 0,\ \ x \in I, \tag{1}$$

$$u_j(0,x) = u_{j0}(x), \quad x \in I, \tag{2}$$

$$D^k u_j(t,-2) = D^k u_j(t,2), \quad t \geqslant 0,\ \ k = 0,1, \tag{3}$$

where $D = \partial/\partial x$, $j = 1,\dots,m$, $I = (-2,2)$, $u = (u_1,\dots,u_m)$ is a vector function, $\overline{u} = (\overline{u}_1,\dots,\overline{u}_m)$, $\overline{u}_j$ is complex conjugate to u_j, and $f_j(u,\overline{u})$ are functions of $2m$ variables. We assume that the functions $f_j(u,\overline{u})$ satisfy the following conditions:

1) $$\operatorname{Im}\sum_{j=1}^m f_j(u,\overline{u})\overline{u}_j = 0, \tag{4}$$

2) there exists a differentiable function $g(u, \overline{u})$ of $2m$ variables such that

(a)
$$\frac{\partial g}{\partial \overline{u}_j} = f_j, \quad \frac{\partial g}{\partial u_j} = \overline{f}_j, \quad j = 1, \dots, m, \tag{5}$$

(b) $g(u, \overline{u})$ is a sixth-order homogeneous function, i.e.,

$$g(\lambda u, \lambda \overline{u}) = \lambda^6 g(u, \overline{u}), \quad \lambda \in \mathbb{R}, \tag{6}$$

(c) the real part of $g(u, \overline{u})$ is nonnegative for all u, i.e.,

$$\operatorname{Re} g(u, \overline{u}) \geqslant 0. \tag{7}$$

We suppose that the solution u of (1)–(3) is in $C^1([0, t_0); H^2(-2, 2))$. An example of system (1) satisfying conditions (4)–(7) is the following system:

$$\begin{aligned} &\mathrm{i}\frac{\partial u_1}{\partial t} + D^2 u_1 = -|u_2|^2 |u_3|^2 u_1, \\ &\mathrm{i}\frac{\partial u_2}{\partial t} + D^2 u_2 = -|u_1|^2 |u_3|^2 u_2, \\ &\mathrm{i}\frac{\partial u_3}{\partial t} + D^2 u_3 = -|u_1|^2 |u_2|^2 u_3, \end{aligned} \tag{8}$$

where $g(u, \overline{u}) = |u_1|^2 |u_2|^2 |u_3|^2$. For $m = 1$, system (1) generalizes the one-dimensional Schrödinger equation

$$\mathrm{i}\frac{\partial u}{\partial t} + D^2 u = -|u|^4 u, \tag{9}$$

where $g(u, \overline{u}) = |u|^6/3$.

In this paper, we obtain a sufficient condition for the blow-up of solutions (1)–(3): the solution of (1)–(3) blows up if

$$\sum_{j=1}^{m} \|D u_j\|_{L^2(I)} \to \infty \quad \text{as } t \to t_0$$

for some finite number $t_0 > 0$.

The problems concerning blow-up and stabilization for nonlinear Schrödinger equations and systems of nonlinear Schrödinger equations were considered in [1–20]. The blow-up problem of (9) in the whole real line $\mathbb{R}$ was considered by many authors, see [6, 7, 11, 16, 19]. System (1) of Schrödinger equations for $I = \mathbb{R}^n$ is considered in [3, 4]. The periodic solutions of Schrödinger equation are considered in [5, 9, 15]. Ogawa and Tsutsumi [15] found a sufficient condition for the blow-up of the periodic solution of the Schrödinger equation (9) for $I = (-2, 2)$. We set

$$E\big(u(t)\big) = \int_I \sum_{j=1}^{m} |D u_j|^2 \,\mathrm{d}x - \operatorname{Re} \int_I g(u, \overline{u}) \,\mathrm{d}x$$

and $E(u_0) = E_0$. In the case $I = \mathbb{R}$, the inequality $E(u_0) < 0$ is a sufficient condition for the solution of (1) and (2) to blow up in finite time $t_0 > 0$ (see [3]). However, in general, the condition $E(u_0) < 0$ is not sufficient for the blow-up of (1)–(3). For example, let us consider the initial-value problem of the following system of ordinary differential equations:

$$\mathrm{i}\frac{\partial z_j(t)}{\partial t} = -f_j(z, \overline{z}), \quad z(0) = z_{0j}, \quad j = 1, \dots, m. \tag{10}$$

For any fixed $z_{0j} \in \mathbb{C}$, problem (10) has a unique global solution. This solution is also a solution of problem (1)–(3), although the condition $E(u_0) < 0$ is satisfied.

