Triple-zero singularity of a Kaldor–Kalecki model of business cycles with delay

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Abstract. In this manuscript, we study triple zero singularity of a Kaldor–Kalecki model of business cycles with delay in both the gross product and the capital stock. By using the frameworks of Campbell–Yuan [1] and Faria–Magalhães [2, 3], the normal form on the center manifold is derived for this singularity and hence the corresponding bifurcation diagrams such as Hopf, BT, zero-Hopf, and homoclinic bifurcations are obtained. An example is given to verify some theoretical results.

Keywords: Kaldor–Kalecki model of business cycles with delay, triple-zero singularities, Hopf, BT, zero-Hopf, and homoclinic bifurcations.

1 Introduction

In 1940, Kaldor [4] first proposed a system of differential equations to model business cycles with nonlinear investment and saving functions so that the system may oscillate cyclically. Krawiec and Szydlowski [5–7] combined Kaldor's model and Kalecki's idea [8] that there is a time delay for investment after a business decision has been made by proposing the following Kaldor–Kalecki model of business cycles:

$$\begin{aligned} \frac{\mathrm{d}Y(t)}{\mathrm{d}t} &= \alpha\bigl[I\bigl(Y(t),K(t)\bigr) - S\bigl(Y(t),K(t)\bigr)\bigr], \\ \frac{\mathrm{d}K(t)}{\mathrm{d}t} &= I\bigl(Y(t-\tau),K(t)\bigr) - qK(t). \end{aligned} \tag{1}$$

Here Y is the gross product, K the capital stock, $\alpha > 0$ the adjustment coefficient in the goods market, $q \in (0,1)$ the depreciation rate of capital stock, $I(Y,K)$ and $S(Y,K)$ investment and saving functions, and $\tau \geqslant 0$ a time lag representing delay for the investment on the capital stock due to the past investment decision.

Kaddar and Talibi Alaoui [9] noted that the past decision on the investment also has influence on the gross product by proposing the following Kaldor–Kalecki model of business cycles:

$$\begin{aligned}\frac{\mathrm{d}Y(t)}{\mathrm{d}t} &= \alpha\big[I\big(Y(t),K(t)\big) - S\big(Y(t),K(t)\big)\big],\\ \frac{\mathrm{d}K(t)}{\mathrm{d}t} &= I\big(Y(t-\tau),K(t-\tau)\big) - qK(t)\end{aligned} \tag{2}$$

with delay in both the gross product and the capital stock. In this research, as in [5–7], we assume that the investment and saving functions have the following forms:

$$I(Y,K) = I(Y) - \beta K, \qquad S(Y,K) = \gamma Y,$$

where $\beta > 0$ and $\gamma \in (0,1)$, respectively. Thus system (2) becomes

$$\begin{aligned}\frac{\mathrm{d}Y(t)}{\mathrm{d}t} &= \alpha\big[I\big(Y(t)\big) - \beta K(t) - \gamma Y(t)\big],\\ \frac{\mathrm{d}K(t)}{\mathrm{d}t} &= I\big(Y(t-\tau)\big) - \beta K(t-\tau) - qK(t).\end{aligned} \tag{3}$$

The dynamical behaviors and bifurcations of system (3) has been studied extensively [9–12]. In [9], Kaddar and Talibi Alaoui found a critical of τ such that system (3) undergoes a Hopf bifurcation. In [10], Wang and Wu refined Kaddar and Talibi Alaoui's result and obtained the normal form of Hopf bifurcation which can be used to determine the stability and the direction of Hopf bifurcation. In [11], Wu studied simple zero, and double zero singularities of system (3) and obtained bifurcation diagrams, from which double limit and heteroclinic bifurcations were obtained. In [12], Wu studied zero-Hopf singularity of system (3) and obtained its corresponding bifurcation diagrams.

Note that all the results mentioned above depend on the distribution of roots of the characteristic equation of the linear part of system (3) at the equilibrium point. If the characteristic equation has a pair of purely imaginary roots, Hopf singularity occurs and hence a limit cycle may bifurcate from the equilibrium point; if the characteristic equation has a simple zero or double zero root, simple zero or double zero singularity occurs; so does zero-Hopf singularity if the characteristic equation has a simple zero root and a pair of purely imaginary roots. However, under certain conditions, the characteristic equation may have a triple zero root and this has not been studied in the literature. For a double zero root or a zero root with algebraic multiplicity 2 and geometric multiplicity 1, the corresponding Jordan matrix is $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$. For a triple zero root or a zero root with algebraic multiplicity 3, there are three cases for geometric multiplicities:

(i) geometric multiplicity 1,
(ii) geometric multiplicity 2,
(iii) geometric multiplicity 3.

Note that, for case (i), the corresponding Jordan norm form is $\left(\begin{smallmatrix}0&1&0\\0&0&1\\0&0&0\end{smallmatrix}\right)$; for case (ii), the corresponding Jordan norm form is $\left(\begin{smallmatrix}0&1&0\\0&0&0\\0&0&0\end{smallmatrix}\right)$; and case (iii), the corresponding Jordan norm form is $\left(\begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix}\right)$. Our study shows that only the first case occurs for system (3).

More specifically, we use (k, β, τ) as bifurcation parameter (where k is the increasing rate of the investment function at the equilibrium point (see the detail in Section 2)) to obtain the critical value (k^*, β^*, τ^*) such that the characteristic equation has a triple zero root with geometric multiplicity 1 and then investigate its corresponding dynamical behaviors. Note that we can find the conditions such that the equilibrium point is asymptotically stable. But this is not practical since business cycles in real world seem to change cyclically. This leads us to study Hopf singularity. But the condition for Hopf singularity is not always satisfied. We show that, for triple zero singularity, we still can obtain limit cycles under small perturbations of (k^*, β^*, τ^*) and certain conditions despise the fact that the condition for Hopf singularity is violated.

The rest of this manuscript is organized as follows. In Section 2, the detailed conditions are given for the linear part of system (3) at an equilibrium point in the (k, β, τ)-parameter space to have a triple zero eigenvalue and other eigenvalues with negative real parts. In Section 3, the normal form of triple zero singularity for system (3) is obtained on the center manifold by using the frameworks from [1] and [2, 3]. In Section 4, the normal form in Section 3 is used to obtain bifurcation diagrams of the original system (3) such as Hopf and homoclinic bifurcations. Finally in Section 5, an example is presented to confirm some theoretical results.

2 Distribution of eigenvalues

Throughout the rest of this paper, we assume that

$$\alpha, \beta > 0, \quad q, \gamma \in (0,1), \quad \text{and } I(s) \text{ is a nonlinear } C^4 \text{ function},$$

and that (Y^*, K^*) is an equilibrium point of system (3). Let $I^* = I(Y^*)$, $u_1 = Y - Y^*$, $u_2 = K - K^*$, and $i(s) = I(s + Y^*) - I^*$. Then system (3) can be transformed as

$$\begin{aligned}\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= \alpha\big[i\big(u_1(t)\big) - \beta u_2(t) - \gamma u_1(t)\big],\\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= i\big(u_1(t-\tau)\big) - \beta u_2(t-\tau) - q u_2(t).\end{aligned} \tag{4}$$

Let the Taylor expansion of i at 0 be

$$i(u) = ku + i^{(2)}u^2 + i^{(3)}u^3 + \mathcal{O}\big(|u|^4\big),$$

where

$$k = i'(0) = I'(Y^*), \quad i^{(2)} = \frac{1}{2}i''(0) = \frac{1}{2}I''(Y^*), \quad i^{(3)} = \frac{1}{3!}i'''(0) = \frac{1}{3!}I'''(Y^*).$$

