Nonlinear generalized cyclic contractions in complete G-metric spaces and applications to integral equations

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Received: 12 August 2012 / **Revised:** 2 December 2012 / **Published online:** 18 April 2013

Abstract. In this paper we introduce generalized cyclic contractions in G-metric spaces and establish some fixed point theorems. The presented theorems extend and unify various known fixed point results. Examples are given in the support of these results. Finally, an application to the study of existence and uniqueness of solutions for a class of nonlinear integral equations is given.

Keywords: cyclic contraction, fixed point, G-metric space, altering distance function.

1 Introduction

Fixed point theory is an important and actual topic of nonlinear analysis. Moreover, it is well known that the contraction mapping principle, formulated and proved in the PhD dissertation of Banach in 1920, which was published in 1922 is one of the most important theorems in classical functional analysis. During the last five decades, this theorem has undergone various generalizations either by relaxing the condition of contractivity or changing the underlying space or sometimes both. Due to the importance, generalizations of Banach fixed point theorem have been investigated heavily by many authors. This celebrated Banach contraction theorem can be stated as follow.

Theorem 1. (*See* [1].) *Let* (X, d) *be a complete metric space and* f *be a mapping of* X *into itself satisfying*:

$$d(fx, fy) \leqslant kd(x, y) \quad \forall x, y \in X, \tag{1}$$

where k *is a constant in* $(0, 1)$. *Then,* f *has a unique fixed point* $x^* \in X$.

[1]The author is grateful to the Ministry of Science and Technological Development of Serbia.

The simplicity of its proof and the possibilities of attaining the fixed point by using successive approximations let this theorem become a very useful tool in analysis and its applications. There is a great number of generalizations of the Banach contraction principle in the literature (see, e.g., [2] and references cited therein).

It is important to note that the inequality (1) implies continuity of f. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity.

The above posed question was answered by Kirk et al. [3] and turned the area of investigation of fixed point by introducing cyclic representations and cyclic contractions, which can be stated as:

Definition 1. (See [3].) Let (X, d) be a metric space. Let p be a positive integer, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_p$ be subsets of X, $Y = \bigcup_{i=1}^{p} \mathcal{A}_i$ and $f: Y \to Y$. Then Y is said to be a cyclic representation of Y with respect to f if

(i) $\mathcal{A}_i$, $i = 1, 2, \ldots, p$, are nonempty closed sets, and
(ii) $f(\mathcal{A}_1) \subseteq \mathcal{A}_2, \ldots, f(\mathcal{A}_{p-1}) \subseteq \mathcal{A}_p$, $f(\mathcal{A}_p) \subseteq \mathcal{A}_1$.

Moreover, f is called a cyclic contraction if there exists $k \in (0, 1)$ such that $d(fx, fy) \leqslant kd(x, y)$ for all $x \in \mathcal{A}_i$ and $y \in \mathcal{A}_{i+1}$ with $\mathcal{A}_{p+1} = \mathcal{A}_1$.

Notice that although a contraction is continuous, cyclic contraction need not to be. This is one of the important gains of this notion. Kirk et al. obtained, among others, cyclic versions of the Banach contraction principle [1], of the Boyd and Wong fixed point theorem [4] and of the Caristi fixed point theorem [5]. Following the paper [3], a number of fixed point theorems on cyclic representation of Y with respect to a self-mapping f have appeared (see, e.g., [6–17]).

On the other hand, in 2006, Mustafa and Sims [18, 19] introduced the notion of generalized metric spaces or simply G-metric spaces. Several authors studied a lot of fixed and common fixed point theorems in such spaces (see, e.g., [20–30]).

An attempt to extend the investigation specified in [3] to G-metric spaces was done by Aydi in [31]. A new variant of cyclic contractive mapping, named as generalized cyclic contraction mapping satisfying generalized altering distance condition in G-metric spaces, is introduced in the present paper. It is followed by the proof of existence and uniqueness of fixed points for such mappings. The obtained result generalizes and improves many existing theorems in the literature. Some examples are given in the support of our results. In conclusion, we apply accomplished fixed point results for generalized cyclic contraction type mappings to the study of existence and uniqueness of solutions for a class of nonlinear integral equations.

2 Preliminaries

For more details on the following definitions and results, we refer the reader to [19].

Definition 2. Let X be a nonempty set and let $G: X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties:

(G1) $G(x,y,z)=0$ if $x=y=z$;
(G2) $0<G(x,x,y)$ for all $x,y\in X$ with $x\neq y$;
(G3) $G(x,x,y)\leqslant G(x,y,z)$ for all $x,y,z\in X$ with $z\neq y$;
(G4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=\cdots$ (symmetry in all three variables);
(G5) $G(x,y,z)\leqslant G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$ (rectangle inequality).

Then the function G is called a G-metric on X and the pair (X,G) is called a G-metric space.

Note that it can be easily deduced from (G4) and (G5) that

$$\left|G(x,y,y)-G(x,z,z)\right|\leqslant 2G(y,z,z) \tag{2}$$

holds for all $x,y,z\in X$.

Definition 3. Let (X,G) be a G-metric space and let (x_n) be a sequence of points in X.

(i) A point $x\in X$ is said to be the limit of sequence (x_n) if $\lim_{n,m\to\infty}G(x,x_n,x_m)=0$, and one says that the sequence (x_n) is G-convergent to x.
(ii) The sequence (x_n) is said to be a G-Cauchy sequence if, for every $\varepsilon>0$, there is a positive integer N such that $G(x_n,x_m,x_l)<\varepsilon$ for all $n,m,l\geqslant N$; that is, if $G(x_n,x_m,x_l)\to 0$ as $n,m,l\to\infty$.
(iii) (X,G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X,G) is G-convergent in X.

Thus, if $x_n\to x$ in a G-metric space (X,G), then for any $\varepsilon>0$, there exists a positive integer N such that $G(x,x_n,x_m)<\varepsilon$ for all $n,m\geqslant N$. It was shown in [19] that the G-metric induces a Hausdorff topology and that the convergence, as described in the above definition, is relative to this topology. The topology being Hausdorff, a sequence can converge to at most one point.