Before stating our result, let us first give some notation. Let $KC^3(a, b)$ be the class of all functions $h : [a, b] \to \mathbb{R}$ satisfying the following conditions: $D^j h \in C(a, b) \cap L^\infty(a, b)$ for $j = 0, 1, 2$, $D^3 h$ may have a finite number of discontinuities in the interval (a, b), $D^3 h \in L^\infty(a, b)$. Let $\phi \in KC^3(\mathbb{R})$ be defined by

$$\phi(x) = \begin{cases} x, & 0 \leqslant x < 1, \\ x - (x-1)^3, & 1 \leqslant x < 1 + 1/\sqrt{3}, \\ h(x), & 1 + 1/\sqrt{3} \leqslant x < 2, \\ 0, & 2 \leqslant x, \end{cases}$$

where $Dh(x) \leqslant 0$ for $x \geqslant 1 + 1/\sqrt{3}$, $D^k h(2) = 0$, $k = 0, 1, 2$, and $\phi(-x) = \phi(-x)$. Set

$$\Phi(x) = \int_0^x \phi(y)\,\mathrm{d}y,$$

$$M_k = \left\|D^k\phi\right\|_{L^\infty}, \quad k = 1, 3, \quad M_2 = \max\left(\sqrt{3}, \frac{\|D^2\phi\|_{L^\infty}}{2}\right), \tag{11}$$

$$c = \max_{|u|=1} \left|g(u, \overline{u})\right|. \tag{12}$$

There it is known that $M_k \leqslant 537 + 297\sqrt{3} = 1051.419\dots$ if $h(x)$ is the sixth order polynomial, see [17]. Note that the maximum (12) always exists because the unit sphere $|u| = 1$ is a compact set and g is a continuous function. For example, $c = 1/27$ for system (8), and $c = 1/3$ for (9). For a positive integer k, we define

$$H^k_{prd} = \left\{v \in H^k(I);\ D^j v(-2) = D^j v(2),\ j = 0, 1, \dots, k-1\right\}.$$

The sufficient conditions of blow up solution is the following theorem in [15].

Theorem 1. *Let $u_0 \in H^1(I)$, $u_0(-2) = u_0(2)$ and $E(u_0) < 0$. In addition we assume that*

$$\eta = -2E(u_0) - 80(1+M)^2\|u_0\|^6_{L^2(I)} - \frac{M}{2}\|u_0\|^2_{L^2(I)} > 0,$$

$$\left(\int_I \Phi(x)\left|u_0(x)\right|^2\mathrm{d}x\right)\left(\frac{2}{\eta}\|Du_0\|^2_{L^2(I)} + 1\right) \leqslant \frac{1}{16},$$

where $M = \sum_{j=1}^{3} \|D^j\phi\|_{L^\infty(I)}$. *Then the solution* $u(t)$ *in* $H^1(\mathbb{R})$ *of* (1), (3), $u_j(t,-2) = u_j(t,2)$ *blows up in a finite time.*

Our main result is the following theorem.

Theorem 2. *Let* $u_{j0} \in H^2_{prd}$, $u(t)$ *be a solution of* (1)–(3) *in* $C^1([0,t_0); H^2(-2,2))$, *and let* f_j, $j = 1,\dots,m$, *satisfy conditions* (4)–(7). *In addition, assume that*

$$\eta = -2E(u_0) - \bigl(16M_2^2 + 32(1+M_1)\bigr)\frac{1}{32c\sqrt{32c}} - \frac{M_3}{2}\frac{1}{\sqrt{32c}} > 0, \tag{13}$$

$$\sum_{j=1}^{m} \int_I \Phi(x)\bigl|u_{j0}(x)\bigr|^2\,\mathrm{d}x\left(\frac{2}{\eta}\|Du_{j0}\|^2_{L^2(I)} + 1\right) \leqslant \frac{1}{4\sqrt{32c}}. \tag{14}$$

Then the solution $u(t)$ *blows up in finite time, i.e.,* $\sum_{j=1}^{m} \|Du_j\|_{L^2(I)} \to \infty$ *as* $t \to t_0$.

Note that the inequalities in Theorem 1 are satisfied if are satisfied the corresponding inequalities (13) and (14) in Theorem 2 for $m = 1$.

2 Proof of Theorem 2

In this section, we state several lemmas and prove Theorem 2.

Lemma 1. *Let* $u_{j0} \in H^2_{prd}$, $u_j(t)$ *be a solution of* (1)–(3), $u_j(t) \in C^1([0,t_0); H^2_{prd}(I))$, *and* f_j *satisfy* (4)–(7), $j = 1,\dots,m$. *Then the following two conservation laws hold for* $0 < t < t_0$:

$$\sum_{j=1}^{m} \bigl\|u_j(t)\bigr\|_{L^2(I)} = \sum_{j=1}^{m} \|u_{j0}\|_{L^2(I)}, \tag{15}$$

$$E\bigl(u(t)\bigr) = E_0. \tag{16}$$

Proof. We multiply the jth equation of (1) by $\overline{u}_j$, integrate over I, take the sum over $j = 1,\dots,m$, and take the imaginary part. Integrating by parts, we get that conditions (3) and (4) yield (15).