The linear part of system (4) at $(0,0)$ is

$$\begin{aligned}\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= \alpha\big[(k-\gamma)u_1(t) - \beta u_2(t)\big],\\ \frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= k u_1(t-\tau) - \beta u_2(t-\tau) - q u_2(t),\end{aligned} \tag{5}$$

and the corresponding characteristic equation is

$$f(\lambda) \equiv \lambda^2 + A\lambda + B + (\beta\lambda + C)\mathrm{e}^{-\lambda\tau} = 0, \tag{6}$$

where

$$A = q - \alpha(k - \gamma), \quad B = -\alpha q(k - \gamma), \quad C = \alpha\beta\gamma.$$

We only consider the case of $\tau > 0$. It is easy to attain

$$\begin{aligned} f'(\lambda) &= 2\lambda + A + \beta\mathrm{e}^{-\lambda\tau} - (\beta\lambda + C)\tau\mathrm{e}^{-\lambda\tau}, \\ f''(\lambda) &= 2 - 2\beta\tau\mathrm{e}^{-\lambda\tau} + \beta\tau^2\lambda\mathrm{e}^{-\lambda\tau} + C\tau^2\mathrm{e}^{-\lambda\tau}, \\ f'''(\lambda) &= -\beta\tau^2\mathrm{e}^{-\lambda\tau}(-3 + \lambda\tau + \alpha\gamma\tau). \end{aligned}$$

It is not hard to check that, if $k = k^*, \beta = \beta^*, \tau = \tau^*$ and $q > \alpha\gamma$, then

$$f(0) = f'(0) = f''(0) = 0, \qquad f'''(0) \neq 0,$$

where

$$\begin{aligned} k^* &= \frac{\gamma(q^2 - \alpha^2\gamma^2 + q\sqrt{q^2 - \alpha^2\gamma^2})}{q^2 - \alpha^2\gamma^2}, \quad \beta^* = \frac{q^2\sqrt{q^2 - \alpha^2\gamma^2}}{q^2 - \alpha^2\gamma^2}, \\ \tau^* &= \frac{q - \alpha\gamma + \sqrt{q^2 - \alpha^2\gamma^2}}{q\alpha\gamma}. \end{aligned}$$

Let $q = \alpha\gamma\sec(\sigma)$ where $\sigma \in (0, \pi/2)$ such that $q \in (0, 1)$. Then k^*, β^* and τ^* can be simply expressed as

$$k^* = \gamma(1 + \csc\sigma), \quad \beta^* = \alpha\gamma\csc\sigma\sec\sigma, \quad \tau^* = \frac{1}{\alpha\gamma}(1 - \cos\sigma + \sin\sigma).$$

Thus we obtain the following result.

Lemma 1. *Suppose $k = k^*$, $\beta = \beta^*$, $\tau = \tau^*$ and $q > \alpha\gamma$. Then Eq.* (6) *has a triple zero root.*

Let $\omega\mathrm{i}$ ($\omega > 0$) be a purely imaginary root of Eq. (6). Then we have

$$-\omega^2 - 2\alpha^2\gamma^2\csc(2\sigma) - \mathrm{i}\mathrm{e}^{\omega\tau^*}\alpha\gamma\omega(\csc\sigma - \sec\sigma) + \alpha\gamma(\alpha\gamma + \mathrm{i}\omega)\csc\sigma\sec\sigma = 0.$$

After separating the real and imaginary parts, we obtain

$$\begin{aligned} \cos(\omega\tau^*) &= \frac{\alpha^2\gamma^2 - \omega^2\sin\sigma + \omega^2\cos\sigma(1 + \sin\sigma)}{\alpha^2\gamma^2 + \omega^2}, \\ \sin(\omega\tau^*) &= -\frac{\omega(-1 + \cos\sigma - \sin\sigma)(2\alpha^2\gamma^2 + \omega^2(1 + \cos\sigma - \sin\sigma))}{2\alpha\gamma(\alpha^2\gamma^2 + \omega^2)}. \end{aligned}$$

It is not hard to check that

$$\cos^2(\omega\tau^*) + \sin^2(\omega\tau^*) - 1 = \frac{\omega^2\cos^2\sigma\sin^2\sigma}{\alpha^2\gamma^2(\alpha^2\gamma^2+\omega^2)} \neq 0$$

for $\omega > 0$. Thus Eq. (6) does not have purely imaginary roots. For the other roots of Eq. (6), we need the following lemma from [11]. Let

$$q_0(\beta) = \frac{1}{2}\bigl(-\beta + \sqrt{\beta^2 + 4\alpha\beta\gamma}\,\bigr).$$

Lemma 2. *Let $k = k^*$ and $\tau > 0$.*

(i) *If $q > q_0(\beta)$, all roots of Eq.* (6) *except* 0 *and purely imaginary roots have negative real parts.*

(ii) *If $0 < q \leqslant q_0(\beta)$, Eq.* (6) *has at least one positive root.*

From this lemma, we know that if $k = k^*$, $\beta = \beta^*$, $\tau = \tau^*$ and $q > \max\{\alpha\gamma, q_0(\beta^*)\}$, Eq. (6) has a triple zero root and all other roots are in the left half plane of the imaginary axis.

Remark. The condition $q > \max\{\alpha\gamma, q_0(\beta^*)\}$ states that if system (3) exhibits triple-zero singularity, the depreciation q can not be very small.

3 The computation of the normal form

In the rest of this manuscript, we always assume

$$k = k^*, \quad \beta = \beta^*, \quad \tau = \tau^*, \quad q > \max\bigl\{\alpha\gamma, q_0(\beta^*)\bigr\}.$$

We treat (k, β, τ) as a bifurcation parameter near (k^*, β^*, τ^*). By scaling $t \to t/\tau$, system (4) can be written as

$$\begin{aligned}
\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= \alpha\tau\bigl[(k-\gamma)u_1(t) - \beta u_2(t) + i^{(2)}u_1^2(t) + i^{(3)}u_1^3(t)\bigr] + \mathcal{O}\bigl(|u_1|^4\bigr),\\
\frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= \tau\bigl[ku_1(t-1) - \beta u_2(t-1) - qu_2(t) + i^{(2)}u_1^2(t-1) + i^{(3)}u_1^3(t-1)\bigr]\\
&\quad + \mathcal{O}\bigl(|u_1|^4\bigr).
\end{aligned}$$

Define $C := C([-1,0],\mathbb{R}^2)$, $C^* := C([0,1],\mathbb{R}^{2*})$, and $C^1 = C^1([-1,0],\mathbb{R}^2)$. Let $\mu_1 = k - k^*$, $\mu_2 = \beta - \beta^*$, $\mu_3 = \tau - \tau^*$. Then on C we have

$$\begin{aligned}
\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} &= \alpha(\tau^*+\mu_3)\bigl[(k^*+\mu_1-\gamma)u_1(t) - (\beta^*+\mu_2)u_2(t) + i^{(2)}u_1^2(t) + i^{(3)}u_1^3(t)\bigr]\\
&\quad + \mathcal{O}\bigl(|\mu|^2|u| + |\mu||u|^3\bigr),\\
\frac{\mathrm{d}u_2(t)}{\mathrm{d}t} &= (\tau^*+\mu_3)\bigl[(k^*+\mu_1)u_1(t-1) - (\beta^*+\mu_2)u_2(t-1) - qu_2(t)\\
&\quad + i^{(2)}u_1^2(t-1) + i^{(3)}u_1^3(t-1)\bigr] + \mathcal{O}\bigl(|\mu|^2|u| + |\mu||u|^3\bigr).
\end{aligned}$$

