Compactons and topological solitons of the Drinfel'd–Sokolov system

Mustafa Inc[a], **Eda Fendoglu**[a], **Houria Triki**[b], **Anjan Biswas**[c]

[a]Department of Mathematics, Firat University
23119 Elazığ, Turkey
minc@firat.edu.tr

[b]Radiation Physics Laboratory, Department of Physics, Badji Mokhtar University
2300 Anaba, Algeria

[c]Department of Mathematical Sciences, Delaware State University
DE 10001-2277 Dover, USA

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Abstract. This paper presents the Drinfel'd–Sokolov system (shortly $D(m,n)$) in a detailed fashion. The Jacobi's elliptic function method is employed to extract the cnoidal and snoidal wave solutions. The compacton and solitary pattern solutions are also retrieved. The ansatz method is applied to extract the topological 1-soliton solutions of the $D(m,n)$ with generalized evolution. There are a couple of constraint conditions that will fall out in order to exist the topological soliton solutions.

Keywords: Drinfel'd–Sokolov system, Jacobi elliptic function, compacton and solitary pattern solutions, topological soliton.

1 Introduction

In 1834, Scott Russell discovered the solitary wave phenomena. Researching for solitary wave solution in nonlinear mathematical physics has been an significant topic. A class of solitary waves with compact support, called compactons by Rosenau and Hyman [1], was inspected. Compactons were shown to knock elastically and also disappeared in finite nuclei of exterior region. In recent years, many powerful methods had been proved such as the homogeneous balance method [2], the hyperbolic function expansion method [3], the Jacobi elliptic function method [4], F-expansion method [5], homotopy analysis method [6], the bifurcation theory method of dynamical system [7] and Weierstrass elliptic function method [8]. In this paper, we consider the nonlinear dispersion $D(m,n)$ [9],

$$\begin{aligned} &u_t + (v^m)_x = 0, \\ &v_t + a(v^n)_{xxx} + bu_xv + cuv_x = 0, \end{aligned} \tag{1}$$

where a, b, c, m, n are parameters. The nonlinear $D(m,n)$ was exposed as a model of water waves. Xie and Yan [9] studied system (1) and they obtained many types of compacton and solitary pattern solutions. Deng et al. [10] obtained more new exact travelling wave solutions of system (1) by using the Weierstrass elliptic function method. Zhang et al. [11] investigated some smooth and non-smooth travelling wave solutions of system (1) by using the bifurcation theory of planar dynamical system. Sweet et al. [12] obtained trigonometric and hyperbolic type solutions to (1) by using the homotopy analysis method. Sweet et al. [13] worked gDS equation by using a Miura-type transformation. Wen et al. [14] investigated some explicit expressions of solutions for the classical Drienfel'd–Sokolov–Wilson equation (DSWE) (1) by using the bifurcation method and qualitative theory of dynamical system

$$\begin{aligned} &u_t + pvv_t = 0,\\ &v_t + ruv_x + su_xv + qv_{xxx} = 0. \end{aligned} \tag{2}$$

Equation has been studied by [15, 16]. Biswas and Triki [17] worked the 1-soliton solution of Eq. (1) with power law nonlinearity by using the solitary wave ansatz method. The paper is organized as follows. In Section 2, a reduction of Eq. (1) is made and the concrete scheme of the approach for solving the equation is presented. New type Jacobi elliptic functions solutions, compactons, solitary pattern and travelling wave solutions of the equation $D(m,n)$ are obtained in Section 3. In Section 4, we get the topological soliton solution of this system by using the ansatz method. Some conclusions are given in Section 5.

2 The solutions to the $D(m,n)$ system

We first consider a nonlinear partial differantial equation of the form

$$F(u, u_t, u_x, u_{xx}, u_{xt}, \dots), \tag{3}$$

where F is a polynomial function with respect to variable changeable or some function which can be reduced to a polynomial function by using some transformations.

Here the developed Jacobi elliptic function method introduces many new traveling wave solutions. We set the solutions in the form

$$u(x,t) = Asn^{\beta}(B\xi,\ell), \quad Acn^{\beta}(B\xi,\ell), \quad Asc^{\beta}(B\xi,\ell), \quad Anc^{\beta}(B\xi,\ell), \tag{4}$$

where A, B, β are parameters that will be determined, sn, cn, sc, nc are the Jacobi elliptic functions and ℓ is the modulus of the Jacobi elliptic functions ($0 < \ell < 1$).

Let

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = k(x - \lambda t), \quad k, \lambda \in \mathbb{R}, \tag{5}$$

where k, λ are constants to be determined.

Substituting (5) into (1), gives

$$\begin{aligned} &\lambda U' - \big(V^m\big)' = 0, \\ &-\lambda V' + ak^2\big(V^n\big)'' + bU'V + cUV' = 0, \end{aligned} \tag{6}$$

where a, b, c, m and n are parameters.

Integrating the first equation of system (6) with respect to ξ, yields

$$U = \frac{1}{\lambda}V^m + C, \tag{7}$$

where C is a integration constant [9].

Substituting (7) into the second equation of system (6) and integrating once, we get

$$(cC - \lambda)V + ak^2\big(V^n\big)'' + \frac{bm + c}{\lambda(m+1)}V^{m+1} = C_0, \tag{8}$$

where C_0 is an integration constant and we accept $C_0 = 0$.