Now we prove (16). Equalities (5) imply

$$\operatorname{Re}\frac{\partial g}{\partial \overline{u}_j} = \operatorname{Re}\frac{\partial g}{\partial u_j}, \quad \operatorname{Im}\frac{\partial g}{\partial \overline{u}_j} = -\operatorname{Im}\frac{\partial g}{\partial u_j}, \tag{17}$$

and

$$\operatorname{Re}\frac{\partial g}{\partial u_j}\frac{\partial u_j}{\partial t} = \operatorname{Re}\frac{\partial g}{\partial \overline{u}_j}\frac{\partial \overline{u}_j}{\partial t}.$$

Hence,

$$\begin{aligned}\sum_{j=1}^{m}\operatorname{Re} f_j\frac{\partial \overline{u}_j}{\partial t} &= \sum_{j=1}^{m}\operatorname{Re}\frac{\partial g}{\partial \overline{u}_j}\frac{\partial \overline{u}_j}{\partial t} = \frac{1}{2}\operatorname{Re}\left(\sum_{j=1}^{m}\frac{\partial g}{\partial \overline{u}_j}\frac{\partial \overline{u}_j}{\partial t} + \sum_{j=1}^{m}\frac{\partial g}{\partial u_j}\frac{\partial u_j}{\partial t}\right)\\ &= \frac{1}{2}\operatorname{Re}\frac{\partial g}{\partial t}.\end{aligned} \tag{18}$$

We multiply the jth equation of (1) by $\partial\overline{u}_j/\partial t$, integrate over I, take the sum over $j = 1,\dots,m$, take the real part, and use (18) to obtain (16). □

The following lemma is Lemma 2.1 in [15].

Lemma 2. *Let $v \in H^1(I)$, $v(-2) = v(2)$, and ρ be a real-valued function such that $D\rho \in L^\infty$ and $\rho(-2) = \rho(2)$. Then we have*

$$\begin{aligned}&\|\rho v\|_{L^\infty(1<|x|<2)}\\ &\quad\leqslant \sqrt{2}\|v\|^{1/2}_{L^2(1<|x|<2)}\bigl[2\|\rho^2 Dv\|_{L^2(1<|x|<2)}\\ &\qquad+\sqrt{2}\|\rho^2 v\|^{1/2}_{L^2(1<|x|<2)}+\|vD\rho^2\|_{L^2(1<|x|<2)}\bigr]^{1/2}.\end{aligned} \tag{19}$$

Lemma 3. *Let $0 < t < t_0$, and $u_j(t)$ be a solution of* (1)–(3) *in $C^1([0,t_0); H^2_{prd})$, $j = 1,\dots,m$. Then we have*

$$\begin{aligned}&-\sum_{j=1}^m \operatorname{Im}\int_I \phi u_j(t) D\overline{u}_j(t)\,\mathrm{d}x + \sum_{j=1}^m \operatorname{Im}\int_I \phi u_{j0} D\overline{u}_{j0}\,\mathrm{d}x\\ &\quad=\int_0^t\Biggl(2\sum_{j=1}^m\int_I D\phi\bigl|Du_j(s)\bigr|^2\,\mathrm{d}x - 2\operatorname{Re}\int_I D\phi g\bigl(u_j(s),\overline{u}_j(s)\bigr)\,\mathrm{d}x\\ &\qquad-\frac{1}{2}\sum_{j=1}^m\int_I D^3\phi\bigl|u_j(s)\bigr|^2\,\mathrm{d}x\Biggr)\,\mathrm{d}s,\end{aligned} \tag{20}$$

$$\begin{aligned}&\int_I \varPhi\bigl|u_j(t)\bigr|^2\,\mathrm{d}x\\ &\quad=\int_I \varPhi|u_{j0}|^2\,\mathrm{d}x - 2\int_0^t\biggl(\operatorname{Im}\int_I \phi u_j(s) D\overline{u}_j(s)\,\mathrm{d}x\biggr)\,\mathrm{d}s, \quad j = 1,\dots,m,\end{aligned} \tag{21}$$

for $0 \leqslant t < t_0$.

