Output feedback control of nonlinear systems with uncertain ISS/iISS supply rates and noises*

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Abstract. This paper considers the problem of global output feedback control for a class of nonlinear systems with inverse dynamics. The main contribution of paper is that: For the inverse dynamics with uncertain ISS/iISS supply rates, and the systems being disturbed by L^2 noises, we construct a reduced-order observer-based output feedback controller, which drives the output of system to zero and maintain other closed-loop signals bounded. Finally, a simulation example shows the effectiveness of the control scheme.

Keywords: nonlinear systems, ISS/iISS, uncertain supply rates, reduced-order observer, L^2 noises.

1 Introduction

Since the notion of input-to-state stability (ISS) was first introduced in [1], it has been recognized as a central concept in nonlinear control systems. [2–5] and the references therein investigated many kinds of properties of ISS. [6–9] and the references therein considered controller design and stability analysis for various classes of nonlinear systems with ISS (or ISpS) inverse dynamics. Subsequently, another important concept, integral input-to-state stability (iISS), was firstly presented in [10], and several characterizations on iISS were investigated in [11], in which iISS is proved to be strictly weaker than ISS. In [12], the authors analyzed nonlinear cascades in which the driven subsystem is iISS, and characterized the admissible iISS-gains for stability. Recently, [13–16] gave several Lyapunov-based small-gain theorems covering iISS systems.

So far, in addition to the above literatures, there are many other results on the design and analysis of controller for nonlinear systems with ISS/iISS inverse dynamics. For

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example, Arcak et al. in [12] applied the admissible iISS-gains for stability of cascade systems to develop a new observer-based backstepping design. Jiang et al. in [17] firstly presented a unifying framework for the robust global regulation via output feedback for nonlinear systems with iISS inverse dynamics. Recently, [18] further studied output feedback regulation for a class of nonlinear systems with iISS inverse dynamics, in which the observer gain is governed by a Riccati differential equation, and Xu and Huang in [19] considered the output regulation problem for output feedback systems with relative degree one and iISS inverse dynamics. In [20], the authors considered reduced-order observer-based output feedback regulation for a class of nonlinear systems with iISS inverse dynamics. Recently, Yu et al. in [21, 22] extended the notion and some properties of iISS to stochastic nonlinear systems.

However, almost of the above papers only consider the ISS/iISS inverse dynamics with known ISS/iISS supply rates. When the inverse dynamics with uncertain ISS/iISS supply rates, how to design a feedback controller for nonlinear systems seems to be an interesting work.

The main contribution of paper is that: For the inverse dynamics with uncertain ISS/iISS supply rates, and the systems being disturbed by L^2 noises, we construct a reduced-order observer-based output feedback controller, which drives the output of system to zero and maintain other closed-loop signals bounded.

The remainder of paper is organized as follows. Section 2 is problem statements. Section 3 gives the design of output feedback controller. Section 4 is the main results. A simulation example is given in Section 5. Section 6 concludes the paper.

Notations

R_+ stands for the set of all nonnegative real numbers, R^n is the n-dimensional Euclidean space, $|x|$ is the usual Euclidean norm of a vector x. $\mathcal{K}$ denotes the set of all functions $\gamma : R_+ \to R_+$, which are continuous, strictly increasing and $\gamma(0) = 0$; $\mathcal{K}_\infty$ is the set of all functions which are of class $\mathcal{K}$ and unbounded, $\mathcal{KL}$ denotes the set of all functions $\beta(s,t) : R_+ \times R_+ \to R_+$, which are of class $\mathcal{K}$ for each fixed t, and decrease to zero as $t \to \infty$ for each fixed s. $\sigma_1(s) = \mathcal{O}(\sigma_2(s))$ as $s \to 0+$ means that $\sigma_1(s) \leqslant c_1\sigma_2(s)$ for some constant $c_1 > 0$ and all s in a small neighborhood of zero, and $\sigma_1(s) = \mathcal{O}(\sigma_2(s))$ as $s \to \infty$ means that $\sigma_1(s) \leqslant c_2\sigma_2(s)$ for some constant $c_2 > 0$ and all large enough s. $L^2(R_+; R)$ is the family of all functions $l : R_+ \to R$ such that $\int_0^\infty l^2(t)\,\mathrm{d}t < \infty$.

2 Problem statements

In this paper, we consider a class of nonlinear systems with the detailed form described as

$$\begin{aligned} \dot{\eta} &= q(t,\eta,y), \\ \dot{x}_i &= x_{i+1} + f_i(t,\bar{x}_i) + g_i(t,\eta,y) + d_i(t), \quad i = 1,\dots,n-1, \\ \dot{x}_n &= u + f_n(t,\bar{x}_n) + g_n(t,\eta,y) + d_n(t), \\ y &= x_1, \end{aligned} \tag{1}$$

where $x = (x_1, \dots, x_n) \in R^n$, $u \in R$, $y \in R$ are the state, the control input, and the measurable output, respectively, $\eta \in R^q$ denotes the inverse dynamics, $(x_2, \dots, x_n)$ and η are unmeasurable signals, $\bar{x}_i = (x_1, \dots, x_i) \in R^i$, $i = 1, \dots, n$. It is assumed that the modeled (or known) dynamics f_i, $i = 1, \dots, n$, are smooth, and the unmodeled (or uncertain) dynamics q and g_i, $i = 1, \dots, n$, are locally Lipschitz. $d_i(t)$, $i = 1, \dots, n$, are uncertain external noise.

The control objective is to design an output feedback controller for system (1) based on a reduced-order observer. Such controller drives the output of systems to zero asymptotically and maintains other closed-loop signals bounded.

The main results of paper are based on the following assumptions.

