Optimal management of a renewable resource utilized by a population with taxation as a control variable*

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Abstract. A dynamical model is proposed and analyzed to discuss the effect of population on a resource biomass by taking into account the crowding effect. It is assumed that the resource biomass, which has commercial importance, is subjected to harvesting. The harvesting effort is assumed to be a dynamical variable and taxation is being used as a control variable. The optimal harvesting policy is discussed using the Pontryagin's maximum principle.

Keywords: renewable resource, harvesting effort, taxation, stability.

1 Introduction

With the rapid growth of industrialization and population, the exploitation of several resources has increased significantly. Although the exploitation of resource is necessary for the growth and development of any country, however, unplanned exploitation will eventually lead to the extinction of these resources and consequently affecting the growth and survival of species dependent on these resources. In the last few decades, many researchers have done work on optimal management of renewable resources. The issues associated with these resources have been discussed in detail by Clark [1, 2]. Clark and De Pee [3] have discussed the implication of restricted malleability of capital for the optimal exploitation of a renewable resource stock. Chaudhuri [4, 5] proposed combined harvesting of two competiting fish species and analyzed the bio-economic and dynamic optimization. Kitabatake [6] discussed a model for fishing resource and proved that if the trawling efficiency in the catch of prey species is improved, then the use of diesel-powered trawling may lead to the extinction of predator as well as prey species. Dai and

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Tang [7] proposed a prey-predator system with constant rate of harvesting. They studied how to approximate the region of asymptotic stability in biological terms in the initial states that ultimately lead to co-existence of two-species. An optimal policy for combined harvest of two ecologically independent species which grow logistically and are harvested at a rate proportional to both stock and effort was discussed by Mesterton-Gibbons [8]. Ragozin and Brown [9] proposed a prey-predator model in which predator is selectively harvested and prey has no commercial value. Pradhan and Chaudhuri [10] analyzed the dynamics of a single species fishery in which fish species follows the Gompertz law of growth. A resource based model in three species fishery consisting of two predators and one prey with competition amongst predators was discussed by Chottopadhyay et al. [11]. They also found the conditions for persistence and global stability of the system. Dubey et al. [12] proposed a model for fishery resource system in an aquatic environment that consists of two zones: a free fishing zone and a reserve zone. This model is further modified by Kar and Misra [13]. Kar et al. [14] proposed a model to study the dynamics of two competing species which are harvested in the presence of a predator. Kar et al. [15] further studied the dynamics of a prey-predator model with non-monotonic functional response where both the species are harvested with constant efforts.

Regulation of renewable resources is a very important problem where an immediate attention is required to be paid. Taxation, license fees, lease of property rights, seasonal harvesting, fishing period control, creating reserve zones, etc. can be used as possible control instruments. In fishery resource management, some investigations have been carried out with taxation as a control instrument. Pradhan and Chaudhuri [16] proposed a mathematical model to study the growth and exploitation of a schooling fish species by imposing a tax on the catch to control the overexploitation of fish species. Dubey et al. [17] discussed a dynamical model for a single-species fishery, which depends partially on a logistically growing resource with functional response and taxation as a control instrument to protect fish population from over-exploitation. Dubey et al. [18] further analyzed a non-linear mathematical model to study the dynamics of an inshore-offshore fishery under variable harvesting by considering taxation as a control instrument. They also proved that the fishery resources can be protected from overexploitation by increasing the tax and discounted rate. Pradhan and Chaudhuri [19] also proposed and analyzed a dynamical reaction model of two species fishery with taxation as a control variable and then discussed its optimal harvesting policy. Ganguly and Chaudhuri [20] also discussed the bionomic exploitation of single-species fishery using taxation as a control variable. Recently, Huo et al. [21] extended the result of Dubey et al. [12] by considering the taxation as a control parameter.

In this present paper, we propose a model of resource biomass and population, where both of them grow logistically and population utilize resource for its own growth and development. The harvesting effort is considered to be a dynamical variable and tax as a control variable. Then we find existence of non-negative equilibria, condition of local as well as global stability analysis. Finally, choosing an appropriate Hamiltonian function the optimal harvesting policy is discussed. The main objective of this paper is to find an optimal taxation policy to give maximum profit to the harvesting community and to sustain the resource biomass at a desied level.