Proof. We multiply the jth equation of (1) by $\phi D\overline{u}_j$, integrate over I, take the sum over $j = 1,\dots,m$, and take the real part. We use (17) and integrate by parts to obtain

$$\begin{aligned}&-\sum_{j=1}^m \frac{\partial}{\partial t}\operatorname{Im}\int_I \phi u_j(t) D\overline{u}_j(t)\,\mathrm{d}x - \sum_{j=1}^m \operatorname{Im}\int_I D\phi u_j(t)\frac{\partial\overline{u}_j}{\partial t}\,\mathrm{d}x\\ &\quad=\sum_{j=1}^m\int_I D\phi\bigl|Du_j(t)\bigr|^2\,\mathrm{d}x + \operatorname{Re}\int_I D\phi g\bigl(u(t),\overline{u}(t)\bigr)\,\mathrm{d}x.\end{aligned} \tag{22}$$

The homogenous function $g(u,\overline{u})$ satisfies the following Euler equality:

$$\sum_{j=1}^m \frac{\partial g}{\partial u_j}u_j + \sum_{j=1}^m \frac{\partial g}{\partial \overline{u}_j}\overline{u}_j = 6g. \tag{23}$$

Equalities (17) and (23) give

$$\sum_{j=1}^{m} \operatorname{Re} \overline{f}_j u_j = \sum_{j=1}^{m} \operatorname{Re} \frac{\partial g}{\partial u_j} u_j = \frac{1}{2} \operatorname{Re} \sum_{j=1}^{m} \left(\frac{\partial g}{\partial \overline{u}_j} u_j + \frac{\partial g}{\partial \overline{u}_j} \overline{u}_j \right) = 3 \operatorname{Re} g. \tag{24}$$

We next multiply the complex conjugate of (1) by $D\phi u_j$, integrate both sides over I, take the sum over $j = 1, \ldots, m$, and take the real part. We use (24) and integrate by parts to obtain

$$\begin{aligned} &\sum_{j=1}^{m} \operatorname{Im} \int_I D\phi u_j(t) \frac{\partial \overline{u}_j}{\partial t} \, \mathrm{d}x \\ &\quad = \sum_{j=1}^{m} \int_I D\phi \big|u_j(t)\big|^2 \, \mathrm{d}x - 3 \operatorname{Re} \int_I D\phi g\big(u(t), \overline{u}(t)\big) \, \mathrm{d}x \\ &\qquad - \frac{1}{2} \sum_{j=1}^{m} \int_I D^3\phi \big|u_j(t)\big|^2 \, \mathrm{d}x. \end{aligned} \tag{25}$$

Substituting (25) into (22) and integrating the both sides of (22) over $(0, t)$, we obtain (20).

We next multiply the complex conjugate of (1) by Φu_j, integrate both sides over I, take the sum over $j = 1, \ldots, m$, and take the imaginary part. We integrate by parts and use the the equality $\Phi(-2) = \Phi(2)$ to obtain (21). □

Lemma 4. *Let $u_j(t) \in H^1(I)$, $j = 1, \ldots, m$, $|u|^2 = \sum_{j=1}^{m} |u_j|^2$, and $I = (-2, 2)$. Then $D|u| \in L^2(I)$ and*

$$\begin{aligned} &\int_{1<|x|<2} (1 - D\phi)|u|^6 \, \mathrm{d}x \\ &\quad \leqslant 32\|u\|^4_{L^2(1<|x|<2)} \sum_{j=1}^{m} \int_{1<|x|<2} (1 - D\phi)|Du_j|^2 \, \mathrm{d}x \\ &\qquad + \big(32 + 32M_1 + 16M_2^2\big)\|u\|^6_{L^2(1<|x|<2)}. \end{aligned} \tag{26}$$

Proof. The inequality

$$\begin{aligned} &\int_I \big|\big(D|u|\big)\big|^2 \, \mathrm{d}x \\ &\quad = \int_I \frac{|\sum_{j=1}^{m} D(|u_j|^2)|^2}{4|u|^2} \, \mathrm{d}x = \int_I \frac{|\sum_{j=1}^{m} u_j D\overline{u}_j|^2}{|u|^2} \, \mathrm{d}x \\ &\quad \leqslant \int_I \frac{\sum_{j=1}^{m} |u_j|^2 \sum_{j=1}^{m} |D\overline{u}_j|^2}{|u|^2} \, \mathrm{d}x = \int_I \sum_{j=1}^{m} |Du_j|^2 \, \mathrm{d}x < \infty \end{aligned} \tag{27}$$

gives $D|u| \in L^2(I)$.