Assumption 1. For η-system of (1), there is a positive definite function $V_0 \in C^1$ such that

$$\underline{\alpha}_0\big(|\eta|\big) \leqslant V_0(\eta) \leqslant \overline{\alpha}_0\big(|\eta|\big), \qquad \frac{\partial V_0}{\partial \eta} q(\eta, y) \leqslant -\pi_0\big(|\eta|\big) + p_0 \gamma_0\big(|y|\big), \tag{2}$$

where $\underline{\alpha}_0, \overline{\alpha}_0, \gamma_0$ are class $\mathcal{K}_\infty$ functions, π_0 is a positive-definite continuous function, and p_0 is an uncertain positive constant.

Remark 1. From [11], one knows that η-subsystem satisfying (2) is iISS, and the functions pairs $(\pi_0, p_0\gamma_0)$ are supply rates. Specially, if π_0 is class $\mathcal{K}_\infty$ function, the η-subsystem is ISS.

Since p_0 in (2) is unknown, the inverse dynamics have uncertain ISS/iISS supply rates.

Assumption 2. The modeled dynamics $f_1(t, y) \leqslant \hat{f}_1(y)$ with $\hat{f}_1(y)$ being smooth function and $\hat{f}_1(0) = 0$, $f_i(t, \bar{x}_i)$, $i = 2, \dots, n$, satisfy that

$$\big|f_i(t, \bar{x}_i) - f_i(t, \bar{\hat{x}}_i)\big| \leqslant \rho_i \big|\bar{x}_i - \bar{\hat{x}}_i\big|, \quad i = 2, \dots, n,$$

where $\bar{x}_i = (x_1, \dots, x_i)$, $\bar{\hat{x}}_i = (x_1, \hat{x}_2, \dots, \hat{x}_i) \in R^i$, and ρ_i are known positive constants with $\rho_0 = (\sum_{i=2}^n \rho_i^2)^{1/2}$ such that the linear matrix inequality

$$\begin{pmatrix} P\bar{A} + \bar{A}^{\mathrm{T}}P + SB + B^{\mathrm{T}}S^{\mathrm{T}} + \rho_0^2\delta_1 I + 2Q & P \\ P & -\delta_1 I \end{pmatrix} \leqslant 0 \tag{3}$$

hods, where $\bar{A} = \left(\begin{smallmatrix} 0 & I_{(n-2)\times(n-2)} \\ 0 & 0 \end{smallmatrix}\right)$, $B = (-1, 0, \dots, 0)_{1\times(n-1)}$, P, Q are positive definite matrices and δ_1 is a positive constant.

Remark 2. Assumption 3 shows that f_i includes not only the output, but also the unmeasured state variables. Moreover, $f_i(\bar{x}_i)$ can be any smooth function with respect to measurable variable x_1, and be Lipschitz function with respect to the unmeasurable variables $x_2, \dots, x_i$ with the Lipschitz constant satisfying LMI (3).

Assumption 3. For each $1 \leqslant i \leqslant n$, there exist unknown positive constants p_{i1}, p_{i2}, and known positive-definite smooth functions ϕ_{i1}, ϕ_{i2} such that

$$\big|g_i(t, \eta, y)\big| \leqslant p_{i1}\phi_{i1}\big(|y|\big) + p_{i2}\phi_{i2}\big(|\eta|\big).$$

Assumption 4. The external noise $d_i(t)$ satisfies $d_i(t) \in L^2(R_+; R)$, $i = 1, \ldots, n$.

Remark 3. Assumption 3 is an usual condition in output feedback control of nonlinear systems (e.g., see [17,18,20,23]). Assumption 4 shows that system (1) is disturbed by L^2 noises.

3 Output feedback controller design

This section gives the design procedure of global output feedback controller by using the method of adaptive backstepping.

3.1 Reduced-order observer design

Firstly, we define a new variable $v = x_2 + g_1(t, \eta, y) + d_1(t)$, which will play an important role in the following design. We construct the following $(n-1)$-dimensional state estimation:

$$\begin{aligned} \dot{\hat{x}}_i &= \hat{x}_{i+1} + f_i(t, \bar{\hat{x}}_i) + l_i(v - \hat{x}_2), \quad i = 2, \ldots, n-1, \\ \dot{\hat{x}}_n &= u + f_n(t, \bar{\hat{x}}_n) + l_n(v - \hat{x}_2), \end{aligned} \tag{4}$$

where observer gain $l = (l_2, \ldots, l_n)^{\mathrm{T}}$ is chosen such that

$$A^{\mathrm{T}}P + PA + \delta_1^{-1}PP + \rho_0^2\delta_1 I \leqslant -2Q, \tag{5}$$

in which $A = \begin{pmatrix} -l & \begin{smallmatrix} I_{n-2} \\ 0 \ldots 0 \end{smallmatrix} \end{pmatrix}$, P, Q are positive definite matrices and $\delta_1 > 0$. Noting that the signal v in (4) is unmeasurable, we introduce the new observation variables

$$\xi_i = \hat{x}_i - l_i y, \quad i = 2, \ldots, n, \tag{6}$$

which, together with (4), leads to

$$\begin{aligned} \dot{\xi}_i &= \hat{x}_{i+1} + f_i(t, \bar{\hat{x}}_i) - l_i\bigl(f_1(t, y) + \hat{x}_2\bigr), \quad i = 2, \ldots, n-1, \\ \dot{\xi}_n &= u + f_n(t, \bar{\hat{x}}_n) - l_n\bigl(f_1(t, y) + \hat{x}_2\bigr). \end{aligned} \tag{7}$$