Lemma 1. *Let (X,G) be a G-metric space, (x_n) a sequence in X and $x\in X$. Then the following are equivalent*:

(i) (x_n) *is G-convergent to* x;
(ii) $G(x_n,x_n,x)\to 0$ *as* $n\to\infty$;
(iii) $G(x_n,x,x)\to 0$ *as* $n\to\infty$.

Lemma 2. *If (X,G) is a G-metric space, then the following are equivalent*:

(i) *the sequence (x_n) is G-Cauchy*;
(ii) *for every $\varepsilon>0$, there exists a positive integer N such that $G(x_n,x_m,x_m)<\varepsilon$ for all $n,m\geqslant N$.*

Lemma 3. *Let (X,G), (X',G') be two G-metric spaces. Then a function $f:X\to X'$ is G-continuous at a point $x\in X$ if and only if it is G-sequentially continuous at x, that is, if (fx_n) is G'-convergent to fx whenever (x_n) is G-convergent to x.*

Definition 4. A G-metric space (X, G) is said to be symmetric if

$$G(x, x, y) = G(x, y, y)$$

holds for arbitrary $x, y \in X$. If this is not the case, the space is called non-symmetric.

To every G-metric on the set X a standard metric can be associated by

$$d_G(x, y) = G(x, x, y) + G(x, y, y).$$

If G is symmetric, then obviously $d_G(x, y) = 2G(x, y, y)$, but in the case of a non-symmetric G-metric, only

$$\frac{3}{2}G(x, y, y) \leqslant d_G(x, y) \leqslant 3G(x, y, y)$$

holds for all $x, y \in X$.

The following are some easy examples of G-metric spaces.

Examples. (a) Let (X, d) be an ordinary metric space. Define G_s by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in X$. Then it is clear that (X, G_s) is a symmetric G-metric space.

(b) Let $X = \{a, b\}$. Define

$$G(a, a, a) = G(b, b, b) = 0, \qquad G(a, a, b) = 1, \qquad G(a, b, b) = 2,$$

and extend G to $X \times X \times X$ by using the symmetry in the variables. Then it is clear that (X, G) is a non-symmetric G-metric space.

Khan et al. [32] introduced the concept of altering distance function:

Definition 5. (See [32].) A function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and non-decreasing;

(ii) $\psi(t) = 0$ if and only if $t = 0$.

3 Main results

First we define the notion of generalized cyclic contraction under an altering distance function in a G-metric space.

Definition 6. Let (X, G) be a G-metric space. Let p be a positive integer, $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_p$ be nonempty subsets of X and $Y = \bigcup_{i=1}^{p} \mathcal{A}_i$. An operator $f : Y \to Y$ satisfies a generalized cyclic contraction, if

(I) $Y = \bigcup_{i=1}^{p} \mathcal{A}_i$ is a cyclic representation of Y with respect to f;

(II) for any $(x,y,z) \in \mathcal{A}_i \times \mathcal{A}_{i+1} \times \mathcal{A}_{i+1}$, $i = 1,2,\dots,p$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$),

$$\begin{aligned}
&\psi\big(G(fx,fy,fz)\big) \\
&\quad \leqslant \beta\psi\bigg(\max\bigg\{G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \\
&\quad \frac{1}{3}\big(G(x,fy,fy) + G(y,fz,fz) + G(z,fx,fx)\big)\bigg\}\bigg),
\end{aligned}$$

where $\beta \in [0,1)$ and $\psi : [0,+\infty) \to [0,+\infty)$ is an altering distance function.

The following is the main result of this paper.

Theorem 2. *Let (X,G) be a G-complete G-metric space, $p \in \mathbb{N}$, $\mathcal{A}_1$, $\mathcal{A}_2$, $\dots$, $\mathcal{A}_p$ nonempty closed subsets of X and $Y = \bigcup_{i=1}^p \mathcal{A}_i$. Suppose $f : Y \to Y$ is a generalized cyclic contraction mapping. Then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^p \mathcal{A}_i$.*

Proof. Let $x_0 \in \mathcal{A}_1$ (such a point exists since $\mathcal{A}_1 \neq \emptyset$). Define the sequence (x_n) in X by:

$$x_{n+1} = fx_n, \quad n = 0,1,2,\dots.$$

Without loss of the generality, we can assume that

$$x_n \neq x_{n+1} \quad \forall n \in \mathbb{N} \cup \{0\}.$$

We shall prove that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3}$$

By the assumption, $G(x_n,x_{n+1},x_{n+1}) > 0$ for all n. From the condition (I), we observe that for all n, there exists $i = i(n) \in \{1,2,\dots,p\}$ such that $(x_n,x_{n+1},x_{n+1}) \in \mathcal{A}_i \times \mathcal{A}_{i+1} \times \mathcal{A}_{i+1}$. Putting $x = x_n$ and $y = x_{n+1} = z$ in condition (II), we have

$$\begin{aligned}
&\psi\big(G(x_{n+1},x_{n+2},x_{n+2})\big) \\
&\quad = \psi\big(G(fx_n,fx_{n+1},fx_{n+1})\big) \\
&\quad \leqslant \beta\psi\bigg(\max\bigg\{G(x_n,x_{n+1},x_{n+1}), G(x_n,fx_n,fx_n), G(x_{n+1},fx_{n+1},fx_{n+1}), \\
&\quad \frac{1}{3}\big(G(x_n,fx_{n+1},fx_{n+1}) + G(x_{n+1},fx_{n+1},fx_{n+1}) + G(x_{n+1},fx_n,fx_n)\big)\bigg\}\bigg) \\
&\quad = \beta\psi\bigg(\max\bigg\{G(x_n,x_{n+1},x_{n+1}), G(x_{n+1},x_{n+2},x_{n+2}), \\
&\quad \frac{1}{3}\big(G(x_n,x_{n+2},x_{n+2}) + G(x_{n+1},x_{n+2},x_{n+2})\big)\bigg\}\bigg).
\end{aligned}$$