We next estimate the integral $\int_{1<|x|<2}(1-D\phi)g\,\mathrm{d}x$. Set $\rho(x)=(1-D\phi(x))^{1/4}$. We use inequalities (19) and

$$(a_1+a_2+a_3)^2\leqslant 2a_1^2+4a_2^2+4a_3^2,\quad a_k\in\mathbb{R},\ k=1,2,3,$$

for the estimate

$$\begin{aligned}\int\limits_{1<|x|<2}\rho^4|u|^6\,\mathrm{d}x &\leqslant \|u\|_{1<|x|<1}^2\|\rho u\|_{L^\infty(1<|x|<1)}^4\\ &\leqslant 4\|u\|_{L^2(1<|x|<2)}^4\big(2\big\|\rho^2D|u|\big\|_{L^2(1<|x|<2)}+\sqrt{2}\|\rho^2 u\|_{L^2(1<|x|<2)}\\ &\quad+\|uD\rho^2\|_{L^2(1<|x|<2)}\big)^2\\ &\leqslant 32\|u\|_{L^2(1<|x|<2)}^4\big\|\rho^2D|u|\big\|_{L^2(1<|x|<2)}^2+32\|u\|_{L^2(1<|x|<2)}^4\|\rho^2u\|_{L^2(1<|x|<2)}^2\\ &\quad+16\|u\|_{L^2(1<|x|<2)}^6\|D\rho^2\|_{L^\infty(1<|x|<2)}^2.\end{aligned}\tag{28}$$

We have (see the proof of Lemma 2.3 in [15])

$$\big\|D\rho^2\big\|_{L^\infty(1<|x|<2)}\leqslant M_2\tag{29}$$

and

$$\big\|\rho^2D|u|\big\|_{L^2(1<|x|<2)}^2\leqslant\sum_{j=1}^m\int\limits_{1<|x|<2}(1-D\phi)|Du_j|^2\,\mathrm{d}x.\tag{30}$$

The proof of (30) is similar to that of (27). Inequalities (28), (29), and (30) yield (26). □

Lemma 5. *Let* $0<t_0\leqslant\infty$*, and* $u_j(t)$ *be a solution of* (1)–(3) *in* $C^1([0,t_0);H^2_{prd})$, $j=1,\dots,m$*. If* $u_j(t)$ *satisfy*

$$\sum_{j=1}^m\big\|u_j(t)\big\|_{L^2(1<|x|<2)}^2<\frac{1}{\sqrt{32c}}\tag{31}$$

for $0\leqslant t<t_0$*, then we have*

$$\begin{aligned}&-\sum_{j=1}^m\operatorname{Im}\int\limits_I\phi u_j(t)D\overline{u}_j(t)\,\mathrm{d}x+\sum_{j=1}^m\operatorname{Im}\int\limits_I\phi u_{j0}D\overline{u}_{j0}\,\mathrm{d}x\\ &\quad\leqslant\left(2E(u_0)+\big(16M_2^2+32(1+M_1)\big)\frac{1}{32c\sqrt{32c}}+\frac{M_3}{2}\frac{1}{\sqrt{32c}}\right)t,\quad 0\leqslant t<t_0,\end{aligned}$$

where M_k*,* $k=1,2,3$*, and* c *are defined in* (11)–(12).

Proof. From the conservation law (16) we have

$$\sum_{j=1}^m\int\limits_{|x|<1}|Du_j|^2\,\mathrm{d}x=E_0-\sum_{j=1}^m\int\limits_{1<|x|<2}|Du_j|^2\,\mathrm{d}x+\operatorname{Re}\int\limits_I g\big(u(t),\overline{u}(t)\big)\,\mathrm{d}x.\tag{32}$$

Combining the equality $D\phi = 1$ for $|x| < 1$ and (20) with (32), we obtain

$$
\begin{aligned}
&-\sum_{j=1}^{m} \operatorname{Im} \int_I \phi u_j(t) D\overline{u}_j(t)\,\mathrm{d}x + \sum_{j=1}^{m} \operatorname{Im} \int_I \phi u_{j0} D\overline{u}_{j0}\,\mathrm{d}x \\
&= \int_0^t \biggl(2E_0 - 2\sum_{j=1}^{m} \int_{1<|x|<2} \bigl|Du_j(t)\bigr|^2\,\mathrm{d}x + 2\operatorname{Re}\int_I g\bigl(u(t),\overline{u}(t)\bigr)\,\mathrm{d}x \\
&\qquad + 2\sum_{j=1}^{m} \int_{1<|x|<2} D\phi\bigl|Du_j(t)\bigr|^2\,\mathrm{d}x - 2\operatorname{Re}\int_I D\phi g\bigl(u(t),\overline{u}(t)\bigr)\,\mathrm{d}x \\
&\qquad - \frac{1}{2}\sum_{j=1}^{m} \int_I D^3\phi\bigl|u_j(t)\bigr|^2\,\mathrm{d}x \biggr)\,\mathrm{d}t \\
&= \int_0^t \biggl(2E_0 - 2\sum_{j=1}^{m} \int_{1<|x|<2} (1-D\phi)\bigl|Du_j(t)\bigr|^2\,\mathrm{d}x \\
&\qquad + 2\operatorname{Re}\int_{1<|x|<2} (1-D\phi) g\bigl(u(t),\overline{u}(t)\bigr)\,\mathrm{d}x - \frac{1}{2}\sum_{j=1}^{m}\int_I D^3\phi\bigl|u_j(t)\bigr|^2\,\mathrm{d}x \biggr)\,\mathrm{d}t. \qquad (33)
\end{aligned}
$$