Let

$$\mathbb{A} = \begin{pmatrix} \alpha\tau^*(k^* - \gamma) & -\alpha\tau^*\beta^* \\ 0 & -\tau^* q \end{pmatrix}, \qquad \mathbb{B} = \begin{pmatrix} 0 & 0 \\ \tau^* k^* & -\tau^*\beta^* \end{pmatrix}.$$

Define

$$\Delta(\lambda) = \lambda I - \bigl(\mathbb{A} + \mathbb{B}\mathrm{e}^{-\lambda}\bigr),$$

and the linear operator

$$\mathcal{L}X_t = \mathbb{A}X(t) + \mathbb{B}X(t-1) \quad \text{for } X \in C.$$

From Section 2, we see that $\mathcal{L}$ has a triple zero eigenvalue and all the other eigenvalues have negative real parts. It is easy to see that

$$\Delta(0) = -\bigl(\mathbb{A} + \mathbb{B}\bigr), \qquad \Delta'(0) = I + \mathbb{B}, \qquad \Delta''(0) = -\mathbb{B}.$$

Let $u = (u_1, u_2, \mu_3)^{\mathrm{T}} \in C$, $\mu = (\mu_1, \mu_2, \mu_3)^{\mathrm{T}}$, and $F(u_t, \mu) = (F^1(u_t, \mu), F^2(u_t, \mu))^{\mathrm{T}}$, where

$$\begin{aligned} F^1(u_t, \mu) &= \alpha\tau^*\mu_1 u_1(0) - \alpha\tau^*\mu_2 u_2(0) + \alpha\mu_3\bigl[(k^* - \gamma)u_1(0) - \beta u_2(0)\bigr] \\ &\quad + \alpha\tau^*\bigl[i^{(2)}u_1^2(0) + i^{(3)}u_1^3(0)\bigr] + \mathcal{O}\bigl(|\mu|^2|u| + |\mu||u|^2\bigr), \\ F^2(u_t, \mu) &= \tau^*\mu_1 u_1(-1) - \tau^*\mu_2 u_2(-1) + k\mu_3 u_1(-1) - \beta^*\mu_3 u_2(-1) \\ &\quad - q\mu_3 u_2(0) + \tau^*\bigl[i^{(2)}u_1^2(-1) + i^{(3)}u_1^3(-1)\bigr] + \mathcal{O}\bigl(|\mu|^2|u| + |\mu||u|^3\bigr). \end{aligned}$$

Then system (7) can be written as

$$\dot{u}(t) = \mathcal{L}u_t + F(u_t, \mu) \tag{7}$$

whose corresponding linear part at 0 is

$$\dot{u}(t) = \mathcal{L}u_t. \tag{8}$$

From [2, 3], the bilinear inner product between C and C^* can be expressed by

$$\langle\psi, \varphi\rangle = \psi(0)\varphi(0) + \int_{-1}^{0} \psi(\xi + 1)\mathbb{B}\varphi(\xi)\,\mathrm{d}\xi.$$

Then $\mathcal{L}$ has a generalized eigenspace P which is invariant under the flow (8). Let P^* be the space adjoint with P in C^*. Then C can be decomposed as $C = P \oplus Q$, where $Q = \{\varphi \in C\colon \langle\psi, \varphi\rangle = 0\ \forall\psi \in P^*\}$. Furthermore, we can choose the bases Φ and Ψ for P and P^*, respectively, such that

$$\langle\Psi, \Phi\rangle = I, \qquad \dot{\Phi} = \Phi J, \qquad \dot{\Psi} = -J\Psi,$$

where I is the identity matrix and $J = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ the Jordan matrix associated with the triple zero eigenvalue with geometric multiplicity 1. This guarantees that cases (ii) and (iii) will not happen for system (3) and hence triple-zero bifurcation occurs.

Next, we obtain the explicit expressions of Φ and Ψ. According to Campbell and Yuan [1], the basis Φ for P can be chosen as

$$\Phi = \begin{bmatrix}\varphi_1 & \varphi_2 & \varphi_3\end{bmatrix} = \begin{bmatrix}v_1 & v_2 + \theta v_1 & v_3 + v_2\theta + v_1\frac{\theta^2}{2}\end{bmatrix}$$

and the basis Ψ for P^* as

$$\Psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \begin{pmatrix}w_1\frac{s^2}{2} - w_2 s + w_3\\ -w_1 s + w_2\\ w_1\end{pmatrix},$$

where $v_1, v_2, v_3 \in \mathbb{R}^3$ and $w_1, w_2, w_3 \in \mathbb{R}^{3*}$ satisfy

$$\begin{aligned}&\Delta(0)v_1 = 0, \qquad \Delta'(0)v_1 + \Delta(0)v_2 = 0,\\ &\frac{1}{2}\Delta''(0)v_1 + \Delta'(0)v_2 + \Delta(0)v_3 = 0,\end{aligned} \tag{9}$$

$$\begin{aligned}&w_1\Delta(0) = 0, \qquad w_1\Delta'(0) + w_2\Delta(0) = 0,\\ &\frac{1}{2}w_1\Delta''(0) + w_2\Delta'(0) + w_3\Delta(0) = 0.\end{aligned} \tag{10}$$

Note that (9) is equivalent to

$$\begin{aligned}&(\mathbb{A}+\mathbb{B})v_1 = 0, \qquad (\mathbb{A}+\mathbb{B})v_2 = (I+\mathbb{B})v_1,\\ &(\mathbb{A}+\mathbb{B})v_3 = -\frac{1}{2}\mathbb{B}v_1 + (I+\mathbb{B})v_2,\end{aligned}$$

from which we obtain

$$\begin{aligned}&v_1 = \begin{pmatrix}1\\ m_1\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\ m_2\end{pmatrix}, \quad v_3 = \begin{pmatrix}0\\ 0\end{pmatrix},\\ &m_1 = \frac{\cos\sigma}{\alpha}, \quad m_2 = -\frac{1+\cos\sigma}{\alpha(1+\sec\sigma+\tan\sigma)}.\end{aligned}$$

Similarly, (10) is equivalent to, respectively,

$$\begin{aligned}&w_1(\mathbb{A}+\mathbb{B}) = 0, \qquad w_2(\mathbb{A}+\mathbb{B}) = w_1(I+\mathbb{B}),\\ &w_3(\mathbb{A}+\mathbb{B}) = -\frac{1}{2}w_1\mathbb{B} + w_2(I+\mathbb{B}).\end{aligned}$$

