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Observer-based robust adaptive control for uncertain stochastic Hamiltonian systems with state and input delays*

Weiwei Sun[a,b], **Lianghong Peng**[a]

[a]Institute of Automation, Qufu Normal University
Qufu 273165, China
wwsun@hotmail.com; plh3971@163.com

[b]School of Electrical Information and Automation, Qufu Normal University
Rizhao 276826, China

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Abstract. This paper investigates the observer-based robust adaptive control problem for a class of stochastic Hamiltonian systems. The systems under consideration relate to parameter uncertainties, unknown state time-delay and input delay. The purpose is to design a delay-dependent observer-based adaptive control law such that for all admissible uncertainties, as well as stochasticity, the closed-loop error system is robustly asymptotically stable in the mean square. Several sufficient conditions are presented to ensure the rationality and validity of the proposed control laws and observers, which are derived based on Lyapunov functional method. Numerical simulations spell out to illustrate the effectiveness of the proposed theories.

Keywords: stochastic Hamiltonian systems, adaptive control, time delay, robustly asymptotically stable.

1 Introduction

Known as a significant class of nonlinear systems, port-controlled Hamiltonian system (PCH), which is proposed by [1], has good structure with clearly physical meaning. A benefit of PCH system is the Hamilton function in PCH system can be taken as the sum of potential energy (excluding gravitational potential energy) and kinetic energy in physical systems. So it often is viewed as a proper candidate of Lyapunov function for many physical systems. On account of this advantage, the Hamilton function approach

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has caused broad attention in the field of nonlinear controller design, as well as in practical control problems (see, e.g., [2–4] and the references therein). Recently, the authors obtain some results on time delay Hamiltonian systems (see, e.g., [5–8]). In [5], the stabilization problem of Hamiltonian systems with state time delay and input saturation is investigated. Sufficient conditions are derived by using Lyapunov–Krasovskii functional theorem to guarantee the systems as well as the resulted closed-loop systems by output feedback asymptotically stable when input saturation effectively occurs. The problem of $\mathcal{L}_2$-disturbance attenuation for time delay port-controlled Hamiltonian systems is studied in [7] and [8]. The case that there are time-invariant uncertainties belonging to some convex bounded polytypic domain is also considered in [7]. Reference [9] studies the finite-time stability and finite-time H_∞ control design for a class of time delay Hamiltonian systems via Razumikhin approach. The Hamilton function considered in [9] belongs to the power functions and the exponents of the Hamilton function have specific form. While, there are few works on the analysis and synthesis for stochastic Hamiltonian systems with time-delay.

For all we know, stochasticity and time delays are usually sources of instability and encountered in many practical systems, including communication systems, engineering systems and process control systems, etc. This is why considerable attentions have been received on the study of stochastic time-delay systems [10–13]. Similar to the analytical methods of general time-delay systems' theories, the analysis and synthesizes of stochastic systems with time-delay and uncertainties are based on the time domain approaches of the Lyapunov stability theories as well. Nevertheless, the disadvantages by applying the Itô differential formula are the appearances of gravitation and the Hessian terms. Furthermore, the system states' derivatives have no definitions on account of the stochastic disturbance (Wiener process). Hence, it is impossible to analyze the stochastic systems by directly choosing Lyapunov functional or function described as which in delay systems. For nonlinear systems, stochastic and delay factors further augment the difficulties of the analysis and synthesis. There is no general controller establishment and observer design method in nonlinear systems. Based on the backstepping method, [14] introduces a systematic design procedure for a memoryless adaptive output-feedback control law. [15] proposes a new stability criterion and a design procedure of observer-based controller for the stochastic nonlinear system with limited communication by using the Lyapunov functional approach. In [16], an adaptive backstepping controller in vector form is designed for a class of stochastic Hamiltonian control systems with unknown drift and diffusion functions. The closed-loop system under the controller has a unique solution that is globally bounded in probability. By using the structural properties of the systems, [17] provides an observer design method called Augment Plus Feedback for PCH systems and two observers are obtained. To the authors' best knowledge, the problem of control design for stochastic nonlinear Hamiltonian systems with time delay has not been discussed yet, which motivates us for the present study.

This paper deals with the observer-based robust adaptive controller design for a class of nonlinear Hamiltonian systems with stochasticity, parameter uncertainties, unknown state varying time-delay and input delay. The uncertainties considered in this paper are some small parameter perturbations. Two cases we considered in this paper: the structure of the systems can not be duplicated and the structure of the systems can be duplicated. We obtain some sufficient conditions for which the uncertain stochastic time-delay Hamiltonian system is robustly asymptotically stable in mean square for all admissible uncertainties and stochasticity. The conditions are acquired by applying Itô differential formula, Young inequality [18] and some properties of norm and trace. The results proposed in this paper, in fact, together with Hamiltonian realization [19] have presented a new approach to the observer-based controller design of stochastic time-delay nonlinear systems. The control problem of a large class of stochastic time-delay nonlinear systems with uncertainties can be worked out through the Hamiltonian system framework. An application of this method is illustrated by two numerical examples in Section 4, which validate our results obtained in this paper.

The paper is organized as follows. Section 2 presents the problem formulation and some preliminaries. The main results are presented in Section 3. Section 4 illustrates the proposed results by two numerical examples, which are followed by the conclusion in Section 5.

Notations. $\mathbb{R}$ is the set of real numbers; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{n\times m}$ is the real matrices with dimension $n \times m$; $\|\cdot\|$ stands for either the Euclidean vector norm or the induced matrix 2-norm. The notation $X \geqslant Y$ (respectively, $X > Y$), where X and Y are symmetric matrices, means that the matrix $X - Y$ is positive semi-definite (respectively, positive definite). $\mathrm{tr}[X]$ denotes the trace for square matrix X; $\lambda_{\max}(P)$ ($\lambda_{\min}(P)$) denotes the maximum (minimum) of eigenvalue of a real symmetric matrix P. $\mathcal{C}_{n,\tau} = \mathcal{C}([-\tau,0],\mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^n$. $\mathbb{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$-measurable bounded $\mathbb{C}([-\tau,0];\mathbb{R}^n)$-valued random variables $\phi = \{\phi(t)\colon\ t \in [-\tau,0]\}$; $\mathbb{C}^i$ denotes the set of all functions with continuous ith partial derivatives; $\mathbb{C}^{2,1}(\mathbb{R}^n \times [-\tau,\infty);\mathbb{R}_+)$ stands for the family of all non-negative functions $V(x,t)$ on $\mathbb{R}^n\times[-\tau,\infty)$, which are $\mathbb{C}^2$ in x and $\mathbb{C}^1$ in t; $\mathbb{C}^{2,1}$ is the family of all functions, which are $\mathbb{C}^2$ in the first argument and $\mathbb{C}^1$ in the second argument. Throughout the paper the superscript 'T' stands for matrix transposition. What is more, for the sake of simplicity, throughout the paper, we denote $\partial H/\partial x$ by ∇H.

2 Problem formulation

Consider a class of stochastic time-delay Hamiltonian systems with unknown parameter perturbations represented by

$$\begin{aligned}
\mathrm{d}x(t) &= \bigl[J(x,p) - R(x,p)\bigr]\nabla H\bigl(x, x\bigl(t-\tau(t)\bigr), p\bigr)\,\mathrm{d}t \\
&\quad + g_1(x)u\bigl(t, t-\tau(t)\bigr)\,\mathrm{d}t + g_2(x)\,\mathrm{d}w(t), \\
y(t) &= g_1^{\mathrm{T}}(x)\nabla H(x,0), \\
x(t) &= \phi(t), \quad t \in [-h,0],
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t,t-\tau(t)) \in \mathbb{R}^q$ is the system control input; $\tau(t) \in \mathcal{C}_{n,\tau}$ is the time-delay; $y(t) \in \mathbb{R}^q$ is the system control output; $w(t) \in \mathbb{R}^r$

is a zero-mean Wiener process, which satisfies $\mathbf{E}\{\mathrm{d}w(t)\} = 0$, $\mathbf{E}\{\mathrm{d}w(t)^2\} = \mathrm{d}t$, where $\mathbf{E}$ is the expectation operator; $p \in \mathbb{R}^l$ stands for the parameter perturbation of the Hamiltonian systems; $\nabla H(x, x(t-\tau(t)), p) \in \mathbb{R}^{n\times 1}$ is the gradient of the Hamilton function $H(x, x(t-\tau(t)), p) : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$, which satisfies $H(x, x(t-\tau(t)), p) \geqslant 0$, $H(0,0,0) = 0$, $\nabla H(0,0,0) = 0$, $\nabla H(x, x(t-\tau(t)), 0) \neq 0$ $(x \neq 0)$; $J(x,p) \in \mathbb{R}^{n\times n}$ is skew-symmetric matrix, that is, $J(x,p) = -J^{\mathrm{T}}(x,p)$; $R(x,p) \in \mathbb{R}^{n\times n}$ is positive semi-definite matrix; $g_1(x) \in \mathbb{R}^{n\times q}$ and $g_2(x) \in \mathbb{R}^{n\times r}$ are gain matrices of appropriate dimensions, $g_1(x)$ has full column rank and $g_2(0) = 0$. In addition, $J(x,p)$, $R(x,p)$, $g_1(x)$ and $g_2(x)$ are locally Lipschitz continuous functions on x.