From (6), one obtains $f_i(t, \bar{\hat{x}}_i) = f_i(t, y, \hat{x}_2, \ldots, \hat{x}_i) = f_i(t, y, \xi_2 + l_2 y, \ldots, \xi_i + l_i y) := \tilde{f}_i(t, y, \bar{\xi}_i)$, where $\bar{\xi}_i = (\xi_2, \ldots, \xi_i)$. Substituting (6) into (7), one obtains the reduced-order observer

$$\begin{aligned} \dot{\xi}_i &= \xi_{i+1} + l_{i+1}y + \tilde{f}_i(t, y, \bar{\xi}_i) - l_i(f_1(t, y) + \xi_2 + l_2 y), \quad i = 2, \ldots, n-1, \\ \dot{\xi}_n &= u + \tilde{f}_n(t, y, \bar{\xi}_n) - l_n(f_1(t, y) + \xi_2 + l_2 y). \end{aligned} \tag{8}$$

Defining the error variables $e_i = x_i - \hat{x}_i$, $2 \leqslant i \leqslant n$, by (1) and (4), one has

$$\begin{aligned} \dot{e}_i &= -l_i e_2 + e_{i+1} + f_i(t, \bar{x}_i) - f_i(t, \bar{\hat{x}}_i) - l_i g_1(t, \eta, y) \\ &\quad - l_i d_1(t) + g_i(t, \eta, y) + d_i(t), \quad i = 2, \ldots, n-1, \\ \dot{e}_n &= -l_n e_2 + f_n(t, \bar{x}_n) - f_n(t, \bar{\hat{x}}_n) - l_n g_1(t, \eta, y) \\ &\quad - l_n d_1(t) + g_n(t, \eta, y) + d_n(t), \end{aligned}$$

which, in compact notation, is rewritten as

$$\dot{e} = Ae + F(t,x) - F(t,\hat{x}) + G(t,\eta,y) + D(t), \tag{9}$$

where $e = (e_2,\ldots,e_n)^{\mathrm{T}}$, $F(t,x) = (f_2(t,\bar{x}_2),\ldots,f_n(t,\bar{x}_n))^{\mathrm{T}}$, $F(t,\hat{x}) = (f_2(t,\bar{\hat{x}}_2), \ldots, f_n(t,\bar{\hat{x}}_n))^{\mathrm{T}}$, $G(t,\eta,y) = (g_2 - l_2 g_1,\ldots,g_n - l_n g_1)^{\mathrm{T}}$, $D(t) = (d_2(t) - l_2 d_1(t),\ldots, d_n(t) - l_n d_1(t))^{\mathrm{T}}$. Setting $\bar{e} = 1/p^* e$, $p^* = \max_{1\leqslant i\leqslant n}\{1, p_{i1}, p_{i2}, p_{12}^2\}$, then (9) becomes

$$\dot{\bar{e}} = A\bar{e} + \frac{1}{p^*}\big(F(t,x) - F(t,\hat{x})\big) + \frac{1}{p^*}G(t,\eta,y) + \frac{1}{p^*}D(t), \tag{10}$$

which together with (1) and (8) consist of the following controlled system for feedback design:

$$\begin{aligned}
\dot{\eta} &= q(t,\eta,y),\\
\dot{\bar{e}} &= A\bar{e} + \frac{1}{p^*}\big(F(t,x) - F(t,\hat{x})\big) + \frac{1}{p^*}G(t,\eta,y) + \frac{1}{p^*}D(t),\\
\dot{y} &= \xi_2 + p^*\bar{e}_2 + l_2 y + f_1(t,y) + g_1(t,\eta,y) + d_1(t),\\
\dot{\xi}_2 &= \xi_3 + l_3 y + \tilde{f}_2(t,y,\bar{\xi}_2) - l_2\big(f_1(t,y) + \xi_2 + l_2 y\big),\\
&\ \ \vdots\\
\dot{\xi}_n &= u + \tilde{f}_n(t,y,\bar{\xi}_n) - l_n\big(f_1(t,y) + \xi_2 + l_2 y\big).
\end{aligned} \tag{11}$$

Remark 4. By Schur compliment lemma in [24], (5) can be solved by the linear matrix inequality (3). P, S and δ_1 in (3) can be solved by using LMI toolbox in MATLAB and the observer gain $l = P^{-1}S$.

Remark 5. In [20], there is a mistake in the choice of observer gain. Here we correct it and give a LMI algorithm of it.

3.2 Adaptive controller design

Now, we give the adaptive controller design procedure by using the backstepping method.

Step 1. Begin with the y-subsystem of (11) and consider ξ_2 as the virtual dynamic control input. We define the 1st dynamic virtual control input

$$\alpha_1 = -c\kappa\psi_1(y)y, \qquad \dot{\kappa} = \Gamma\psi_1(y)y^2, \tag{12}$$

where Γ, c are two positive parameters and ψ_1 is a smooth positive design function, introduce a new intermediate variable $v_2 = \xi_3 + l_3 y + \tilde{f}_2(y,\bar{\xi}_2) - l_2(f_1(y) + \xi_2 + l_2 y) - \partial\alpha_1/\partial\kappa\Gamma\psi_1(y)y^2$, and set $z_1 = \xi_2 - \alpha_1(\kappa,y)$, obviously,

$$\dot{z}_1 = v_2 - \frac{\partial\alpha_1}{\partial y}\big(\xi_2 + l_2 y + p^*\bar{e}_2 + f_1 + g_1 + d_1(t)\big). \tag{13}$$

Step 2. Denoting $V_1 = (1/2)y^2$, viewing ξ_2 as the virtual control input, and considering the Lyapunov function $V_2 = (1/2)y^2 + (1/2)z_1^2$, with the use of (11)–(13), one has

$$\dot{V}_2 = z_1\left(v_2 + y - \frac{\partial \alpha_1}{\partial y}(\xi_2 + l_2 y + p^*\bar{e}_2 + f_1 + g_1 + d_1)\right) - c\kappa\psi_1 y^2 + y(p^*\bar{e}_2 + l_2 y + f_1 + g_1 + d_1). \tag{14}$$

