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Smoothness of solutions with respect to multi-strip integral boundary conditions for nth order ordinary differential equations

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Abstract. Under certain conditions, solutions of the boundary value problem, $y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$, $a < x < b$, $y^{(i-1)}(x_1) = y_i$, $i = 1, \dots, n-1$, $y(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} y(x)\,\mathrm{d}x = y_n$, $a < x_1 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \dots < \xi_m < \eta_m < x_2 < b$, are differentiated with respect to the boundary conditions.

Keywords: nth order ordinary differential equation, multi-strip integral boundary conditions, smooth dependence, boundary data.

1 Introduction

In this paper, we will be concerned with differentiating solutions of boundary value problems with respect to boundary data for the nth order ordinary differential equation,

$$y^{(n)} = f\bigl(x, y, y', \dots, y^{(n-1)}\bigr), \quad a < x < b, \tag{1}$$

satisfying the Dirichlet and multi-strip integral boundary conditions,

$$y^{(i-1)}(x_1) = y_i, \quad i = 1, \dots, n-1, \qquad y(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} y(x)\,\mathrm{d}x = y_n, \tag{2}$$

where $a < x_1 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \dots < \xi_m < \eta_m < x_2 < b$, $\gamma_i \in \mathbb{R}$, $i = 1, \dots, m$, and $y_1, \dots, y_n \in \mathbb{R}$, and where we assume:

(i) $f(x, s_1, s_2, \dots, s_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous,
(ii) $\partial f/\partial s_i(x, s_1, s_2, \dots, s_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous, $i = 1, 2, \dots, n$, and
(iii) Solutions of initial value problems for (1) extend to (a, b).

Condition (iii) is not necessary for the results of this paper, yet, by assuming (iii), we avoid continually making statements in terms of solutions' maximal intervals of existence.

Under uniqueness assumptions on solutions of (1), (2), we will establish analogues of a result that Hartman [1] attributes to Peano concerning differentiation of solutions of (1) with respect to initial conditions. For our differentiation with respect to boundary conditions results, given a solution $y(x)$ of (1), we will give much attention to the *variational equation for (1) along* $y(x)$, which is defined by

$$z^{(n)} = \sum_{i=1}^{n} \frac{\partial f}{\partial s_i}\bigl(x, y(x), y'(x), \dots, y^{(n-1)}(x)\bigr) z^{(i-1)}. \tag{3}$$

Interest in nonlocal boundary value problems for differential equations involving integral boundary conditions has been ongoing for several years. To see only few of these papers, we refer the reader to the papers [2–9]. And very recently, Ahmad and Ntouyas [10] initiated research regarding multipoint nonlocal integral boundary conditions such as seen in (2). In describing such boundary conditions, they coined the term "multi-strip integral boundary conditions." Such boundary conditions can be interpreted in the sense that a controller at the right end of the interval under consideration is influenced by a discrete distribution of finite many sensors (or strips) of arbitrary length expressed in terms of integral boundary conditions. Subsequent to that paper, Ahmad and Ntouyas have put forth a couple of additional papers devoted to solutions of boundary value problems involving multi-strip integral boundary conditions for both fractional differential equations and fractional differential inclusions; see [11, 12]. It can also be pointed out that, under suitable measures, the boundary conditions can be considered in the form of Stieltjes integrals; readers can find of interest the papers, [13–15] and [16–18].

In the same way, there have been many papers devoted to smoothness of solutions of boundary value problems in regard to smoothness of the differential equation's nonlinearity, as well as in regard to the smoothness of the boundary conditions. For a view of how this work has evolved, involving not only boundary value problems for ordinary differential equations, but also discrete versions, we suggest the manifold results in the classical papers [19–28] and [8, 9, 29–34], as well as the more current papers [35, 36] and [37–39].

The theorem for which we seek an analogue and attributed to Peano by Hartman can be stated in the context of (1) as follows:

Theorem 1 (Peano). *Assume that with respect of* (1)*, conditions* (i)–(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) := y(x, x_0, c_1, \dots, c_n)$ *denote the solution of* (1) *satisfying the initial conditions* $y^{(i-1)}(x_0) = c_i$*,* $i = 1, \dots, n$*. Then,*

(a) *For* $j = 1, \dots, n$*,* $\alpha_j := \partial y/\partial c_j$ *exists on* (a, b)*, is the solution of the variational equation* (3) *along* $y(x)$*, and satisfies the initial conditions,*

$$\alpha_j^{(i-1)}(x_0) = \delta_{ij}, \quad i = 1, \dots, n.$$

(b) $\partial y/\partial x_0$ *exists on* (a,b)*, and* $\beta := \partial y/\partial x_0$ *is the solution of the variational equation* (3) *along* $y(x)$ *satisfying the initial conditions,*

$$\beta^{(i-1)}(x_0) = -y^{(i)}(x_0), \quad i = 1,\dots,n.$$

(c) $\partial y/\partial x_0(x) = -\sum_{j=1}^{n} y^{(j)}(x_0)\partial y/\partial c_j(x)$.

In addition, our analogue of Theorem 1 depends on uniqueness of solutions of (1), (2), a condition we list as an assumption:

(iv) Given $a < x_1 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < x_2 < b$, if $y^{(i-1)}(x_1) = z^{(i-1)}(x_1)$, $i = 1,\dots,n-1$, and $y(x_2) - \sum_{i=1}^{m}\gamma_i\int_{\xi_i}^{\eta_i} y(x)\,\mathrm{d}x = z(x_2) - \sum_{i=1}^{m}\gamma_i\int_{\xi_i}^{\eta_i} z(x)\,\mathrm{d}x$, where $y(x)$ and $z(x)$ are solutions of (1), then $y(x) \equiv z(x)$.