By (G5), we have

$$G(x_n,x_{n+2},x_{n+2}) \leqslant G(x_n,x_{n+1},x_{n+1}) + G(x_{n+1},x_{n+2},x_{n+2}).$$

Thus

$$\frac{1}{3}\big(G(x_n, x_{n+2}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+2})\big) \\ \leqslant \max\big\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\big\}.$$

If

$$\max\big\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\big\} = G(x_{n+1}, x_{n+2}, x_{n+2}),$$

then

$$\psi\big(G(x_{n+1}, x_{n+2}, x_{n+2})\big) \leqslant \beta\psi\big(G(x_{n+1}, x_{n+2}, x_{n+2})\big).$$

Since $\beta < 1$, we get that $G(x_{n+1}, x_{n+2}, x_{n+2}) = 0$ and hence $x_{n+1} = x_{n+2}$ which is a contradiction. Thus

$$\max\big\{G(x_n, x_{n+1}, x_{n+1}), G(x_{n+1}, x_{n+2}, x_{n+2})\big\} = G(x_n, x_{n+1}, x_{n+1}).$$

Therefore, we have

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leqslant G(x_n, x_{n+1}, x_{n+1}) \quad \forall n \in \mathbb{N} \tag{4}$$

and

$$\psi\big(G(x_{n+1}, x_{n+2}, x_{n+2})\big) \leqslant \beta\psi\big(G(x_n, x_{n+1}, x_{n+1})\big) \quad \forall n \in \mathbb{N}. \tag{5}$$

By (4), we get that $(G(x_n, x_{n+1}, x_{n+1}))_{n\in\mathbb{N}}$ is a non-increasing sequence. Hence there is $r \geqslant 0$ such that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = r.$$

Passing to the limit as $n \to \infty$ in (5), we get

$$\psi(r) \leqslant \beta\psi(r).$$

Therefore $\psi(r) = 0$ and hence $r = 0$. Thus, (3) is proved.

Next, we show that, (x_n) is a G-Cauchy sequence in X. Suppose the contrary, that is, (x_n) is not G-Cauchy. Then there exists $\epsilon > 0$ for which we can find two subsequences $(x_{m(k)})$ and $(x_{n(k)})$ of (x_n) such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \qquad G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \geqslant \epsilon. \tag{6}$$

This means that

$$G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) < \epsilon. \tag{7}$$

From (6), (7) and (G5), we get

$$\begin{aligned}\epsilon &\leqslant G(x_{m(k)}, x_{n(k)}, x_{n(k)}) \\ &\leqslant G(x_{m(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}) \\ &< \epsilon + G(x_{n(k)-1}, x_{n(k)}, x_{n(k)}).\end{aligned}$$

Passing to the limit as $k \to \infty$ and using (3), we get

$$\lim_{k\to\infty} G(x_{m(k)}, x_{n(k)}, x_{n(k)}) = \epsilon. \tag{8}$$

On the other hand, for all k, there exists $j(k) \in \{1, \dots, p\}$ such that $n(k) - m(k) + j(k) \equiv 1[p]$. Then $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labelled sets $\mathcal{A}_i$ and $\mathcal{A}_{i+1}$ for certain $i \in \{1, \dots, p\}$. Using (2) and (G5), we have

$$\begin{aligned}
&\bigl|G(x_{m(k)-j(k)}, x_{m(k)-j(k)}, x_{n(k)}) - G(x_{n(k)}, x_{m(k)}, x_{m(k)})\bigr| \\
&\quad \leqslant 2G(x_{m(k)-j(k)}, x_{m(k)-j(k)}, x_{m(k)}) \\
&\quad \leqslant 2\sum_{l=0}^{j(k)-1} G(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \\
&\quad \leqslant 2\sum_{l=0}^{p-1} G(x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l}, x_{m(k)-j(k)+l+1}) \to 0
\end{aligned}$$

as $k \to \infty$ (from (3)), which, by (8), implies that

$$\lim_{k\to\infty} G(x_{m(k)-j(k)}, x_{m(k)-j(k)}, x_{n(k)}) = \varepsilon. \tag{9}$$

Using (3), we have

$$\lim_{k\to\infty} G(x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)}) = 0 \tag{10}$$

and

$$\lim_{k\to\infty} G(x_{n(k)+1}, x_{n(k)+1}, x_{n(k)}) = 0. \tag{11}$$

Again, using (2), we get

$$\begin{aligned}
&\bigl|G(x_{m(k)-j(k)}, x_{m(k)-j(k)}, x_{n(k)+1}) - G(x_{m(k)-j(k)}, x_{n(k)}, x_{n(k)})\bigr| \\
&\quad \leqslant 2G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}).
\end{aligned}$$

Passing to the limit as $k \to \infty$ in the above inequality, and using (11) and (9), we get

$$\lim_{k\to\infty} G(x_{m(k)-j(k)}, x_{m(k)-j(k)}, x_{n(k)+1}) = \varepsilon. \tag{12}$$

Similarly, we have

$$\begin{aligned}
&\bigl|G(x_{n(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}) - G(x_{m(k)-j(k)}, x_{m(k)-j(k)}, x_{n(k)})\bigr| \\
&\quad \leqslant 2G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}).
\end{aligned}$$

Passing to the limit as $k \to \infty$, and using (3) and (9), we obtain

$$\lim_{k\to\infty} G(x_{n(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}) = \varepsilon. \tag{13}$$

Similarly, we have

$$\lim_{k\to\infty} G(x_{m(k)-j(k)+1}, x_{n(k)+1}, x_{n(k)+1}) = \varepsilon. \tag{14}$$