Remark 1. $\nabla H(0,0,0) = 0$, $g_2(0) = 0$ and $\nabla H(x, x(t-\tau(t)), 0) \neq 0$ $(x \neq 0)$ guarantee that $x = 0$ is an isolated equilibrium point of system (1). These conditions are not conservative, since if the equilibrium point of system (1) is not at the origin, we can change isolated equilibrium point to origin by applying coordinate transformations.

The delay $\tau(t)$ in (1) is a time-varying continuous function, which satisfies

$$0 \leqslant \tau(t) \leqslant h < \infty \quad \text{and} \quad \dot{\tau}(t) \leqslant \bar{h} < 1,$$

where h and $\bar{h}$ are known positive scalars.

p in system (1) is small parameter perturbation, which maintains the dissipativeness of the structure matrix. The perturbations usually bring a direct effect on the states, but an indirect effect on the output of the system. Thus, the output can be chosen independent on p. Obviously, all the functions in system (1) related to p can be decomposed as follows:

$$\begin{aligned}
\nabla H(x, x_\tau, p) &= \nabla H(x, x_\tau, 0) + \Delta_H(x, x_\tau, p),\\
J(x,p) &= J(x,0) + \Delta_J(x,p),\\
R(x,p) &= R(x,0) + \Delta_R(x,p),
\end{aligned}$$

where $\Delta_H(x, x_\tau, 0) = 0$; $\Delta_J(x,0) = \Delta_R(x,0) = 0$; $\|\Delta_J(x,p) - \Delta_R(x,p)\| \leqslant m$, m is a known bounded constant; $x_\tau := x(t-\tau(t))$. For the sake of simplicity of expressions, in the rest of the paper, we will always denote $x(t-\tau(t))$ by x_τ. What is more, $H(x,0) := H(x, x(t-\tau(t)), 0)|_{\tau(t)=0}$ and $\nabla H(x,0) := \nabla H(x, x(t-\tau(t)), 0)|_{\tau(t)=0}$.

Assumption 1. There exists an appropriate dimensioned function matrix $\psi(x, x_\tau)$ such that

$$2\big[J(x,p) - R(x,p)\big]\Delta_H(x, x_\tau, p) = g_1(x)\psi^{\mathrm{T}}(x, x_\tau)\theta, \tag{2}$$

where θ is a constant parameter vector subjecting to p.

Remark 2. Assumption 1 is the matched condition, an usual assumption in the adaptive control of Hamiltonian systems. Commonly $\psi(x, x_\tau)$ and θ can be easy found such that (2) holds (see [20]).

Under Assumption 1, system (1) can be rewritten as

$$\begin{aligned} \mathrm{d}x(t) &= \big[J(x,p) - R(x,p)\big]\nabla H(x,x_\tau,0)\,\mathrm{d}t + \frac{1}{2}g_1(x)\psi^{\mathrm{T}}(x,x_\tau)\theta\,\mathrm{d}t \\ &\quad + g_1(x)u\big(t,t-\tau(t)\big)\,\mathrm{d}t + g_2(x)\,\mathrm{d}w(t), \\ y &= g_1^{\mathrm{T}}(x)\nabla H(x,0), \\ x(t) &= \phi(t), \quad t \in [-h,0]. \end{aligned} \tag{3}$$

Suppose that the states of system (1) are not fully measured and the structure of (3) can not be duplicated, we can design an adaptive observer of the following form to estimate the states of system (3):

$$\begin{aligned} \mathrm{d}\hat{x}(t) &= \big[J(\hat{x},0) - R(\hat{x},0)\big]\nabla H(\hat{x},\hat{x}_\tau,0)\,\mathrm{d}t + \frac{1}{2}g_1(\hat{x})\psi^{\mathrm{T}}(x,x_\tau)\hat{\theta}\,\mathrm{d}t \\ &\quad + g_1(\hat{x})u\big(t,t-\tau(t)\big)\,\mathrm{d}t + K^{\mathrm{T}}(\hat{x})\big[y - g_1^{\mathrm{T}}(\hat{x})\nabla H(\hat{x},0)\big]\,\mathrm{d}t \\ &\quad + g_2(\hat{x})\,\mathrm{d}w(t), \\ \dot{\hat{\theta}} &= \Gamma\psi(x,x_\tau)g_1^{\mathrm{T}}(x)\nabla H(x,0), \\ \hat{x}(t) &= \hat{\phi}(t), \quad t \in [-h,0], \end{aligned} \tag{4}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimation of the state vector, $\hat{x}_\tau := \hat{x}(t-\tau(t)) \in \mathbb{R}^n$ is the time-delay observe state, $K(x)$ is nonzero function matrix, $\hat{\theta}$ is the estimate vector of θ, $\Gamma > 0$ is a constant adaptive gain matrix.

Definition 1. The uncertain stochastic time-delay Hamiltonian system (3) (system (4)) is said to be robustly asymptotically stable in mean square, if there exists a controller $u(t,t-\tau(t))$ such that

$$\lim_{t\to\infty} \mathbf{E}\big\{\|x(t)\|^2\big\} = 0 \qquad \Big(\lim_{t\to\infty} \mathbf{E}\big\{\|\hat{x}(t)\|^2\big\} = 0\Big),$$

where $x(t)$ and $\hat{x}(t)$ are the solutions of system (3) and (4) at time t under the initial condition $x(t) = \phi(t)$ and $\hat{x}(t) = \hat{\phi}(t)$, respectively.

The observer-based robust adaptive control design problem of the uncertain stochastic time-delay Hamiltonian system (1) can be formulated as follows: find an adaptive controller

$$u = \alpha(x,x_\tau,\hat{x}_\tau,\hat{\theta}), \qquad \dot{\hat{\theta}} = \beta(x,\hat{x},\hat{x}_\tau,\hat{\theta})$$

such that system (4) is a global asymptotically observer of system (3) and the error dynamics remain mean square asymptotically stable for any admissible uncertain parameters. To this end, some assumptions should be imposed on system (1).

Assumption 2. The Hamilton function $H(x,x_\tau,0)$ satisfies:

(A1) $H(x,x_\tau,0) \in \mathbb{C}^2$;

(A2) $H(x,0) \geqslant \alpha\|x\|^2$;

(A3) $\nabla^{\mathrm{T}} H(x,0)\nabla H(x,0) \geqslant \beta\|x\|^2$,

where α, β are all known positive scalars.

Remark 3. Assumption 2 not only guarantees the existence of $\nabla H(x,0)$ and $Hess(H(x,0))$ but also guarantees that $H(x,0)$ and $\nabla H(x,0)$ are bounded below in terms of x. We shall note that the assumption is not very conservative to Hamilton functions and the majority of Hamilton functions in Hamiltonian systems can easily satisfy these conditions.

We conclude this section by recalling some auxiliary results to be used in this paper.