We use (12) to estimate the integral

$$
\operatorname{Re}\int_{1<|x|<2} (1-D\phi) g\bigl(u(t),\overline{u}(t)\bigr)\,\mathrm{d}x \leqslant c \int_{1<|x|<2} (1-D\phi)|u|^6\,\mathrm{d}x. \qquad (34)
$$

The inequalities $D\phi \leqslant 1$ and (7) give us that the left-hand side of inequality (34) is nonnegative. Inequalities (26), (31), and (34) imply

$$
\begin{aligned}
&\operatorname{Re}\int_{1<|x|<2} (1-D\phi) g\bigl(u(t),\overline{u}(t)\bigr)\,\mathrm{d}x \\
&\quad \leqslant 32c\|u\|^4_{L^2(1<|x|<2)} \sum_{j=1}^{m} \int_{1<|x|<2} (1-D\phi)|Du_j|^2\,\mathrm{d}x \\
&\qquad + (32 + 32M_1 + 16M_2)\frac{1}{32\sqrt{32c}}. \qquad (35)
\end{aligned}
$$

Inequalities (31) and (35) and Eq. (33) yield

$$
-\sum_{j=1}^{m} \operatorname{Im} \int_I \phi u_j(t) D\overline{u}_j(t)\,\mathrm{d}x + \sum_{j=1}^{m} \operatorname{Im} \int_I \phi u_{j0} D\overline{u}_{j0}\,\mathrm{d}x
$$

$$\leqslant \int_0^t \Bigg(2E_0 - 2\big(1 - 32c\|u\|^4_{L^2(1<|x|<2)}\big) \sum_{j=1}^m \int\limits_{1<|x|<2} \big|Du_j(t)\big|^2 \,\mathrm{d}x + (32 + 32M_1 + 16M_2)\frac{1}{32\sqrt{32c}} + \frac{M_3}{2\sqrt{32c}}\Bigg)\,\mathrm{d}t$$
$$\leqslant \int_0^t \bigg(2E_0 + (32 + 32M_1 + 16M_2)\frac{1}{32\sqrt{32c}} + \frac{M_3}{2\sqrt{32c}}\bigg)\,\mathrm{d}t$$
$$= \bigg(2E_0 + (32 + 32M_1 + 16M_2)\frac{1}{32\sqrt{32c}} + \frac{M_3}{2\sqrt{32c}}\bigg)t. \qquad \square$$

Proof of Theorem 2. Suppose, on the contrary, that the solution of (1)–(3) does not blow up for all $t \geqslant 0$. We first prove that condition (31) holds for all $t \geqslant 0$, while the solution $u(t)$ exists (does not blow up) if (14) is satisfied. Inequalities (14) and $1 \leqslant 2\Phi$ for $1 < |x| < 2$ yield

$$\sum_{j=1}^m \|u_{j0}\|^2_{L^2(1<|x|<2)} < \frac{1}{2\sqrt{32c}}.$$

The continuity of $\|u_j(t)\|_{L^2(1<|x|<2)}$ gives us that inequality (31) holds in the interval $[0, t_0)$ for some $t_0 > 0$. Suppose, on the contrary, that

$$\sum_{j=1}^m \big\|u_j(t_0)\big\|^2_{L^2(1<|x|<2)} = \frac{1}{\sqrt{32c}}. \tag{36}$$

The assumptions of Lemma 5 are satisfied for $t \in [0,\ t_0)$. Inequalities (13), (20), and (21) and Lemma 5 imply

$$\begin{aligned}
&\sum_{j=1}^m \int_I \Phi\big|u_j(t)\big|^2 \,\mathrm{d}x \\
&\quad= \sum_{j=1}^m \int_I \Phi|u_{j0}|^2 \,\mathrm{d}x - 2t\sum_{j=1}^m \mathrm{Im} \int_I \phi u_{j0} D\overline{u}_{j0} \,\mathrm{d}x - \eta t^2 \\
&\quad= -\eta\bigg(t + \frac{1}{\eta}\sum_{j=1}^m \mathrm{Im} \int_I \phi u_{j0} D\overline{u}_{j0} \,\mathrm{d}x\bigg)^2 + \frac{1}{\eta}\bigg(\sum_{j=1}^m \mathrm{Im} \int_I \phi u_{j0} D\overline{u}_{j0} \,\mathrm{d}x\bigg)^2 \\
&\qquad+ \sum_{j=1}^m \int_I \Phi|u_{j0}|^2 \,\mathrm{d}x \\
&\quad\leqslant \sum_{j=1}^m \bigg(\frac{1}{\eta}\|u_{j0}\|^2_{L^2(I)}\|Du_{j0}\|^2_{L^2(I)} + \int_I \Phi|u_{j0}|^2 \,\mathrm{d}x\bigg), \quad 0 \leqslant t < t_0.
\end{aligned} \tag{37}$$