Using (II), we obtain

$$\begin{aligned}
&\psi\big(G(x_{n(k)+1}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1})\big)\\
&\quad = \psi\big(G(fx_{n(k)}, fx_{m(k)-j(k)}, fx_{m(k)-j(k)})\big)\\
&\quad \leqslant \beta\psi\bigg(\max\bigg\{G(x_{n(k)}, x_{m(k)-j(k)}, x_{m(k)-j(k)}), G(x_{n(k)}, fx_{n(k)}, fx_{n(k)}),\\
&\qquad G(x_{m(k)-j(k)}, fx_{m(k)-j(k)}, fx_{m(k)-j(k)}),\\
&\qquad \frac{1}{3}\big(G(x_{n(k)}, fx_{m(k)-j(k)}, fx_{m(k)-j(k)})\\
&\qquad + G(x_{m(k)-j(k)}, fx_{m(k)-j(k)}, fx_{m(k)-j(k)})\\
&\qquad + G(x_{m(k)-j(k)}, fx_{n(k)}, fx_{n(k)})\big)\bigg\}\bigg)\\
&\quad = \beta\psi\bigg(\max\bigg\{G(x_{n(k)}, x_{m(k)-j(k)}, x_{m(k)-j(k)}), G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}),\\
&\qquad G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}),\\
&\qquad \frac{1}{3}\big(G(x_{n(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1})\\
&\qquad + G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1})\\
&\qquad + G(x_{m(k)-j(k)}, x_{n(k)+1}, x_{n(k)+1})\big)\bigg\}\bigg)\\
&\quad \leqslant \beta\psi\bigg(\max\bigg\{G(x_{n(k)}, x_{m(k)-j(k)}, x_{m(k)-j(k)}), G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}),\\
&\qquad G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1}),\\
&\qquad \frac{1}{3}\big(G(x_{n(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1})\\
&\qquad + G(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}, x_{m(k)-j(k)+1})\\
&\qquad + 2G(x_{m(k)-j(k)}, x_{m(k)-j(k)}, x_{n(k)+1})\big)\bigg\}\bigg).
\end{aligned}$$

Passing to the limit as $k \to \infty$, using (8)–(14) and the continuity of ψ, we get

$$\psi(\epsilon) \leqslant \beta\psi(\epsilon).$$

Since $0 \leqslant \beta < 1$, we get $\psi(\epsilon) = 0$. Hence $\epsilon = 0$, a contradiction. Thus, (x_n) is a G-Cauchy sequence in X.

Since (X, G) is G-complete, there exists $x^* \in X$ such that

$$\lim_{n\to\infty} x_n = x^*. \tag{15}$$

We shall prove that

$$x^* \in \bigcap_{i=1}^{p} \mathcal{A}_i. \tag{16}$$

From condition (I), and since $x_0 \in \mathcal{A}_1$, we have $(x_{np})_{n\geqslant 0} \subseteq \mathcal{A}_1$. Since $\mathcal{A}_1$ is closed, from (15), we get that $x^* \in \mathcal{A}_1$. Again, from condition (I), we have $(x_{np+1})_{n\geqslant 0} \subseteq \mathcal{A}_2$. Since $\mathcal{A}_2$ is closed, from (15), we get that $x^* \in \mathcal{A}_2$. Continuing this process, we obtain (16).

Now, we shall prove that x^* is a fixed point of f. Indeed, from (16), since for all n, there exists $i(n) \in \{1, 2, \dots, p\}$ such that $x_n \in \mathcal{A}_{i(n)}$, applying (II) with $x = y = x^*$ and $z = x_n$, we obtain

$$\begin{aligned}
&\psi\big(G(fx^*, fx^*, x_{n+1})\big) \\
&\quad = \psi\big(G(fx^*, fx^*, fx_n)\big) \\
&\quad \leqslant \beta\psi\bigg(\max\bigg\{G(x^*, x^*, x_n), G(x^*, fx^*, fx^*), G(x_n, fx_n, fx_n), \\
&\qquad \frac{1}{3}\big(G(x^*, fx^*, fx^*) + G(x^*, fx_n, fx_n) + G(x_n, fx^*, fx^*)\big)\bigg\}\bigg) \\
&\quad = \beta\psi\bigg(\max\bigg\{G(x^*, x^*, x_n), G(x_n, x_{n+1}, x_{n+1}), \\
&\qquad \frac{1}{3}\big(G(x^*, x_{n+1}, x_{n+1}) + G(x_n, x^*, x^*)\big)\bigg\}\bigg).
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the above inequality and using (15), we obtain that

$$\psi\big(G(fx^*, fx^*, x^*)\big) \leqslant \beta\psi(0) = 0,$$

hence $fx^* = x^*$, that is, x^* is a fixed point of f.

Finally, we prove that x^* is the unique fixed point of f. Assume that y^* is another fixed point of f, that is, $fy^* = y^*$. By the condition (I), this implies that $y^* \in \bigcap_{i=1}^{p} \mathcal{A}_i$. Then we can apply (II) for $x = x^*$ and $y = z = y^*$. We obtain

$$\begin{aligned}
\psi\big(G(fx^*, fy^*, fy^*)\big) &\leqslant \beta\psi\bigg(\max\bigg\{G(x^*, y^*, y^*), G(x^*, fx^*, fx^*), G(y^*, fy^*, fy^*), \\
&\qquad \frac{1}{3}\big(G(x^*, fy^*, fy^*) + G(y^*, fy^*, fy^*) + G(y^*, fx^*, fx^*)\big)\bigg\}\bigg).
\end{aligned}$$

Since x^* and y^* are fixed points of f, we get that

$$\begin{aligned}
\psi\big(G(x^*, y^*, y^*)\big) &\leqslant \beta\psi\bigg(\max\bigg\{G(x^*, y^*, y^*), \frac{1}{3}\big(G(x^*, y^*, y^*) + G(y^*, x^*, x^*)\big)\bigg\}\bigg) \\
&\leqslant \beta\psi\big(G(x^*, y^*, y^*)\big).
\end{aligned}$$

Since $0 \leqslant \beta < 1$, we get $\psi(G(x^*, y^*, y^*)) = 0$. Hence $G(x^*, y^*, y^*) = 0$, that is, $x^* = y^*$. Thus, we have proved the uniqueness of the fixed point. $\square$

Following the proof of Theorem 2, we can derive the similar conclusion if condition (II) is replaced by the following condition (II′)

$$\psi\big(G(fx,fy,fz)\big) \leqslant \beta\psi\bigg(\max\bigg\{G(x,y,z), G(x,x,fx), G(y,y,fy), G(z,z,fz), \frac{1}{3}\big(G(x,x,fy)+G(y,y,fz)+G(z,z,fx)\big)\bigg\}\bigg),$$

where $\beta \in [0,1)$ and $\psi : [0,+\infty) \to [0,+\infty)$ is an altering distance function.