From $w_1(\mathbb{A}+\mathbb{B}) = 0$ we obtain $w_1 = (a_1, a_2)$, a_1 and a_2 will be determined later. Since $\varphi_3(\theta) = v_3 + v_2\theta + v_1\theta^2/2$ and $\psi_3(s) = w_1$. Setting $w_2 = (a_3, a_4)$ in $w_2(\mathbb{A}+\mathbb{B}) = w_1(I+\mathbb{B})$, $w_3 = (a_5, a_6)$ in $w_3(\mathbb{A}+\mathbb{B}) = -(1/2)w_1\mathbb{B} + w_2(I+\mathbb{B})$ and then using $\langle\Psi, \Phi\rangle = I$, we obtain the expressions of a_1, a_2, a_3, a_4, a_5 and a_6 after long and tedious calculations

$$\begin{aligned}&a_1 = n\bigl(q+\beta^*\bigr), \quad a_2 = -n\alpha\beta^*,\\ &n = -\frac{24\alpha^2\cos\frac{\sigma}{2}(1+\sin\sigma)}{(2\cos\frac{\sigma}{2} - \sin\frac{\sigma}{2})(4\alpha^2 - \cos 3\sigma + \cos\sigma(5+8\sin\sigma))},\end{aligned}$$

$$
\begin{aligned}
a_3 &= \big[12\alpha^4\sec^3\sigma(1+\sec\sigma)(1+\sin\sigma)\tan\sigma\big] \\
&\quad\times\big[\big(-3+2\alpha^2+\cos 2\sigma-4\sin\sigma\big)(2+2\sec\sigma-\tan\sigma)(1+\sec\sigma+\tan\sigma) \\
&\quad\times\big(-1+2\sec^2\sigma+\alpha^2\sec^3\sigma+2\sec\sigma\tan\sigma\big)\big]^{-1}, \\
a_4 &= \big[6\alpha^3\sec\sigma(1+\sec\sigma-\tan\sigma)\big(-1+2\sec^2\sigma+2\sec\sigma\tan\sigma\big)\big] \\
&\quad\times\big[\big(-3+2\alpha^2+\cos 2\sigma-4\sin\sigma\big)(2+2\sec\sigma-\tan\sigma) \\
&\quad\times\big(-1+2\sec^2\sigma+\alpha^2\sec^3\sigma+2\sec\sigma\tan\sigma\big)\big]^{-1} \\
a_5 &= -\big[6\alpha^2(1+\cos\sigma)^3\cot^2\sigma(-1+\sec\sigma) \\
&\quad\times\big(-1+16\alpha^2\sec^7\sigma+\tan\sigma+\sec\sigma(5+6\tan\sigma) \\
&\quad-2\sec^6\sigma\big(-16+4\alpha^2+\alpha^4-8\alpha^2\tan\sigma\big) \\
&\quad+\sec^2\sigma\big(18-\alpha^2+\big(-12+\alpha^2\big)\tan\sigma\big)+\sec^3\big(5\big(\alpha^2-4\big)+4\big(\alpha^2-8\big)\tan\sigma\big) \\
&\quad-4\sec^4\sigma\big(-2\big(\alpha^2-6\big)+\big(3\alpha^2-4\big)\tan\sigma\big) \\
&\quad-2\sec^5\sigma\big(10\alpha^2-8+\big(\alpha^4+4\alpha^2-16\big)\tan\sigma\big)\big)\big] \\
&\quad\times\big[\big(-3+2\alpha^2+\cos 2\sigma-4\sin\sigma\big)(1+\sin\sigma)\big(\alpha^2+\cos\sigma(1+\sin\sigma)\big) \\
&\quad\times(2+2\sec\sigma-\tan\sigma)(1+\sec\sigma+\tan\sigma)^2 \\
&\quad\times\big(-1+2\sec^2\sigma+\alpha^2\sec^3\sigma+2\sec\sigma\tan\sigma\big)\big]^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
a_6 &= -\bigg[3\alpha^3\cos^4\frac{\sigma}{2}\sec^6\frac{\sigma}{2}\bigg(\cos\frac{\sigma}{2}+\sin\frac{\sigma}{2}\bigg)^3\bigg(\big(-84-64\alpha^2+64\alpha^4\big)\cos\frac{\sigma}{2} \\
&\quad+12\big(4\alpha^2-3\big)\cos\frac{3\sigma}{2}+6\cos\frac{5\sigma}{2}16\alpha^2\cos\frac{5\sigma}{2}+46\cos\frac{7\sigma}{2}+6\cos\frac{9\sigma}{2} \\
&\quad-2\cos\frac{11\sigma}{2}-42\sin\frac{\sigma}{2}-76\alpha^2\sin\frac{\sigma}{2}-54\sin\frac{3\sigma}{2}-36\alpha^2\sin\frac{3\sigma}{2}-69\sin\frac{5\sigma}{2} \\
&\quad+12\alpha^2\sin\frac{5\sigma}{2}-11\sin\frac{7\sigma}{2}+4\alpha^2\sin\frac{7\sigma}{2}+15\sin\frac{9\sigma}{2}+\sin\frac{11\sigma}{2}\bigg)\bigg] \\
&\quad\times\big[2\big(-3+2\alpha^2+\cos 2\sigma-4\sin\sigma\big)(1+\sin\sigma)^2\big(\alpha^2+\cos\sigma(1+\sin\sigma)\big) \\
&\quad\times(2+2\sec\sigma-\tan\sigma)(1+\sec\sigma+\tan\sigma)^2 \\
&\quad\times\big(-1+2\sec^2\sigma+\alpha^2\sec^3\sigma+2\sec\sigma\tan\sigma\big)\big]^{-1}.
\end{aligned}
$$

Thus we obtain the bases Φ and Ψ of P and P^* such that $\dot{\Phi}=\Phi J$ and $\dot{\Psi}=-J\Psi$.

Next we compute the corresponding normal form. Let $u=\Phi x+y$ (here $x=(x_1,x_2,x_3)^{\mathrm{T}}\in\mathbb{R}^3$ and $y=(y_1,y_2)^{\mathrm{T}}\in C$); namely

$$
\begin{aligned}
u_1(\theta) &= x_1+\theta x_2+\frac{\theta^2}{2}x_3+y_1(\theta), \\
u_2(\theta) &= m_1x_1+(\theta m_1+m_2)x_2+\bigg(\frac{m_1}{2}\theta^2+\theta m_2\bigg)x_3+y_2(\theta).
\end{aligned}
$$

Then, on the center manifold $y = g(x(t), \theta)$, system (7) becomes

$$\begin{aligned}\dot{x} &= Jx + \Psi(0)F\big(\Phi x + g(x,\theta), \mu\big)\\ &= Jx + \frac{1}{2}f_2^1(x,0,\mu) + \frac{1}{3!}f_3^1(x,0,\mu) + \mathcal{O}\big(|\mu||x|^2 + |\mu|^2|x| + |x|^4\big),\end{aligned}$$

which is

$$\begin{aligned}\dot{x}_1 &= x_2 + \sum_{j=1}^{3}\alpha_j x_j\\ &\quad + \frac{1}{4}\tau^* i^{(2)}\big[4\alpha a_5\big(x_1 + g_1(x,-1)\big)^2 + a_6\big(2x_1 - 2x_2 + x_3 + 2g_2(x,-1)\big)^2\big]\\ &\quad + \frac{1}{8}\tau^* i^{(3)}\big[8\alpha a_5\big(x_1 + g_1(x,-1)\big)^3 + a_6\big(2x_1 - 2x_2 + x_3 + 2g_2(x,-1)\big)^3\big]\\ &\quad + \mathcal{O}\big(|\mu||x|^2 + |\mu|^2|x| + |x|^4\big),\\ \dot{x}_2 &= x_3 + \sum_{j=1}^{3}\beta_j x_j\\ &\quad + \frac{1}{4}\tau^* i^{(2)}\big[4\alpha a_3\big(x_1 + g_1(x,-1)\big)^2 + a_4\big(2x_1 - 2x_2 + x_3 + 2g_2(x,-1)\big)^2\big]\\ &\quad + \frac{1}{8}\tau^* i^{(3)}\big[8\alpha a_3\big(x_1 + g_1(x,-1)\big)^3 + a_4\big(2x_1 - 2x_2 + x_3 + 2g_2(x,-1)\big)^3\big]\\ &\quad + \mathcal{O}\big(|\mu||x|^2 + |\mu|^2|x| + |x|^4\big),\\ \dot{x}_3 &= \sum_{j=1}^{3}\gamma_j x_j\\ &\quad + \frac{1}{4}\tau^* i^{(2)}\big[4\alpha a_1\big(x_1 + g_1(x,-1)\big)^2 + a_2\big(2x_1 - 2x_2 + x_3 + 2g_2(x,-1)\big)^2\big]\\ &\quad + \frac{1}{8}\tau^* i^{(3)}\big[8\alpha a_1\big(x_1 + g_1(x,-1)\big)^3 + a_2\big(2x_1 - 2x_2 + x_3 + 2g_2(x,-1)\big)^3\big]\\ &\quad + \mathcal{O}\big(|\mu||x|^2 + |\mu|^2|x| + |x|^4\big),\end{aligned} \tag{11}$$