We will also make extensive use of a similar uniqueness condition on (3) along solutions $y(x)$ of (1).

(v) Given $a < x_1 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < x_2 < b$ and a solution $y(x)$ of (1), if $u^{(i-1)}(x_1) = 0$, $i = 1,\dots,n-1$, and $u(x_2) - \sum_{i=1}^{m}\gamma_i\int_{\xi_i}^{\eta_i} u(x)\,\mathrm{d}x = 0$, where $u(x)$ is a solution of (3) along $y(x)$, then $u(x) \equiv 0$.

Remark 1. We observe that, if (v) is assumed, then for $\alpha_n(x)$ in Theorem 1,

$$\alpha_n(x_2) - \sum_{i=1}^{m}\gamma_i\int_{\xi_i}^{\eta_i}\alpha_n(x)\,\mathrm{d}x \neq 0. \tag{4}$$

2 An analogue of Peano's theorem for (1), (2)

In this section, we state and prove our analogue of Theorem 1 for boundary value problem (1), (2). Continuous dependence of solutions on boundary conditions plays a fundamental role for such a differentiation result. Proof of continuous dependence usually makes application of the Brouwer theorem on invariance of domain. The spirit of such arguments can be found in [36] or [38]; we state the continuity result here, but we omit the details.

Theorem 2. *Assume* (i)–(iv) *are satisfied with respect to* (1)*. Let* $u(x)$ *be a solution of* (1) *on* (a,b)*, and let* $a < c < x_1 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < x_2 < d < b$ *be given. Then, there exists a* $\delta > 0$ *such that, for* $|x_j - t_j| < \delta$*,* $j = 1,2$*,* $|\xi_i - \rho_i| < \delta$ *and* $|\eta_i - \sigma_i| < \delta$*,* $i = 1,\dots,m$*,* $|u^{(j-1)}(x_1) - y_j| < \delta$*,* $j = 1,\dots,n-1$*,* $|u(x_2) - \sum_{i=1}^{m}\gamma_i\int_{\xi_i}^{\eta_i} u(x)\,\mathrm{d}x - y_n| < \delta$*, and* $|\gamma_k - \zeta_k| < \delta$*,* $k = 1,\dots,m$*, there exists a unique solution* $u_\delta(x)$ *of* (1) *such that* $u_\delta^{(j-1)}(t_1) = y_j$*,* $j = 1,\dots,n-1$*,* $u_\delta(t_2) - \sum_{i=1}^{m}\zeta_i\int_{\rho_i}^{\sigma_i} u_\delta(x)\,\mathrm{d}x = y_n$*, and* $\{u_\delta^{(j)}(x)\}$ *converges uniformly to* $u^{(j)}(x)$*, as* $\delta \to 0$*, on* $[c,d]$*, for* $j = 0,1,\dots,n-1$.

We now present the result of this paper.

Theorem 3. *Assume conditions* (i)–(v) *are satisfied. Let* $u(x)$ *be a solution of* (1) *on* (a,b)*. Let* $a < x_1 < \xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_m < \eta_m < x_2 < b$ *be given, so that* $u(x) = u(x, x_1, x_2, \xi_1, \eta_1, \ldots, \xi_m, \eta_m, u_1, \ldots, u_n, \gamma_1, \ldots, \gamma_m)$*, where* $u^{(j-1)}(x_1) = u_j$*,* $j = 1, \ldots, n-1$*, and* $u(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} u(x)\,\mathrm{d}x = u_n$*. Then,*

(a) *For* $j = 1, \ldots, n-1$*,* $r_j := \partial u/\partial u_j$ *exists on* (a,b)*, is the solution of the variational equation* (3) *along* $u(x)$*, and satisfies the boundary conditions,*

$$r_j^{(i-1)}(x_1) = \delta_{ij}, \quad i = 1, \ldots, n-1, \qquad r_j(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_1}^{\eta_i} r_j(x)\,\mathrm{d}x = 0.$$

(b) $r_n := \partial u/\partial u_n$ *exists on* (a,b)*, is the solution of* (3) *along* $u(x)$*, and satisfies the boundary conditions,*

$$r_n^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1, \qquad r_n(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_1}^{\eta_i} r_n(x)\,\mathrm{d}x = 1.$$

(c) $z_1 := \partial u/\partial x_1$ *and* $z_2 := \partial u/\partial x_2$ *exist on* (a,b), *are solutions of* (3) *along* $u(x)$*, and satisfy the respective boundary conditions,*

$$z_1^{(i-1)}(x_1) = -u^{(i)}(x_1), \quad i = 1, \ldots, n-1, \qquad z_1(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} z_1(x)\,\mathrm{d}x = 0,$$

$$z_2^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1, \qquad z_2(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} z_2(x)\,\mathrm{d}x = -u'(x_2).$$

(d) *For each* $j = 1, \ldots, m$*,* $w_j := \partial u/\partial \xi_j$ *exists on* (a,b)*, is the solution of* (3) *along* $u(x)$*, and satisfies the boundary conditions,*

$$w_j^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1, \qquad w_j(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} w_j(x)\,\mathrm{d}x = -\gamma_j u(\xi_j).$$