2 Mathematical model

Consider a habitat when a renewable resource is growing logistically. Then the dynamics of this resource biomass is governed by

$$\frac{\mathrm{d}B}{\mathrm{d}t} = a_0 B - a_1 B^2 = a_0 B\left(1 - \frac{B}{M}\right), \tag{1a}$$

where $B(t)$ is the resource biomass density, a_0 is its intrinsic growth rate and $M = a_0/a_1$ is the corresponding carrying capacity which the environment can support.

Now we assume that the resource biomass is being utilized by population of density $N(t)$ at any time $t \geqslant 0$ which may affect the intrinsic growth rate and carrying capacity of the resource biomass, and a_0 and M in Eq. (1a) may be regarded as a function of N. Thus, Eq. (1a) can be written as

$$\frac{\mathrm{d}B}{\mathrm{d}t} = a_0(N)B - \frac{f_0 B^2}{M(N)}, \tag{1b}$$

where f_0 is a positive constant. We take the following assumptions:

(i) The intrinsic growth rate $a_0(N)$ is a decreasing function of N and it satisfies $a_0(0) = r > 0$, $a_0'(N) \leqslant 0$ for $N \geqslant 0$.

We take a particular from of $a_0(N)$ as

$$a_0(N) = r - \alpha_1 N.$$

(ii) The carrying capacity $M(N)$ is also a decreasing function of N and it satisfies $M(0) = m_0 > 0$, $M'(N) \leqslant 0$ for $N \geqslant 0$.

We take a particular form of $M(N)$ as

$$M(N) = \frac{m_0}{1 + m_1 N}.$$

Let us denote $K = rm_0/f_0$ and $\alpha_2 = m_1 f_0/m_0$.

Then Eq. (1b) can be rewritten as

$$\frac{\mathrm{d}B}{\mathrm{d}t} = rB\left(1 - \frac{B}{K}\right) - \alpha_1 NB - \alpha_2 NB^2.$$

Now we assume that the population of density $N(t)$ is growing logistically and its growth rate as well as carrying capacity increases as the resource biomass density increases. Then in a similar way, the dynamics of population may be governed by the following differential equation:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = sN\left(1 - \frac{N}{L}\right) + \beta_1 NB + \beta_2 NB^2.$$

In this equation, s is the intrinsic growth rate of population, L is its carrying capacity in the absence of resource biomass, β_1 and β_2 are the growth rates of population in the presence of resource biomass.

Next, we assume that the resource biomass is subjected to a harvesting rate $h(t) = qEB$, where q is a positive constant and in fishery resource it is known as catchability coefficient, E is the harvesting effort. Let p be the fixed selling price per unit biomass and c the fixed cost of harvesting per unit of effort. Then the economic revenue is

$$R_0(t) = pqEB - cE.$$

In order to conserve the resource biomass, the regulating agency imposes a tax $\tau > 0$ per unit harvested resource biomass. Then the net economic revenue to the harvesting agency is

$$R(t) = (p - \tau)qEB - cE.$$

Thus, the dynamics of the harvesting effort can be governed by the following differential equation:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \alpha_0 E\{(p - \tau)qB - c\},$$

where α_0 is the stiffness parameter measuring the strength of reaction of effort to the perceived rent.

Keeping the above aspect in view, the dynamics of the system can be governed by the following system of differential equations:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = rB\left(1 - \frac{B}{K}\right) - \alpha_1 NB - \alpha_2 NB^2 - qEB, \tag{2a}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = sN\left(1 - \frac{N}{L}\right) + \beta_1 NB + \beta_2 NB^2, \tag{2b}$$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \alpha_0 E\{(p - \tau)qB - c\}, \tag{2c}$$

$$B(0) > 0, \quad N(0) > 0, \quad E(0) > 0.$$

In the next section, we shall discuss the stability analysis of model (2).

3 Stability analysis

First of all, we state the following lemma which represents a region of attraction of the model system (2a)–(2c).

Lemma 1. *The set*

$$\Omega = \left\{(B, N, E) \in R_3^+ \colon 0 \leqslant B \leqslant K,\ 0 \leqslant N \leqslant L_0,\ 0 \leqslant B(t) + E(t) \leqslant \frac{2rK}{\delta_0}\right\}$$

is a region of attraction for all solutions initiating in the interior of positive orthant, where $L_0 = (L/s)(s + \beta_1 K + \beta_2 K^2)$ *and* $\delta_0 = \min\{r, \alpha_0(p - \tau)qK - \alpha_0 c\}$, $p > c/(qK) + \tau$.

The above lemma shows that all solutions of the model (2a)–(2c) are non-negative and bounded, thus the model is biologically well-behaved.