where α_j, β_j, γ_j are linear functions of μ ($j = 1, 2, 3$) and will be given later. Denote the coefficient of $x_1^j x_2^k x_3^l$ by $(a_{jkl}, b_{jkl}, c_{jkl})^{\mathrm{T}}$ in the right side of system (11). Then

$$\begin{aligned}\frac{1}{2}f_2^1(x,0,\mu) &= \sum_{1\leqslant j\leqslant 3}\begin{pmatrix}\alpha_j(\mu)\\ \beta j(\mu)\\ \gamma_j(\mu)\end{pmatrix}x_j + \sum_{j+k+l=2}\begin{pmatrix}a_{jkl}\\ b_{jkl}\\ c_{jkl}\end{pmatrix}x_1^j x_2^k x_3^l.\\ \frac{1}{3!}f_3^1(x,0,\mu) &= \sum_{j+k+l=3}\begin{pmatrix}a_{jkl}\\ b_{jkl}\\ c_{jkl}\end{pmatrix}x_1^j x_2^k x_3^l.\end{aligned}$$

If $i^{(2)} \neq 0$, after projection on the center manifold and truncation up to the second order, system (11) can be written as the following:

$$\dot{x} = Jx + \frac{1}{2}f_2^1(x,0,\mu) + \mathcal{O}\big(|\mu|^2|x| + |x|^3\big), \tag{12}$$

If $i^{(2)} = 0$, $i^{(3)} \neq 0$, after projection on the center manifold and truncation up to the third order, then system (11) can be written as the following:

$$\dot{x} = Jx + \sum_{1\leqslant k\leqslant 3} \begin{pmatrix} \alpha_k(\mu) \\ \beta_k(\mu) \\ \gamma_k(\mu) \end{pmatrix} x_k + \frac{1}{3!} f_3^1(x,0,0) + \mathcal{O}\big(|\mu|^2|x| + |x|^4\big). \tag{13}$$

From [1], system (12) can be transformed as the following normal form,

$$\dot{x} = Jx + \frac{1}{2!} g_2^1(x,0,\mu) + \mathcal{O}\big(|\mu|^2|x| + |x|^3\big),$$

or

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \\ &\dot{x}_3 = \sum_{j=1}^{3} \chi_j x_j + A_{200}x_1^2 + A_{110}x_1x_2 + A_{101}x_1x_3 + A_{020}x_2^2 + \mathcal{O}\big(|\mu|^2|x| + |x|^3\big), \end{aligned} \tag{14}$$

in which χ_j and A_{jkl} are given by

$$\begin{aligned} &\chi_1 = \gamma_1, \quad \chi_2 = \beta_1 + \gamma_2, \quad \chi_3 = \alpha_1 + \beta_2 + \gamma_3, \\ &A_{200} = c_{200}, \quad A_{020} = 2a_{200} + b_{110} + c_{020}, \\ &A_{110} = 2b_{200} + c_{110}, \quad A_{101} = 2a_{200} + b_{110} + c_{101}. \end{aligned}$$

After long computation, we obtain the explicit expressions of α_1, β_1, γ_1, β_2, γ_3, and A_{jkl} in (14)

$$\begin{aligned} \alpha_1 &= \tau^*(\alpha a_5 + a_6)(\mu_1 - m_1\mu_2), \\ \beta_1 &= \tau^*(\alpha a_3 + a_4)(\mu_1 - m_1\mu_2) \\ &\quad + \left[\alpha a_3\left(\frac{\beta^*\gamma}{q} - \beta m_1\right) + a_4\left(\gamma + \frac{\beta^*\gamma}{q} - (q+\beta^*)m_1\right)\right]\mu_3, \\ \gamma_1 &= \tau^*(\alpha a_1 + a_2)(\mu_1 - m_1\mu_2) + \frac{1}{q}\big[\alpha\beta^* a_1 + (q+\beta^*)a_2\big](\gamma - qm_1)\mu_3, \\ \beta_2 &= \tau^*\big[-a_4\mu_1 + \big(-\alpha a_3 m_2 + a_4(m_1 - m_2)\big)\mu_2\big] \\ &\quad + \left[-\alpha\beta^* a_3 m_2 + a_4\left(-\gamma - \frac{\beta^*\gamma}{q} - qm_2 + \beta^*(m_1 - m_2)\right)\right]\mu_3, \\ \gamma_2 &= \tau^*\big[-a_2\mu_1 + \big(-\alpha a_1 m_2 + a_2(m_1 - m_2)\big)\mu_2\big] \\ &\quad - \frac{1}{q}\big[q\alpha\beta^* a_1 m_2 + a_2\big(-q\alpha\beta^* a_1 m_2 + a_2\big(-q\beta^* m_1 + (q+\beta)(\gamma + qm_2)\big)\big)\big]\mu_3, \\ \gamma_3 &= \frac{1}{2q}\tau^* a_2\big[q\mu_1 + q(-m_1 + 2m_2)\mu_2 + \big((q+\beta^*)\gamma - q\beta^*(m_1 - 2m_2)\big)\mu_3\big], \end{aligned}$$

and also the explicit expressions of A_{ljk}

$$\begin{aligned} &A_{200} = \tau^* i^{(2)}(\alpha a_1 + a_2), \quad A_{020} = \tau^* i^{(2)}(a_2 - 2a_4 + 2\alpha a_5 + 2a_6), \\ &A_{110} = 2\tau^* i^{(2)}(-a_2 + \alpha a_3 + a_4), \quad A_{101} = \tau^* i^{(2)}(a_2 - 2a_4 + 2\alpha a_5 + 2a_6). \end{aligned}$$

Since

$$\left|\frac{\partial \chi}{\partial \mu}\right| = \det\begin{pmatrix} \frac{\partial \chi_1}{\partial \mu_1} & \frac{\partial \chi_1}{\partial \mu_2} & \frac{\partial \chi_1}{\partial \mu_3} \\ \frac{\partial \chi_2}{\partial \mu_1} & \frac{\partial \chi_2}{\partial \mu_2} & \frac{\partial \chi_2}{\partial \mu_3} \\ \frac{\partial \chi_3}{\partial \mu_1} & \frac{\partial \chi_3}{\partial \mu_2} & \frac{\partial \chi_3}{\partial \mu_3} \end{pmatrix}$$
$$= \frac{1728\alpha^6 \cos\sigma \csc^3\frac{\sigma}{2} \sin^7\sigma}{\gamma(2\cos\frac{\sigma}{2} - \sin\frac{\sigma}{2})^3[5\cos\sigma - \cos 3\sigma + 4(\alpha^2 + \sin 2\sigma)]^3} \neq 0,$$

we have that $(\mu_1, \mu_2, \mu_3) \to (\chi_1, \chi_2, \chi_3)$ is regular and hence the transversality condition holds.