To study compactons of (8), we suppose that (8) has the solution

$$V(\xi) = Asn^{\beta}(B\xi, \ell), \tag{9}$$

with the aid of symbolic computation, we have

$$\begin{aligned} &(cC - \lambda)Asn^{\beta}(B\xi, \ell) + ak^2B^2A^n(n\beta - 1)n\beta sn^{n\beta - 2}(B\xi, \ell) \\ &\quad - ak^2B^2A^n n\beta\big[(n\beta - 1)\big(1 + \ell^2\big) + (\ell + 1)\big]sn^{n\beta}(B\xi, \ell) \\ &\quad \times ak^2B^2A^n n\beta\big[\ell^2(n\beta - 1) + \ell(\ell + 1)\big]sn^{n\beta + 2}(B\xi, \ell) \\ &\quad + \frac{bm + c}{\lambda(m+1)}A^{m+1}sn^{(m+1)\beta}(B\xi, \ell) = 0. \end{aligned} \tag{10}$$

Therefore, we can obtain two possible systems of nonlinear algebraic equations from (10) as

$$\begin{aligned} &n = m + 1, \qquad \beta = \frac{2}{n-1}, \\ &A^{n-1} = \frac{\lambda(m+1)(\lambda - cC)[(2m+1)(1+\ell^2) + m(1+\ell)]}{(bm+c)(m+2)}, \\ &k = \frac{m}{(m+1)B}\sqrt{\frac{bm+c}{2\lambda a[(2m+1)(1+\ell^2) + m(1+\ell)]}}, \end{aligned} \tag{11}$$

$$\begin{aligned} &n = 1, \qquad \beta = -\frac{2}{m}, \\ &A^{-m} = \frac{(bm+c)[(2+m)(1+\ell^2) - m(1+\ell)]}{\lambda(\lambda - cC)(m+1)(m+2)}, \\ &k = \frac{m}{B}\sqrt{\frac{(cC - \lambda)}{2(m+2)(1+\ell^2) - 2m(1+\ell))}}. \end{aligned} \tag{12}$$

Type 1. If we write (11) into (9), then we get the following new Jacobi elliptic function solutions for the $D(n-1,n)$ equation with $n>1$:

$$v_1(\xi)=\left[\frac{\lambda(m+1)(\lambda-cC)[(2m+1)(1+\ell^2)+m(1+\ell)]}{(bm+c)(m+2)}\times sn^2\left(\frac{m}{m+1}\sqrt{\frac{bm+c}{2\lambda a[(2m+1)(1+\ell^2)+m(1+\ell)]}}(x-\lambda t),\ell\right)\right]^{1/(n-1)}, \tag{13}$$

$$u_1(\xi)=\left[\frac{(m+1)(\lambda-cC)[(2m+1)(1+\ell^2)+m(1+\ell)]}{(bm+c)(m+2)}\times sn^2\left(\frac{m}{m+1}\sqrt{\frac{bm+c}{2\lambda a[(2m+1)(1+\ell^2)+m(1+\ell)]}}(x-\lambda t),\ell\right)\right]+C. \tag{14}$$

Remark 1. When the modulus $\ell\to 0$, solutions (13) and (14) become compacton solutions

$$v_{1,1}(\xi)=\left[\frac{\lambda(m+1)(\lambda-cC)(3m+1)}{(bm+c)(m+2)}\times \sin^2\left(\frac{m}{m+1}\sqrt{\frac{bm+c}{2\lambda a(3m+1)}}(x-\lambda t)\right)\right]^{1/(n-1)} \tag{15}$$

for $0\leqslant |m/(m+1)\sqrt{(bm+c)/(2\lambda a(3m+1))}\xi|\leqslant\pi$ and otherwise $v_{1,1}(\xi)=0$,

$$u_{1,1}(\xi)=\left[\frac{(m+1)(\lambda-cC)(3m+1)}{(bm+c)(m+2)}\times \sin^2\left(\frac{m}{m+1}\sqrt{\frac{bm+c}{2\lambda a(3m+1)}}(x-\lambda t)\right)\right]+C \tag{16}$$

for $0\leqslant |m/(m+1)\sqrt{(bm+c)/(2\lambda a(3m+1))}\xi|\leqslant\pi$ and otherwise $u_{1,1}(\xi)=0$.

Remark 2. When the modulus $\ell\to 1$, we obtain the following kink solutions:

$$v_{1,2}(\xi)=\left[\frac{\lambda(m+1)(\lambda-cC)(6m+1)}{(bm+c)(m+2)}\times \tanh^2\left(\frac{m}{m+1}\sqrt{\frac{bm+c}{2\lambda a(6m+1)}}(x-\lambda t)\right)\right]^{1/(n-1)}, \tag{17}$$

$$u_{1,2}(\xi)=\left[\frac{(m+1)(\lambda-cC)(6m+1)}{(bm+c)(m+2)}\times \tanh^2\left(\frac{m}{m+1}\sqrt{\frac{bm+c}{2\lambda a(6m+1)}}(x-\lambda t)\right)\right]+C. \tag{18}$$

Type 2. If we write (12) into (9), then we obtain other new exact Jacobi elliptic function solutions for the $D(m,1)$ with $m < 0$

$$v_2(\xi) = \Bigg[\frac{(bm+c)[(2+m)(1+\ell^2)-m(1+\ell)]}{\lambda(\lambda-cC)(m+1)(m+2)} \times sn^2\bigg(m\sqrt{\frac{cC-\lambda}{2[(m+2)(1+\ell^2)-m(1+\ell)]}}(x-\lambda t),\ell\bigg)\Bigg]^{-1/m}, \tag{19}$$

$$u_2(\xi) = \Bigg[\frac{(bm+c)[(2+m)(1+\ell^2)-m(1+\ell)]}{\lambda^2(\lambda-cC)(m+1)(m+2)} \times sn^2\bigg(m\sqrt{\frac{cC-\lambda}{2[(m+2)(1+\ell^2)-m(1+\ell)]}}(x-\lambda t),\ell\bigg)\Bigg] + C. \tag{20}$$