The proof of this lemma is similar to [22, 23], so omitted.

To study the behavior of equilibrium points, we note that the model system (2a)–(2c) has the following six equilibrium points:

$$P_0(0,0,0), \quad P_1(K,0,0), \quad P_2(0,L,0),$$
$$P_3(\tilde{B},\tilde{N},0), \quad P_4(\bar{B},0,\bar{E}) \quad \text{and} \quad P^*(B^*,N^*,E^*).$$

The equilibria P_0, P_1 and P_2 always exist. We show the existence of other equilibria as follows.

Existence of $P_3(\tilde{B},\tilde{N},0)$. Here $\tilde{B}$ and $\tilde{N}$ are the positive solutions of the following algebraic equations:

$$r\left(1-\frac{B}{K}\right)-\alpha_1 N-\alpha_2 NB=0, \tag{3a}$$

$$s\left(1-\frac{N}{L}\right)+\beta_1 B+\beta_2 B^2=0. \tag{3b}$$

From Eq. (3b), we get

$$N=\frac{L}{s}\left(s+\beta_1 B+\beta_2 B^2\right).$$

Putting the value of N, in Eq. 3a, we get a cubic equation in B, i.e.,

$$a_1 B^3+a_2 B^2+a_3 B+a_4=0, \tag{3c}$$

where

$$\begin{aligned}
a_1 &= \alpha_2\beta_2 LK,\\
a_2 &= (\alpha_2\beta_1+\alpha_1\beta_2)LK,\\
a_3 &= rs+\alpha_1\beta_1 LK+s\alpha_2 LK,\\
a_4 &= (\alpha_1 L-r)sK.
\end{aligned}$$

Equation (3c) has a positive solution in B if

$$r>\alpha_1 L. \tag{3d}$$

Knowing the value of $B=\tilde{B}$, the value of $\tilde{N}$ can then be calculated from (3b).

Existence of $P_4(\bar{B},0,\bar{E})$. Here $\bar{B}$ and $\bar{E}$ are the positive solutions of the algebraic equations

$$r\left(1-\frac{B}{K}\right)-qE=0, \tag{4a}$$

$$\alpha_0\left((p-\tau)qB-c\right)=0. \tag{4b}$$

Solving the above equation, we get

$$\bar{B} = \frac{c}{(p-\tau)q} \quad \text{and} \quad \bar{E} = \frac{r}{q}\left\{1 - \frac{c}{Kq(p-\tau)}\right\}.$$

This shows that $\bar{E}$ exists if

$$0 < \tau < p - \frac{c}{Kq} \quad \text{and} \quad p > \tau \tag{4c}$$

hold.

Existence of $P^*(B^*, N^*, E^*)$. Here B^*, N^* and E^* are the positive solution of following algebraic equations:

$$r\left(1 - \frac{B}{K}\right) - \alpha_1 N - \alpha_2 NB - qE = 0, \tag{5a}$$

$$s\left(1 - \frac{N}{L}\right) + \beta_1 B + \beta_2 B^2 = 0, \tag{5b}$$

$$(p-\tau)qB - c = 0. \tag{5c}$$

Solving the above equations, we get

$$B^* = \frac{c}{q(p-\tau)}, \tag{6a}$$

$$N^* = \frac{L}{s}\left(s + \beta_1 B^* + \beta_2 B^{*2}\right), \tag{6b}$$

and

$$E^* = \frac{1}{q}\left\{r - \left(\frac{rB^*}{K} + \alpha_1 N^* + \alpha_2 N^* B^*\right)\right\}. \tag{6c}$$

Thus, E^* exists if the following hold:

$$p > \tau, \quad r > \frac{(\alpha_1 + \alpha_2 B^*)N^* K}{K - B^*}, \quad B^* < K. \tag{7}$$

From these conditions, we can conclude that, the non-zero equilibrium point exists if the intrinsic growth rate of the resource biomass must be larger than a threshold value.

Now we discuss the local and global stability behavior of these equilibrium points. For local stability analysis, first we find variational matrices at each equilibrium point. Then by using eigenvalue method and Routh–Hurwitz criteria, we get the following results:

(i) The point P_0 is a saddle point with unstable manifold in the $B-N$ plane and stable manifold in the E-direction.

(ii) If $0 < \tau < p - c/(Kq)$, then the point P_1 is always a saddle point with stable manifold in the B-direction and unstable manifold in the $N-E$ plane.