Corollary 1. *Let (X,G) be a G-complete G-metric space, $p \in \mathbb{N}$, $\mathcal{A}_1$, $\mathcal{A}_2$, ..., $\mathcal{A}_p$ nonempty closed subsets of X, $Y = \bigcup_{i=1}^p \mathcal{A}_i$ and $f : Y \to Y$ such that:*

(I) *$Y = \bigcup_{i=1}^p \mathcal{A}_i$ is a cyclic representation of Y with respect to f;*

(II′) *for any $(x,y,z) \in \mathcal{A}_i \times \mathcal{A}_{i+1} \times \mathcal{A}_{i+1}$, $i = 1,2,\dots,p$ (with $\mathcal{A}_{p+1} = \mathcal{A}_1$),*

$$G(fx,fy,fz) \leqslant \beta\max\bigg\{G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \frac{1}{3}\big(G(x,fy,fy)+G(y,fz,fz)+G(z,fx,fx)\big)\bigg\}$$

or

$$G(fx,fy,fz) \leqslant \beta\max\bigg\{G(x,y,z), G(x,x,fx), G(y,y,fy), G(z,z,fz), \frac{1}{3}\big(G(x,x,fy)+G(y,y,fz)+G(z,z,fx)\big)\bigg\},$$

where $\beta \in [0,1)$.

Then f has a unique fixed point. Moreover, the fixed point of f belongs to $\bigcap_{i=1}^p \mathcal{A}_i$.

Proof. Take $\psi : [0,+\infty) \to [0,+\infty)$ given by $\psi(t) = t$. Then f and ψ satisfy all the hypotheses of Theorem 2 with condition (II) or (II′). □

4 Examples

Now we present some examples showing how our Theorem 2 can be used.

Example 1. Let $X = \mathbb{R}$ and $G : X \times X \times X \to X$ be defined by

$$G(x,y,z) = \max\big\{|x-y|, |x-z|, |y-z|\big\}$$

for all $x,y,z \in X$. Clearly, (X,G) is a G-complete G-metric space. Consider the closed subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ of X defined by

$$\mathcal{A}_1 = \bigg\{\frac{1}{2n}\colon n \in \mathbb{N}\bigg\} \cup \{0\} \quad \text{and} \quad \mathcal{A}_2 = \bigg\{-\frac{1}{2n-1}\colon n \in \mathbb{N}\bigg\} \cup \{0\}.$$

Define the selfmap f on X by

$$fx = \begin{cases} -x/(x+8) & \text{if } x \in \mathcal{A}_1, \\ -x/8 & \text{if } x \in \mathcal{A}_2. \end{cases}$$

Clearly, we have $f(\mathcal{A}_1) \subseteq \mathcal{A}_2$ and $f(\mathcal{A}_2) \subseteq \mathcal{A}_1$.

Next we show that f satisfies condition (II). Let $(x, y, z) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2$ with $x, y, z \neq 0$. We have

$$\begin{aligned}
&G(fx, fy, fz) \\
&\quad = \max\bigl\{|fx - fy|, |fx - fz|, |fy - fz|\bigr\} \\
&\quad = \max\left\{\left|\frac{x}{x+8} - \frac{y}{8}\right|, \left|\frac{x}{x+8} - \frac{z}{8}\right|, \left|\frac{y}{8} - \frac{z}{8}\right|\right\} \\
&\quad \leqslant \frac{1}{8}\max\left\{|x|\left|1 + \frac{1}{x+8}\right| + |y|\left|1 + \frac{1}{8}\right|, |x|\left|1 + \frac{1}{x+8}\right| + |z|\left|1 + \frac{1}{8}\right|, |y - z|\right\} \\
&\quad = \frac{1}{8}\max\left\{\left|\frac{-x}{x+8} - x\right| + \left|\frac{-y}{8} - y\right|, \left|\frac{-x}{x+8} - x\right| + \left|\frac{-z}{8} - z\right|, |y - z|\right\} \\
&\quad \leqslant \frac{1}{4}\max\bigl\{\max\bigl\{|x - fx|, |y - fy|\bigr\}, \max\bigl\{|x - fx|, |z - fz|\bigr\}, \\
&\qquad \max\bigl\{|x - y|, |x - z|, |y - z|\bigr\}\bigr\} \\
&\quad \leqslant \frac{1}{4}\max\bigl\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz)\bigr\} \\
&\quad \leqslant \frac{1}{4}\max\biggl\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \\
&\qquad \frac{1}{3}\bigl(G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)\bigr)\biggr\}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\psi\bigl(G(fx, fy, fz)\bigr) \leqslant \beta\psi\biggl(\max\biggl\{&G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \\
&\frac{1}{3}\bigl(G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)\bigr)\biggr\}\biggr)
\end{aligned}$$

for $\beta = 1/4$ and $\psi(t) = t$. Similarly, one can show that the previous inequality holds for $(x, y, z) \in \mathcal{A}_2 \times \mathcal{A}_1 \times \mathcal{A}_1$, as well as when some of x, y, z is equal to 0.

Hence, all the conditions of Theorem 2 are satisfied (with $p = 2$), and we deduce that f has a unique fixed point $x^* \in \mathcal{A}_1 \cap \mathcal{A}_2 = \{0\}$.