Remark 3. If the modulus $\ell \to 0$, then we get the solitary wave solutions with compact support

$$v_{2,1}(\xi) = \bigg[\frac{2(bm+c)}{\lambda(\lambda-cC)(m+1)(m+2)}\sin^2\bigg(\frac{m}{2}\sqrt{(cC-\lambda)}\bigg)(x-\lambda t)\bigg]^{-1/m} \tag{21}$$

for $0 \leqslant |(m/2)\sqrt{(cC-\lambda)}\xi| \leqslant \pi$ and otherwise $v_{2,1}(\xi) = 0$,

$$u_{2,1}(\xi) = \bigg[\frac{2(bm+c)}{\lambda^2(\lambda-cC)(m+1)(m+2)}\sin^2\bigg(\frac{m}{2}\sqrt{(cC-\lambda)}\bigg)(x-\lambda t)\bigg] + C \tag{22}$$

for $0 \leqslant |(m/2)\sqrt{(cC-\lambda)}\xi| \leqslant \pi$ and otherwise $u_{2,1}(\xi) = 0$.

Remark 4. When the modulus $\ell \to 1$, then we obtain the other kink solutions

$$v_{2,2}(\xi) = \bigg[\frac{4(bm+c)}{\lambda(\lambda-cC)(m+1)(m+2)}\tanh^2\bigg(\frac{m}{2}\sqrt{\frac{cC-\lambda}{2}}\bigg)(x-\lambda t)\bigg]^{-1/m}, \tag{23}$$

$$u_{2,2}(\xi) = \bigg[\frac{4(bm+c)}{\lambda^2(\lambda-cC)(m+1)(m+2)}\tanh^2\bigg(\frac{m}{2}\sqrt{\frac{cC-\lambda}{2}}\bigg)(x-\lambda t)\bigg] + C. \tag{24}$$

Similarly, if we use another transformation for (8),

$$V(\xi) = Acn^\beta(B\xi,\ell), \tag{25}$$

and substituting (25) into (8) yields

$$(cC-A)Acn^\beta(B\xi,\ell) + ak^2B^2A^n(n\beta-1)n\beta(1-\ell^2)cn^{n\beta-2}(B\xi,\ell) + ak^2B^2A^nn\beta\big[(\ell^2+\ell-1)+(n\beta-1)(2\ell^2-1)\big]cn^{n\beta}(B\xi,\ell) - ak^2B^2A^nn\beta\ell \times \big[\ell(n\beta-1)+\ell+1\big]cn^{n\beta+2}(B\xi,\ell)\frac{bm+c}{\lambda(m+1)}A^{m+1}cn^{(m+1)\beta}(B\xi,\ell) = 0. \tag{26}$$

So we can obtain two possible systems of nonlinear algebraic equations from (26) as

$$
\begin{aligned}
&n = m+1, \qquad \beta = \frac{2}{n-1},\\
&A^{n-1} = \frac{\lambda(m+1)(cC-\lambda)[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}{(bm+c)(m+2)(1-\ell^2)},\\
&k = \frac{m}{(m+1)B}\sqrt{-\frac{bm+c}{2\lambda a[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}},
\end{aligned}
$$

$$
\begin{aligned}
&n = 1, \qquad \beta = -\frac{2}{m},\\
&A^{-m} = \frac{(bm+c)[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}{\lambda(\lambda-cC)(m+2)(1-\ell^2)},\\
&k = \frac{m}{B}\frac{(\lambda-cC)}{2a[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}.
\end{aligned}
$$

Type 3. We obtain the following another type Jacobi elliptic function solutions for $D(n-1,n)$ equation:

$$
\begin{aligned}
v_3(\xi) = \bigg[&\frac{\lambda(m+1)(cC-\lambda)[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}{(bm+c)(m+2)(1-\ell^2)}\\
&\times cn^2\bigg(\frac{m}{m+1}\sqrt{\frac{-(bm+c)}{2\lambda a[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}}(x-\lambda t),\ell\bigg)\bigg]^{1/(n-1)}, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
u_3(\xi) = \bigg[&\frac{(m+1)(cC-\lambda)[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}{(bm+c)(m+2)(1-\ell^2)}\\
&\times cn^2\bigg(\frac{m}{m+1}\sqrt{\frac{-(bm+c)}{2\lambda a[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}}(x-\lambda t),\ell\bigg)\bigg] + C. \qquad (28)
\end{aligned}
$$

Remark 5. If the modulus $\ell \to 0$, then the exact compacton solution of $D(n-1,n)$ equation is given by

$$
v_{3,1}(\xi) = \bigg[\frac{2\lambda(m+1)(\lambda-cC)}{(bm+c)(m+2)}\cos^2\bigg(\frac{m}{2(m+1)}\sqrt{\frac{bm+c}{\lambda a(m+1)}}(x-\lambda t)\bigg)\bigg]^{1/(n-1)} \qquad (29)
$$

for $|m/(2(m+1))\sqrt{(bm+c)/(\lambda a(m+1))}\xi| \leqslant \pi/2$ and otherwise $v_{3,1}(\xi) = 0$,

$$
u_{3,1}(\xi) = \bigg[\frac{2(m+1)(\lambda-cC)}{(bm+c)(m+2)}\cos^2\bigg(\frac{m}{2(m+1)}\sqrt{\frac{bm+c}{\lambda a(m+1)}}(x-\lambda t)\bigg)\bigg] + C \qquad (30)
$$

for $|m/(2(m+1))\sqrt{(bm+c)/(\lambda a(m+1))}\xi| \leqslant \pi/2$ and otherwise $u_{3,1}(\xi) = 0$.