(iii) (a) If the point P_3 exists, then the point P_2 is a saddle point with unstable manifold in the B-direction and stable manifold in the $N-E$ plane.

(b) If the point P_3 does not exists, then P_2 is always locally asymptotically stable.

(iv) (a) If $0 < \tau < p - c/(q\tilde{B})$, then P_3 is a saddle point with stable manifold in the $B - N$ plane and unstable manifold in the E-direction.
(b) If $\tau > p - c/(q\tilde{B})$, then P_3 is locally asymptotically stable.

(v) The point P_4, whenever it exists, is a saddle point with stable manifold in the $B - E$ plane and unstable manifold in the N-direction.

To study the local stability behavior of the interior equilibrium P^*, we note that the characteristic equation of the variational matrix computed at P^* is given by

$$\lambda^3 + \bar{a}_1\lambda^2 + \bar{a}_2\lambda + \bar{a}_3 = 0, \tag{8a}$$

where

$$\begin{aligned}
\bar{a}_1 &= \frac{rB^*}{K} + \alpha_2 N^* B^* + \frac{sN^*}{L},\\
\bar{a}_2 &= \left(\frac{rB^*}{K} + \alpha_2 N^* B^*\right)\frac{sN^*}{L} + \left(\alpha_1 B^* + \alpha_2 {B^*}^2\right)(\beta_1 N^* + 2\beta_2 N^* B^*)\\
&\quad + \alpha_0 q^2 B^* E^*(p - \tau),\\
\bar{a}_3 &= \frac{\alpha_0 s q^2 B^* N^* E^*}{L}(p - \tau).
\end{aligned}$$

By Routh–Hurwitz criteria, it follows that all roots of Eq. (8a) have negative real parts iff

$$\bar{a}_1 > 0, \quad \bar{a}_3 > 0 \quad \text{and} \quad \bar{a}_1\bar{a}_2 > \bar{a}_3. \tag{8b}$$

Clearly $\bar{a}_1$is always positive, $\bar{a}_3 > 0$ iff $p > \tau$. It is easy to cheek that $\bar{a}_1\bar{a}_2 > \bar{a}_3$ holds true. Thus we can now state the following theorem.

Theorem 1. *The interior equilibrium P^*, whenever exists, is locally asymptotically stable.*

In the next theorem, we show that P^* is globally stable.

Theorem 2. *The interior equilibrium P^*, whenever exists, is globally asymptotically stable.*

Proof of Theorem 2 is given in Appendix.

In the next section, we discuss the bionomical equilibrium of the model system (2).

4 Bionomical equilibrium

The bionomical equilibrium is said to be achieved when the total revenue (TR) obtained by selling the harvested biomass is equal to the total cost (TC) for effort, i.e., the economic rent is completely dissipated.

Then net economic revenue at time t is given by

$$\pi(B, E, t) = (pqB - c)E.$$

The bionomical equilibrium is $P_\infty(B_\infty, N_\infty, E_\infty)$, where $B_\infty, N_\infty, E_\infty$ are the positive solutions of

$$\dot{B} = \dot{N} = \dot{E} = \pi = 0. \tag{9a}$$

Solving (9a), we get

$$B_\infty = \frac{c}{pq}, \tag{9b}$$

$$N_\infty = \frac{L}{s}\left[s + \frac{\beta_1 c}{pq} + \frac{\beta_2 c^2}{p^2 q^2}\right], \tag{9c}$$

$$E_\infty = \frac{r}{q}\left(1 - \frac{c}{Kpq}\right) - \left(\alpha_1 + \frac{c\alpha_2}{pq}\right)\frac{N_\infty}{q}. \tag{9d}$$

It is clear that $E_\infty > 0$ if

$$\frac{r}{q}\left(1 - \frac{c}{Kpq}\right) > \left(\alpha_1 + \frac{c\alpha_2}{pq}\right)\frac{N_\infty}{q}. \tag{10}$$

Thus the bionomical equilibrium $P_\infty(B_\infty, N_\infty, E_\infty)$ exists under condition (10).

If $E > E_\infty$, then the total costs exceed the total revenues. In such a case, some users will lose money and eventually some will drop out, thus reducing the level of harvesting effort. If $E < E_\infty$, then the total revenues exceed the total costs. In such a case, it attracts additional user and thus increasing the level of harvesting effort.

Remark 1. From (9b) and (10), it may be noted that $B_\infty = c/(pq) < K$.