Lemma 1. (*See* [18, *Young inequality*].) *If there exists a positive constant* ϵ*, two constants* $a > 1$*,* $b > 1$*, which satisfy* $(a-1)(b-1) = 1$*, there holds*

$$x^{\mathrm{T}} y \leqslant \frac{\epsilon^a}{a}\|x\|^a + \frac{1}{b\epsilon^b}\|y\|^b \quad \forall (x,y) \in \mathbb{R}^n.$$

Lemma 2. *For any given matrices* $A \in \mathbb{R}^{n\times r}$ *and* $B \in \mathbb{R}^{n\times r}$*, it follows that*

$$\operatorname{tr}\bigl(A^{\mathrm{T}} B\bigr) \leqslant \frac{1}{2}\bigl[\operatorname{tr}\bigl(A^{\mathrm{T}} A\bigr) + \operatorname{tr}\bigl(B^{\mathrm{T}} B\bigr)\bigr].$$

Proof. This proof can be achieved by using the properties of matrix's trace. □

Lemma 3. (*See* [14].) *For system*

$$\mathrm{d}x(t) = f\bigl(x(t), x\bigl(t-\tau(t)\bigr)\bigr)\,\mathrm{d}t + g\bigl(x(t), x\bigl(t-\tau(t)\bigr)\bigr)\,\mathrm{d}w(t) \quad \forall t \geqslant 0,$$

assume that $f(x,y)$ *and* $g(x,y)$ *are locally Lipschitz in* (x,y)*. If there exists a function* $V(x,t) \in \mathbb{C}^{2,1}(\mathbb{R}^n \times [-\tau,\infty); \mathbb{R}_+)$ *such that, for some constant* $K > 0$ *and any* $t \geqslant 0$*,*

$$\begin{aligned}
&\mathcal{L}V \leqslant K\bigl(1 + V\bigl(x(t),t\bigr) + V\bigl(x\bigl(t-\tau(t)\bigr), t-\tau(t)\bigr)\bigr),\\
&\lim_{|x|\to\infty} \inf_{t\geqslant 0} V(x,t) = \infty,
\end{aligned}$$

then, there exists a unique solution on $[-h,\infty)$ *for any initial data* $x(t)$*:* $t \in [-h,0]$.

Lemma 4. (*See* [21].) *Let* $V(x,t) \in \mathbb{C}^{2,1}(\mathbb{R}^n \times \mathbb{R}_+)$ *and* τ_1, τ_2 *be bounded stopping times such that* $0 \leqslant \tau_1 \leqslant \tau_2$ *a.s. If* $V(x,t)$ *and* $\mathcal{L}V(x,t)$ *are bounded on* $t \in [\tau_1,\tau_2]$ *a.s., then*

$$\mathbf{E}\bigl\{V\bigl(x(\tau_2),\tau_2\bigr) - V\bigl(x(\tau_1),\tau_1\bigr)\bigr\} = \mathbf{E}\Biggl\{\int_{\tau_1}^{\tau_2} \mathcal{L}V(x,t)\,\mathrm{d}t\Biggr\}.$$

3 Main results

We investigate the observer-based robust adaptive control design for system (3) with varying time-delay, thus the results developed following will be dependent of the size of delay. The following result gives a robust adaptive observer and controller structure for system (3).

Theorem 1. *Consider system* (3) *and Assumptions* 1 *and* 2 *are satisfied. If there exist positive define matrices* Q_1, Q_2, Q_3 *and a nonzero function matrix* $K(x)$ *such that*

$$(1-\bar{h})Q_1 \geqslant \left(\left\|J(x,0)-R(x,0)\right\|+m\right)^2 I, \tag{5}$$

$$(1-\bar{h})Q_2 \geqslant \left\|J(x,0)-R(x,0)\right\|^2 I, \tag{6}$$

$$Q_3 \geqslant S(x)\left[\frac{1}{2}(\lambda+1)\mu I+2Q_1+I\right]^{\mathrm{T}} S^{\mathrm{T}}(x), \tag{7}$$

and

$$\begin{aligned} P(x) = {} & -2K^{\mathrm{T}}(x)g_1^{\mathrm{T}}(x)-2g_1(x)K(x)+\frac{1}{2}(\lambda+1)\mu I+Q_2+I+g_1(x)Q_3g_1^{\mathrm{T}}(x) \\ & < 0 \end{aligned} \tag{8}$$

hold, then system (4) *under the feedback adaptive control law*

$$\begin{aligned} u = {} & -\frac{1}{2}\psi^{\mathrm{T}}(x,x_\tau)\hat{\theta}-K(\hat{x})\nabla H(\hat{x},0) \\ & -\frac{1}{2}S(x)\left[\frac{1}{2}(\lambda+1)\mu I+2Q_1+I\right]\nabla H(x,0) \\ & +\frac{1}{2}S(x)\left[\frac{1}{2}(\lambda+1)\mu I+2Q_1+I\right]^{\mathrm{T}} S^{\mathrm{T}}(x)g_1^{\mathrm{T}}(\hat{x})\nabla H(\hat{x},0), \\ \dot{\hat{\theta}} = {} & \Gamma\psi(x,x_\tau)g_1^{\mathrm{T}}(x)\nabla H(x,0), \end{aligned} \tag{9}$$

can be taken as a global asymptotically observer of system (3) *under the feedback control law* (9)*, where* $S(x)=[g_1^{\mathrm{T}}(x)g_1(x)]^{-1}g_1^{\mathrm{T}}(x)$*,* $\lambda=\sup_{t\geqslant 0}\|Hess(H(x,0))\|^2$*,* μ *is a scalar, which satisfies* $\mu \geqslant (1/\beta)\operatorname{tr}[g_2^{\mathrm{T}}(x)g_2(x)]/\|x\|^2$.

Proof. For any continuous function $f(x)=g_1\bar{u}$, there has

$$\left[g_1^{\mathrm{T}}g_1\right]^{-1}g_1^{\mathrm{T}}g_1\bar{u}=\bar{u}=\left[g_1^{\mathrm{T}}g_1\right]^{-1}g_1^{\mathrm{T}}f(x),$$

therefore,

$$\begin{aligned} g_1(x)u = {} & -\frac{1}{2}g_1(x)\psi^{\mathrm{T}}(x,x_\tau)\hat{\theta}-g_1(x)K(\hat{x})\nabla H(\hat{x},0) \\ & -\frac{1}{2}\left[\frac{1}{2}(\lambda+1)\mu I+2Q_1+I\right]\nabla H(x,0) \\ & +\frac{1}{2}\left[\frac{1}{2}(\lambda+1)\mu I+2Q_1+I\right]^{\mathrm{T}} S^{\mathrm{T}}(x)g_1^{\mathrm{T}}(\hat{x})\nabla H(\hat{x},0), \end{aligned}$$

under which, system (3) and (4) can be rewritten as an augmented system as follows:

$$\begin{aligned}
\mathrm{d}x(t) &= \big[J(x,p) - R(x,p)\big]\nabla H(x,x_\tau,0)\,\mathrm{d}t \\
&\quad + \frac{1}{2}g_1(x)\psi^{\mathrm{T}}(x,x_\tau)(\theta-\hat{\theta})\,\mathrm{d}t - g_1(x)K(\hat{x})\nabla H(\hat{x},0)\,\mathrm{d}t \\
&\quad + \frac{1}{2}\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]^{\mathrm{T}} S^{\mathrm{T}}(x)g_1^{\mathrm{T}}(\hat{x})\nabla H(\hat{x},0)\,\mathrm{d}t \\
&\quad - \frac{1}{2}\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]\nabla H(x,0)\,\mathrm{d}t + g_2(x)\,\mathrm{d}w(t), \\
\mathrm{d}\hat{x}(t) &= \big[J(\hat{x},0) - R(\hat{x},0)\big]\nabla H(\hat{x},\hat{x}_\tau,0)\,\mathrm{d}t - g_1(\hat{x})K(\hat{x})\nabla H(\hat{x},0)\,\mathrm{d}t \\
&\quad - \frac{1}{2}g_1(\hat{x})S(x)\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]\nabla H(x,0)\,\mathrm{d}t \\
&\quad + \frac{1}{2}g_1(\hat{x})S(x)\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]^{\mathrm{T}} S^{\mathrm{T}}(x)g_1^{\mathrm{T}}(\hat{x})\nabla H(\hat{x},0)\,\mathrm{d}t \\
&\quad + K^{\mathrm{T}}(\hat{x})g_1^{\mathrm{T}}(x)\nabla H(x,0)\,\mathrm{d}t - K^{\mathrm{T}}(\hat{x})g_1^{\mathrm{T}}(\hat{x})\nabla H(\hat{x},0)\,\mathrm{d}t + g_2(\hat{x})\,\mathrm{d}w(t), \\
x(t) &= \phi(t), \quad \hat{x}(t) = \hat{\phi}(t), \quad t\in[-h,0].
\end{aligned} \tag{10}$$