We use the inequalities $\phi^2 \leqslant 2\Phi$ and (37) to obtain

$$\sum_{j=1}^{m} \int_I \Phi \left| u_j(t) \right|^2 \mathrm{d}x \\ \leqslant \sum_{j=1}^{m} \int_I \Phi(x) \left| u_{j0}(x) \right|^2 \mathrm{d}x \left(\frac{2}{\eta} \| D u_{j0} \|^2_{L^2(I)} + 1 \right), \quad 0 \leqslant t < t_0. \tag{38}$$

Inequalities (14), (38), and $1 \leqslant 2\Phi$ for $1 < |x| < 2$ yield

$$\sum_{j=1}^{m} \left\| u_j(t) \right\|^2_{L^2(1<|x|<2)} \leqslant 2 \sum_{j=1}^{m} \int_I \Phi \left| u_j(t) \right|^2 \mathrm{d}x < \frac{1}{2\sqrt{32c}}$$

for $0 \leqslant t < t_0$. The continuity of $\| u_j(t) \|_{L^2(1<|x|<2)}$ gives that

$$\sum_{j=1}^{m} \left\| u_j(t_0) \right\|^2_{L^2(1<|x|<2)} \leqslant \frac{1}{2\sqrt{32c}}.$$

It is a contradiction to (36). Hence, inequality (31) is satisfied for $t \geqslant 0$, while the solution $u(t)$ exists.

Finally, we prove that the solution $u(t)$ blows up. Inequality (37) implies that

$$\sum_{j=1}^{m} \int_I \Phi \left| u_j(t) \right|^2 \mathrm{d}x$$

becomes negative in finite time. Hence,

$$\sum_{j=1}^{m} \int_I \Phi \left| u_j(t) \right|^2 \mathrm{d}x \to 0, \quad t \to t_0, \tag{39}$$

for some $t_0 > 0$. The inequality $1 < 2\Phi$ for $1 < |x| < 2$ and the limit (39) give

$$\lim_{t \to t_0} \int_{1<|x|<2} \left| u_j(t) \right|^2 \mathrm{d}x \leqslant \lim_{t \to t_0} \int_{1<|x|<2} 2\Phi \left| u_j(t) \right|^2 \mathrm{d}x = 0 \tag{40}$$

and

$$\lim_{t \to t_0} \int_{1<|x|<2} |D\phi| \left| u_j(t) \right|^2 \mathrm{d}x \\ \leqslant M_1 \lim_{t \to t_0} \int_{1<|x|<2} \left| u_j(t) \right|^2 \mathrm{d}x = 0, \quad j = 1, \dots, m. \tag{41}$$

The equality

$$\int_I D\phi \left|u_j(t)\right|^2 \mathrm{d}x = -2\,\mathrm{Re}\int_I \phi u_j(t) D\overline{u}_j(t)\,\mathrm{d}x$$

yields

$$\left|\int_I D\phi\left|u_j(t)\right|^2 \mathrm{d}x\right| \leqslant 2\biggl(\int_I \phi^2\left|u_j(t)\right|^2 \mathrm{d}x\biggr)^{1/2}\biggl(\int_I \left|Du_j(t)\right|^2 \mathrm{d}x\biggr)^{1/2}, \quad j = 1,\dots,m. \tag{42}$$

The conservation law (15) and inequalities (40), (41), and (42) imply

$$\begin{aligned}
&\sum_{j=1}^m \int_I |u_{j0}|^2\,\mathrm{d}x \\
&\quad = \sum_{j=1}^m \lim_{t\to t_0} \int_{|x|<1} \left|u_j(t)\right|^2 \mathrm{d}x = \sum_{j=1}^m \lim_{t\to t_0} \int_{|x|<1} D\phi\left|u_j(t)\right|^2 \mathrm{d}x \\
&\quad = \sum_{j=1}^m \lim_{t\to t_0} \int_I D\phi\left|u_j(t)\right|^2 \mathrm{d}x \\
&\quad \leqslant \sum_{j=1}^m \lim_{t\to t_0} 2\biggl(\int_I \phi^2\left|u_j(t)\right|^2 \mathrm{d}x\biggr)^{1/2}\biggl(\int_I \left|Du_j(t)\right|^2 \mathrm{d}x\biggr)^{1/2}.
\end{aligned} \tag{43}$$

The inequalities $\phi^2 \leqslant 2\Phi$ and (43) give

$$\sum_{j=1}^m \int_I |u_{j0}|^2\,\mathrm{d}x \leqslant \sum_{j=1}^m \lim_{t\to t_0} 2\biggl(\int_I 2\Phi\left|u_j(t)\right|^2 \mathrm{d}x\biggr)^{1/2}\biggl(\int_I \left|Du_j(t)\right|^2 \mathrm{d}x\biggr)^{1/2}. \tag{44}$$