Type 4. From Eqs. (27) and (28), we obtain the following other type Jacobi elliptic function solutions for $D(m,1)$ equation for $m<0$:

$$v_4(\xi)=\Bigg[\frac{(bm+c)[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}{\lambda(\lambda-cC)(m+2)(1-\ell^2)} \times cn^2\bigg(m\sqrt{\frac{(\lambda-cC)}{2a[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}}(x-\lambda t),\ell\bigg)\Bigg]^{-1/m}, \quad (31)$$

$$u_4(\xi)=\Bigg[\frac{(bm+c)[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}{\lambda^2(\lambda-cC)(m+2)(1-\ell^2)} \times cn^2\bigg(m\sqrt{\frac{(\lambda-cC)}{2a[m(\ell^2+\ell-1)+(m+2)(2\ell^2-1)]}}(x-\lambda t),\ell\bigg)\Bigg]+C. \quad (32)$$

Remark 6. If the modulus of Jacobi elliptic function $\ell\to 0$, then we have the compacton solutions

$$v_{4,1}(\xi)=\bigg[\frac{-2(bm+c)(m+1)}{\lambda^2(\lambda-cC)(m+2)}\times cos^2\bigg(\frac{m}{2}\sqrt{\frac{(cC-\lambda)}{a}}(x-\lambda t)\bigg)\bigg]^{-1/m} \quad (33)$$

for $|(m/2)\sqrt{(cC-\lambda)/a}\xi|\leqslant\pi/2$ and otherwise $v_{4,1}(\xi)=C$,

$$u_{4,1}(\xi)=\bigg[\frac{-2(bm+c)(m+1)}{\lambda^2(\lambda-cC)(m+2)}\cos^2\bigg(\frac{m}{2}\sqrt{\frac{(cC-\lambda)}{a}}(x-\lambda t)\bigg)\bigg]+C \quad (34)$$

for $|(m/2)\sqrt{cC-\lambda)/a}\xi|\leqslant\pi/2$ and otherwise $u_{4,1}(\xi)=C$.

3 Exact soliton and solitary pattern solutions of nonlinear $D(m,n)$ equation

To work soliton and solitary pattern solutions of (1), we deal with another transformation

$$V(\xi)=Asc^{\beta}(B\xi,\ell). \quad (35)$$

If we substitution (35) into (8), then we have the following solutions.

Type 5. $D(n-1,n)$ equation has the Jacobi elliptic function solutions for $n>1$

$$v_5(\xi)=\Bigg[\frac{\lambda(m+1)(cC-\lambda)[(m+2)(2-\ell^2)+m(2-\ell)]}{(bm+c)(m+2)} \times sc^2\bigg(\frac{m}{(m+1)}\sqrt{\frac{-(bm+c)}{2\lambda a[(m+2)(2-\ell^2)+m(2-\ell)]}}(x-\lambda t),\ell\bigg)\Bigg]^{1/(n-1)}, \quad (36)$$

$$u_5(\xi) = \left[\frac{(m+1)(cC-\lambda)[(m+2)(2-\ell^2)+m(2-\ell)]}{(bm+c)(m+2)} \times sc^2\left(\frac{m}{(m+1)}\sqrt{\frac{-(bm+c)}{2\lambda a[(m+2)(2-\ell^2)+m(2-\ell)]}}(x-\lambda t), \ell\right)\right] + C. \tag{37}$$

Remark 7. When the modulus in rows $\ell \to 0$ and $\ell \to 1$, respectively, we get the following travelling wave and solitary pattern solutions:

$$v_{5,1}(\xi) = \left[\frac{4\lambda(m+1)^2(cC-\lambda)}{(bm+c)(m+2)} \times \tan^2\left(\frac{m}{2\sqrt{2}(m+1)}\sqrt{-\frac{bm+c}{\lambda a(m+1)}}(x-\lambda t)\right)\right]^{1/(n-1)}, \tag{38}$$

$$u_{5,1}(\xi) = \left[\frac{4(m+1)^2(cC-\lambda)}{(bm+c)(m+2)} \times \tan^2\left(\frac{m}{2\sqrt{2}(m+1)}\sqrt{-\frac{bm+c}{\lambda a(m+1)}}(x-\lambda t)\right)\right] + C, \tag{39}$$

$$v_{5,2}(\xi) = \left[\frac{2\lambda(m+1)^2(cC-\lambda)}{(bm+c)(m+2)} \times \sinh^2\left(\frac{m}{2(m+1)}\sqrt{-\frac{bm+c}{\lambda a(m+1)}}(x-\lambda t)\right)\right]^{1/(n-1)} \tag{40}$$

for $0 \leqslant |m/(2(m+1))\sqrt{-(bm+c)/(\lambda a(m+1))}\xi| \leqslant \pi$ and otherwise $v_{5,2}(\xi) = 0$,

$$u_{5,2}(\xi) = \left[\frac{2(m+1)^2(cC-\lambda)}{(bm+c)(m+2)} \times \sinh^2\left(\frac{m}{2(m+1)}\sqrt{-\frac{bm+c}{\lambda a(m+1)}}(x-\lambda t)\right)\right] + C \tag{41}$$

for $0 \leqslant |m/(2(m+1))\sqrt{-(bm+c)/(\lambda a(m+1))}\xi| \leqslant \pi$ and otherwise $u_{5,2}(\xi) = 0$.