(e) *For each* $j = 1, \ldots, m$*,* $q_j := \partial u/\partial \eta_j$ *exists on* (a,b)*, is the solution of* (3) *along* $u(x)$*, and satisfies the boundary conditions,*

$$q_j^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1, \qquad q_j(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} q_j(x)\,\mathrm{d}x = \gamma_j u(\eta_j).$$

(f) *For each* $j = 1, \ldots, m$*,* $p_j := \partial u/\partial \gamma_j$ *exists on* (a,b)*, is the solution of* (3) *along* $u(x)$*, and satisfies the boundary conditions,*

$$p_j^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1, \qquad p_j(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} p_j(x)\,\mathrm{d}x = \int_{\xi_j}^{\eta_j} u(x)\,\mathrm{d}x.$$

(g) *The partial derivatives satisfy,*

$$\frac{\partial u}{\partial x_1}(x) = -\sum_{j=1}^{n-1} u^{(j)}(x_1)\frac{\partial u}{\partial u_j}(x), \qquad \frac{\partial u}{\partial x_2}(x) = -u'(x_2)\frac{\partial u}{\partial u_n}(x),$$
$$\frac{\partial u}{\partial \xi_j}(x) = -\gamma_j u(\xi_j)\frac{\partial u}{\partial u_n}(x), \qquad \frac{\partial u}{\partial \eta_j}(x) = \gamma_j u(\eta_j)\frac{\partial u}{\partial u_n}(x),$$
$$\frac{\partial u}{\partial \gamma_j}(x) = \int_{\xi_j}^{\eta_j} u(x)\,\mathrm{d}x \frac{\partial u}{\partial u_n}(x).$$

Proof. With $u(x) = u(x, x_1, x_2, \xi_1, \eta_1, \dots, \xi_m, \eta_m, u_1, \dots, u_n, \gamma_1, \dots, \gamma_m)$, as given in the statement of the theorem, many of the results will be established by considering $u(x)$ as a solution of an initial value problem for (1). In particular, from the boundary value notation, $u^{(i-1)}(x_1) = u_i$, $i = 1, \dots, n-1$, and we will let

$$\beta_n = u^{(n-1)}(x_1). \tag{5}$$

Then, using the notation of Theorem 1 for solutions of initial value problems for (1) and viewing $u(x)$ as a solution of an initial value problem, we will frequently interchange notation by using,

$$u(x) = u(x, x_1, x_2, \xi_1, \eta_1, \dots, \xi_m, \eta_m, u_1, \dots, u_n, \gamma_1, \dots, \gamma_m) \\ = y(x, x_1, u_1, \dots, u_{n-1}, \beta_n).$$

For part (a), we fix $j = 1, \dots, n-1$, and for notational shorthand purposes, we denote $u(x)$ by $u(x, \cdot, u_j)$. Let $\delta > 0$ be as in Theorem 2. Let $0 < |h| < \delta$ be given and define

$$r_{jh}(x) = \frac{1}{h}\big[u(x, \cdot, u_j + h) - u(x, \cdot, u_j)\big].$$

Then, for every $h \neq 0$,

$$r_{jh}^{(j-1)}(x_1) = \frac{1}{h}[u_j + h - u_j] = 1,$$
$$r_{jh}^{(i-1)}(x_1) = \frac{1}{h}[u_i - u_i] = 0, \quad i \in \{1, \dots, n-1\} \setminus \{j\},$$

and

$$r_{jh}(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} r_{jh}(x)\,\mathrm{d}x \\ = \frac{1}{h}\big[u(x_2, \cdot, u_j + h) - u(x_2, \cdot, u_j)\big] \\ - \frac{1}{h}\sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} \big[u(x, \cdot, u_j + h) - u(x, \cdot, u_j)\big]\,\mathrm{d}x \\ = \frac{1}{h}[u_n - u_n] = 0.$$

With β_n as defined in (5), let

$$\epsilon_n = \epsilon_n(h) = u^{(n-1)}(x_1, \cdot, u_1 + h) - \beta_n.$$

By Theorem 2, $\epsilon_n = \epsilon_n(h) \to 0$, as $h \to 0$. Using the notation of Theorem 1 for solutions of initial value problems for (1) and viewing the solutions u as solutions of initial value problems, we have

$$\begin{aligned} r_{jh}(x) = \frac{1}{h}\big[&y(x, x_1, u_1, u_2, \dots, u_j + h, \dots, u_{n-1}, \beta_n + \epsilon_n) \\ &- y(x, x_1, u_1, u_2, \dots, u_j, \dots, u_{n-1}, \beta_n)\big]. \end{aligned}$$

Then, by utilizing a telescoping sum, we have

$$\begin{aligned} r_{jh}(x) = \frac{1}{h}\big[&\big\{y(x, x_1, u_1, u_2, \dots, u_j + h, \dots, u_{n-1}, \beta_n + \epsilon_n) \\ &- y(x, x_1, u_1, u_2, \dots, u_j, \dots, u_{n-1}, \beta_n + \epsilon_n)\big\} \\ &+ \big\{y(x, x_1, u_1, u_2, \dots, u_j, \dots, u_{n-1}, \beta_n + \epsilon_n) \\ &- y(x, x_1, u_1, u_2, \dots, u_j, \dots, u_{n-1}, \beta_n)\big\}\big]. \end{aligned}$$