If $i^{(2)} = 0$ and $i^{(3)} \neq 0$, then $A_{ijk} = 0$ for $i + j + k = 2$. Using a result from [13], we transform system (13) into the following normal form:

$$\dot{x} = Jx + \frac{1}{2}g_2^1(x, 0, \mu) + \frac{1}{3!}g_3^1(x, 0, 0) + \mathcal{O}\big(|\mu|^2|x| + |x|^4\big),$$

or equivalently

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \\ &\dot{x}_3 = \chi_1 x_1 + \chi_2 x_2 + \chi_3 x_3 + A_{300}x_1^3 + A_{210}x_1^2x_2 + A_{120}x_1x_2^2 \\ &\qquad + A_{030}x_2^3 + A_{201}x_1^2x_3 + A_{102}x_1x_3^2 + \mathcal{O}(|\mu|^2|x| + |x|^4), \end{aligned} \tag{15}$$

where

$$\begin{aligned} &A_{300} = c_{300}, \quad A_{030} = \frac{1}{3}b_{210} + \frac{1}{3}b_{120} - \frac{2}{3}b_{102} - \frac{2}{3}b_{201} + \frac{2}{3}b_{021} - \frac{1}{3}c_{111} + c_{030}, \\ &A_{210} = 3b_{300} + c_{210}, \quad A_{120} = 6a_{111} + c_{120}, A_{201} = 3a_{300} + c_{201}, \\ &A_{102} = -3a_{030} + a_{201} - \frac{1}{2}a_{120} + 3a_{111} - \frac{3}{2}b_{030} + \frac{1}{2}b_{111} - \frac{1}{2}c_{201} - \frac{1}{2}c_{012} + c_{102}. \end{aligned}$$

Using the expressions of a_{ijk}, b_{ijk} and c_{ijk}, we have

$$\begin{aligned} &A_{300} = \tau^* i^{(3)}(\alpha a_1 + a_2), \quad A_{210} = 3\tau^* i^{(3)}(-a_2 + \alpha a_3 + a_4), \\ &A_{120} = 3\tau^* i^{(3)}(a_2 + 2\alpha a_5 + 2a_6), \quad A_{210} = \frac{3}{2}\tau^* i^{(3)}(a_2 + 2\alpha a_5 + 2a_6), \\ &A_{102} = \frac{3}{8}\tau^* i^{(3)}(a_2 + 8\alpha a_5 + 12a_6), \quad A_{030} = -\frac{1}{2}\tau^* i^{(3)}a_4. \end{aligned}$$

Note that

$$\alpha a_1 + a_2 = \frac{6\alpha^3\sec^2\sigma(1 + \sin\sigma)\tan\sigma}{(-2 - 2\sin\sigma + \tan\sigma)(-1 + 2\sin^2\sigma + \alpha^2\sec^3\sigma + 2\sec\sigma\tan\sigma)} < 0.$$

4 Bifurcation diagrams

In this section, we truncate higher order terms from system (14) and (15) to obtain bifurcation diagrams of system (3).

4.1 $i^{(2)} \neq 0$

If $i^{(2)} \neq 0$, we consider the truncated system of (14)

$$\begin{aligned} &\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_3, \\ &\dot{x}_3 = \chi_1 x_1 + \chi_2 x_2 + \chi_3 x_3 + A_{200} x_1^2 + A_{110} x_1 x_2 + A_{101} x_1 x_3 + A_{020} x_2^2, \end{aligned} \tag{16}$$

where A_{200}, A_{110}, A_{101} and A_{020} are in Section 3. Note that

$$(A_{200}, A_{110}, A_{101}, A_{020}) \to (-A_{200}, -A_{110}, -A_{101}, -A_{020})$$

under the transformation $(x_1, x_2, x_3) \to (-x_1, -x_2, -x_3)$. We may assume that $i^{(2)} > 0$. The complete bifurcation diagrams of system (16) can be found in [1]. Here, we just briefly list some results.

Lemma 3. *For* (χ_1, χ_2, χ_3) *small enough*:

(i) *The origin is stable when* $\chi_i < 0$ $(i = 1, 2, 3)$ *and* $\chi_1 > -\chi_2\chi_3$.

(ii) *The origin undergoes a Hopf bifurcation on the curve*

$$C_1 = \left\{ (\chi_1, \chi_2, \chi_3) : \chi_3 = -\frac{\chi_1}{\chi_2}, \chi_2 < 0 \right\}.$$

(iii) *For* $\chi_1 \neq 0$*, there is a nontrivial equilibrium point at* $x^* = (-\chi_1/A_{200}, 0, 0)$. *Moreover, the nontrivial equilibrium point* x^* *undergoes a Hopf bifurcation on the curve*

$$\begin{aligned} C_2 = \bigg\{ &(\chi_1, \chi_2, \chi_3)\colon\ \chi_3 = \chi_1 \left(\frac{A_{101}}{A_{200}} - \frac{A_{200}}{A_{110}\chi_1 - A_{200}\chi_2} \right), \\ &\frac{A_{200}}{A_{110}\chi_1 - A_{200}\chi_2} > 0 \bigg\}. \end{aligned}$$

(iv) *The origin undergoes a BT bifurcation on the curve*

$$C_3 = \{ (\chi_1, \chi_2, \chi_3)\colon\ \chi_1 = \chi_2 = 0 \}.$$

(v) *The origin and the nontrivial equilibrium point* x^* *undergo zero-Hopf bifurcation on the curve*

$$C_4 = \{ (\chi_1, \chi_2, \chi_3)\colon\ \chi_1 = \chi_3 = 0,\ \chi_2 < 0 \}.$$

Note that for case (iv), when $\chi_1 = \chi_2 = 0$ and $\chi_3 < 0$, system (16) undergoes BT bifurcation. Using the technique in [14] (we omit the detail), it is not hard to obtain that (16) is equivalent to the following normal form:

$$\begin{aligned} &\dot{x}_1 = x_2, \\ &\dot{x}_2 = \rho_1 x_1 + \rho_2 x_2 + r_1 x_1^2 + s_1 x_1 x_2, \end{aligned} \tag{17}$$

where

$$\rho_1 = \chi_1, \quad \rho_2 = -\frac{\chi_1 + \chi_3\chi_2}{\chi_3^2}, \quad r_1 = \frac{A_{200}}{\chi_3^2}, \quad s_1 = -\frac{2A_{200} + A_{110}\chi_3}{\chi_3^2}.$$

Since χ_3 is small, we can see that $r_1 s_1 < 0$. For the bifurcation diagram of this system, we need the following result from [15].

Lemma 4. *Assume that* $r_1 s_1 \neq 0$ *and* (ρ_1, ρ_2) *are small enough. Then*:

(i) *The curve* $l_h = \{(\rho_1, \rho_2)\colon\ \rho_2 = (s_1/r_1)\rho_1 + o(\rho_1),\ \rho_1 > 0\}$ *is a Hopf point branch of system* (17).

(ii) *The curve* $l_\infty = \{(\rho_1, \rho_2)\colon\ \rho_2 = (6s_1/7r_1)\rho_1 + o(\rho_1),\ \rho_1 > 0\}$ *is a homoclinic branch of system* (17).

Moreover if (ρ_1, ρ_2) *is between* l_h *and* l_∞*, there is a unique stable limit cycle.*

Then using the expressions of χ_1, χ_2, χ_3 and hence μ_1, μ_2, μ_3, we have the following result.

Theorem 1. *Assume that* $i^{(2)} \neq 0$ *and* (μ_1, μ_2, μ_3) *are small enough and* $\chi_3 \neq 0$*. Then*:

(i) *The curve* $\tilde{l}_h = \{(\mu_1, \mu_2, \mu_3)\colon\ \chi_1 = A_{200}\chi_2\chi_3/(A_{200} + A_{110}\chi_3) + o(|\chi|^2),$ $\chi_1 > 0\}$ *is a Hopf point branch of system* (7).