We choose a candidate Lyapunov functional as follows:

$$\begin{aligned}
V(x,\hat{x},\tilde{\theta}) &= 2H(x,0) + 2H(\hat{x},0) + \frac{1}{2}\tilde{\theta}^{\mathrm{T}}\Gamma^{-1}\tilde{\theta} \\
&\quad + \int\limits_{t-\tau(t)}^{t} \nabla^{\mathrm{T}} H\big(x,x(\xi_1),0\big)Q_1\nabla H\big(x,x(\xi_1),0\big)\,\mathrm{d}\xi_1 \\
&\quad + \int\limits_{t-\tau(t)}^{t} \nabla^{\mathrm{T}} H\big(\hat{x},\hat{x}(\xi_2),0\big)Q_2\nabla H\big(\hat{x},\hat{x}(\xi_2),0\big)\,\mathrm{d}\xi_2,
\end{aligned}$$

where $\tilde{\theta} = \theta - \hat{\theta}$. According to Itô differential formula, one has

$$\mathrm{d}V(x,\hat{x},\tilde{\theta}) = \mathcal{L}V(x,\hat{x},\tilde{\theta})\,\mathrm{d}t + \nabla V(x,\hat{x},\tilde{\theta})\big[g_2(x)+g_2(\hat{x})\big]\,\mathrm{d}w(t),$$

where

$$\begin{aligned}
&\mathcal{L}V(x,\hat{x},\tilde{\theta}) \\
&\quad = 2\nabla^{\mathrm{T}} H(x,0)\big[J(x,p) - R(x,p)\big]\nabla H(x,x_\tau,0) \\
&\qquad + \operatorname{tr}\big[g_2^{\mathrm{T}}(x)Hess\big(H(x,0)\big)g_2(x)\big] - 2\nabla^{\mathrm{T}} H(x,0)g_1(x)K(\hat{x})\nabla H(\hat{x},0) \\
&\qquad - \nabla^{\mathrm{T}} H(x,0)\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]\nabla H(x,0)
\end{aligned}$$

$$
\begin{aligned}
&+ \nabla^{\mathrm{T}} H(x,0)\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]^{\mathrm{T}} S^{\mathrm{T}}(x) g_1^{\mathrm{T}}(\hat{x}) \nabla H(\hat{x},0) \\
&+ \nabla^{\mathrm{T}} H(x,0) Q_1 \nabla H(x,0) - \bigl(1-\dot{\tau}(t)\bigr) \nabla^{\mathrm{T}} H(x,x_\tau,0) Q_1 \nabla H(x,x_\tau,0) \\
&+ 2\nabla^{\mathrm{T}} H(\hat{x},0)\bigl[J(\hat{x},0) - R(\hat{x},0)\bigr] \nabla H(\hat{x},\hat{x}_\tau,0) \\
&+ \operatorname{tr}\bigl[g_2^{\mathrm{T}}(\hat{x}) \mathit{Hess}\bigl(H(\hat{x},0)\bigr) g_2(\hat{x})\bigr] - 2\nabla^{\mathrm{T}} H(\hat{x},0) g_1(\hat{x}) K(\hat{x}) \nabla H(\hat{x},0) \\
&- \nabla^{\mathrm{T}} H(\hat{x},0) g_1(\hat{x}) S(x)\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right] \nabla H(x,0) \\
&+ \nabla^{\mathrm{T}} H(\hat{x},0) g_1(\hat{x}) S(x)\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]^{\mathrm{T}} S^{\mathrm{T}}(x) g_1^{\mathrm{T}}(\hat{x}) \nabla H(\hat{x},0) \\
&+ 2\nabla^{\mathrm{T}} H(\hat{x},0) K^{\mathrm{T}}(\hat{x}) g_1^{\mathrm{T}}(x) \nabla H(x,0) - 2\nabla^{\mathrm{T}} H(\hat{x},0) K^{\mathrm{T}}(\hat{x}) g_1^{\mathrm{T}}(\hat{x}) \nabla H(\hat{x},0) \\
&+ \nabla^{\mathrm{T}} H(\hat{x},0) Q_2 \nabla H(\hat{x},0) - \bigl(1-\dot{\tau}(t)\bigr) \nabla^{\mathrm{T}} H(\hat{x},\hat{x}_\tau,0) Q_2 \nabla H(\hat{x},\hat{x}_\tau,0). \qquad (11)
\end{aligned}
$$

According to Lemma 2, one has

$$
\begin{aligned}
&\operatorname{tr}\bigl[g_2^{\mathrm{T}}(x) \mathit{Hess}\bigl(H(x,0)\bigr) g_2(x)\bigr] \\
&\quad \leqslant \frac{1}{2} \operatorname{tr}\bigl[g_2^{\mathrm{T}}(x) \mathit{Hess}\bigl(H(x,0)\bigr) \mathit{Hess}^{\mathrm{T}}\bigl(H(x,0)\bigr) g_2(x)\bigr] + \frac{1}{2} \operatorname{tr}\bigl[g_2^{\mathrm{T}}(x) g_2(x)\bigr] \\
&\quad \leqslant \nabla^{\mathrm{T}} H(x,0)\left[\frac{1}{2}(\lambda+1)\mu I\right] \nabla H(x,0) \qquad (12)
\end{aligned}
$$

and

$$
\begin{aligned}
&\operatorname{tr}\bigl[g_2^{\mathrm{T}}(\hat{x}) \mathit{Hess}\bigl(H(\hat{x},0)\bigr) g_2(\hat{x})\bigr] \\
&\quad \leqslant \frac{1}{2} \operatorname{tr}\bigl[g_2^{\mathrm{T}}(\hat{x}) \mathit{Hess}\bigl(H(\hat{x},0)\bigr) \mathit{Hess}^{\mathrm{T}}\bigl(H(\hat{x},0)\bigr) g_2(\hat{x})\bigr] + \frac{1}{2} \operatorname{tr}\bigl[g_2^{\mathrm{T}}(\hat{x}) g_2(\hat{x})\bigr] \\
&\quad \leqslant \nabla^{\mathrm{T}} H(\hat{x},0)\left[\frac{1}{2}(\lambda+1)\mu I\right] \nabla H(\hat{x},0). \qquad (13)
\end{aligned}
$$

Substituting (12) and (13) into (11), we get

$$
\begin{aligned}
&\mathcal{L}V(x,\hat{x},\tilde{\theta}) \\
&\quad = 2\nabla^{\mathrm{T}} H(x,0)\bigl[J(x,p) - R(x,p)\bigr] \nabla H(x,x_\tau,0) \\
&\qquad - \nabla^{\mathrm{T}} H(x,0)[Q_1 + I] \nabla H(x,0) - (1-\bar{h}) \nabla^{\mathrm{T}} H(x,x_\tau,0) Q_1 \nabla H(x,x_\tau,0) \\
&\qquad + 2\nabla^{\mathrm{T}} H(\hat{x},0)\bigl[J(\hat{x},0) - R(\hat{x},0)\bigr] \nabla H(\hat{x},\hat{x}_\tau,0) \\
&\qquad + \nabla^{\mathrm{T}} H(\hat{x},0)\left[-2g_1(\hat{x})K(\hat{x}) - 2K^{\mathrm{T}}(\hat{x}) g_1^{\mathrm{T}}(\hat{x}) + \frac{1}{2}(\lambda+1)\mu I + Q_2\right] \nabla H(\hat{x},0) \\
&\qquad + \nabla^{\mathrm{T}} H(\hat{x},0) g_1(\hat{x}) S(x)\left[\frac{1}{2}(\lambda+1)\mu I + 2Q_1 + I\right]^{\mathrm{T}} S^{\mathrm{T}}(x) g_1^{\mathrm{T}}(\hat{x}) \nabla H(\hat{x},0) \\
&\qquad - (1-\bar{h}) \nabla^{\mathrm{T}} H(\hat{x},\hat{x}_\tau,0) Q_2 \nabla H(\hat{x},\hat{x}_\tau,0).
\end{aligned}
$$