By Young's inequality, one leads to

$$z_1 y \leqslant \frac{1}{2}z_1^2 + \frac{1}{2}y^2, \tag{15}$$

$$-\frac{\partial \alpha_1}{\partial y}z_1(p^*\bar{e}_2 + g_1 + d_1) \leqslant \frac{2\epsilon_2 p^* + p^{*2}}{2\epsilon_2}\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_1^2 + \epsilon_2\bar{e}_2^2 + \frac{g_1^2}{2p^*} + \frac{d_1^2}{2p^*},$$

where ϵ_2 is a small design parameter to be determined in Appendix. We define an unknown constant θ such that $\theta \geqslant (2\epsilon_2 p^* + p^{*2})/(2\epsilon_2)$, and set $\Phi_1(t, \bar{e}_2, \eta, y) = y(p^*\bar{e}_2 + l_2 y + f_1 + g_1 + d_1) + (1/2)y^2 + \epsilon_2\bar{e}_2^2 + g_1^2/2p^* + d_1^2/2p^*$, by (14) and (15), some simple manipulations lead to

$$\dot{V}_2 \leqslant z_1\left(v_2 + \frac{1}{2}z_1 - \frac{\partial \alpha_1}{\partial y}(\xi_2 + l_2 y + f_1) + \theta\left(\frac{\partial \alpha_1}{\partial y}\right)^2 z_1\right) - c\kappa\psi_1(y)y^2 + \Phi_1.$$

Letting $\hat{\theta}$ be the estimate of the unknown parameter θ, choosing $\bar{V}_2 = V_2 + 1/(2\Gamma_\theta) \times (\hat{\theta} - \theta)^2$, where $\Gamma_\theta > 0$ is a parameter, and setting $z_2 = \xi_3 - \alpha_2(\kappa, y, \xi_2, \hat{\theta})$, $\tau_1 = \Gamma_\theta(\partial\alpha_1/\partial y)^2 z_1^2$, and $\alpha_2 = -c_1 z_1 - (1/2)z_1 - l_3 y - \tilde{f}_2 + l_2(f_1 + \xi_2 + l_2 y) + (\partial\alpha_1/\partial\kappa) \times \Gamma\psi_1 y^2 + (\partial\alpha_1/\partial y)(\xi_2 + l_2 y + f_1) - \hat{\theta}(\partial\alpha_1/\partial y)^2 z_1$, in which $c_1 > 0$ is a constant, one can verify that

$$\dot{\bar{V}}_2 \leqslant -c\kappa\psi_1(y)y^2 + \Phi_1 + z_1 z_2 - c_1 z_1^2 + \frac{1}{\Gamma_\theta}(\hat{\theta} - \theta)(\dot{\hat{\theta}} - \tau_1), \tag{16}$$

and the variable z_2 satisfies

$$\dot{z}_2 = v_3 - \frac{\partial \alpha_2}{\partial \hat{\theta}}\dot{\hat{\theta}} - \frac{\partial \alpha_2}{\partial y}(\xi_2 + p^*\bar{e}_2 + l_2 y + f_1 + g_1 + d_1),$$

where $v_3 = \xi_4 + l_4 y + \tilde{f}_3 - l_3(f_1 + \xi_2 + l_2 y) - (\partial\alpha_2/\partial\kappa)\Gamma\psi_1 y^2 - (\partial\alpha_2/\partial\xi_2)(\xi_3 + l_3 y + \tilde{f}_2 - l_2(f_1 + \xi_2 + l_2 y))$.

Step i ($3 \leqslant i \leqslant n$). At step i, one can obtain the similar property to (16). Such a result is presented by the following lemma, for notational coherence, denote $u = \xi_{n+1}$.

Lemma 1. *For each $i = 3, \dots, n$, there exist smooth functions α_i, τ_{i-1}, Φ_{i-1}, variable $z_i = \xi_{i+1} - \alpha_i$, and positive constant c_{i-1} such that $\bar{V}_i = \bar{V}_{i-1} + (1/2)z_{i-1}^2$ satisfies*

$$\dot{\bar{V}}_i \leqslant -c\kappa\psi_1(y)y^2 + \Phi_{i-1} + z_{i-1}z_i - \sum_{j=1}^{i-1} c_j z_j^2 + \frac{1}{\Gamma_\theta}\left(\hat{\theta} - \theta - \sum_{j=1}^{i-1}\Gamma_\theta z_j\frac{\partial \alpha_j}{\partial \hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_{i-1}). \tag{17}$$

Proof. See Appendix. □

Hence at step n, by Lemma 1, there is a smooth dynamic output feedback controller

$$u = \alpha_n(\kappa, y, \xi_2, \dots, \xi_n, \hat{\theta}), \qquad \dot{\kappa} = \mathit{\Gamma}\psi_1(y)y^2, \qquad \dot{\hat{\theta}} = \tau_{n-1}, \tag{18}$$

such that $\bar{V}_n = (1/2)y^2 + (1/2)\sum_{j=1}^{n-1} z_j^2 + (1/2)\mathit{\Gamma}_\theta(\hat{\theta} - \theta)^2$ satisfies

$$\dot{\bar{V}}_n \leqslant -c\kappa\psi_1(y)y^2 + \mathit{\Phi}_{n-1} - \sum_{j=1}^{n-1} c_j z_j^2. \tag{19}$$

4 Main result

Before giving the main result of paper, we need the following lemmas.

Lemma 2. *Consider the η-subsystem satisfying Assumption* 1.