Type 6. From the algebraic system, we get the other new type Jacobi elliptic function solutions for the $D(m,1)$ equation with $m < 0$

$$v_6(\xi) = \left[\frac{(bm+c)[(m+2)(2-\ell^2)-m(2-\ell)]}{\lambda(m+1)(cC-\lambda)(m+2)} \times sc^2\left(m\sqrt{\frac{\lambda-cC}{2[(m+2)(2-\ell^2)-m(2-\ell)]}}(x-\lambda t), \ell\right)\right]^{-1/m}, \tag{42}$$

$$u_6(\xi) = \left[\frac{(bm+c)[(m+2)(2-\ell^2)-m(2-\ell)]}{\lambda^2(m+1)(cC-\lambda)(m+2)} \times sc^2\left(m\sqrt{\frac{\lambda-cC}{2[(m+2)(2-\ell^2)-m(2-\ell)]}}(x-\lambda t),\ell\right)\right]+C. \tag{43}$$

Remark 8. In Eqs. (42) and (43), when the modulus of Jacobi elliptic function in rows $\ell \to 0$ and $\ell \to 1$, we get soliton and solitary pattern solutions, respectively,

$$v_{6,1}(\xi) = \left[\frac{4(bm+c)}{\lambda(m+1)(cC-\lambda)(m+2)}\tan^2\left(\frac{m}{2\sqrt{2}}\sqrt{(\lambda-cC)}(x-\lambda t)\right)\right]^{-1/m}, \tag{44}$$

$$vu_{6,1}(\xi) = \left[\frac{4(bm+c)}{\lambda^2(m+1)(cC-\lambda)(m+2)}\tan^2\left(\frac{m}{2\sqrt{2}}\sqrt{(\lambda-cC)}(x-\lambda t)\right)\right]+C, \tag{45}$$

$$v_{6,2}(\xi) = \left[\frac{2(bm+c)}{\lambda(m+1)(cC-\lambda)(m+2)}\sinh^2\left(\frac{m}{2}\sqrt{(\lambda-cC)}(x-\lambda t)\right)\right]^{-1/m} \tag{46}$$

for $0 \leqslant |(m/2)\sqrt{\lambda-cC}\xi| \leqslant \pi$ and otherwise $u_{6,2}(\xi)=0$,

$$u_{6,2}(\xi) = \left[\frac{2(bm+c)}{\lambda^2(m+1)(cC-\lambda)(m+2)}\sinh^2\left(\frac{m}{2}\sqrt{(\lambda-cC)}(x-\lambda t)\right)\right]+C \tag{47}$$

for $0 \leqslant |(m/2)\sqrt{\lambda-cC}\xi| \leqslant \pi$ and otherwise $u_{6,2}(\xi)=0$. We deal with last another transformation

$$V(\xi) = Anc^\beta(B\xi,\ell). \tag{48}$$

Type 7. From the algebraic system, we obtain the other new type Jacobi elliptic function solutions for the $D(n-1,n)$ equation with $n>1$

$$v_7(\xi) = \left[\frac{\lambda(m+1)(\lambda-cC)[2(m+1)(2\ell^2-1)+m(\ell-\ell^2)]}{(bm+c)(m+2)\ell^2} \times nc^2\left(\frac{m}{m+1}\sqrt{\frac{-(bm+c)}{2\lambda a[2(m+1)(2\ell^2-1)+m(\ell-\ell^2)]}}(x-\lambda t),\ell\right)\right]^{1/(n-1)}, \tag{49}$$

$$u_7(\xi) = \left[\frac{(m+1)(\lambda-cC)[2(m+1)(2\ell^2-1)+m(\ell-\ell^2)]}{(bm+c)(m+2)\ell^2} \times nc^2\left(\frac{m}{m+1}\sqrt{\frac{-(bm+c)}{2\lambda a[2(m+1)(2\ell^2-1)+m(\ell-\ell^2)]}}(x-\lambda t),\ell\right)\right]+C. \tag{50}$$

Remark 9. When the modulus $\ell \to 1$, so we get new type solitary pattern solutions

$$v_{7,2}(\xi) = \left[\frac{2\lambda(m+1)^2(\lambda-cC)}{(bm+c)(m+2)}\cosh^2\left(\frac{m}{2(m+1)}\sqrt{\frac{-(bm+c)}{\lambda a(m+1)}}(x-\lambda t)\right)\right]^{1/(n-1)} \quad (51)$$

for $|(m/(2(m+1))\sqrt{-(bm+c)/(\lambda a(m+1))}\xi| \leqslant \pi/2$ and otherwise $v_{7,2}(\xi) = 0$,

$$u_{7,2}(\xi) = \left[\frac{2(m+1)^2(\lambda-cC)}{(bm+c)(m+2)}\cosh^2\left(\frac{m}{2(m+1)}\sqrt{\frac{-(bm+c)}{\lambda a(m+1)}}(x-\lambda t)\right)\right] + C \quad (52)$$

for $|m/(2(m+1))\sqrt{-(bm+c)/(\lambda a(m+1))}\xi| \leqslant \pi/2$ and otherwise $u_{7,2}(\xi) = C$.