(ii) *The curve* $\tilde{l}_\infty = \{(\mu_1, \mu_2, \mu_3)\colon\ \chi_1 = 7A_{200}\chi_2\chi_3/(5A_{200} + 6A_{110}\chi_3) + o(|\chi|^2),$ $\chi_1 > 0\}$ *is a homoclinic branch of system* (7).

Case (v) is very interesting and we can further find the normal form for this bifurcation:

$$\begin{aligned}
\dot{x}_1 &= \left(\frac{\chi_1}{4\chi_2} + \frac{\chi_3}{4}\right)x_1 + \frac{2A_{200} + iA_{110}\sqrt{-\chi_2} + A_{101}\chi_2}{4\chi_2}x_1x_3, \\
\dot{x}_2 &= \left(\frac{\chi_1}{4\chi_2} + \frac{\chi_3}{4}\right)x_2 + \frac{2A_{200} - iA_{110}\sqrt{-\chi_2} + A_{101}\chi_2}{4\chi_2}x_2x_3, \\
\dot{x}_3 &= -\frac{\chi_1}{2\chi_2}x_3 + \frac{-A_{200} + (A_{020} - A_{101})\chi_2}{\chi_2^3}x_1x_2 - \frac{A_{200}}{2\chi_2}x_3^2,
\end{aligned} \tag{18}$$

Since $x_1 = \bar{x}_2$, through the change of variables $x_1 = w_1 - iw_2$, $x_2 = w_1 + iw_2$, $x_3 = w_3$, and then a change to cylindrical coordinates according to $w_1 = r\cos\xi$, $w_2 = r\sin\xi$, $w_3 = \zeta$, system (18) becomes

$$\begin{aligned}
\dot{r} &= \left(\frac{\chi_1}{4\chi_2} + \frac{\chi_3}{4}\right)r + \frac{2A_{200} + A_{101}\chi_2}{4\chi_2}r\zeta, \\
\dot{\zeta} &= -\frac{\chi_1}{2\chi_2}\zeta + \frac{-A_{200} + (A_{020} - A_{101})\chi_2}{\chi_2^3}r^2 - \frac{A_{200}}{2\chi_2}\zeta^2, \\
\dot{\xi} &= -\sqrt{-\chi_2}.
\end{aligned}$$

Discarding the third equation above, we have

$$\begin{aligned}\dot r &= \left(\frac{\chi_1}{4\chi_2}+\frac{\chi_3}{4}\right)r+\frac{2A_{200}+A_{101}\chi_2}{4\chi_2}r\zeta,\\ \dot\zeta &= -\frac{\chi_1}{2\chi_2}\zeta+\frac{-A_{200}+(A_{020}-A_{101})\chi_2}{\chi_2^3}r^2-\frac{A_{200}}{2\chi_2}\zeta^2.\end{aligned}\tag{19}$$

After shifting $r\to r$ and $\zeta\to\zeta+\chi_1/A_{200}$ and rescaling

$$r\to-\frac{1}{\chi_2^2}\sqrt{\frac{1}{2}\big|A_{200}\big(-A_{200}+(A_{020}-A_{101})\chi_2\big)\big|}r,\qquad \zeta\to\frac{A_{200}}{2\chi_2}\zeta,$$

system (19) can be written as

$$\begin{aligned}\dot r &= (-\frac{A_{200}+A_{101}\chi_2}{4\chi_2A_{200}}\chi_1+\frac{\chi_3}{4})r+\frac{2A_{200}+A_{101}\chi_2}{2A_{200}}r\zeta,\\ \dot\zeta &= \frac{\chi_1^2}{4\chi_2^2}+sr^2-\zeta^2,\end{aligned}\tag{20}$$

where $s=\operatorname{sgn}[A_{200}(-A_{200}+(A_{020}-A_{101})\chi_2)]$. Note that if $\mu=(\mu_1,\mu_2,\mu_3)$ is small, so is χ_2 and hence $s=-1$. The complete bifurcation diagrams of system (19) are very complicated and the interested reader can find in [16].

4.2 $i^{(2)}=0$ and $i^{(3)}\neq 0$

If $i^{(2)}=0$ and $i^{(3)}\neq0$, we consider the truncated system of (15)

$$\begin{aligned}&\dot x_1=x_2,\qquad \dot x_2=x_3,\\ &\dot x_3=\chi_1x_1+\chi_2x_2+\chi_3x_3+A_{300}x_1^3+A_{210}x_1^2x_2+A_{120}x_1x_2^2\\ &\qquad+A_{003}x_2^3+A_{201}x_1^2x_3+A_{102}x_1x_3^2.\end{aligned}\tag{21}$$

The complete bifurcation diagrams of system (21) are similar to those for $i^{(2)}\neq0$ and summarized in the following lemma.

Lemma 5. (i)′ *The origin is stable when* $\chi_i<0$ $(i=1,2,3)$ *and* $\chi_1>-\chi_2\chi_3$.

(ii)′ *The origin undergoes a Hopf bifurcation on the curve*

$$C_1'=\left\{(\chi_1,\chi_2,\chi_3)\colon\ \chi_3=-\frac{\chi_1}{\chi_2},\ \chi_2<0\right\}.$$

(iii)′ *For* $\chi_1<0$*, there are two nontrivial equilibrium points* $x_\pm^*=(\pm\sqrt{\chi_1/A_{300}},0,0)$. *Moreover, the nontrivial equilibrium points* $x_\pm^*$ *undergo a Hopf bifurcation on the curve*

$$C_2'=\left\{(\chi_1,\chi_2,\chi_3)\colon\ \chi_3=\frac{\chi_1(2A_{300}^2-A_{201}A_{210}\chi_1+A_{201}A_{300}\chi_2)}{A_{300}(-A_{210}\chi_1+A_{300}\chi_2)},\ \frac{A_{210}\chi_1}{A_{300}}-\chi_2>0\right\}.$$

(iv)′ *The origin undergoes a BT bifurcation on the curve*

$$C_3' = \{(\chi_1, \chi_2, \chi_3)\colon \chi_1 = \chi_2 = 0\}.$$

(v)′ *The origin and the nontrivial equilibrium points $x^*_\pm$ undergo zero-Hopf bifurcation on the curve*

$$C_4' = \{(\chi_1, \chi_2, \chi_3)\colon \chi_1 = \chi_3 = 0,\ \chi_2 < 0\}.$$

We can use the same technique as in the previous subsection to study BT and zero-Hopf bifurcations and here we omit the details. Note that for case (iv)′, when $\chi_1 = \chi_2 = 0$ and $\chi_3 < 0$, the system undergoes BT bifurcation. Using the technique in [14], it is not hard to obtain that (21) is equivalent to the following normal form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \rho_1 x_1 + \rho_2 x_2 + r_2 x_1^3 + s_2 x_1^2 x_2, \end{aligned} \tag{22}$$

where

$$\rho_1 = -\frac{\chi_1}{\chi_3}, \quad \rho_2 = -\frac{\chi_1 + \chi_3\chi_2}{\chi_3^2}, \quad r_2 = A_{300}, \quad s_2 = -\frac{3A_{300} + A_{210}\chi_3}{\chi_3^2}.$$

Since χ_3 is small, we can see that $r_2 s_2 < 0$. Without loss of generality, we assume $i^{(3)} < 0$ so that $r_2 > 0$ and $s_2 < 0$. System (22) can be transformed as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \varepsilon_1 x_1 + \varepsilon_2 x_2 + x_1^3 - x_1^2 x_2, \end{aligned} \tag{23}$$

where

$$\begin{aligned} \varepsilon_1 &= \left(\frac{s_2}{r_2}\right)^2 \chi_1 = \frac{\chi_1 (3A_{300} + A_{210}\chi_3)^2}{A_{300}^2 \chi_3^4}, \\ \varepsilon_2 &= \frac{s_2}{r_2}\chi_2 = \frac{\chi_1 (3A_{300} + A_{210}\chi_3)(\chi_1 + \chi_2\chi_3)}{A_{300}\chi_3^4}. \end{aligned}$$

The complete bifurcation diagrams of system (23) can be found, for example, in [14, 17]. Here, we just list two results.