(i) *If $\liminf_{s\to\infty} \pi_0(s) = \infty$, then, for any positive-definite continuous function ϕ with*

$$\phi(s) = \mathcal{O}\big(\pi_0(s)\big) \quad \text{as } s \to 0+,$$

there always exist a positive-definite function σ and a class $\mathcal{K}_\infty$ function φ such that

$$\int_0^t \phi\big(|\eta(\tau)|\big)\,\mathrm{d}\tau \leqslant \sigma\big(|\eta_0|\big) + \tilde{p}_0 \int_0^t \varphi\big(|y(\tau)|\big)\,\mathrm{d}\tau,$$

where $\tilde{p}_0$ is unknown positive constant. Moreover, if γ_0 is such that $\gamma_0(s) = \mathcal{O}(s^2)$ as $s \to 0+$, so is φ.

(ii) *If $\liminf_{s\to\infty} \pi_0(s) < \infty$, then, for any positive-definite continuous function ϕ with*

$$\phi(s) = \mathcal{O}\big(\pi_0(s)\big) \quad \text{as } s \to 0+ \text{ and } s \to \infty,$$

the same conclusion of (i) *also holds.*

Proof. The proof of Lemma 2 is similar to the proof of Proposition 2 in [20]. □

Lemma 3. *There are unknown positive constant θ_0, which is dependent on ϵ_2, p^*, p_{i1}, p_{i2} $(1 \leqslant i \leqslant n)$, l_2 and relative degree n, and uncertain $L^2(R_+; R)$ functions $D_1(t)$, $D_2(t)$, $D_3(t)$ such that*

$$|\mathit{\Phi}_{n-1}| \leqslant n\epsilon_2\bar{e}_2^2 + \theta_0\big(y^2 + \hat{f}_1^2(y) + \phi_{11}^2\big(|y|\big) + \phi_{12}^2\big(|\eta|\big)\big) + D_1^2(t),$$

$$\frac{1}{{p^*}^2}|G|^2 \leqslant \sum_{i=2}^n 4\big(l_i^2\phi_{11}^2(|y|) + \phi_{i1}^2(|y|)\big) + \sum_{i=2}^n 4\big(l_i^2\phi_{12}^2\big(|\eta|\big) + \phi_{i2}^2\big(|\eta|\big)\big) + D_2^2(t),$$

$$\frac{1}{{p^*}^2}\big|D(t)\big|^2 \leqslant D_3^2(t).$$

Proof. With the aid of the completion of squares, it follows directly from the definitions of Φ_{n-1}, G and $D(t)$. □

Theorem 1. *Suppose that Assumptions* 1–4 *hold with the following properties*:

$$\phi_{i2}^2(s) = \begin{cases} \mathcal{O}(\pi_0(s)) \text{ as } s \to 0+ & \text{if } \liminf_{s\to\infty} \pi_0(s) = \infty, \\ \mathcal{O}(\pi_0(s)) \text{ as } s \to 0+,\ s \to \infty & \text{if } \liminf_{s\to\infty} \pi_0(s) < \infty \end{cases} \tag{20}$$

for all $i = 1, \dots, n$, *and* $\gamma_0(s) = \mathcal{O}(s^2)$ *as* $s \to 0+$. *Then by choosing the design function* ψ_1, *one has*:

(i) *The solutions of* (1), (8) *and* (18) *are well-defined and bounded over* $[0, \infty)$.

(ii) $\lim_{t\to\infty}(|y(t)| + |\eta(t)|) = 0$.

Proof. (i) Choosing the Lyapunov function $V_e = \bar{e}^{\mathrm{T}} P \bar{e}$, where $P = P^{\mathrm{T}} > 0$ is defined in (5), and $V_c = V_e + \bar{V}_n$, by (10) and Lemma 3, one has

$$\begin{aligned}
\dot{V}_e &= \bar{e}^{\mathrm{T}}\big(A^{\mathrm{T}}P + PA\big)\bar{e} + \frac{2}{p^*}\bar{e}^{\mathrm{T}}P\big(F(t,x) - F(t,\hat{x})\big) + \frac{2}{p^*}\bar{e}^{\mathrm{T}}PG + \frac{2}{p^*}\bar{e}^{\mathrm{T}}PD \\
&\leqslant \bar{e}^{\mathrm{T}}\big(A^{\mathrm{T}}P + PA\big)\bar{e} + \delta_1^{-1}\bar{e}^{\mathrm{T}}PP\bar{e} + \frac{\delta_1}{p^{*2}}\big|F(t,x) - F(t,\hat{x})\big|^2 \\
&\quad + 2\delta_2^{-1}\bar{e}^{\mathrm{T}}PP\bar{e} + \frac{\delta_2}{p^{*2}}|G|^2 + \frac{\delta_2}{p^{*2}}|D|^2 \\
&\leqslant \bar{e}^{\mathrm{T}}\big(A^{\mathrm{T}}P + PA + \delta_1^{-1}PP + \delta_1\rho_0^2 I + 2\delta_2^{-1}PP\big)\bar{e} \\
&\quad + 4\delta_2\sum_{i=2}^{n}\big(l_i^2\phi_{11}^2\big(|y|\big) + \phi_{i1}^2\big(|y|\big)\big) \\
&\quad + 4\delta_2\sum_{i=2}^{n}\big(l_i^2\phi_{12}^2\big(|\eta|\big) + \phi_{i2}^2\big(|\eta|\big)\big) + \delta_2\big(D_2^2(t) + D_3^2(t)\big).
\end{aligned} \tag{21}$$