Example 2. Let $X = \mathbb{R}^3$ and let $G : X^3 \to \mathbb{R}^+$ be given as

$$G\bigl((x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)\bigr) = \sum_{i=1}^{3} \bigl[|x_i - x_{i+1}| + |y_i - y_{i+1}| + |z_i - z_{i+1}|\bigr],$$

where $(x_4, y_4, z_4) = (x_1, y_1, z_1)$. It is easy to see that (X, G) is a G-complete G-metric space. Consider the following closed subsets of X:

$$\mathcal{A}_1 = \{(a, 0, 0)\colon 0 \leqslant a \leqslant 1\}, \qquad \mathcal{A}_2 = \{(0, b, 0)\colon 0 \leqslant b \leqslant 1\},$$
$$\mathcal{A}_3 = \{(0, 0, c)\colon 0 \leqslant c \leqslant 1\}, \qquad Y = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$$

and the mapping $f : Y \to Y$ given by

$$f(a, 0, 0) = \left(0, \frac{a}{2}, 0\right), \qquad f(0, b, 0) = \left(0, 0, \frac{b}{4}\right), \qquad f(0, 0, c) = \left(\frac{c}{8}, 0, 0\right).$$

It is clear that $Y = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ is a cyclic representation of Y with respect to f. We will check that f satisfies the contraction condition (II).

Take $\psi(t) = t^2$ (which is an altering distance function) and $\beta = 1/4$. Let, e.g., $(x, y, z) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_2$ (the other two cases are treated analogously), and let $x = (a, 0, 0)$, $y = (0, b, 0)$, $z = (0, c, 0)$. Without loss of generality, we can assume that $b \geqslant c$. Then $fx = (0, a/2, 0)$, $fy = (0, 0, b/4)$, $fz = (0, 0, c/4)$ and

$$G(fx, fy, fz) = a + \frac{b}{2}, \qquad G(x, y, z) = 2(a + b), \qquad G(x, fx, fx) = 3a,$$
$$G(y, fy, fy) = \frac{5}{2}b, \qquad G(z, fz, fz) = \frac{5}{2}c,$$
$$\frac{1}{3}\big(G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)\big) = \frac{1}{3}\left(2a + \frac{5}{2}b + \frac{c}{2} + 2\left|c - \frac{a}{2}\right|\right).$$

Hence,

$$\begin{aligned}\psi\big(G(fx, fy, fz)\big) &= \left(a + \frac{b}{2}\right)^2 \leqslant \frac{1}{4}\big(2(a + b)\big)^2 = \beta\psi\big(G(x, y, z)\big) \\ &\leqslant \beta\psi\bigg(\max\bigg\{G(x, y, z), G(x, fx, fx), G(y, fy, fy), G(z, fz, fz), \\ &\qquad \frac{1}{3}\big(G(x, fy, fy) + G(y, fz, fz) + G(z, fx, fx)\big)\bigg\}\bigg).\end{aligned}$$

Thus, conditions of Theorem 2 are fulfilled and f has a unique fixed point $(0, 0, 0) \in \bigcap_{i=1}^{3} \mathcal{A}_i$.

Example 3. Define a G-metric G on the set $X = \{a, b, c\}$ by

$$G(x, x, x) = 0, \quad x \in X, \qquad G(a, a, b) = G(a, a, c) = G(a, c, c) = 1,$$
$$G(a, b, b) = G(b, b, c) = G(b, c, c) = 2, \qquad G(a, b, c) = 2$$

with symmetry in all variables (see [25]). Note that G is non-symmetric since $G(a, a, b) \neq G(a, b, b)$. Let $\mathcal{A}_1 = \{a, b\}$ and $\mathcal{A}_2 = \{a, c\}$ and consider the mapping $f : X \to X$ given by $fa = fc = a$ and $fb = c$. Obviously, $\mathcal{A}_1 \cup \mathcal{A}_2 = X$ is a cyclic representation with respect to f.

Let $\beta = 1/2$ and $\psi(t) = t$, and for $x, y, z \in X$ denote $L(x,y,z) = \psi(G(x,y,z))$ and

$$R(x,y,z) = \beta\psi\biggl(\max\biggl\{G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \frac{1}{3}\bigl(G(x,fy,fy) + G(y,fz,gz) + G(z,fx,fx)\bigr)\biggr\}\biggr).$$

We have to check the inequality $L(x,y,z) \leqslant R(x,y,z)$ in the following cases:

1. $x \in \{a,b\}$ and $y, z \in \{a,c\}$.
   (a) If $x = a$ and $y, z \in \{a,c\}$, then $fx = fy = fz = a$ and $L = 0$; hence $L \leqslant R$ holds true.
   (b) If $x = b$ and $y, z \in \{a,c\}$, then $fx = c$, $fy = fz = a$; hence $L = \psi(G(c,a,a)) = 1$. Since $G(x,fx,fx) = G(b,c,c) = 2$, it is $R \geqslant (1/2)\cdot 2 = 1$ and $L \leqslant R$ holds true.
2. $x \in \{a,c\}$ (then $fx = a$) and $y, z \in \{a,b\}$.
   (a) If $y = z = a$, then $fy = fz = a$ and $L = 0$; hence $L \leqslant R$ holds true.
   (b) If one of y, z is equal to b (e.g., $y = b$), then $L = G(a,a,c)$ or $L = G(a,c,c)$, i.e., $L = 1$, and $G(y,fy,fy) = G(b,c,c) = 2$. Hence, $L = 1 \leqslant (1/2)\cdot 2 \leqslant R$.

All the conditions of Corollary 1 are fulfilled and f has a unique fixed point $a \in \mathcal{A}_1 \cap \mathcal{A}_2$.

5 An application to integral equations

In this section we apply Theorem 2 to study the existence and uniqueness of solutions for a class of nonlinear integral equations.

We consider the nonlinear integral equation

$$u(t) = \int_a^b \mathcal{G}(t,s)K\bigl(s,u(s)\bigr)\,\mathrm{d}s \quad \text{for } t \in [a,b], \tag{17}$$

where $K : [a,b] \times \mathbb{R}^+ \to \mathbb{R}^+$ and $\mathcal{G} : [a,b] \times [a,b] \to \mathbb{R}^+$ are continuous functions.

Denote by $X = C([a,b], \mathbb{R}^+)$ the set of non-negative real continuous functions on $[a,b]$. We endow X with the G-metric

$$G(u,v,w) = \max_{t\in[a,b]} \bigl|u(t) - v(t)\bigr| + \max_{t\in[a,b]} \bigl|u(t) - w(t)\bigr| + \max_{t\in[a,b]} \bigl|v(t) - w(t)\bigr|,$$

for $u, v, w, \in X$. Then (X, G) is a G-complete metric space.