5 The maximum sustainable yield

The maximum rate of harvesting of any biological resource biomass is called maximum sustainable yield (MSY) and any larger harvest rate will lead to the depletion of resource eventually to zero. In absence of any population, the value of MSY is given by [1]

$$h^0_{\mathrm{MSY}} = \frac{rK}{4}.$$

If the resource biomass is subjected to the harvesting by a population, the sustainable yield is given by

$$h = qEB^* = rB^*\left(1 - \frac{B^*}{K}\right) - \alpha_1 N^* B^* - \alpha_2 N^* {B^*}^2.$$

We note that $\partial h/\partial B^* = 0$ yields $B^* = K(r - \alpha_1 N^*)/(2(r + \alpha_2 K N^*))$ and $\partial^2 h/\partial {B^*}^2 < 0$.

Thus, $h_{\mathrm{MSY}} = K(r - \alpha_1 N^*)^2/(4(r + \alpha_2 N^* K))$, when $B^* = K(r - \alpha_1 N^*)/(2(r + \alpha_2 K N^*))$.

From the above equations, it is interesting to note that, when $N^* = 0$, then $B^* = K/2$ and

$$h_{\mathrm{MSY}} = \frac{rK}{4} = h^0_{\mathrm{MSY}}.$$

This result matches with the result of Clark [1].

If $h > h_{\mathrm{MSY}}$, then it denotes the overexploitation of the resource and if $h < h_{\mathrm{MSY}}$, then the resource biomass is under exploitation.

6 Optimal harvesting policy

A regulatory agency adopts the optimal harvesting policy to maximize the total discounted net revenue using taxation as a control instrument on the resource biomass.

The present value J of a continuous time-stream of revenues is given by

$$J = \int_0^\infty \mathrm{e}^{-\delta t}\bigl(pqB(t) - c\bigr)E(t)\,\mathrm{d}t,$$

where δ is the instantaneous rate of annual discount. Thus our objective is to

$$\max J$$

subject to the state equation (5a)–(5c) and to the control constraint

$$\tau_{\min} < \tau < \tau_{\max}.$$

To find the optimal level of equilibrium, we use Pontryagins's maximum principle. The associated Hamiltonian function is given by

$$\begin{aligned} H &= \mathrm{e}^{-\delta t}(pqB - c)E + \lambda_1\left[rB\left(1 - \frac{B}{K}\right) - \alpha_1 NB - \alpha_2 NB^2 - qEB\right] \\ &\quad + \lambda_2\left[sN\left(1 - \frac{N}{L}\right) + \beta_1 NB + \beta_2 NB^2\right] + \lambda_3\bigl[\alpha_0 E\bigl\{(p-\tau)qB - c\bigr\}\bigr], \end{aligned} \tag{11}$$

where λ_i, $i = 1, 2, 3$, are adjoint variables.

From Eq. (11), we note that H is linear in the control variable τ, hence the optimal control will be a combination of bang-bang control and singular control.

For H to be maximum on the control set $\tau_{\min} < \tau < \tau_{\max}$, we must have

$$\frac{\partial H}{\partial \tau} = 0, \quad \text{i.e.,} \quad \lambda_3 = 0. \tag{12}$$

This gives a necessary condition for a singular control to be optimal.

Note from the maximum principle,

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = -\frac{\partial H}{\partial B}, \quad \frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = -\frac{\partial H}{\partial N}, \quad \frac{\mathrm{d}\lambda_3}{\mathrm{d}t} = -\frac{\partial H}{\partial E}.$$

The above equation can be written as

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = -\mathrm{e}^{-\delta t} pqE - \lambda_1\left[r\left(1 - \frac{2B}{K}\right) - \alpha_1 N - 2\alpha_2 NB - qE\right] - \lambda_2[\beta_1 N + 2\beta_2 NB] - \lambda_3\alpha_0 E(p - \tau)q,$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \lambda_1\left[\alpha_1 B + \alpha_2 B^2\right] - \lambda_2\left[s\left(1 - \frac{2N}{L}\right) + \beta_1 B + \beta_2 B^2\right],$$

$$\frac{\mathrm{d}\lambda_3}{\mathrm{d}t} = -\mathrm{e}^{-\delta t}(pqB - c) + \lambda_1(qB) = 0.$$

Using (5a)–(5c) and (12), these three previous equations can be re-written as

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = -\mathrm{e}^{-\delta t} pqE + \lambda_1\left[\frac{rB}{K} + \alpha_2 NB\right] - \lambda_2[\beta_1 N + 2\beta_2 NB], \tag{13a}$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \lambda_1\left[\alpha_1 B + \alpha_2 B^2\right] + \lambda_2\frac{sN}{L}, \tag{13b}$$

$$\lambda_1 = \mathrm{e}^{-\delta t}\left(p - \frac{c}{qB}\right). \tag{13c}$$

Thus $\mu(t) = \lambda_1(t)\mathrm{e}^{\delta t} = p - c/(qB)$ is the usual shadow price along the singular path.