By Theorem 1 and the Mean Value theorem, we obtain

$$\begin{aligned} r_{jh}(x) = {}&\frac{1}{h}\alpha_j\big(x, y(x, x_1, u_1, \dots, u_j + \bar{h}, \dots, u_{n-1}, \beta_n + \epsilon_n)\big)(u_j + h - u_j) \\ &+ \frac{1}{h}\alpha_n\big(x, y(x, x_1, u_1, \dots, u_{n-1}, \beta_n + \bar{\epsilon}_n)\big)(\beta_n + \epsilon_n - \beta_n), \end{aligned}$$

where $\alpha_j(x, y(\cdot))$ is the solution of the variational equation (3) along $y(\cdot)$ and satisfies

$$\alpha_j^{(i-1)}(x_1) = \delta_{ji}, \quad i = 1, \dots, n,$$

and $\alpha_n(x, y(\cdot))$ is the solution of the variational equation (3) along $y(\cdot)$ and satisfies

$$\alpha_n^{(i-1)}(x_1) = \delta_{ni}, \quad i = 1, \dots, n.$$

Furthermore, $u_j + \bar{h}$ is between u_j and $u_j + h$, and $\beta_n + \bar{\epsilon}_n$ is between β_n and $\beta_n + \epsilon_n$. Now simplifying,

$$\begin{aligned} r_{jh}(x) = {}&\alpha_j\big(x, y(x, x_1, u_1, \dots, u_j + \bar{h}, \dots, u_{n-1}, \beta_n + \epsilon_n)\big) \\ &+ \frac{\epsilon_n}{h}\alpha_n\big(x, y(x, x_1, u_1, \dots, u_{n-1}, \beta_n + \bar{\epsilon}_n)\big). \end{aligned}$$

Thus, to show $\lim_{h\to 0} r_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon_n/h$ exists.

Now, by Remark 1,

$$\alpha_n\big(x_2, y(\cdot)\big) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n\big(x, y(\cdot)\big)\,\mathrm{d}x \neq 0.$$

However, we derived above that $r_{jh}(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} r_{jh}(x)\,\mathrm{d}x = 0$, from which we obtain

$$\frac{\epsilon_n}{h} = \frac{-\alpha_j(x_2, y(x, x_1, \cdot, u_j + \bar{h}, \cdot, \beta_n + \epsilon_n))}{\alpha_n(x_2, y(x, x_1, u_1, \cdot, \beta_n + \bar{\epsilon}_n)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, y(x, x_1, u_1, \cdot, \beta_n + \bar{\epsilon}_n))\,\mathrm{d}x}$$
$$+ \frac{\sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_j(x, y(x, x_1, \cdot, u_j + \bar{h}, \cdot, \beta_n + \epsilon_n))\,\mathrm{d}x}{\alpha_n(x_2, y(x, x_1, u_1, \cdot, \beta_n + \bar{\epsilon}_n)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, y(x, x_1, u_1, \cdot, \beta_n + \bar{\epsilon}_n))\,\mathrm{d}x}.$$

As a consequence of continuous dependence, we can let $h \to 0$, so that

$$\lim_{h\to 0} \frac{\epsilon_n}{h} = \frac{-\alpha_j(x_2, y(x, x_1, u_1, \cdot, \beta_n)) + \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_j(x, y(x, x_1, u_1, \cdot, \beta_n))\,\mathrm{d}x}{\alpha_n(x_2, y(x, x_1, u_1, \cdot, \beta_n)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, y(x, x_1, u_1, \cdot, \beta_n))\,\mathrm{d}x}$$
$$= \frac{-\alpha_j(x_2, u(x)) + \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_j(x, u(x))\,\mathrm{d}x}{\alpha_n(x_2, u(x)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, u(x))\,\mathrm{d}x} := A.$$

Let $r_j(x) = \lim_{h\to 0} r_{jh}(x)$, and note by construction of $r_{jh}(x)$,

$$r_j(x) = \frac{\partial u}{\partial u_j}(x, x_1, x_2, \xi_1, \eta_1, \ldots, \xi_m, \eta_m, u_1, \ldots, u_n, \gamma_1, \ldots, \gamma_m).$$

Furthermore,

$$r_j(x) = \lim_{h\to 0} r_{jh}(x) = \alpha_j\bigl(x, y(x, x_1, u_1, \cdot, \beta_n)\bigr) + A\alpha_2\bigl(x, y(x, x_1, u_1, \cdot, \beta_n)\bigr)$$
$$= \alpha_1\bigl(x, u(x)\bigr) + A\alpha_2\bigl(x, u(x)\bigr),$$

which is a solution of the variational equation (3) along $u(x)$. In addition because of the boundary conditions satisfied by $r_{jh}(x)$, we also have,

$$r_j^{(i-1)}(x_1) = \delta_{ji}, \quad i = 1, \ldots, n-1, \quad \text{and} \quad r_j(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} r_j(x)\,\mathrm{d}x = 0.$$

This completes the argument for $\partial u/\partial u_j$.