Type 8. From the algebraic system, we have the other new type Jacobi elliptic function solutions for the $D(m,1)$ equation with $m < 0$

$$v_8(\xi) = \left[\frac{(bm+c)[2(1-2\ell^2)+m(\ell-\ell^2)]}{\lambda(cC-\lambda)(m+1)(m+2)\ell^2} \times nc^2\left(m\sqrt{\frac{(cC-\lambda)}{2a[2(1-2\ell^2)+m(\ell-\ell^2)]}}(x-\lambda t),\ell\right)\right]^{-1/m}, \quad (53)$$

$$u_8(\xi) = \left[\frac{(bm+c)[2(1-2\ell^2)+m(\ell-\ell^2)]}{\lambda^2(cC-\lambda)(m+1)(m+2)\ell^2} \times nc^2\left(m\sqrt{\frac{(cC-\lambda)}{2a[2(1-2\ell^2)+m(\ell-\ell^2)]}}(x-\lambda t),\ell\right)\right] + C. \quad (54)$$

Remark 10. If the modulus of Jacobi elliptic function in row $\ell \to 1$, then we have

$$v_{8,2}(\xi) = \left[\frac{-2(bm+c)}{\lambda(cC-\lambda)(m+2)(m+1)}\cosh^2\left(\frac{m}{2}\sqrt{\frac{(\lambda-cC)}{\lambda a}}(x-\lambda t)\right)\right]^{-1/m} \quad (55)$$

for $|(m/2)\sqrt{-(\lambda-cC)/(\lambda a)}\xi| \leqslant \pi/2$ and otherwise $v_{8,2}(\xi) = 0$,

$$u_{8,2}(\xi) = \left[\frac{-2(bm+c)}{\lambda^2(cC-\lambda)(m+2)(m+1)}\cosh^2\left(\frac{m}{2}\sqrt{\frac{(\lambda-cC)}{\lambda a}}(x-\lambda t)\right)\right] + C \quad (56)$$

for $|(m/2)\sqrt{-(\lambda-cC)/(\lambda a)}\xi| \leqslant \pi/2$ and otherwise $u_{8,2}(\xi) = 0$.

Corollary 1. (i) *When $n > 1$, the obtained solutions* (23) *and* (24) *are full agreement with the results* (2.11a) *and* (2.11b) *described in* [9].

(ii) *When $m < 0$, the obtained solutions* (21), (33) *and* (46) *agree well with the results* (29), (34) *and* (30) *in* [10], *respectively.*

(iii) *Many solutions that obtained by Xie and Yan* [9] *by using sine-cosine method are gotten by Jacobi elliptic function method which we have present in this study.*

4 Topological 1-soliton solution of the $D(m, n)$ equation

The $D(m, n)$ equation with generalized evolution studied in this section is given by [17]

$$\big(q^l\big)_t + k\big(r^m\big)_x = 0, \tag{57}$$

$$\big(r^l\big)_t + a\big(r^n\big)_{xxx} + bq_x r + cqr_x = 0, \tag{58}$$

where k, a, b and c are constants while the parameters l, m and n are all positive integers. If setting $l = 1$, Eqs. (57) and (58) collapse to the commonly known $D(m, n)$ equation studied in [18–20].

Note that the first term in Eq. (57), represents the generalized evolution term while the second term is the dispersive term. In Eq. (58), the first term is again the generalized evolution term while the second term is the nonlinear dispersion and the last two terms are the nonlinear coupling terms.

In this section, we interest to find the topological 1-soliton solution to the $D(m, n)$ equation given by Eqs. (57) and (58) as it appears, namely, for general values of l, m and n. Note that it is not possible to integrate Eqs. (57) and (58) for any general values of l, m and n by the classical method of integration. However, a solitary wave ansatz method leads to a closed form soliton solution to this model equation.

To start off, the hypothesis is given by [21, 22]

$$q(x, t) = A_1 \tanh^{p_1} \tau \tag{59}$$

and

$$r(x, t) = A_2 \tanh^{p_2} \tau, \tag{60}$$

where

$$\tau = B(x - vt), \tag{61}$$

A_1 and A_2 and B are free parameters while v is the velocity of the solitons in (59)–(61). The unknown exponents $p_1 > 0$ and $p_2 > 0$ will be determined as a function of l, m and n.

From (59) it is possible to obtain

$$\big(q^l\big)_t = p_1 lvA_1^l B\big(\tanh^{lp_1+1}\tau - \tanh^{lp_1-1}\tau\big), \tag{62}$$

$$q_x = p_1 A_1 B\big(\tanh^{p_1-1}\tau - \tanh^{p_1+1}\tau\big), \tag{63}$$

and, similarly, from (60) we get

$$\big(r^l\big)_t = p_2 lvA_2^l B\big(\tanh^{lp_2+1}\tau - \tanh^{lp_2-1}\tau\big), \tag{64}$$

$$r_x = p_2 A_2 B\big(\tanh^{p_2-1}\tau - \tanh^{p_2+1}\tau\big), \tag{65}$$

$$\big(r^m\big)_x = mp_2 A_2^m B\big(\tanh^{mp_2-1}\tau - \tanh^{mp_2+1}\tau\big), \tag{66}$$

$$\begin{aligned}\big(r^n\big)_{xxx} = np_2 A_2^n B^3 \big[&(np_2 - 1)(np_2 - 2) \tanh^{np_2-3}\tau \\ &- 2p_2^2 n^2 + (np_2 - 1)(np_2 - 2) \tanh^{np_2-1}\tau \\ &+ 2p_2^2 n^2 + (np_2 + 1)(np_2 + 2) \tanh^{np_2+1}\tau \\ &- (np_2 + 1)(np_2 + 2) \tanh^{np_2+3}\tau\big].\end{aligned} \tag{67}$$