From (19), (21) and Lemma 3, it follows that

$$\begin{aligned}
c\dot{V}_c &\leqslant -c\kappa\psi_1(y)y^2 + \theta_0\big(y^2 + \hat{f}_1^2(y) + \phi_{11}^2\big(|y|\big) + \phi_{12}^2\big(|\eta|\big)\big) \\
&\quad + D_1^2(t) + \delta_2\big(D_2^2(t) + D_3^2(t)\big) \\
&\quad + \bar{e}^{\mathrm{T}}\big(A^{\mathrm{T}}P + PA + \delta_1^{-1}PP + \delta_1\rho_0^2 I + 2\delta_2^{-1}PP + n\epsilon_2 I\big)\bar{e} \\
&\quad + 4\delta_2\sum_{i=2}^{n}\big(l_i^2\phi_{11}^2\big(|y|\big) + \phi_{i1}^2\big(|y|\big)\big) + 4\delta_2\sum_{i=2}^{n}\big(l_i^2\phi_{12}^2\big(|\eta|\big) + \phi_{i2}^2\big(|\eta|\big)\big).
\end{aligned} \tag{22}$$

One can choose sufficiently large δ_2 and sufficiently small ϵ_2 such that $2\delta_2^{-1}PP + n\epsilon_2 I \leqslant Q$, which together with (5) imply that

$$A^{\mathrm{T}}P + PA + \big(\delta_1^{-1} + 2\delta_2^{-1}\big)PP + \big(\rho_0^2\delta_1 + n\epsilon_2\big)I \leqslant -Q, \tag{23}$$

where Q is defined in (5). By Assumption 1, Lemma 2 and (20), one has

$$\int_0^t \phi_{i2}^2\big(|\eta(s)|\big)\,\mathrm{d}s \leqslant \sigma_i\big(|\eta(0)|\big) + \tilde{p}_{i0}\int_0^t \varphi_{i1}\big(|y(s)|\big)\,\mathrm{d}s, \tag{24}$$

where σ_i are positive definite functions, $\varphi_{i1} \in \mathcal{K}_\infty$ with $\varphi_{i1}(s) = \mathcal{O}(s^2)$ as $s \to 0+$, and $\tilde{p}_{i0}$ are unknown positive constants.

Choose a smooth design function ψ_1 to satisfy

$$\psi_1(y)y^2 \geqslant \max\big\{y^2 + \hat{f}_1^2(y) + \phi_{11}^2\big(|y|\big),\ l_i^2\phi_{11}^2\big(|y|\big) + \phi_{i1}^2\big(|y|\big),\\ \varphi_{i1}\big(|y|\big),\ \gamma_0\big(|y|\big),\ 1 \leqslant i \leqslant n\big\}. \tag{25}$$

Such a function ψ_1 always exists due to the fact that f_1, ϕ_{i1} are smooth near zero with $f_1(0) = \phi_{i1}(0) = 0$, and $\varphi_{i1}(s) = \mathcal{O}(s^2)$ as $s \to 0+$. Then it follows from (22) and (25) that

$$\dot{V}_c \leqslant -c\kappa\psi_1 y^2 + \big(\theta_0 + 4(n-1)\delta_2\big)\psi_1 y^2 + \theta_0\phi_{12}^2\big(|\eta|\big)\\ + 4\delta_2\sum_{i=2}^n \big(l_i^2\phi_{12}^2\big(|\eta|\big) + \phi_{i2}^2\big(|\eta|\big)\big) + D_1^2(t) + \delta_2\big(D_2^2(t) + D_3^2(t)\big). \tag{26}$$

Integrating on both sides of (26) from 0 to t, and noting $\dot{\kappa} = \varGamma\psi_1(y)y^2$ in (18), by (24) and (25), one gets

$$V_c(t) - V_c(0) \leqslant -\frac{c}{2\varGamma}\kappa^2(t) + d_1\kappa(t) + d_2 + \int_0^t D_4^2(s)\,\mathrm{d}s, \tag{27}$$

where $d_1 = (1/\varGamma)(2\theta_0\tilde{p}_{10} + 8(n-1)\delta_2 + 4\delta_2\sum_{i=2}^n l_i^2\tilde{p}_{i0})$, $d_2 = -d_1\kappa(0) + c/(2\varGamma) \times \kappa^2(0) + \theta_0\sigma_1(|\eta(0)|) + 4\delta_2\sum_{i=2}^n(\sigma_i(|\eta(0)|) + l_i^2\sigma_1(|\eta(0)|))$, $D_4^2(t) = D_1^2(t) + \delta_2(D_2^2(t) + D_3^2(t))$.

Assume that the solutions of the closed-loop system are defined on a right-maximal interval $[0, T)$ with $0 < T \leqslant \infty$. Next, we will prove that $\kappa(t)$ is bounded on $[0, T)$ by contradiction. Suppose that $\kappa(t)$ is unbounded, since $\dot{\kappa} = \varGamma\psi_1(y)y^2 \geqslant 0$, so $\kappa(t)$ is increasing and tends to ∞ as $t \to T$. Dividing both sides of (27) by $\kappa(t)$ for sufficiently large t (where $t < T$), one gets

$$\frac{-V_c(0) - d_2 - \int_0^t D_4^2(s)\,\mathrm{d}s}{\kappa(t)} \leqslant -\frac{c}{2\varGamma}\kappa(t) + d_1. \tag{28}$$

Since $D_1, D_2, D_3 \in L^2(R_+; R)$, so $D_4 \in L^2(R_+; R)$. As $t \to T$, the right side of (28) converges to $-\infty$, while the left side of (28) converges to zero, which is a contradiction. Consequently, $\kappa(t)$ is bounded on $[0, T)$.

By (12), (25) and the boundedness of $\kappa(t)$, we obtain that $\int_0^t \gamma_0(|y(s)|)\,\mathrm{d}s$ is bounded on $[0, T)$, which together with (2) imply that $V_0(\eta(t))$ and $\eta(t)$ remain bounded on $[0, T)$.

Using (27) and the boundedness of $\kappa(t)$, one also concludes that $V_c(t)$ is bounded over $[0, T)$. By definition of $V_c(t)$ in (21) above, it holds that the closed-loop signals $y(t)$, $z_1(t), \dots, z_{n-1}(t)$, $\hat{\theta}(t)$ and $\bar{e}(t)$ are all bounded over $[0, T)$. From the definition of $z_i(t)$ and $\alpha_i(t)$, it is not hard to prove that $\xi_i(t), x_i(t), u(t)$ are bounded over $[0, T)$. Therefore, $T = \infty$, and conclusion (i) holds.