For part (b), there are similarities with the previous argument, yet there are significant enough differences for us to include the details concerning the characterization of $\partial u/\partial u_n$. For this consideration, we denote $u(x)$ by $u(x, \cdot, u_n)$. Again, let $\delta > 0$ be as in Theorem 2. Let $0 < |h| < \delta$ be given and define

$$r_{nh}(x) = \frac{1}{h}\bigl[u(x, \cdot, u_n + h) - u(x, \cdot, u_n)\bigr].$$

This time, for $h \neq 0$,

$$r_{nh}^{(i-1)}(x_1) = \frac{1}{h}[u_i - u_i] = 0, \quad i = 1, \ldots, n-1,$$

and

$$
\begin{aligned}
& r_{nh}(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} r_{nh}(x)\,\mathrm{d}x \\
&\quad = \frac{1}{h}\big[u(x_2,\cdot,u_n+h) - u(x_2,\cdot,u_n)\big] \\
&\qquad - \frac{1}{h}\sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} \big[u(x,\cdot,u_n+h) - u(x,\cdot,u_n)\big]\,\mathrm{d}x \\
&\quad = \frac{1}{h}[u_n + h - u_n] = 1.
\end{aligned}
$$

Again with β_n defined in (5), let

$$\epsilon_n = \epsilon_n(h) = u^{(n-1)}(x_1,\cdot,u_n+h) - \beta_n.$$

As before, $\epsilon_n = \epsilon_n(h) \to 0$, as $h \to 0$. Employing the notation of Theorem 1 for solutions of initial value problems for (1) and viewing the solutions u as solutions of initial value problems, we have

$$r_{nh}(x) = \frac{1}{h}\big[y(x,x_1,u_1,\dots,u_{n-1},\,\beta_n+\epsilon_n) - y(x,x_1,u_1,\dots,u_{n-1},\beta_n)\big].$$

By Theorem 1 and the Mean Value theorem, we obtain

$$
\begin{aligned}
r_{nh}(x) &= \frac{1}{h}\alpha_n\big(x, y(x,x_1,u_1,\dots,u_{n-1},\beta_n+\bar{\epsilon}_n)\big)(\beta_n+\epsilon_n-\beta_n) \\
&= \frac{\epsilon_n}{h}\alpha_n\big(x, y(x,x_1,u_1,\dots,u_{n-1},\beta_n+\bar{\epsilon}_n)\big),
\end{aligned}
$$

where $\beta_n+\bar{\epsilon}_n$ is between β_n and $\beta_n+\epsilon_n$, and $\alpha_n(x,y(\cdot))$ is the solution of the variational equation (3) along $y(\cdot)$ and satisfies,

$$\alpha_n^{(i-1)}(x_1) = \delta_{ni}, \quad i = 1,\dots,n.$$

Thus, to show $\lim_{h\to 0} r_{nh}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon_n/h$ exists.

By Remark 1,

$$\alpha_n\big(x_2,y(\cdot)\big) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n\big(x,y(\cdot)\big)\,\mathrm{d}x \neq 0,$$

and we also have above that $r_{nh}(x_2) - \sum_{i=1}^{m}\gamma_i \int_{\xi_i}^{\eta_i} r_{nh}(x)\,\mathrm{d}x = 1$, from which we obtain

$$\frac{\epsilon_n}{h} = \frac{1}{\alpha_n(x_2,y(x,x_1,u_1,\cdot,\beta_n+\bar{\epsilon}_n)) - \sum_{i=1}^{m}\gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x,y(x,x_1,u_1,\cdot,\beta_n+\bar{\epsilon}_n))\,\mathrm{d}x}.$$

By continuous dependence, we can let $h \to 0$, so that

$$\lim_{h\to 0}\frac{\epsilon_n}{h} = \frac{1}{\alpha_n(x_2, y(x, x_1, u_1, \cdot, \beta_n)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, y(x, x_1, u_1, \cdot, \beta_2))\,\mathrm{d}x}$$
$$= \frac{1}{\alpha_n(x_2, u(x)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, u(x))\,\mathrm{d}x} = B.$$

Let $r_n(x) = \lim_{h\to 0} r_{nh}(x)$, and then by construction of $r_{nh}(x)$,

$$r_n(x) = \frac{\partial u}{\partial u_n}(x, x_1, x_2, \xi_1, \eta_1, \ldots, \xi_m, \eta_m, u_1, \ldots, u_n, \gamma_1, \ldots, \gamma_m).$$

Moreover,

$$r_n(x) = \lim_{h\to 0} r_{nh}(x) = B\alpha_n\bigl(x, y(x, x_1, u_1, \cdot, u_{n-1}, \beta_n)\bigr) = B\alpha_n\bigl(x, u(x)\bigr),$$

which is a solution of the variational equation (3) along $u(x)$. Because of the boundary conditions satisfied by $r_{nh}(x)$, we also have,

$$r_n^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1, \quad \text{and} \quad r_n(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} r_n(x)\,\mathrm{d}x = 1.$$

And this completes the argument for $\partial u/\partial u_n$.

For part (c) of the theorem, we will produce the details for $\partial u/\partial x_2$, with the arguments for $\partial u/\partial x_1$ being somewhat along the same lines. For this case, we denote $u(x)$ by $u(x, x_2, \cdot)$. With $\delta > 0$ as in Theorem 2, let $0 < |h| < \delta$ be given, and define

$$z_{2h}(x) = \frac{1}{h}\bigl[u(x, x_2 + h, \cdot) - u(x, x_2, \cdot)\bigr].$$

First, we consider boundary conditions. We have

$$z_{2h}^{(i-1)}(x_1) = \frac{1}{h}[u_i - u_i] = 0, \quad i = 1, \ldots, n-1.$$

Next, by employing the Mean Value theorem,

$$z_{2h}(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} z_{2h}(x)\,\mathrm{d}x$$
$$= \frac{1}{h}\Biggl[u(x_2, x_2 + h, \cdot) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} u(x, x_2 + h, \cdot)\,\mathrm{d}x\Biggr]$$
$$- \frac{1}{h}\Biggl[u(x_2, x_2, \cdot) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} u(x, x_2, \cdot)\,\mathrm{d}x\Biggr]$$