In view of condition (7), it follows that

$$\begin{aligned}
&\mathcal{L}V(x,\hat{x},\tilde{\theta})\\
&\quad\leqslant 2\nabla^{\mathrm T}H(x,0)\big[J(x,p)-R(x,p)\big]\nabla H(x,x_\tau,0)\\
&\qquad-\nabla^{\mathrm T}H(x,0)[Q_1+I]\nabla H(x,0)-(1-\bar{h})\nabla^{\mathrm T}H(x,x_\tau,0)Q_1\nabla H(x,x_\tau,0)\\
&\qquad+2\nabla^{\mathrm T}H(\hat{x},0)\big[J(\hat{x},0)-R(\hat{x},0)\big]\nabla H(\hat{x},\hat{x}_\tau,0)\\
&\qquad+\nabla^{\mathrm T}H(\hat{x},0)\bigg[-2g_1(\hat{x})K(\hat{x})-2K^{\mathrm T}(\hat{x})g_1^{\mathrm T}(\hat{x})+\frac{1}{2}(\lambda+1)\mu I\\
&\qquad+Q_2+g_1(\hat{x})Q_3g_1^{\mathrm T}(\hat{x})\bigg]\nabla H(\hat{x},0)\\
&\qquad-(1-\bar{h})\nabla^{\mathrm T}H(\hat{x},\hat{x}_\tau,0)Q_2\nabla H(\hat{x},\hat{x}_\tau,0).
\end{aligned}\tag{14}$$

Noticing Lemma 1 and setting $\epsilon=1$, $a=b=2$, it yields

$$\begin{aligned}
&2\nabla^{\mathrm T}H(x,0)\big[J(x,p)-R(x,p)\big]\nabla H(x,x_\tau,0)\\
&\quad\leqslant\big\|\nabla^{\mathrm T}H(x,0)\big\|^2+\big\|\big[J(x,p)-R(x,p)\big]\nabla H(x,x_\tau,0)\big\|^2.
\end{aligned}$$

Using the properties of the norms that

$$\|Ez\|\leqslant\|E\|\|z\|\quad\forall E\in\mathbb{R}^{n\times n},\ \forall z\in\mathbb{R}^n$$

and

$$\|E+F\|\leqslant\|E\|+\|F\|\quad\forall E\in\mathbb{R}^{n\times n},\ \forall F\in\mathbb{R}^{n\times n},$$

we have

$$\begin{aligned}
&2\nabla^{\mathrm T}H(x,0)\big[J(x,p)-R(x,p)\big]\nabla H(x,x_\tau,0)\\
&\quad\leqslant\big\|\nabla^{\mathrm T}H(x,0)\big\|^2+\big\|J(x,p)-R(x,p)\big\|^2\big\|\nabla H(x,x_\tau,0)\big\|^2
\end{aligned}$$

and

$$\big\|J(x,p)-R(x,p)\big\|\leqslant\big\|J(x,0)-R(x,0)\big\|+\big\|\Delta_J(x,p)-\Delta_R(x,p)\big\|.$$

Introducing $\|\Delta_J(x,p)-\Delta_R(x,p)\|\leqslant m$, it follows that

$$\begin{aligned}
&\big\|J(x,p)-R(x,p)\big\|^2\\
&\quad\leqslant\big[\big\|J(x,0)-R(x,0)\big\|+m\big]^2\\
&\quad=\|J(x,0)-R(x,0)\big\|^2+2m\big\|J(x,0)-R(x,0)\big\|+m^2.
\end{aligned}$$

As a consequence, one has

$$\begin{aligned}
&2\nabla^{\mathrm T}H(x,0)\big[J(x,p)-R(x,p)\big]\nabla H(x,x_\tau,0)\\
&\quad\leqslant\nabla^{\mathrm T}H(x,0)\nabla H(x,0)+\big[\big\|J(x,0)-R(x,0)\big\|^2+m^2\\
&\qquad+2m\big\|J(x,0)-R(x,0)\big\|\big]\nabla^{\mathrm T}H(x,x_\tau,0)\nabla H(x,x_\tau,0).
\end{aligned}\tag{15}$$

Similarly, the following inequality holds:

$$
\begin{aligned}
&2\nabla^{\mathrm{T}} H(\hat{x},0)\big[J(\hat{x},0)-R(\hat{x},0)\big]\nabla H(\hat{x},\hat{x}_\tau,0)\\
&\quad\leqslant \nabla^{\mathrm{T}} H(\hat{x},0)\nabla H(\hat{x},0)+\big\|J(\hat{x},0)-R(\hat{x},0)\big\|^2\nabla^{\mathrm{T}} H(\hat{x},\hat{x}_\tau,0)\nabla H(\hat{x},\hat{x}_\tau,0). \qquad (16)
\end{aligned}
$$

Combining inequalities (14), (15) and (16), it yields

$$
\begin{aligned}
&\mathcal{L}V(x,\hat{x},\tilde{\theta})\\
&\quad\leqslant -\nabla^{\mathrm{T}} H(x,0)Q_1\nabla H(x,0)\\
&\qquad+\nabla^{\mathrm{T}} H(x,x_\tau,0)\big[\big(\big\|J(x,0)-R(x,0)\big\|+m\big)^2 I-(1-\bar{h})Q_1\big]\nabla H(x,x_\tau,0)\\
&\qquad+\nabla^{\mathrm{T}} H(\hat{x},0)\big[-2g_1(\hat{x})K(\hat{x})-2K^{\mathrm{T}}(\hat{x})g_1^{\mathrm{T}}(\hat{x})\\
&\qquad+\frac{1}{2}(\lambda+1)\mu I+Q_2+I+g_1(\hat{x})Q_3g_1^{\mathrm{T}}(\hat{x})\big]\nabla H(\hat{x},0)\\
&\qquad+\nabla^{\mathrm{T}} H(\hat{x},\hat{x}_\tau,0)\big[\big\|J(\hat{x},0)-R(\hat{x},0)\big\|^2 I-(1-\bar{h})Q_2\big]\nabla H(\hat{x},\hat{x}_\tau,0). \qquad (17)
\end{aligned}
$$

Substituting (5), (6) and (8) into (17), leads to

$$
\mathcal{L}V(x,\hat{x},\tilde{\theta})\leqslant -\nabla^{\mathrm{T}} H(x,0)Q_1\nabla H(x,0)+\nabla^{\mathrm{T}} H(\hat{x},0)P(\hat{x})\nabla H(\hat{x},0)\leqslant 0.
$$

Set $c_1=\lambda_{\min}(Q_1)>0$, $c_2=\inf_{t\geqslant 0}\{\lambda_{\min}(-P(x))\}>0$, $c=\min\{c_1,c_2\}$, we further obtain

$$
\mathcal{L}V(x,\hat{x},\tilde{\theta})\leqslant -c\big[\nabla^{\mathrm{T}} H(x,0)\nabla H(x,0)+\nabla^{\mathrm{T}} H(\hat{x},0)\nabla H(\hat{x},0)\big].
$$

In addition, based on (A3) in Assumption 2, there has

$$
\mathcal{L}V(x,\hat{x},\tilde{\theta})\leqslant -c\beta\big(\|x\|^2+\|\hat{x}\|^2\big).
$$

In view of the properties of norm, we further get

$$
\mathcal{L}V(x,\hat{x},\tilde{\theta})\leqslant -\frac{c\beta}{2}\big(\|x\|+\|\hat{x}\|\big)^2\leqslant -\frac{c\beta}{2}\|x-\hat{x}\|^2,
$$

or more compactly

$$
\mathbf{E}\big\{\mathcal{L}V(x,\hat{x},\tilde{\theta})\big\}\leqslant -\frac{c\beta}{2}\mathbf{E}\big\{\|x-\hat{x}\|^2\big\}.
$$

From Lemma 4, it is true that, for all $t>t_0$, $t_0\in[-h,0]$,

$$
\begin{aligned}
&\mathbf{E}\big\{V(t)\big\}-\mathbf{E}\big\{V(t_0)\big\}\\
&\quad=\int_{t_0}^{t}\mathbf{E}\big\{\mathcal{L}V(s)\big\}\,\mathrm{d}s\leqslant\int_{t_0}^{t}\mathbf{E}\bigg\{-\frac{c\beta}{2}\big\|x(s)-\hat{x}(s)\big\|^2\bigg\}\,\mathrm{d}s.
\end{aligned}
$$