Substituting (62)–(67) into (57) and (58), respectively, yields

$$p_1 lvA_1^l B\big(\tanh^{lp_1+1}\tau - \tanh^{lp_1-1}\tau\big) + kmp_2 A_2^m B\big(\tanh^{mp_2-1}\tau - \tanh^{mp_2+1}\tau\big) = 0 \tag{68}$$

and

$$\begin{aligned} &p_2 lvA_2^l B\big(\tanh^{lp_2+1}\tau - \tanh^{lp_2-1}\tau\big) \\ &\quad + anp_2 A_2^n B^3\big[(np_2-1)(np_2-2)\tanh^{np_2-3}\tau \\ &\quad - \big\{2p_2^2n^2 + (np_2-1)(np_2-2)\big\}\tanh^{np_2-1}\tau \\ &\quad + \big\{2p_2^2n^2 + (np_2+1)(np_2+2)\big\}\tanh^{np_2+1}\tau \\ &\quad - (np_2+1)(np_2+2)\tanh^{np_2+3}\tau\big] \\ &\quad + (bp_1+cp_2)A_1A_2B\big(\tanh^{p_1+p_2-1}\tau - \tanh^{p_1+p_2+1}\tau\big) = 0. \end{aligned} \tag{69}$$

Now, from (68), matching the exponents of $\tanh^{lp_1+1}\tau$ and $\tanh^{mp_2+1}\tau$, gives

$$lp_1 = mp_2, \tag{70}$$

which is also obtained by equating the exponents of $\tanh^{lp_1-1}\tau$ and $\tanh^{mp_2-1}\tau$. Setting their respective coefficients to zero, yields

$$lp_1 vA_1^l = kmp_2 A_2^m. \tag{71}$$

Also, from (69), equating the exponent pairs of $\tanh^{np_2+3}\tau$, $\tanh^{p_1+p_2+1}\tau$ and $\tanh^{lp_2+1}\tau$, $\tanh^{np_2+1}\tau$, respectively, yields

$$np_2 + 2 = p_1 + p_2 \tag{72}$$

and

$$np_2 = lp_2. \tag{73}$$

The latter gives

$$n = l. \tag{74}$$

By inserting (70) and (74) into (72), we obtain

$$p_2 = \frac{2n}{m - n(n-1)}. \tag{75}$$

Substituting (75) into (70), we get

$$p_1 = \frac{2m}{m - n(n-1)}. \tag{76}$$

It should be remarked that the soliton solutions (59) and (60) exist only when $p_1 > 0$ and $p_2 > 0$. These conditions imply that $m > n(n-1)$ in Eqs. (75) and (76).

Also the substitution of (75) and (76) into (71) gives

$$v = \frac{kA_2^m}{A_1^l}, \tag{77}$$

which determine the velocity of the soliton. Again, from (69), the linearly independent functions are $\tanh^{p_2+j}\tau$ for $j=\pm1,\pm3$. Therefore, setting their respective coefficients to zero, yields the following parametric equations:

$$p_2 lvA_2^l B + anp_2 A_2^n B^3\{2p_2^2n^2 + (np_2+1)(np_2+2)\} + (bp_1+cp_2)A_1A_2B = 0, \tag{78}$$

$$anp_2A_2^nB^3(np_2+1)(np_2+2) + (bp_1+cp_2)A_1A_2B = 0, \tag{79}$$

$$p_2 lvA_2^l B + anp_2 A_2^n B^3\{2p_2^2n^2 + (np_2-1)(np_2-2)\} = 0, \tag{80}$$

$$anp_2A_2^nB^3(np_2-1)(np_2-2) = 0. \tag{81}$$

To solve (81), we have considered the two cases.

Case 1: $np_2 - 1 = 0$. This yields

$$p_2 = \frac{1}{n}. \tag{82}$$

Further substitution of (82) into (75) and (78)–(80), respectively, gives

$$m + n = 3n^2, \tag{83}$$

$$p_1 = \frac{m}{n^2}, \tag{84}$$

$$B = \sqrt{-\frac{(mb+nc)A_1}{6an^2A_2^{n-1}}}, \tag{85}$$

$$v = -2aB^2. \tag{86}$$

Equating the two values of the velocity v from (77) and (85), gives

$$B = \sqrt{-\frac{kA_2^m}{2aA_1^l}}, \tag{87}$$

which implies that it is necessary to have

$$ak < 0 \tag{88}$$

for solitons to exist.

Besides, Eq. (85) shows that the solitons exist for

$$a(mb+c) < 0. \tag{89}$$

Further, equating the two values of the free parameter B from (85) and (87), gives

$$3kn^2A_2^{m+n-1} = (mb+nc)A_1^{l+1}. \tag{90}$$

Case 2: $np_2 - 2 = 0$. From it, we get

$$p_2 = \frac{2}{n}. \tag{91}$$

By substituting (91) into (75) and (78)–(80), respectively, we obtain

$$m + n = 2n^2, \tag{92}$$

$$p_1 = \frac{2m}{n^2}, \tag{93}$$

$$B = \sqrt{-\frac{(mb + nc)A_1}{12an^2 A_2^{n-1}}}, \tag{94}$$

$$v = -8aB^2. \tag{95}$$

Equating the two values of the velocity v from (77) and (95), gives

$$B = \sqrt{-\frac{kA_2^m}{8aA_1^l}}, \tag{96}$$

which implies that it is necessary to have

$$ak < 0 \tag{97}$$

for existing solitons.