$$= \frac{1}{h}\left[u(x_2+h, x_2+h, \cdot) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} u(x, x_2+h, \cdot)\,\mathrm{d}x\right]$$
$$- \frac{1}{h}\left[u(x_2, x_2, \cdot) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} u(x, x_2, \cdot)\,\mathrm{d}x\right]$$
$$- \frac{1}{h}\bigl[u(x_2+h, x_2+h, \cdot) - u(x_2, x_2+h, \cdot)\bigr]$$
$$= \frac{1}{h}[u_n - u_n] - \frac{1}{h}u'(\nu_h, x_2+h, \cdot)h = -u'(\nu_h, x_2+h, \cdot),$$

where ν_h is between x_2 and x_2+h. In passing to the limit, we have

$$\lim_{h\to 0}\left\{z_{2h}(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} z_{2h}(x)\,\mathrm{d}x\right\} = -u'(x_2, x_2, \cdot) = -u'(x_2).$$

Next, we deal with the existence of $\lim_{h\to 0} z_{2h}(x)$. With β_n as defined in (5), this time we let

$$\epsilon_n = \epsilon_n(h) = u^{(n-1)}(x_1, x_2+h, \cdot) - \beta_n,$$

and by Theorem 2, $\epsilon_n = \epsilon_n(h) \to 0$, as $h \to 0$. As in parts (a) and (b), we use the notation of Theorem 1 for solutions of initial value problems for (1) and viewing the solutions u as solutions of initial value problems, we have

$$\begin{aligned} z_{2h}(x) &= \frac{1}{h}\bigl[u(x, x_2+h, \cdot) - u(x, x_2, \cdot)\bigr] \\ &= \frac{1}{h}\bigl[y(x, x_1, u_1, \dots, u_{n-1}, \beta_n+\epsilon_n) - y(x, x_1, u_1, \dots, u_{n-1}, \beta_n)\bigr] \\ &= \frac{1}{h}\alpha_n\bigl(x, y(x, x_1, u_1, \dots, u_{n-1}, \beta_n+\bar\epsilon_n)\bigr)\cdot\epsilon_n \\ &= \frac{\epsilon_n}{h}\alpha_n\bigl(x, y(x, x_1, u_1, \dots, u_{n-1}, \beta_n+\bar\epsilon_n)\bigr), \end{aligned}$$

where $\alpha_n(x, y(\cdot))$ is the solution of (3) along $y(\cdot)$ and satisfies

$$\alpha_n^{(i-1)}(x_1) = \delta_{ni}, \quad i = 1, \dots, n,$$

and $\beta_n + \bar\epsilon_n$ lies between β_n and $\beta_n + \epsilon_n$. As before, to show $\lim_{h\to 0} z_{2h}(x)$ exists, it suffices to show $\lim_{h\to 0} \epsilon_n/h$ exists.

Now, recalling from above that $z_{2h}(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} z_{2h}(x)\,\mathrm{d}x = -u'(\nu_h, x_2 + h, \cdot)$, it follows from Remark 1 that,

$$\frac{\epsilon_n}{h} = \frac{-u'(\nu_h, x_2+h, \cdot)}{\alpha_n(x_2, y(x, x_1, u_1, \cdot, \beta_n+\bar\epsilon_n)) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, y(x, x_1, u_1, \cdot, \beta_n+\bar\epsilon_n))\,\mathrm{d}x}.$$

And passing to the limit due to continuous dependence, we have,

$$\lim_{h\to 0}\frac{\epsilon_n}{h}=\frac{-u'(x_2)}{\alpha_n(x_2,u(x))-\sum_{i=1}^m\gamma_i\int_{\xi_i}^{\eta_i}\alpha_n(x,u(x))\,\mathrm{d}x}:=C.$$

From above,

$$z_{2h}(x)=\frac{\epsilon_n}{h}\alpha_n\bigl(x,y(x,x_1,u_1,\dots,\beta_n+\bar{\epsilon}_n)\bigr),$$

from which we can evaluate the limit as $h\to 0$, and if we let $z_2(x)=\lim_{h\to 0}z_{2h}(x)$, we have $z_2(x)=\partial u/\partial x_2$. That is, we obtain

$$\begin{aligned}z_2(x)&=\frac{\partial u}{\partial x_2}(x,x_1,x_2,\xi_1,\eta_1,\dots,\xi_m,\eta_m,u_1,\dots,u_n,\gamma_1,\dots,\gamma_m)\\&=\lim_{h\to 0}z_{2h}(x)=C\alpha_n\bigl(x,u(x)\bigr),\end{aligned}$$

which is a solution of (3) along $u(x)$. In addition, from above computations, $z_2(x)$ satisfies the boundary conditions,

$$z_2^{(i-1)}(x_1)=\lim_{h\to 0}z_{2h}^{(i-1)}(x_1)=0,\quad i=1,\dots,n-1,$$

$$z_2(x_2)-\sum_{i=1}^m\gamma_i\int_{\xi_i}^{\eta_i}z_2(x)\,\mathrm{d}x=\lim_{h\to 0}\Biggl[z_{2h}(x_2)-\sum_{i=1}^m\gamma_i\int_{\xi_i}^{\eta_i}z_{2h}(x)\,\mathrm{d}x\Biggr]=-u'(x_2).$$

This completes the proof of part (c).