Consider the self-map $f : X \to X$ defined by

$$fu(t) = \int_a^b \mathcal{G}(t,s)K\bigl(s,u(s)\bigr)\,\mathrm{d}s \quad \text{for } t \in [a,b].$$

Clearly, u is a solution of (17) if and only if u is a fixed point of f.

We will prove the existence and uniqueness of the fixed point of f under the following conditions:

(I) $\int_a^b \mathcal{G}(t,s)\,\mathrm{d}s \leqslant \beta$ (with $\beta < 1$) for all $t \in [a,b]$.

(II) $K(s,\cdot)$ is a non-increasing function for any fixed $s \in [0,1]$, that is,

$$x, y \in \mathbb{R}^+, \quad x \geqslant y \quad \Longrightarrow \quad K(s,x) \leqslant K(s,y).$$

(III) $|K(s,x) - K(s,y)| \leqslant |x-y|$ for all $s \in [a,b]$ and $x, y \in \mathbb{R}^+$.

(IV) There exist $(\alpha,\gamma) \in X^2$ such that $\alpha(t) \leqslant \gamma(t)$ for $t \in [a,b]$ and that

$$f\alpha(t) \leqslant \gamma(t) \quad \text{and} \quad f\gamma(t) \geqslant \alpha(t) \quad \text{for } t \in [a,b].$$

Theorem 3. *Under the conditions* (I)–(IV)*, equation* (17) *has a unique solution* $u^* \in X$ *and it belongs to* $\mathcal{C} = \{u \in X\colon\ \alpha(t) \leqslant u(t) \leqslant \gamma(t),\ t \in [a,b]\}$.

Proof. In order to prove the existence of a (unique) fixed point of f we construct the closed subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ of X as follows:

$$\mathcal{A}_1 = \bigl\{u \in X\colon\ u(t) \leqslant \gamma(t),\ t \in [a,b]\bigr\}$$

and

$$\mathcal{A}_2 = \bigl\{u \in X\colon\ u(t) \geqslant \alpha(t),\ t \in [a,b]\bigr\}.$$

We shall prove that

$$f(\mathcal{A}_1) \subseteq \mathcal{A}_2 \quad \text{and} \quad f(\mathcal{A}_2) \subseteq \mathcal{A}_1.$$

Let $u \in \mathcal{A}_1$, that is,

$$u(s) \leqslant \gamma(s) \quad \text{for all } s \in [a,b].$$

Since $\mathcal{G}(t,s) \geqslant 0$ for all $t, s \in [a,b]$, we deduce from (II) and (IV) that

$$\int_a^b \mathcal{G}(t,s) K\bigl(s,u(s)\bigr)\,\mathrm{d}s \geqslant \int_a^b \mathcal{G}(t,s) K\bigl(s,\gamma(s)\bigr)\,\mathrm{d}s \geqslant \alpha(t)$$

for all $t \in [a,b]$. Then we have $fu \in \mathcal{A}_2$. Similarly, the other inclusion is proved. Hence, $Y = \mathcal{A}_1 \cup \mathcal{A}_2$ is a cyclic representation of Y with respect to f.

Finally we will show that, for each $u \in \mathcal{A}_1$ and $v, w \in \mathcal{A}_2$ one has $G(fu,fv,fw) \leqslant \beta G(u,v,w)$.

To this end, let $u \in \mathcal{A}_1$ and $(v,w) \in \mathcal{A}_2 \times \mathcal{A}_2$. Therefore by (III), we deduce that, for each $t \in [a,b]$,

$$\begin{aligned} &G(fu,fv,fw) \\ &\quad = \max_{t\in[a,b]} \bigl|fu(t) - fv(t)\bigr| + \max_{t\in[a,b]} \bigl|fu(t) - fw(t)\bigr| + \max_{t\in[a,b]} \bigl|fv(t) - fw(t)\bigr| \end{aligned}$$

$$
\begin{aligned}
&= \max_{t\in[a,b]} \int_a^b \mathcal{G}(t,s)\bigl|K\bigl(s,u(s)\bigr) - K\bigl(s,v(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad + \max_{t\in[a,b]} \int_a^b \mathcal{G}(t,s)\bigl|K\bigl(s,u(s)\bigr) - K\bigl(s,w(s)\bigr)\bigr|\,\mathrm{d}s \\
&\quad + \max_{t\in[a,b]} \int_a^b \mathcal{G}(t,s)\bigl|K\bigl(s,v(s)\bigr) - K\bigl(s,w(s)\bigr)\bigr|\,\mathrm{d}s \\
&\leqslant \max_{t\in[a,b]} \int_a^b \mathcal{G}(t,s)\bigl(\bigl|u(s)-v(s)\bigr| + \bigl|u(s)-w(s)\bigr| + \bigl|v(s)-w(s)\bigr|\bigr)\,\mathrm{d}s \\
&\leqslant G(u,v,w) \max_{t\in[a,b]} \int_a^b \mathcal{G}(t,s)\,\mathrm{d}s \leqslant \beta G(u,v,w) \\
&\leqslant \beta \max\Bigl\{G(u,v,w), G(u,fu,fu), G(v,fv,fv), G(w,fw,fw), \\
&\quad \frac{1}{3}\bigl(G(u,fv,fv) + G(v,fw,fw) + G(w,fu,fu)\bigr)\Bigr\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
G(fu,fv,fw) \leqslant \beta \max\Bigl\{&G(u,v,w), G(u,fu,fu), G(v,fv,fv), G(w,fw,fw), \\
&\frac{1}{3}\bigl(G(u,fv,fv) + G(v,fw,fw) + G(w,fu,fu)\bigr)\Bigr\}.
\end{aligned}
$$

Using the same technique, we can show that the above inequality also holds if we take $(u,v,w) \in \mathcal{A}_2 \times \mathcal{A}_1 \times \mathcal{A}_1$.

Consequently, by Corollary 1, f has a unique fixed point $u^* \in \mathcal{A}_1 \cap \mathcal{A}_2$, that is, $u^* \in \mathcal{C}$ is the unique solution to (17). $\square$

Acknowledgment. The authors are indebted to the referees for careful reading and suggestions that helped us to improve the text.