Besides, Eq. (94) shows that the solitons exist for

$$a(mb + c) < 0. \tag{98}$$

Further, equating the two values of the free parameter B from (94) and (96), gives

$$3kn^2 A_2^{m+n-1} = 2(mb + nc)A_1^{l+1}. \tag{99}$$

Lastly, we can determine the dark soliton solutions for the $D(m, n)$ equation with generalized evolution (57) and (58) when we substitute (77), (82), (84) and (85) or (87) in (59) and (60) with the respective constraints (83) and (90) for the first case of solution or we substitute (77), (91), (93) and (94) or (96) in (59) and (60) with the respective constraints (92) and (99) for the second case of solution as

$$q(x, t) = A_1 \tanh^{2m/(m-n(n-1))}\big[B(x - vt)\big], \tag{100}$$

$$r(x, t) = A_2 \tanh^{2n/(m-n(n-1))}\big[B(x - vt)\big], \tag{101}$$

which provided that $a(mb + nc) < 0$ and $ak < 0$.

5 Conclusions

We presented the various form of the Drinfel'd–Sokolov equation in this paper. Its generalized version D(m,n) equation was also addressed. A furthermore generalized version of this manuscript was studied, where the generalized evolution was taken into consideration. The Jacobi's elliptic function method was applied and several forms of solutions including the cnoidal and snoidal wave solutions were obtained. The compacton, solitary pattern, periodic wave solutions were also obtained. Finally, the topological 1-soliton solution was also retrieved for the $D(m, n)$ equation with genreralized evolution. In this case a couple of constraint conditions fell out during the course of derivation of the soliton solution that must hold in order to exist the solution.

References

1. P. Rosenau, J.M. Hyman, Compactons: Solitons with finite wave-lengths, *Phys. Rev. Lett.*, **70**:564–567, 1993.
2. M.L. Wang, Solitary wave solutions for variant Boussinesq equations, *Phys. Lett. A*, **199**:169–172, 1995.
3. E.J. Parkes, B.R. Duffy, Travelling solitary wave solutions to a compound KdV–Burgers equation, *Phys. Lett. A*, **229**:217–220, 1997.
4. Z.T. Fu, S.K. Liu, S.D. Liu, Q. Zhao, New Jacobi elliptic function expansion and new periodic solutions of nonlinear wave equations, *Phys. Lett. A*, **290**:72–76, 2001.
5. E.G. Fan, H.Q. Zhang, A new algebraic method for finding the line solitons and doubly periodic wave solutions to a two-dimensional perturbed KdV equation, *Chaos Solitons Fractals*, **15**:567–574, 2003.
6. S.J. Liao, *Beyond Perturbation: Introduction to Homotopy Analysis Method*, Chapman & Hall, Boca Raton, FL, 2003.
7. J.B. Li, Z.R. Liu, Smooth and non-smooth travelling wave in a nonlinearly dispersive equation, *Appl. Math. Model.*, **25**:41–56, 2000.
8. H.W. Schurmann, Travelling wave solutions for cubic-quantic nonlinear Schrödinger equation, *Phys. Rev. E*, **54**:4312–4320, 1996.
9. F. Xie, Z. Yan, New exact solution profiles of the nonlinear dispersion Drinfel'd–Sokolov system, *Chaos Solitons Fractals*, **39**:866–872, 2009.
10. X. Deng, J. Cao, X. Li, Travelling wave solutions for the nonlinear dispersion Drinfel'd–Sokolov system, *Commun. Nonlinear Sci. Numer. Simul.*, **15**:281–290, 2010.
11. K. Zhang, S. Tang, Bifurcations of travelling wave solutions for the nonlinear Drinfel'd–Sokolov system *Appl. Math. Comput.*, **217**:1620–1631, 2010.
12. E. Sweet, R.A. Van Gorder, Trigonometric and hyperbolic type solutions to a generalized Drinfel'd–Sokolov equation, *Appl. Math. Comput.*, **217**:4147–4166, 2010.
13. E. Sweet, R.A. Van Gorder, Analytical solutions to a generalized Drinfel'd–Sokolov equation related to DSSH and KdV6, *Appl. Math. Comput.*, **216**:2783–2791, 2010.
14. Z. Wen, Z. Liu, M. Song, New exact solutions for the classical Drinfel'd–Sokolov–Wilson equation, *Appl. Math. Comput.*, **215**:2349–2358, 2009.
15. C.P. Liu, X.P. Liu, Exact solutions of the classical DSWE and the relations among the solutions, *Phys. Lett. A*, **303**:197–203, 2002.

16. Z. Fu, N. Yuan, Z. Chen, J. Mao, S. Liu, Multi-order exact solutions to the Drinfel'd–Sokolov–Wilson equations, *Phys. Lett. A*, **373**:3710–3714, 2009.

17. A. Biswas, H. Triki, 1-soliton solution of the $D(m,n)$ equation with generalized evolution, *Appl. Math. Comput.*, **217**:8482–8488, 2011.

18. A.M. Wazwaz, Exact and explicit travelling wave solutions for the nonlinear Drinfeld–Sokolov system, *Commun. Nonlinear Sci. Numer. Simul.*, **11**:311–325, 2006.

19. A.M. Wazwaz, The Cole–Hopf transformation and multiple soliton solutions for the integrable sixth order Drinfeld–Sokolov–Satsuma–Hirota equation, *Appl. Math. Comput.*, **207**:248–255, 2009.

20. K. Zhang, S. Tang, Z. Wang, Bifurcation of travelling wave solutions for the nonlinear dispersion Drinfel'd–Sokolov $D(m,n)$ system, *Appl. Math. Comput.*, **217**(6):2433–2447, 2010.

21. M. Saha, A.K. Sarma, A. Biswas, Dark optical solitons in power law media with time-dependent coefficients, *Phys. Lett A*, **373**:4438–4441, 2009.

22. G. Ebadi, E.V. Krishnan, S. Johnson, A. Biswas, Cnoidal wave, snoidal wave and soliton solutions of the $D(m,n)$ equation, *Arab. J. Math.*, **2**(1):19–31, 2013.