(ii) By the boundedness of $y(t)$ and $\dot{y}(t)$, then $\gamma_0(|y(t)|)$ is uniformly continuous in $[0, \infty)$. Using $\int_0^\infty \gamma_0(|y(t)|)\,\mathrm{d}t < \infty$ and Barbalat's lemma in [23], one has $\lim_{t\to\infty} \gamma_0(|y(t)|) = 0$ and $\lim_{t\to\infty} y(t) = 0$. By Assumption 1, $\int_0^\infty \gamma_0(|y(t)|)\,\mathrm{d}t < \infty$ and Proposition 6 in [10], one has $\lim_{t\to\infty} \eta(t) = 0$. By (21), (23) and (24), one can obtain that $\int_0^\infty \bar{e}^{\mathrm{T}} Q \bar{e}(t)\,\mathrm{d}t < \infty$, so by Barbalat's lemma, $\lim_{t\to\infty} \bar{e}(t) = 0$. This concludes the proof. □

5 A simulation example

Consider the following nonlinear system with inverse dynamics and noises:

$$\begin{aligned}
\dot{\eta} &= -\arctan\eta + d_0 y^2, \\
\dot{x}_1 &= x_2 + f_1(x_1) + p_{11} y + p_{12}\frac{\eta}{1+|\eta|} + \frac{d_1}{1+t}, \\
\dot{x}_2 &= u + f_2(\bar{x}_2) + p_{21} y^2 + p_{22}\frac{\eta^2}{1+\eta^2} + d_2 e^{-t}, \\
y &= x_1,
\end{aligned} \tag{29}$$

where $f_1(x_1) = x_1^2$, $f_2(\bar{x}_2) = x_1 + \cos x_2$, and p_{11}, p_{12}, p_{21}, p_{22}, d_0, d_1 and d_2 are unknown constants. Choosing $V(\eta) = \eta\arctan\eta$, it is easy to verify that $\dot{V}(\eta) \leqslant -\arctan^2|\eta| + 3d_0 y^2$.

With the notations of Assumptions 1-4, one can take $\pi_0(|\eta|) = \arctan^2|\eta|$, $\gamma_0(|y|) = 3y^2$, $\phi_{11}(|y|) = |y|$, $\phi_{12}(|\eta|) = |\eta|/(1+|\eta|)$, $\phi_{21}(|y|) = y^2$, $\phi_{22}(|\eta|) = \eta^2/(1+\eta^2)$. Then $\phi_{i2}^2(s) = \mathcal{O}(\pi_0(s))$ as $s \to 0+$ and $s \to \infty$, $i = 1, 2$, the conditions of Theorem 1 are satisfied.

By (8), the reduced-order observer is given by

$$\dot{\xi}_2 = u + f_2(y, \xi_2 + l_2 y) - l_2\big(f_1(y) + \xi_2 + l_2 y\big). \tag{30}$$

According to Section 3, the dynamic output feedback control law can be designed as

$$\begin{aligned}
&\dot{\kappa} = \varGamma\psi_1(y)y^2, \qquad \dot{\hat{\theta}} = \varGamma_\theta\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_1^2, \\
&u = -c_1 z_1 - \frac{1}{2}z_1 - f_2(y, \xi_2 + l_2 y) + l_2\big(f_1(y) + \xi_2 + l_2 y\big) \\
&\qquad + \frac{\partial\alpha_1}{\partial\kappa}\varGamma\psi_1(y)y^2 + \frac{\partial\alpha_1}{\partial y}(\xi_2 + l_2 y + f_1) - \hat{\theta}\left(\frac{\partial\alpha_1}{\partial y}\right)^2 z_1,
\end{aligned} \tag{31}$$

where $\alpha_1 = -c\kappa\psi_1(y)y$, $z_1 = \xi_2 - \alpha_1$, and $\varGamma$, $\varGamma_\theta$, c, c_1 are positive parameters.

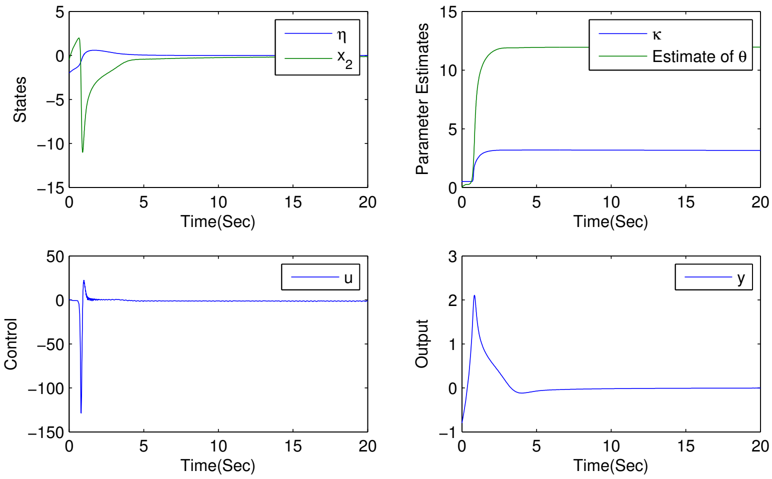

Fig. 1. The responses of closed-loop system (29)–(31).

By the proof of Proposition 2 in [20], one can take $\varphi_{11}(|y|) = \varphi_{21}(|y|) = 3y^2$ in (24). So $\psi_1(y)$ in (25) can be chosen as $\psi_1(y) = 2y^2 + l_2^2 + 3$.