Putting the values of λ_1 in Eq. (13b), we get

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} - A_1\lambda_2 = -A_2\mathrm{e}^{-\delta t}, \tag{14}$$

where

$$A_1 = \frac{sN}{L}, \quad A_2 = -\left(p - \frac{c}{qB}\right)\left(\alpha_1 B + \alpha_2 B^2\right).$$

The solution of (14) is

$$\lambda_2 = \frac{A_2}{A_1 + \delta}\mathrm{e}^{-\delta t} + K_0\mathrm{e}^{A_1 t}.$$

Now when $t \to \infty$, then the shadow price $\lambda_2\mathrm{e}^{\delta t}$ is bounded if $K_0 = 0$.

Thus the solution is

$$\lambda_2 = \frac{A_2}{A_1 + \delta}\mathrm{e}^{-\delta t}.$$

Substituting the value of λ_1 and λ_2 in (13a), a little algebraic manipulation yields

$$E = \frac{p - \frac{c}{qB}}{pq}\left\{\left(\delta + \frac{rB}{K} + \alpha_2 NB\right) + \frac{(\alpha_1 B + \alpha_2 B^2)(\beta_1 N + 2\beta_2 NB)}{\frac{sN}{L} + \delta}\right\}, \tag{15}$$

$$\tau_\delta = p - \frac{c}{qB}. \tag{16}$$

Hence solving Eqs. (5a)–(5b) with the help of Eqs. (15) and (16), we get an optimal solution $(B_\delta, N_\delta, E_\delta)$ and the optimal tax τ_δ.

7 Numerical simulation

For the numerical simulation part of the model system (2a)–(2c), we choose the following set of values of parameters (others sets of values of parameters may exist):

$$r = 1.6,\quad s = 1.2,\quad K = 100,\quad L = 100,\quad p = 0.5,\quad q = 0.01,\quad \alpha_0 = 0.1,$$
$$\alpha_1 = 0.001,\quad \alpha_2 = 0.0001,\quad \beta_1 = 0.01,\quad \beta_2 = 0.0001,\quad c = 0.001,\quad \tau = 0.1. \tag{17}$$

For the above set of values of parameters, we note that the positive equilibrium $P^*(B^*, N^*, E^*)$ exists and is given by

$$B^* = 0.25,\quad N^* = 100.2089,\quad E^* = 149.3286.$$

It may also be noted that P^* is locally as well as globally asymptotically stable.

Now we plot the dynamics of the system for the set of values of parameters given in (17) with the help of MATLAB 6.1. The behavior of B, N and E with respect to time t is plotted in Fig. 1. From this figure, we note that B and N increase for a very short time and then they decrease and finally settle down at its steady state. However, E increases with time and attains its equilibrium level.

Figure 2 shows the behavior of B, N and E with different initial values. From Fig. 2, we see that all trajectories starting with different initial points converge to $P^*(0.25, 100.2089, 149.3286)$. Thus P^* is globally asymptotically stable.

Again we observe that α_2 and β_2 are important parameter in the model. We plot B, N and E with respect to time t for different values of α_2 and β_2. Here we observe the change of the behavior of B, N and E for different values of α_2 and β_2 as shown in Fig. 3 and Fig. 4, respectively. From these figures we note that if α_2 increases, then B and N initially decrease. But after a threshold value of α_2, the behavior is changed. If α_2 increases beyond this threshold value, then B and N decrease as α_2 decreases. Again B tends to zero level and N tends to its equilibrium level.

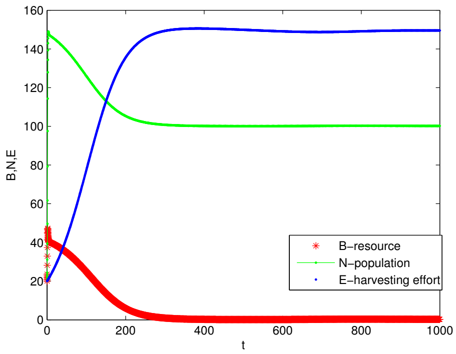

Fig. 1. Plot of B, N and E verses t for the values of parameters given in Eq. (17).