Hence, one has

$$
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\big\{V(x,\hat{x},\tilde{\theta})\big\}\leqslant -\frac{c\beta}{2}\mathbf{E}\big\{\big\|x(t)-\hat{x}(t)\big\|^2\big\}.
$$

From Assumption 2, we can get that

$$V(x,\hat{x},\tilde{\theta}) \geqslant \alpha\|x-\hat{x}\|^2.$$

Set $k=-c\beta/(2\alpha)$, then it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\} \leqslant k\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\}. \tag{18}$$

Multiplying e^{-kt} to the two sides of the inequality (18), then yields

$$\mathrm{e}^{-kt}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\} - \mathrm{e}^{-kt}k\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\} \leqslant 0,$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\mathrm{e}^{-kt}\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\}\bigr) \leqslant 0. \tag{19}$$

Integrating the inequality (19) from t_0 to t, we have

$$\mathrm{e}^{-kt}\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\} - \mathrm{e}^{-kt_0}\mathbf{E}\bigl\{\bigl\|x(t_0)-\hat{x}(t_0)\bigr\|^2\bigr\} \leqslant 0,$$

i.e.,

$$\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\} \leqslant \mathrm{e}^{k(t-t_0)}\mathbf{E}\bigl\{\bigl\|x(t_0)-\hat{x}(t_0)\bigr\|^2\bigr\} \quad \forall t>t_0.$$

Due to $k<0$, there has

$$\lim_{t\to\infty}\mathbf{E}\bigl\{\|x-\hat{x}\|^2\bigr\} = 0.$$

According to Definition 1, we can conclude that the closed-loop system (4) is global asymptotically observer of system (3) under the control law (9). This completes the proof. □

Remark 4. Owing to the fact of $H(x,x_\tau,0)\in\mathbb{C}^2$ and Lemma 3, the solution of the closed-loop system (10) is existent and unique on $[0,\infty)$ for any initial data in some neighborhood of equilibrium.

System (4) is an adaptive observer of system (3) while the structure of system (3) can not be duplicated. Furthermore, if the structure of system (3) could be duplicated, we can design an adaptive observer in the form of

$$\begin{aligned}
\mathrm{d}\hat{x}(t) &= \bigl[J(\hat{x},p)-R(\hat{x},p)\bigr]\nabla H(\hat{x},\hat{x}_\tau,0)\,\mathrm{d}t + \frac{1}{2}g_1(\hat{x})\psi^{\mathrm{T}}(x,x_\tau)\hat{\theta}\,\mathrm{d}t \\
&\quad + g_1(\hat{x})u\bigl(t,t-\tau(t)\bigr)\,\mathrm{d}t + K^{\mathrm{T}}(\hat{x})\bigl[y-g_1^{\mathrm{T}}(\hat{x})\nabla H(\hat{x},0)\bigr]\,\mathrm{d}t \\
&\quad + g_2(\hat{x})\,\mathrm{d}w(t), \\
\dot{\hat{\theta}} &= \varGamma\psi(x,x_\tau)g_1^{\mathrm{T}}(x)\nabla H(x,0), \\
\hat{x}(t) &= \hat{\phi}(t), \quad t\in[-h,0].
\end{aligned} \tag{20}$$

Thus, we have the following corollary.

Corollary 1. *Consider system* (3) *and Assumptions* 1 *and* 2 *are satisfied. If there exist positive define matrices* Q_1*,* Q_2*,* Q_3 *and a nonzero function matrix* $K(x)$ *such that*

$$Q_1 = Q_2, \tag{21}$$

and the conditions (5)*,* (7) *and* (8) *hold, then system* (20) *under the feedback adaptive control law* (9) *can be acted as a global asymptotically observer of system* (3) *under the feedback adaptive control law* (9)*, where* $S(x) = [g_1^{\mathrm{T}}(x)g_1(x)]^{-1}g_1^{\mathrm{T}}(x)$*.*

Remark 5. If the delay $\tau(t)$ in system (1) is a constant, i.e., $\bar{h} = 0$, conditions (5) and (6) will be independent of the time-delay, namely, they become delay-independent conditions. Therefore, Theorem 1 and Corollary 1 still hold when the time delay under consideration is a constant.

Remark 6. For the state matrices of system (1) contain delay, Theorem 1 still holds provided the state matrices of system (4) be replaced by $J(\hat{x}, \hat{x}_\tau, 0)$ and $R(\hat{x}, \hat{x}_\tau, 0)$. Accordingly, conditions (5), (6) become

$$\begin{aligned}
&(1-\bar{h})Q_1 \geqslant \bigl(\bigl\|J(x, x_\tau, 0) - R(x, x_\tau, 0)\bigr\| + m\bigr)^2 I,\\
&(1-\bar{h})Q_2 \geqslant \bigl\|J(x, x_\tau, 0) - R(x, x_\tau, 0)\bigr\|^2 I.
\end{aligned}$$

This result is established on the premise that Assumption 1 holds with $J(x_\tau, p)$, $R(x_\tau, p)$ instead of $J(x, p)$, $R(x, p)$ in (2). An example is given to verify this statement.

Example 1. Consider a time delay Hamiltonian systems (1) with

$$\begin{aligned}
H(x, x_\tau, p) &= \frac{1}{2}\bigl[x_1^2\bigl(t-\tau(t)\bigr) + x_2^2 + \sin^2 x_2 + (1+p)x_3^2\bigl(t-\tau(t)\bigr)\bigr],\\
J(x_\tau, p) &= \begin{pmatrix} 0 & 0 & p + x_3(t-\tau(t)) \\ 0 & 0 & 0 \\ -p - x_3(t-\tau(t)) & 0 & 0 \end{pmatrix},\\
R(x_\tau, p) &= \begin{pmatrix} x_1^2(t-\tau(t)) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^2 \end{pmatrix}, \qquad g_1(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3^2+1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

we may testify Assumption 1 by finding out the following $\psi(x, x_\tau)$ and θ:

$$\psi(x, x_\tau) = \begin{pmatrix} 2x_3(t-\tau(t)) & 0 & 0 \\ 2x_3^2(t-\tau(t)) & 0 & 0 \\ 0 & 0 & 2x_3(t-\tau(t)) \end{pmatrix}, \qquad \theta = \begin{pmatrix} p^2 \\ p \\ -p^3 \end{pmatrix}.$$

4 Illustrative examples

In this section, we give an example to show how to apply the results proposed in this paper to design the robust adaptive observer for some classes of nonlinear systems with stochasticity and varying time-delay based on Hamiltonian function method.

Let us consider the following stochastic time-delay nonlinear systems with uncertainties

$$\begin{aligned}
\mathrm{d}x_1 &= \bigl[-x_1^2(t)x_1\bigl(t-\tau(t)\bigr) + p(1+p)x_3\bigl(t-\tau(t)\bigr) \\
&\quad + (1+p)x_3(t)x_3\bigl(t-\tau(t)\bigr) + u_1\bigr]\,\mathrm{d}t, \\
\mathrm{d}x_2 &= \biggl[-x_2(t) - \frac{1}{2}\sin 2x_2(t) + p(1+p)x_3\bigl(t-\tau(t)\bigr) \\
&\quad + \bigl(x_3^2(t)+1\bigr)u_2\biggr]\,\mathrm{d}t + x_2\,\mathrm{d}w(t), \\
\mathrm{d}x_3 &= \biggl[-\bigl(x_3(t)+p\bigr)x_1\bigl(t-\tau(t)\bigr) - px_2(t) - \frac{p}{2}\sin 2x_2(t) \\
&\quad - p^2(1+p)x_3\bigl(t-\tau(t)\bigr) + u_3\biggr]\,\mathrm{d}t,
\end{aligned} \tag{22}$$

where p is uncertain parameter, and $|p| < 1$; $\tau(t) = 0.5\sin^2(t)$is the time varying delay. Since $\dot\tau(t) = 0.5\sin(2t)$, we may take $\bar h = 0.5$.