Lemma 6. *For small $\varepsilon_1, \varepsilon_2$:*

(i) *System* (23) *undergoes a Hopf bifurcation for the trivial equilibrium point on the line*

$$H_1 = \{(\varepsilon_1, \varepsilon_2)\colon \varepsilon_2 = 0,\ \varepsilon_1 < 0\}.$$

(ii) *On the curve*

$$C = \left\{(\varepsilon_1, \varepsilon_2)\colon \varepsilon_2 = -\frac{1}{5}\varepsilon_1 + o(\varepsilon_1),\ \varepsilon_1 < 0\right\},$$

system (23) *undergoes a heteroclinic bifurcation. Moreover, if $(\varepsilon_1, \varepsilon_2)$ is in the region between the curves H_1 and C system* (23) *has a unique stable periodic orbit.*

For (v)', we can use the same technique in [14] to find the following normal form for zero-Hopf bifurcation at the origin (for simplicity):

$$\begin{aligned} \dot r &= r(\eta_1 + p_{11} r + p_{12}\zeta),\\ \dot\zeta &= \zeta(\eta_2 + p_{21} r + p_{22}\zeta), \end{aligned} \tag{24}$$

where

$$\begin{aligned} \eta_1 &= \frac{\chi_1 + \chi_2\chi_3}{2\chi_2}, \quad \eta_2 = -\frac{\chi_1}{\chi_2},\\ p_{11} &= \frac{3A_{300} + \chi_2(-A_{120} + 3(A_{201} + A_{102}\chi_2))}{2\chi_2^3}, \quad p_{12} = \frac{\chi_2 A_{201} + 3A_{300}}{2\chi_2},\\ p_{21} &= -\frac{2[3A_{300} + \chi_2(-A_{120} + 2A_{201} + A_{102}\chi_2)]}{\chi_2^3}, \quad p_{22} = -\frac{A_{300}}{\chi_2}. \end{aligned}$$

This system is a so-called predator-prey system which has been well studied and here we omit the detail.

5 Numerical simulations

In this section, we give some example to verify some theoretical results obtained Section 4. For simplicity, we assume that (0,0) is one of the equilibrium points.

Let $\alpha = 1$, $\gamma = 3/5$, $q = 1$. Then $k^* = 27/20$, $\beta^* = 5/4$, $\tau^* = 2$. Take

$$I(s) = \tanh(ks) + 0.1s^2.$$

Then $(0,0)$ is the trivial equilibrium point and $i^{(2)} = 0.1 > 0$. Simple calculation shows

$$\begin{aligned} \chi_1 &= -\frac{10}{23}(5\mu_1 - 3\mu_2), \quad \chi_2 = -\frac{5}{644}(275\mu_1 - 53\mu_2 + 105\mu_3),\\ \chi_3 &= \frac{5(174830\mu_1 - 413778\mu_2 - 93015\mu_3)}{1808352}, \end{aligned}$$

$$A_{200} = -\frac{5}{23}, \quad A_{020} = \frac{87415}{904176}, \quad A_{110} = -\frac{275}{644}, \quad A_{101} = \frac{2335}{4914}.$$

1. Take $\mu_1 = -0.00001$, $\mu_2 = 0.00001$, $\mu_3 = 0.012970922388922591$ and hence $k = k^* + \mu_1 = 1.34999$, $\beta = \beta^* + \mu_2 = 1.25001$, $\tau = \tau^* + \mu_3 = 2.01297$. Easy calculation shows that

$$(\chi_1, \chi_2, \chi_3) = (0.0000347826, -0.0105487, -0.00335216)$$

is on the curve C_2 so that there is a unique limit cycle bifurcation from the nontrivial equilibrium point. It is not hard to use the technique in [14] to show that the first Lyapunov coefficient $\ell_1(0) = -2.26094 \times 10^{10} < 0$ in this setting. Thus the limit cycle is stable (see Fig. 1).

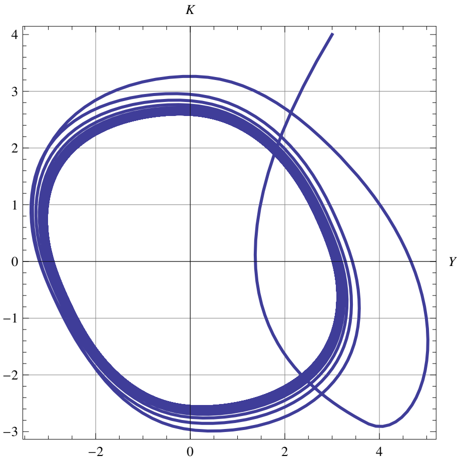

Fig. 1. A stable limit cycle is generated when (μ_1, μ_2) is near to C_2.

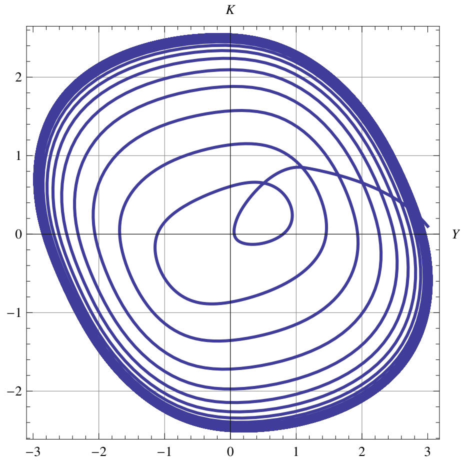

Fig. 2. A stable limit cycle is generated when (μ_1, μ_2, μ_3) is such that (ρ_1, ρ_2) is between two curves l_h and l_∞.

2. Take $\mu_1 = 0.0001$, $\mu_2 = 0.00016663748512537344$, $\mu_3 = 0.0001$ and hence $k = k^* + \mu_1 = 1.3501$, $\beta = \beta^* + \mu_2 = 1.2501666374851255$, $\tau = \tau^* + \mu_3 = 2.0001$. Easy calculation shows that

$$(\chi_1, \chi_2, \chi_3) = \bigl(-3.80629 \times 10^{-8}, -0.000226461, -0.000168024\bigr)$$

and hence

$$\rho_1 = -0.00022653195983350339, \quad \rho_2 = 0.00042063278506487O7.$$

It is easy to check that (ρ_1, ρ_2) is between two curves l_h and l_∞ and thus there is a unique stable limit cycle bifurcating from the trivial equilibrium point (see Fig. 2).

6 Conclusion

Since Krawiec and Szydlowski [5–7] introduced Kalecki's idea that there is a time delay for investment before a business decision to Kaldor's model, the Kaldor–Kalecki model has been studied extensively. In this presentation, we studied the triple-zero singularity for the Kaldor–Kalecki model of business cycle with delay in both the gross product and the capital stock. First we analyzed the characteristic equation at the equilibrium point and gave the condition such that it has a triple-zero root. The normal form for this singularity was presented. By using this normal form, the bifurcation diagrams were given. Some examples were given to confirm the theoretic result.

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