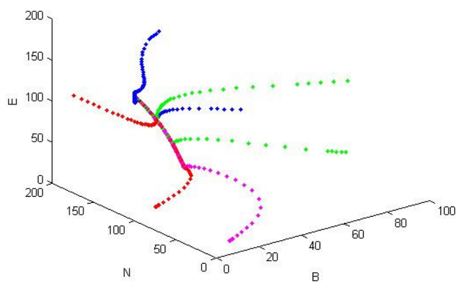

Fig. 2. Global stability of P^*.

For the set of parameters given in (17) with $\delta = 5$, solving Eqs. (15) and (16) with the help of Eqs. (5a)–(5c), we get optimal level of solution

$$B_\delta = 0.28491, \quad N_\delta = 100.2381, \quad E_\delta = 149.2348 \quad \text{and} \quad \tau_\delta = 0.149.$$

We observe that τ is also an important parameter which governs the dynamics of the system. The behavior of B, N and E with respect to time t for different values of τ are shown in Figs. 5–7, respectively. From these figure, we note that the densities of the resource biomass and population increase as τ increases, but the density of effort decreases as τ increases. For an optimal level of the tax imposed on per unit of harvested biomass, the resource biomass, the population and the effort settle down at their respective optimal level.

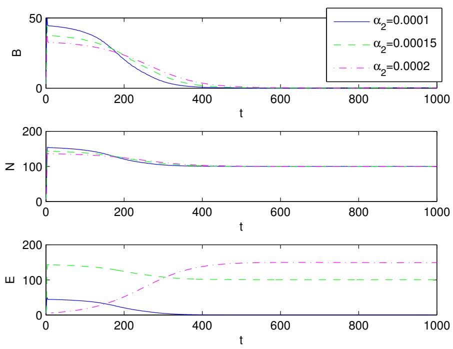

Fig. 3. Plot of B, N and E with respect to time t for different values of α_2, others values of parameters are same as given in Eq. (17).

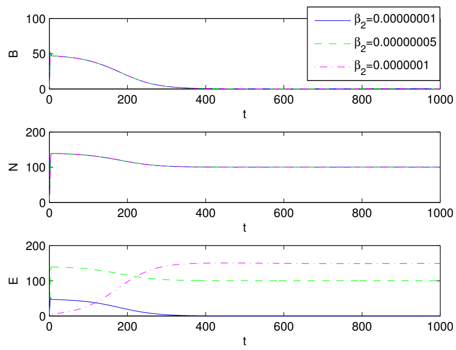

Fig. 4. Plot of B, N and E with respect to time t for different values of β_2, others values of parameters are same as given in Eq. (17).

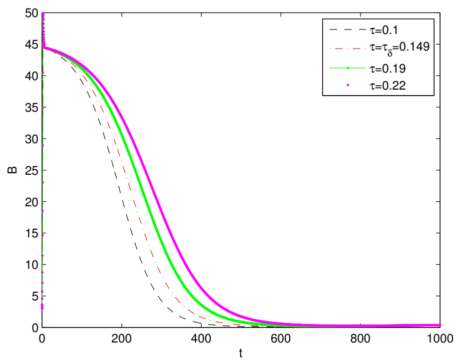

Fig. 5. Plot of B with respect to time t for different values of τ, others values of parameters are same as given in Eq. (17).

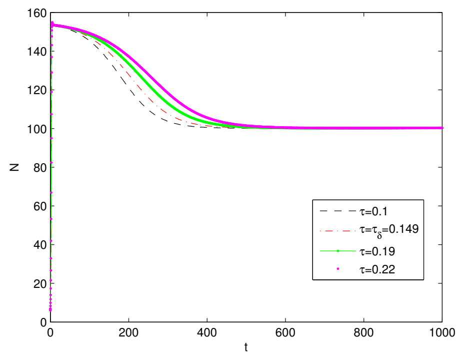

Fig. 6. Plot of N with respect to time t for different values of τ, others values of parameters are same as given in Eq. (17).