Let $x = [x_1,\, x_2,\, x_3]^{\mathrm T}$, $u = [u_1,\, u_2,\, u_3]^{\mathrm T}$, system (22) can be realized into the following Hamiltonian systems form:

$$\begin{aligned}
\mathrm{d}x(t) &= \bigl[J(x,p) - R(x,p)\bigr]\nabla H(x,x_\tau,p)\,\mathrm{d}t + g_1(x)u\bigl(t,t-\tau(t)\bigr)\,\mathrm{d}t \\
&\quad + g_2(x)\,\mathrm{d}w(t)
\end{aligned}$$

with the initial condition

$$x(t_0) = \phi(t_0), \quad t_0 \in [-0.5, 0],$$

where

$$H(x,x_\tau,p) = 0.5\bigl[x_1^2\bigl(t-\tau(t)\bigr) + x_2^2 + \sin^2 x_2 + (1+p)x_3^2\bigl(t-\tau(t)\bigr)\bigr],$$

$$J(x,p) = \begin{pmatrix} 0 & 0 & p+x_3(t) \\ 0 & 0 & p \\ -p-x_3(t) & -p & 0 \end{pmatrix}, \qquad g_1(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_3^2(t)+1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$R(x,p) = \begin{pmatrix} x_1^2(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^2 \end{pmatrix}, \qquad g_2(x) = \begin{pmatrix} 0 \\ x_2(t) \\ 0 \end{pmatrix}.$$

It is easy to verify that the Hamilton function $H(x,0)$, its gradient $\nabla H(x,0)$, as well as $Hess(H(x,0))$ satisfy Assumptions 1 and 2. Meanwhile, β can be taken as $\beta = 1$.

To illustrate the result, we carry simulations with the the initial condition $\phi(x_0) = [-0.8,\, -0.5,\, 0.8]^{\mathrm T}$, $\hat\phi(x_0) = [-0.5,\, -0.8,\, 0.5]^{\mathrm T}$ and $\theta_0 = [0.36,\, 0.6,\, 0.216]^{\mathrm T}$. Since $\|\Delta_J(x,p) - \Delta_R(x,p)\| = \max_{t\geqslant 0}\{1, x_3(t)\}$, we may take $m = \sqrt{2}$. Besides, since $\mu \geqslant \operatorname{tr}[g_2^{\mathrm T}(x)g_2(x)]/\beta\|x\|^2$, we can choose $\mu = 1$. Furthermore, from $\|Hess(H(x,0))\|^2 = \max_{t\geqslant 0}\{1, (1+\cos(2x_2(t)))^2\}$, we get $\lambda = 4$.

First, assuming that the states of system (22) are not fully measured and the structure can not be duplicated, we will design an observer-based controller by using Theorem 1. So we can work out the inequalities (5), (6), (7) and (8) to find the following matrices:

$$Q_1 = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{pmatrix}, \qquad K(x) = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8(x_3^2(t)+1) & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Thus according to Theorem 1, the observer

$$\begin{aligned} \mathrm{d}\hat{x}_1 &= \big[-\hat{x}_1^2(t)\hat{x}_1\big(t-\tau(t)\big) + x_3\big(t-\tau(t)\big)\hat{\theta}_1 + x_3(t)x_3\big(t-\tau(t)\big)\hat{\theta}_2 \\ &\quad + \hat{x}_3(t)\hat{x}_3\big(t-\tau(t)\big) + 8x_1(t) - 8\hat{x}_1(t) + u_1\big]\,\mathrm{d}t, \\ \mathrm{d}\hat{x}_2 &= \bigg[-\hat{x}_2(t) - \frac{1}{2}\sin 2\hat{x}_2(t) + \frac{(\hat{x}_3^2(t)+1)}{x_3^2(t)+1}x_3\big(t-\tau(t)\big)\hat{\theta}_1 \\ &\quad - 8\big(x_3^2(t)+1\big)\big(\hat{x}_3^2(t)+1\big)\bigg(\hat{x}_2(t) + \frac{1}{2}\sin 2\hat{x}_2(t)\bigg) \\ &\quad + 8\big(x_3^2(t)+1\big)^2\bigg(x_2(t) + \frac{1}{2}\sin 2x_2(t)\bigg) \\ &\quad + \big(\hat{x}_3^2(t)+1\big)u_2\bigg]\,\mathrm{d}t + \hat{x}_2(t)\,\mathrm{d}w(t), \\ \mathrm{d}\hat{x}_3 &= \big[-\hat{x}_3(t)\hat{x}_1\big(t-\tau(t)\big) - x_3\big(t-\tau(t)\big)\hat{\theta}_3 + 8x_3(t) - 8\hat{x}_3(t) + u_3\big]\,\mathrm{d}t \end{aligned} \tag{23}$$

is a globally asymptotically observer of system (22) under the feedback adaptive control law

$$\begin{aligned} u &= \begin{pmatrix} -x_3(t-\tau(t))\hat{\theta}_1 - x_3(t)x_3(t-\tau(t))\hat{\theta}_2 + 3.75\hat{x}_1(t) \\ -\frac{x_3(t-\tau(t))}{x_3^2(t)+1}\hat{\theta}_1 - 8(\hat{x}_3^2(t)+1)[\hat{x}_2(t) + \frac{1}{2}\sin 2\hat{x}_2(t)] \\ x_3(t-\tau(t))\hat{\theta}_3 - 11.75x_3(t) + 3.75\hat{x}_3(t) \end{pmatrix} \\ &\quad - \begin{pmatrix} 11.75x_1(t) \\ 11.75\frac{1}{(x_3^2(t)+1)}[x_2(t) + \frac{1}{2}\sin 2x_2(t)] \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ 11.75\frac{(\hat{x}_3^2(t)+1)}{(x_3^2(t)+1)^2}[\hat{x}_2(t) + \frac{1}{2}\sin 2\hat{x}_2(t)] \\ 0 \end{pmatrix}, \end{aligned}$$

$$\dot{\hat{\theta}} = \begin{pmatrix} 2x_3(t-\tau(t))x_1(t) + 2x_3(t-\tau(t))[x_2(t) + \frac{1}{2}\sin 2x_2(t)] \\ 2x_3(t)x_3(t-\tau(t))x_1(t) \\ -2x_3(t)x_3(t-\tau(t)) \end{pmatrix}.$$

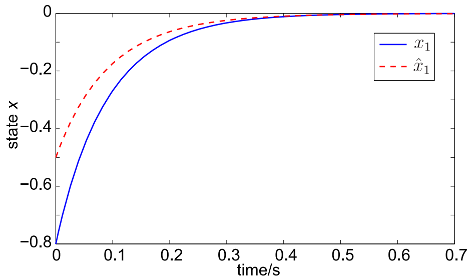

Fig. 1. Responses of states x_1, $\hat{x}_1$.

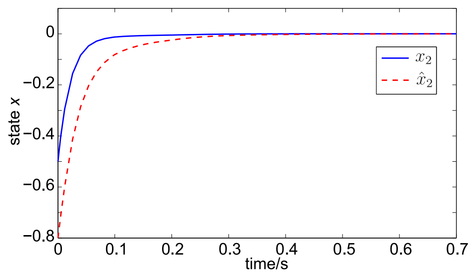

Fig. 2. Responses of states x_2, $\hat{x}_2$.

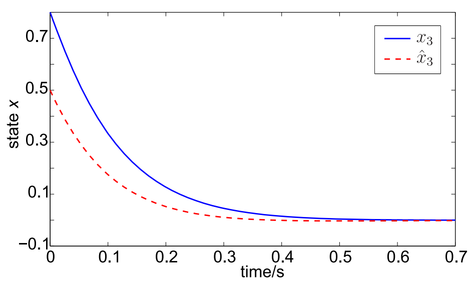

Fig. 3. Responses of states x_3, $\hat{x}_3$.

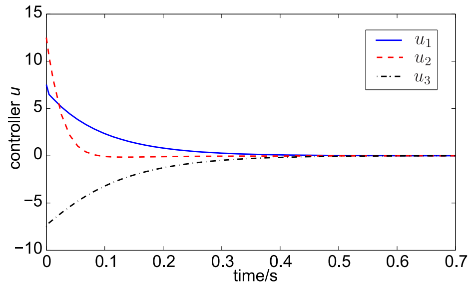

Fig. 4. Control input u.

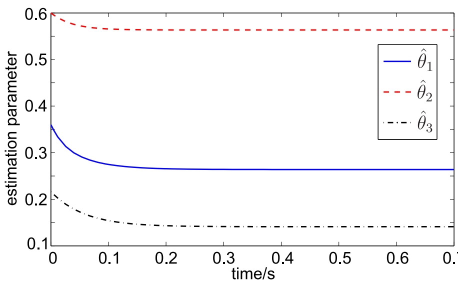

Fig. 5. Estimation parameter $\hat{\theta}$.