For part (d), fix $j\in\{1,\dots,m\}$, and define

$$J:=\{1,\dots,m\}\setminus\{j\}.$$

In dealing with characterization of $\partial u/\partial\xi_j$, we denote $u(x)$ by $u(x,\cdot,\xi_j)$. Let $\delta>0$ as in Theorem 2 and let $0<|h|<\delta$ be given. Define

$$w_{jh}(x)=\frac{1}{h}\bigl[u(x,\cdot,\xi_j+h)-u(x,\cdot,\xi_j)\bigr].$$

We first look at boundary conditions satisfied by $w_{jh}(x)$. To begin with,

$$\begin{aligned}w_{jh}^{(i-1)}(x_1)&=\frac{1}{h}\bigl[u^{(i-1)}(x_1,\cdot,\xi_j+h)-u^{(i-1)}(x_1,\cdot,\xi_j)\bigr]\\&=\frac{1}{h}[u_i-u_i]=0,\quad i=1,\dots,n-1.\end{aligned}$$

Next, by employing the Mean Value theorem for integrals,

$$
\begin{aligned}
&w_{jh}(x_2) - \sum_{i=1}^{m} \gamma_i \int\limits_{\xi_i}^{\eta_i} w_{jh}(x)\,\mathrm{d}x \\
&\quad = \frac{1}{h}\Bigg[-\gamma_j \int\limits_{\xi_j}^{\xi_j+h} u(x,\cdot,\xi_j+h)\,\mathrm{d}x + \Bigg\{u(x_2,\cdot,\xi_j+h) - \sum_{i\in J} \gamma_i \int\limits_{\xi_i}^{\eta_i} u(x,\cdot,\xi_j+h)\,\mathrm{d}x \\
&\qquad - \gamma_j \int\limits_{\xi_j+h}^{\eta_j} u(x,\cdot,\xi_j+h)\,\mathrm{d}x\Bigg\} - \Bigg\{u(x_2,\cdot,\xi_j) - \sum_{i=1}^{m} \gamma_i \int\limits_{\xi_i}^{\eta_i} u(x,\cdot,\xi_j)\,\mathrm{d}x\Bigg\}\Bigg] \\
&\quad = \frac{1}{h}\Bigg[-\gamma_j \int\limits_{\xi_j}^{\xi_j+h} u(x,\cdot,\xi_j+h)\,\mathrm{d}x + u_n - u_n\Bigg] \\
&\quad = -\frac{1}{h}\gamma_j u(c_h,\cdot,\xi_j+h)\cdot h = -\gamma_j u(c_h,\cdot,\xi_j+h),
\end{aligned}
$$

for some c_h inclusively between ξ_j and $\xi_j + h$. By Theorem 2, we can compute the limit,

$$
\lim_{h\to 0}\Bigg[w_{jh}(x_2) - \sum_{i=1}^{m} \gamma_i \int\limits_{\xi_i}^{\eta_i} w_{jh}(x)\,\mathrm{d}x\Bigg] = -\gamma_j u(\xi_j).
$$

Now, we deal with the existence of $\lim_{h\to 0} w_{jh}(x)$. Let β_n be as in (5), and set

$$
\epsilon_n = u^{(n-1)}(x_1,\cdot,\xi_j+h) - \beta_n.
$$

By Theorem 2, $\epsilon_n \to 0$, as $h \to 0$, and upon employing initial value solutions notation,

$$
\begin{aligned}
w_{jh}(x) &= \frac{1}{h}\big[y(x,x_1,u_1,\dots,u_{n-1},\beta_n+\epsilon_n) - y(x,x_1,u_1,\dots,u_{n-1},\beta_n)\big] \\
&= \frac{\epsilon_n}{h}\alpha_n\big(x, y(x,x_1,u_1,\dots,u_{n-1},\beta_n+\bar{\epsilon}_n)\big),
\end{aligned}
$$

where $\beta_n + \bar{\epsilon}_n$ lies between β_n and $\beta_n + \epsilon_n$, and $\alpha_n(x, y(\cdot))$ is as in the cases above. From Remark 1, we can solve for

$$
\frac{\epsilon_n}{h} = \frac{w_{jh}(x_2) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} w_{jh}(x)\,\mathrm{d}x}{\alpha_n(x_2, y(\cdot)) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, y(\cdot))\,\mathrm{d}x},
$$

from which, using the above limit, we can calculate,

$$
\lim_{h\to 0}\frac{\epsilon_n}{h} = \frac{-\gamma_j u(\xi_j)}{\alpha_n(x_2, u(x)) - \sum_{i=1}^{m} \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, u(x))\,\mathrm{d}x} := D.
$$

As as consequence, $\lim_{h\to 0} w_{jh}(x)$ exists, and we define $w_j(x) := \lim_{h\to 0} w_{jh}(x)$. In particular,

$$\begin{aligned} w_j(x) &= \frac{\partial u}{\partial \xi_j}(x, x_1, x_2, \xi_1, \eta_1, \ldots, \xi_m, \eta_m, u_1, \ldots, u_n, \gamma_1, \ldots, \gamma_m) \\ &= \lim_{h\to 0} w_{jh}(x) = D\alpha_n\bigl(x, u(x)\bigr), \end{aligned}$$

which is a solution of (3) along $u(x)$. In addition, $w_j(x)$ satisfies the boundary conditions,

$$w_j^{(i-1)}(x_1) = \lim_{h\to 0} w_{jh}^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1,$$

$$w_j(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} w_j(x)\,\mathrm{d}x = \lim_{h\to 0}\left[w_{jh}(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} w_{jh}(x)\,\mathrm{d}x\right] = -\gamma_j u(\xi_j).$$

The proof of part (d) is complete.