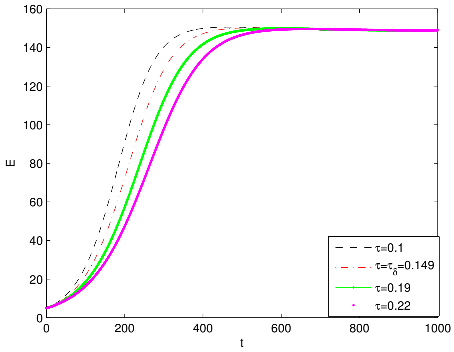

Fig. 7. Plot of E with respect to time t for different values of τ, others values of parameters are same as given Eq. (17).

8 Conclusions

In this paper, a mathematical model has been discussed where the resource biomass, which has commercial importance, is harvested according to catch-per-unit-effort hypothesis. The harvesting effort has been considered as a dynamical variable. The population utilizes the resource for its own growth and development. The population and the resource both are growing logistically. The existence of equilibrium points has been discussed and stability analysis has been carried out by eigenvalue method, Routh–Hurwitz criteria and Liapunov direct method. A threshold level of the intrinsic growth rate of resource biomass has been found and it has been shown that if the intrinsic growth rate of the resource biomass is larger than a threshold value and price of the per unit harvested biomass is larger than the tax imposed on it, then the non-zero equilibrium point P^* exists. And whenever P^* exists, it is always locally and globally asymptotically stable. When the population does not have any direct effect on the resource biomass, and the resource biomass is being continuously harvested, then Eq. (4c) gives a range of tax. This range of tax may be very useful by the regulatory agency at the time of formulating tax structure on per unit harvested resource biomass. But when the resource biomass is utilized by a population and it is also harvested, then the range of tax should be slightly modified as given in Eq. (7) keeping in view some other thresholds on the intrinsic growth rate of resource biomass. It has been also found that if the price of the harvested resource increases faster than cost of harvesting, then the resource biomass density shifts to a lower equilibrium level. This shows that price of the per unit harvested resource should not increase beyond a critical level, otherwise the survival of resource biomass will be threatened. It has also been observed that q (harvesting coefficient or catchability coefficient) increases with the advancement of technology due to which the resource biomass may further shift to a lower equilibrium level. Equation (6a) shows that the equilibrium level of resource biomass may be increased by increasing tax to a certain level on the harvested resource biomass. The bionomical equilibrium of the model has been found and it has been shown that the bionomical equilibrium of the resource biomass does not depend upon the growth rate and carrying capacity of population utilizing the resource biomass. The maximum sustainable yield (MSY) of the model has been obtained and it has been observed that if $h > h_{\mathrm{MSY}}$, then the resource biomass will tend to zero and if $h < h_{\mathrm{MSY}}$, then the resource biomass and population can be maintained at appropriate levels. In Section 5, we have derived a formula for h_{MSY}, which shows that maximum sustainable yield depends upon the carrying capacity of the resource biomass and the equilibrium level of population. The optimal harvesting policy has been discussed using Pontryagin's maximum principle. Constructing an appropriate Hamiltonian function the optimal tax policy has been found. A computer simulation has been performed to illustrate all theoretical results. We have used tax on the per unit harvested resource biomass as a regulatory instrument to derive the optimal tax trajectory. It has been shown that if resource biomass, population and harvesting effort all kept along this path, then resource biomass and population both can be maintained at an appropriate level.

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Appendix: Proof of Theorem 2

Consider the positive definite function about P^*:

$$W = \left(B - B^* - B^* \ln\frac{B}{B^*}\right) + c_1\left(N - N^* - N^* \ln\frac{N}{N^*}\right) + c_2\left(E - E^* - E^* \ln\frac{E}{E^*}\right).$$

Differentiating W with respect to time t along the solutions of model (2) and by choosing $c_2 = 1/(\alpha_0(p-\tau))$, a little algebraic manipulation yields

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\left(\frac{r}{K} + \alpha_2 N^*\right)(B-B^*)^2 - \frac{c_1 s}{L}(N-N^*)^2 + \left[c_1\beta_2(B+B^*) + c_1\beta_1 - \alpha_2 B - \alpha_1\right](B-B^*)(N-N^*).$$

Sufficient condition for $\mathrm{d}W/\mathrm{d}t$ to be negative definite is that the following condition holds:

$$a_{12}^2 < 4a_{11}a_{22}. \tag{A.1}$$

If we choose $c_1 = \alpha_1/(\beta_1 + \beta_2(K + B^*))$, then condition (A.1) is satisfied. Thus, W is a Liapunov function with respect to all solutions initiating in the interior of the positive orthan, proving the theorem.

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