Figures 1–5 are the responses of the states, the control input and the parameter estimation $\hat{\theta}$ of the system with delay $\tau(t) = 0.5\sin^2(t)$ respectively. It is obvious that the stochastic varying time-delay nonlinear system (22) and (23) with uncertainties converges to its equilibrium very quickly.

Next, assuming that the states of system (22) are not fully measured and the structure can be duplicated, we will demonstrate the application of Corollary 1. So we may work out inequalities (5), (7), (8) and (21) to find out the following matrices:

$$Q_1 = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{pmatrix}, \qquad K(x) = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10(x_3^2(t)+1) & 0 \\ 0 & 0 & 10 \end{pmatrix}.$$

According to Corollary 1, system

$$\begin{aligned}
\mathrm{d}\hat{x}_1 &= \bigl[-\hat{x}_1^2(t)\hat{x}_1\bigl(t-\tau(t)\bigr) + p(1+p)\hat{x}_3\bigl(t-\tau(t)\bigr) + (1+p)\hat{x}_3(t)\hat{x}_3\bigl(t-\tau(t)\bigr) \\
&\quad + x_3\bigl(t-\tau(t)\bigr)\hat{\theta}_1 + x_3(t)x_3\bigl(t-\tau(t)\bigr)\hat{\theta}_2 + 10x_1(t) - 10\hat{x}_1(t) + u_1\bigr]\,\mathrm{d}t, \\
\mathrm{d}\hat{x}_2 &= \biggl[-\hat{x}_2(t) - \frac{1}{2}\sin 2\hat{x}_2(t) + p(1+p)\hat{x}_3\bigl(t-\tau(t)\bigr) \\
&\quad + \frac{\hat{x}_3^2(t)+1}{x_3^2(t)+1}x_3\bigl(t-\tau(t)\bigr)\hat{\theta}_1 + \bigl(\hat{x}_3^2(t)+1\bigr)u_2 \\
&\quad - 10\bigl(x_3^2(t)+1\bigr)\bigl(\hat{x}_3^2(t)+1\bigr)\biggl(\hat{x}_2(t) + \frac{1}{2}\sin 2\hat{x}_2(t)\biggr) \\
&\quad + 10\bigl(x_3^2(t)+1\bigr)^2\biggl(x_2(t) + \frac{1}{2}\sin 2x_2(t)\biggr)\biggr]\,\mathrm{d}t + \hat{x}_2(t)\,\mathrm{d}w(t), \\
\mathrm{d}\hat{x}_3 &= \biggl[-\bigl(\hat{x}_3(t)+p\bigr)\hat{x}_1\bigl(t-\tau(t)\bigr) - p\hat{x}_2(t) - \frac{p}{2}\sin 2\hat{x}_2(t) \\
&\quad - p^2(1+p)\hat{x}_3\bigl(t-\tau(t)\bigr) + x_3\bigl(t-\tau(t)\bigr)\hat{\theta}_3 \\
&\quad + 10x_3(t) - 10\hat{x}_3(t) + u_3\biggr]\,\mathrm{d}t
\end{aligned} \tag{24}$$

is a globally asymptotically observer of systems (22) under the feedback adaptive control law

$$\begin{aligned}
u &= \begin{pmatrix} -x_3(t-\tau(t))\hat{\theta}_1 - x_3(t)x_3(t-\tau(t))\hat{\theta}_2 + 1.75\hat{x}_1(t) \\ -\frac{x_3(t-\tau(t))}{x_3^2(t)+1}\hat{\theta}_1 - 10(\hat{x}_3^2(t)+1)[\hat{x}_2(t) + \frac{1}{2}\sin 2\hat{x}_2(t)] \\ x_3(t-\tau(t))\hat{\theta}_3 - 11.75x_3(t) + 1.75\hat{x}_3(t) \end{pmatrix} \\
&\quad - \begin{pmatrix} 11.75x_1(t) \\ 11.75\frac{1}{(x_3^2(t)+1)}[x_2(t) + \frac{1}{2}\sin 2x_2(t)] \\ 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 \\ 11.75\frac{(\hat{x}_3^2(t)+1)}{(x_3^2(t)+1)^2}[\hat{x}_2(t) + \frac{1}{2}\sin 2\hat{x}_2(t)] \\ 0 \end{pmatrix}, \\
\dot{\hat{\theta}} &= \begin{pmatrix} 2x_3(t-\tau(t))x_1(t) + 2x_3(t-\tau(t))[x_2(t) + \frac{1}{2}\sin 2x_2(t)] \\ 2x_3(t)x_3(t-\tau(t))x_1(t) \\ -2x_3(t)x_3(t-\tau(t)) \end{pmatrix}.
\end{aligned}$$

To illustrate conclusion, we take the same initial conditions as mentioned above. Figures 6–10 are the responses of the states, the control input u and the parameter

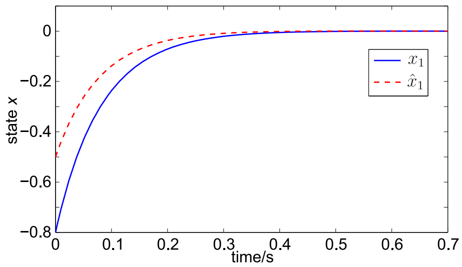

Fig. 6. Responses of states x_1, $\hat{x}_1$.

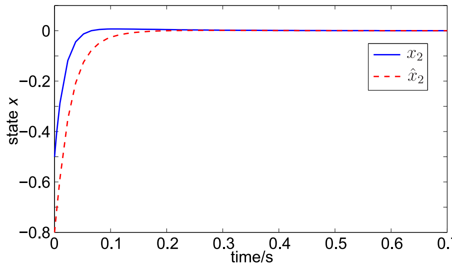

Fig. 7. Responses of states x_2, $\hat{x}_2$.

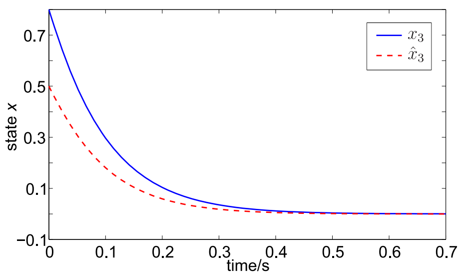

Fig. 8. Responses of states x_3, $\hat{x}_3$.

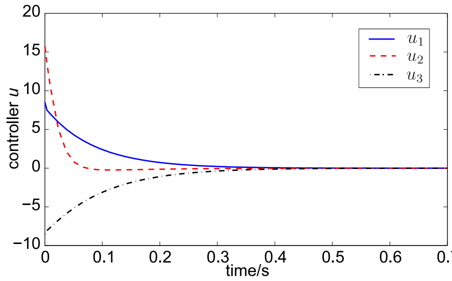

Fig. 9. Control input u.

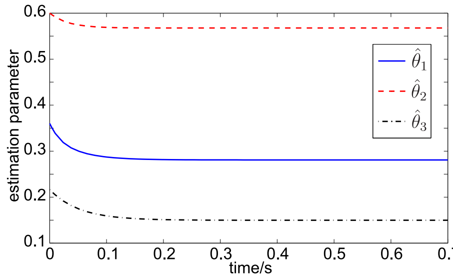

Fig. 10. Estimation parameter $\hat{\theta}$.

estimation $\hat{\theta}$ of the system with delay $\tau(t) = 0.5\sin^2(t)$, respectively. It is obvious from Figs. 6–10 that the stochastic varying time-delay nonlinear system (24) with uncertainties converges to its equilibrium more quickly than system (23).

5 Conclusion

In this paper, the observer-based robust adaptive control problem of a class of stochastic Hamiltonian systems with time-delay and parameter uncertainties has been investigated by using Young inequality and some properties of norm and trace. The designed controller ensures that the closed-loop error system is asymptotically stable in mean square. The results are achieved through well-chosen Lyapunov functional depending on the special structural properties of the Hamiltonian systems. Simulations show that the results acquired in this paper are practicable and validate in analyzing the observer-based robust adaptive control of some classes of stochastic time-delay nonlinear systems.

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