Note that (13) yields $\sum_{j=1}^m \int_I |u_{j0}|^2\,\mathrm{d}x > 0$. From (39) and (44) we have

$$\int_I \left|Du_j(t)\right|^2 \mathrm{d}x \to \infty, \quad t \to t_0,$$

for some $j = 1,\dots,m$, i.e., the solution blows up. □

References

1. T. Akahori, S. Ibrahim, H. Kikuchi, H. Nawa, Existence of a ground state and blow-up problem for a nonlinear Schrödinger equation with critical growth, *Diff. Integral Equ.*, **25**, pp. 383–402, 2012.

2. C. Besse, R. Carles, N.J. Mauser, H.P. Stimming, Monotonicity properties of the blow-up time for nonlinear Schrödinger equations: Numerical evidence, *Discrete Contin. Dyn. Syst., Ser. B*, **9**(1), pp. 11–36, 2008.

3. A. Domarkas, Collapse of solutions of a system of nonlinear Schrödinger equations, *Lith. Math. J.*, **31**(4), pp. 412–417, 1991.

4. A. Domarkas, F. Ivanauskas, Solvability of a mixed problem for a nonlinear system of equations of Schrödinger type, *Lith. Math. J.*, **27**(3), pp. 217–224, 1987.

5. M.B. Erdoğan, V. Zharnitsky, Quasi-linear dynamics in nonlinear Schrödinger equation with periodic boundary conditions, *Commun. Math. Phys.*, **281**(3), pp. 655–673, 2008.

6. R.T. Glassey, On the blowing up of solutions to the Cauchy problem for the nonlinear Schrödinger equations, *J. Math. Phys.*, **18**, pp. 1794–1797, 1977.

7. O. Kavian, A remark on the blowing-up of solutions to the Cauchy problem for nonlinear Schrödinger equations, *Trans. Am. Math. Soc.*, **299**, pp. 193–203, 1987.

8. P.G. Kevrekidis, D.E. Pelinovsky, A. Stefanov, Nonlinearity management in higher dimensions, *J. Phys. A, Math. Gen.*, **39**(3), pp. 479–488, 2006.

9. Z. Liang, Quasi-periodic solutions for 1D Schrödinger equation with the nonlinearity $|u|^{2p}u$, *J. Differ. Equations*, **244**, pp. 2185–2225, 2008.

10. Z. Lü, Z. Liu, L^2-concentration of blow-up solutions for two-coupled nonlinear Schrödinger equations, *J. Math. Anal. Appl.*, **380**(2), pp. 531–539, 2011.

11. F. Merle, Limit of the solution of the nonlinear Schrödinger equation at the blow-up time, *J. Funct. Anal.*, **84**, pp. 201–214, 1989.

12. C. Miao, G. Xu, L. Zhao, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equations of fourth order in dimensions $d \geqslant 9$, *J. Differ. Equations*, **251**(2), pp. 3381–3402, 2011.

13. D. Mohamad, Blow up for the damped L^2-critical nonlinear Schrödinger equation, *Adv. Differ. Equ.*, **17**(3–4), pp. 337–367, 2012.

14. G.D. Montesinos, D. Gaspar, V.M. Pérez-García, P.J. Torres, Stabilization of solitons of the multidimensional nonlinear Schrödinger equation: Matter-wave breathers, *Physica D*, **191**(3–4), pp. 193–210, 2004.

15. T. Ogawa, Y. Tsutsumi, Blow-up of solutions for the nonlinear Schrödinger equation with quartic potential and periodic boundary condition, in: *Functional-Analytic Methods for Partial Differential Equations, Proceedings of a Conference and a Symposium held in Tokyo, Japan, July 3–9, 1989*, Lect. Notes Math., Vol. 1450, Springer, Berlin, Heidelberg, 1990, pp. 236–251.

16. T. Ogawa, Y. Tsutsumi, Blow-up of H^1 solution for the nonlinear Schrödinger equation, *J. Differ. Equations*, **92**, pp. 317–330, 1991.

17. G. Puriuškis, An estimate for certain functional and its application, *Šiauliai Math. Semin.*, **6**(14), pp. 47–52, 2011.

18. T. Tao, M. Visan, X. Zhang, Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions, *Duke Math. J.*, **140**(1), pp. 165–202, 2007.

19. M.I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Commun. Math. Phys.*, **87**, pp. 567–576, 1983.

20. S. Zhu, H. Yang, J. Zhang, Blow-up of rough solutions to the fourth-order nonlinear Schrödinger equation, *Nonlinear Anal., Theory Methods Appl., Ser. A*, **74**(17), pp. 6186–6201, 2011.