References

1. S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux equations integrales, *Fundam. Math.*, **3**:133–181, 1922.

2. H.K. Nashine, New fixed point theorems for mappings satisfying generalized weakly contractive condition with weaker control functions, *Ann. Pol. Math.*, **104**:109–119, 2012.

3. W.A. Kirk, P.S. Srinavasan, P. Veeramani, Fixed points for mapping satisfying cyclical contractive conditions, *Fixed Point Theory*, **4**:79–89, 2003.

4. D.W. Boyd, S.W. Wong, On nonlinear contractions, *Proc. Am. Math. Soc.*, **20**:458–464, 1969.

5. J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, *Trans. Am. Math. Soc.*, **215**:241–251, 1976.

6. I.A. Rus, Cyclic representations and fixed points, *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity*, **3**:171–178, 2005.

7. C. Di Bari, T. Suzuki, C. Vetro, Best proximity points for cyclic Meir–Keeler contractions, *Nonlinear Anal., Theory Methods Appl.*, **69**:3790–3794, 2008.

8. M. Păcurar, I.A. Rus, Fixed point theory for cyclic φ-contractions, *Nonlinear Anal., Theory Methods Appl.*, **72**:1181–1187, 2010.

9. B. Piatek, On cyclic Meir–Keeler contractions in metric spaces, *Nonlinear Anal., Theory Methods Appl.*, **74**:35–40, 2011.

10. E. Karapinar, Fixed point theory for cyclic weaker ϕ-contraction, *Appl. Math. Lett.*, **24**:822–825, 2011.

11. E. Karapinar, K. Sadarangani, Fixed point theory for cyclic $(\phi\text{-}\varphi)$-contractions, *Fixed Point Theory Appl.*, **2011**, 8 pp., 2011, doi:10.1186/1687-1812-2011-69.

12. H.K. Nashine, Cyclic generalized ψ-weakly contractive mappings and fixed point results with applications to integral equations, *Nonlinear Anal., Theory Methods Appl.*, **75**:6160–6169, 2012.

13. C.M. Chen, Fixed point theorems for cyclic Meir–Keeler type mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**, 13 pp., 2012, doi:10.1186/1687-1812-2012-41.

14. H. Aydi, E. Karapinar, B. Samet, Fixed point theorems for various classes of cyclic mappings, *J. Appl. Math.*, **2012**, Article ID 867216, 15 pp., 2012.

15. H. Aydi, C. Vetro, W. Sintunavarat, P. Kumam, Coincidence and fixed points for contractions and cyclical contractions in partial metric spaces, *Fixed Point Theory Appl.*, **2012**, 18 pp., 2012, doi:10.1186/1687-1812-2012-124.

16. H. Aydi, E. Karapinar, A fixed point result of Boyd–Wong cyclic contractions in partial metric spaces, *Int. J. Math. Math. Sci.*, **2012**, Article ID 597074, 11 pp., 2012.

17. M. Jleli, E. Karapinar, B. Samet, Fixed point results for almost generalized cyclic (ψ, ϕ)-weak contractive type mappings with applications, *Abstr. Appl. Anal.*, **2012**, Article ID 917831, 17 pp., 2012.

18. Z. Mustafa, B. Sims, Some remarks concerning D-metric spaces, in: *Proceedings of the International Conference on Fixed Point Theory and Applications, Valencia, Spain, July 2003*, pp. 189–198.

19. Z. Mustafa, B. Sims, A new approach to generalized metric spaces, *J. Nonlinear Convex Anal.*, **7**:289–297, 2006.

20. Z. Mustafa, H. Obiedat, F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory Appl.*, **2008**, Article ID 189870, 12 pp., 2008.

21. Z. Mustafa, B. Sims, Fixed point theorems for contractive mapping in complete G-metric spaces, *Fixed Point Theory Appl.*, **2009**, Article ID 917175, 10 pp., 2009.

22. Z. Mustafa, W. Shatanawi, M. Bataineh, Existence of fixed point results in G-metric spaces, *Int. J. Math. Math. Sci.*, **2009**, Article ID 283028, 10 pp., 2009.

23. R. Chugh, T. Kadian, A. Rani, B.E. Rhoades, Property P in G-metric spaces, *Fixed Point Theory Appl.*, **2010**, Article ID 401684, 12 pp., 2010.

24. W. Shatanawi, Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces, *Fixed Point Theory Appl.*, **2010**, Article ID 181650, 9 pp., 2010.

25. M. Abbas, S.H. Khan, T. Nazir, Commnon fixed points of R-weakly commuting maps in generalized metric spaces, *Fixed Point Theory Appl.*, **2011**, 11 pp., 2011, doi:10.1186/1687-1812-2011-41.

26. Z. Kadelburg, H.K. Nashine, S. Radenović, Common coupled fixed point results in partially ordered G-metric spaces, *Bull. Math. Anal. Appl.*, **4**:51–63, 2012.

27. H.K. Nashine, Z. Kadelburg, R.P. Pathak, S. Radenović, Coincidence point results in ordered G-cone metric space, *Math. Comput. Modelling*, **57**(3–4):701–709.

28. E. Karapinar, B. Samet, Generalized $\alpha - \psi$ contractive type mappings and related fixed point theorems with applications, *Abstract Appl. Anal.*, **2012**, Article ID 793486, 17 pp., 2012.

29. M. Abbas, H. Aydi, On common fixed point of generalized contractive mappings in metric spaces, *Surv. Math. Appl.*, **7**:39–47, 2012.

30. H. Aydi, E. Karapinar, H. Lakzian, Fixed point results on a class of generalized metric spaces, *Mathematical Sciences*, **6**:46, 2012.

31. H. Aydi, Generalized cyclic contractions in G-metric space, *J. Nonlinear Analysis Appl.* (in press).

32. M.S. Khan, M. Swaleh, S. Sessa, Fixed point theorems by altering distancces between the points, *Bull. Aust. Math. Soc.*, **30**:1–9, 1984.