For part (e), the arguments are completely analogous to those just given for part (d).

For part (f), fix $j \in \{1, \ldots, m\}$, and let J be as defined in the proof of part (d). In characterizing $\partial u/\partial\gamma_j$, we will designate $u(x)$ by $u(x, \cdot, \gamma_j)$. With $\delta > 0$ and $0 < |h| < \delta$ given as usual, define

$$p_{jh}(x) = \frac{1}{h}\bigl[u(x, \cdot, \gamma_j + h) - u(x, \cdot, \gamma_j)\bigr].$$

As in the previous cases, we first consider boundary conditions satisfied by $p_{jh}(x)$. To begin with,

$$\begin{aligned} p_{jh}^{(i-1)}(x_1) &= \frac{1}{h}\bigl[u^{(i-1)}(x_1, \cdot, \gamma_j + h) - u^{(i-1)}(x_1, \cdot, \gamma_j)\bigr] \\ &= \frac{1}{h}\bigl[u_i - u_i\bigr] = 0, \quad i = 1, \ldots, n-1, \end{aligned}$$

and

$$\begin{aligned} &p_{jh}(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} p_{jh}(x)\,\mathrm{d}x \\ &\quad = \frac{1}{h}\left[h\int_{\xi_j}^{\eta_j} u(x, \cdot, \gamma_j + h)\,\mathrm{d}x + u(x_2, \cdot, \gamma_j + h) - (\gamma_j + h)\int_{\xi_j}^{\eta_j} u(x, \cdot, \gamma_j + h)\,\mathrm{d}x \right. \\ &\qquad \left. - \sum_{i\in J} \gamma_i \int_{\xi_i}^{\eta_i} u(x, \cdot, \gamma_j + h)\,\mathrm{d}x - \left\{u(x_2, \cdot, \gamma_j) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} u(x, \cdot, \gamma_j)\,\mathrm{d}x\right\}\right] \\ &\quad = \frac{1}{h}\left[h\int_{\xi_j}^{\eta_j} u(x, \cdot, \gamma_j + h)\,\mathrm{d}x + u_n - u_n\right] = \int_{\xi_j}^{\eta_j} u(x, \cdot, \gamma_j + h)\,\mathrm{d}x, \end{aligned}$$

from which we can take the limit,

$$\lim_{h\to 0}\left[p_{jh}(x_2)-\sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} p_{jh}(x)\,\mathrm{d}x\right] = \int_{\xi_j}^{\eta_j} u(x)\,\mathrm{d}x.$$

Finally, in considering the existence of $\lim_{h\to 0} p_{jh}(x)$, let β_n be as in (5), and again, define

$$\epsilon_n = u^{(n-1)}(x_1, \cdot, \gamma_j + h) - \beta_n,$$

for which by continuous dependence $\epsilon_n \to 0$, as $h \to 0$. Using initial value solutions notation, we have

$$\begin{aligned} p_{jh}(x) &= \frac{1}{h}\bigl[y(x, x_1, u_1, \ldots, u_{n-1}, \beta_n + \epsilon_n) - y(x, x_1, u_1, \ldots, u_{n-1}, \beta_n)\bigr] \\ &= \frac{\epsilon_n}{h}\alpha_2\bigl(x, y(x, x_1, u_1, \ldots, u_{n-1}, \beta_n + \bar{\epsilon}_n)\bigr), \end{aligned}$$

with $\beta_n + \bar{\epsilon}_n$ between β_n and $\beta_n + \epsilon_n$, and $\alpha_n(x, y(\cdot))$ as usual. In view of Remark 1, we can solve for

$$\frac{\epsilon_n}{h} = \frac{p_{jh}(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} p_{jh}(x)\,\mathrm{d}x}{\alpha_n(x_2, y(\cdot)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, y(\cdot))\,\mathrm{d}x}.$$

Using the last above limit, we can calculate,

$$\lim_{h\to 0}\frac{\epsilon_n}{h} = \frac{\int_{\xi_j}^{\eta_j} u(x)\,\mathrm{d}x}{\alpha_n(x_2, u(x)) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} \alpha_n(x, u(x))\,\mathrm{d}x} := E.$$

It follows that $p_j(x) := \lim_{h\to 0} p_{jh}(x)$ exists, and

$$\begin{aligned} p_j(x) &= \frac{\partial u}{\partial \gamma_j}(x, x_1, x_2, \xi_1, \eta_1, \ldots, \xi_m, \eta_m, u_1, \ldots, u_n, \gamma_1, \ldots, \gamma_m) \\ &= E\alpha_n\bigl(x, u(x)\bigr), \end{aligned}$$

is a solution of (3) along $u(x)$. And $p_j(x)$ satisfies the boundary conditions,

$$p_j^{(i-1)}(x_1) = 0, \quad i = 1, \ldots, n-1, \quad \text{and} \quad p_j(x_2) - \sum_{i=1}^m \gamma_i \int_{\xi_i}^{\eta_i} p_j(x)\,\mathrm{d}x = \int_{\xi_j}^{\eta_j} u(x)\,\mathrm{d}x.$$

This completes the proof of part (f).

Part (g) of the theorem is immediate by verifying that each side of the respective equations are solutions of (3) along $u(x)$ and satisfy the same boundary conditions, and then assumption (v) establishes the equalities.

The proof is complete. □

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