In simulation, we choose the parameters $c = 0.1$, $c_1 = 0.1$, $l_2 = 1$, $\Gamma_\theta = 0.8$, $\Gamma = 0.6$, $p_{11} = 1$, $p_{12} = 1$, $p_{21} = 0.5$, $p_{22} = 0.5$, $d_0 = 1$, $d_1 = 3$, $d_2 = 5$, the initial values $\eta(0) = -2$, $x_1(0) = -0.8$, $x_2(0) = -0.5$, $\xi_2(0) = 0.1$, $\kappa(0) = 0$, $\hat{\theta}(0) = 0.5$. Fig. 1 gives the responses of closed-loop system (29)–(31).

6 Conclusions

This paper considers global output feedback control for a class of nonlinear systems with inverse dynamics and L^2 noise. For the inverse dynamics with uncertain supply rates, the reduced-order observer based output feedback controller is constructed, which drives the output of system to zero asymptotically and maintains other closed-loop signals bounded.

Appendix. The proof of Lemma 1

Assuming that $\bar{V}_{i-1}$ satisfies the similar properties to (17), noticing that

$$\begin{aligned}
\dot{z}_{i-1} &= v_i - \frac{\partial \alpha_{i-1}}{\partial \hat{\theta}} \dot{\hat{\theta}} - \frac{\partial \alpha_{i-1}}{\partial y} (\xi_2 + p^* \bar{e}_2 + l_2 y + f_1 + g_1 + d_1), \\
v_i &= \xi_{i+1} + l_{i+1} y + \tilde{f}_i - l_i (f_1 + \xi_2 + l_2 y) - \frac{\partial \alpha_{i-1}}{\partial \kappa} \Gamma \psi_1 y^2 \\
&\quad - \sum_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \xi_j} \big(\xi_{j+1} + l_{j+1} y + \tilde{f}_j - l_j (f_1 + \xi_2 + l_2 y)\big),
\end{aligned} \tag{32}$$

there holds

$$\dot{V}_i \leqslant -c\kappa\psi_1(y)y^2 + \Phi_{i-2} + z_{i-2}z_{i-1} - \sum_{j=1}^{i-2} c_j z_j^2 + \frac{1}{\Gamma_\theta}\left(\hat{\theta} - \theta - \sum_{j=1}^{i-2} \Gamma_\theta z_j \frac{\partial\alpha_j}{\partial\hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_{i-2}) + z_{i-1}\left(v_i - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\dot{\hat{\theta}} - \frac{\partial\alpha_{i-1}}{\partial y}(\xi_2 + p^*\bar{e}_2 + l_2 y + f_1 + g_1 + d_1)\right). \tag{33}$$

Using Young's inequality, it follows that

$$-\frac{\partial\alpha_{i-1}}{\partial y} z_{i-1}(p^*\bar{e}_2 + g_1 + d_1) \leqslant \epsilon_2\bar{e}_2^2 + \frac{2\epsilon_2 p^* + p^{*2}}{2\epsilon_2}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_{i-1}^2 + \frac{g_1^2}{2p^*} + \frac{d_1^2}{2p^*}. \tag{34}$$

Define

$$\tau_{i-1} = \tau_{i-2} + \Gamma_\theta\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_{i-1}^2, \qquad \Phi_{i-1} = \Phi_{i-2} + \epsilon_2\bar{e}_2^2 + \frac{g_1^2}{2p^*} + \frac{d_1^2}{2p^*},$$

$$\alpha_i = -c_{i-1}z_{i-1} - z_{i-2} - l_{i+1}y + l_i(f_1 + \xi_2 + l_2 y) - \tilde{f}_i + \frac{\partial\alpha_{i-1}}{\partial y}(\xi_2 + l_2 y + f_1) + \sum_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\xi_j}\big(\xi_{j+1} + l_{j+1}y + \tilde{f}_j - l_j(f_1 + \xi_2 + l_2 y)\big) + \frac{\partial\alpha_{i-1}}{\partial\kappa}\Gamma\psi_1 y^2 + \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}\tau_{i-1} - \hat{\theta}\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_{i-1} + \sum_{j=1}^{i-2} z_j\frac{\partial\alpha_j}{\partial\hat{\theta}}\Gamma_\theta\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_{i-1},$$

where $l_{n+1} = 0$, which together with (32)–(34) and $z_i = \xi_{i+1} - \alpha_i$ imply that

$$\begin{aligned}\dot{V}_i &\leqslant -c\kappa\psi_1(y)y^2 + \Phi_{i-1} + z_{i-1}z_{i-2} - \sum_{j=1}^{i-1} c_j z_j^2 + \frac{1}{\Gamma_\theta}\left(\hat{\theta} - \theta - \sum_{j=1}^{i-2}\Gamma_\theta z_j\frac{\partial\alpha_j}{\partial\hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_{i-2}) + z_{i-1}\left((\theta - \hat{\theta})\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_{i-1}\right.\\ &\quad \left. + \sum_{j=1}^{i-2} z_j\frac{\partial\alpha_j}{\partial\hat{\theta}}\Gamma_\theta\left(\frac{\partial\alpha_{i-1}}{\partial y}\right)^2 z_{i-1} - \frac{\partial\alpha_{i-1}}{\partial\hat{\theta}}(\dot{\hat{\theta}} - \tau_{i-1})\right)\\ &= -c\kappa\psi_1(y)y^2 + \Phi_{i-1} + z_{i-1}z_i - \sum_{j=1}^{i-1} c_j z_j^2 + \frac{1}{\Gamma_\theta}\left(\hat{\theta} - \theta - \sum_{j=1}^{i-1}\Gamma_\theta z_j\frac{\partial\alpha_j}{\partial\hat{\theta}}\right)(\dot{\hat{\theta}} - \tau_{i-1}).\end{aligned} \tag{35}$$

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