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A note on the tail behavior of randomly weighted and stopped dependent sums*

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Abstract. In this paper, we deal with the tail behavior of the maximum of randomly weighted and stopped sums. We assume that primary random variables (with a certain dependence structure) are identically distributed with heavy-tailed distribution function and random weights are nonnegative. In this note, we specify some conditions for the (weak) asymptotics of the tail of random maximum.

Keywords: heavy tails, extended negative dependence, asymptotic tail probability.

1 Introduction

Let $X, X_1, X_2, \dots$ be a sequence of identically distributed random variables (r.v.s), having a certain dependence structure, with heavy-tailed distribution function (d.f.) F_X. Denote the weighted sums $S_k := \sum_{i=1}^{k} \theta_i X_i$, $k \geqslant 1$, where $\theta_1, \theta_2, \dots$ are some nonnegative r.v.s, and consider the random maximum of these sums

$$M_\tau = \max_{0 \leqslant k \leqslant \tau} S_k,$$

where $S_0 := 0$ and τ is a nonnegative nondegenerate at zero integer-valued r.v. Assume that $\{X, X_1, X_2, \dots\}$, $\{\theta_1, \theta_2, \dots\}$ and τ are mutually independent. We are interested in the asymptotics of tail probability $\mathbf{P}(M_\tau > x)$ as $x \to \infty$. Clearly, since M_τ is driven by three sets of random variables, $\{X_k, k \geqslant 1\}$, $\{\theta_k, k \geqslant 1\}$ and τ, in order to get the asymptotics of tail probability $\mathbf{P}(M_\tau > x)$, some relations between the corresponding d.f.s must be postulated.

Recently, [16] studied the asymptotic tail behavior of random sum $S_\tau = \sum_{k=1}^{\tau} \theta_k X_k$ and random maximum M_τ when $X_1, X_2, \dots$ are independent and identically distributed

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(i.i.d.) r.v.s with consistently varying common d.f. F_X (see definition below). [23] generalized the results of [16] to a certain extent. The main results in both papers state that under assumption $\mathbf{P}(Z_\tau > x) = o(\overline{F_X}(x))$, where

$$Z_\tau := \theta_1 + \cdots + \theta_\tau,$$

and some other conditions on the distributions of r.v.s $\{\theta_k,\, k \geqslant 1\}$, X and τ (see, e.g., Theorem 1 below), probability $\mathbf{P}(M_\tau > x)$ is weakly tail-equivalent to $\mathbf{E}\sum_{k=1}^{\tau} \mathbf{P}(\theta_k X_k > x)$, i.e.

$$0 < \liminf_{x\to\infty} \frac{\mathbf{P}(M_\tau > x)}{\mathbf{E}\sum_{k=1}^{\tau} \mathbf{P}(\theta_k X_k > x)} \leqslant \limsup_{x\to\infty} \frac{\mathbf{P}(M_\tau > x)}{\mathbf{E}\sum_{k=1}^{\tau} \mathbf{P}(\theta_k X_k > x)} < \infty. \tag{1}$$

In the present note, we aim to specify the conditions, under which relation $\mathbf{P}(Z_\tau > x) = o(\overline{F_X}(x))$ holds for the wide class of heavy tailed distribution functions and dependence structures. Together, we extend the result in (1) to a wider dependence class.

In Section 2, we introduce some classes of heavy-tailed d.f.s and dependence structures used in the paper. In Section 3, we present our main results, which are based on propositions in Section 4. Auxiliary lemmas are given in Section 5.

2 Preliminaries

Throughout the paper, for two positive functions $f(x)$ and $g(x)$, we write $f(x) \lesssim g(x)$ if $\limsup_{x\to\infty} f(x)/g(x) \leqslant 1$; $f(x) \sim g(x)$ if $\lim_{x\to\infty} f(x)/g(x) = 1$; $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$; $f(x) \asymp g(x)$ if $0 < \liminf_{x\to\infty} f(x)/g(x) \leqslant \limsup_{x\to\infty} f(x)/g(x) < \infty$.

2.1 Heavy-tailed distribution classes

A distribution of r.v. X, supported on $[0,\infty)$, is said to be heavy-tailed if $\mathbf{E}\mathrm{e}^{\delta X} = \infty$ for all $\delta > 0$ and light-tailed otherwise. We recall the definitions of some classes of heavy-tailed d.f.s. Let $\overline{F}(x) := 1 - F(x)$ for all real x. A d.f. F supported on $[0,\infty)$ belongs to the consistently varying-tailed class ($F \in \mathscr{C}$) if

$$\lim_{y\nearrow 1} \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1,$$

belongs to the dominatedly varying-tailed class ($F \in \mathscr{D}$) if, for any fixed $y \in (0,1)$,

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty,$$

is long-tailed ($F \in \mathscr{L}$) if, for every fixed positive $y > 0$,

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = 1,$$

is subexponential ($F \in \mathscr{S}$) if

$$\lim_{x\to\infty} \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} = 2,$$

where F^{*2} denotes convolution of $F(x)$ with itself, and belongs to the class $\mathscr{S}^*$ (is strongly subexponential) if $m := \int_{[0,\infty)} x \mathrm{d}F(x) < \infty$ and

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\,\mathrm{d}y \sim 2m\overline{F}(x), \quad x \to \infty.$$

If a d.f. F is supported on $\mathbb{R}$, then F belongs to any of these classes if the d.f. $F(x)\mathbf{1}_{\{x\geqslant 0\}}$ belongs to the corresponding class. In the case of finite mean, it holds that

$$\mathscr{C} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{S}^* \subset \mathscr{S} \subset \mathscr{L}$$

(see [11, 12]). For the example of d.f. which is subexponential but does not belong to $\mathscr{S}^*$, see [7]; for d.f. which is dominatedly varying-tailed but not long-tailed (hence, not in $\mathscr{S}$ and $\mathscr{S}^*$), see [6].

Denote

$$\overline{F}_*(y) := \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \quad \overline{F}^*(y) := \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}, \quad y > 1,$$

and define the upper and lower Matuszewska indices of d.f. F, respectively:

$$J_F^+ := -\lim_{y\to\infty} \frac{\log \overline{F}_*(y)}{\log y}, \qquad J_F^- := -\lim_{y\to\infty} \frac{\log \overline{F}^*(y)}{\log y}.$$

Additionally, let

$$L_F := \lim_{y\searrow 1} \overline{F}_*(y).$$

Parameter L_F and the Matuszewska indices are important quantities for the characterization of the classes of heavy-tailed d.f.s. In particular (see, e.g., [2]), the following four statements are equivalent:

$$\text{(i) } F \in \mathscr{D}, \quad \text{(ii) } \overline{F}_*(y) > 0 \quad \text{for some } y > 1, \quad \text{(iii) } L_F > 0, \quad \text{(iv) } J_F^+ < \infty.$$

Also, $F \in \mathscr{C}$ if and only if $L_F = 1$.

2.2 Dependence structures

A sequence of real-valued r.v.s $\xi_1, \xi_2, \dots$ is said to be *upper extended negatively dependent* (UEND) if there exists a positive constant κ such that, for each $n \geqslant 1$ and real $x_1, x_2, \dots, x_n$, the following inequality holds:

$$\mathbf{P}\left(\bigcap_{i=1}^n \{\xi_i > x_i\}\right) \leqslant \kappa \prod_{i=1}^n \mathbf{P}(\xi_i > x_i). \tag{2}$$

If $\mathbf{P}(\xi_i > x_i, \xi_j > x_j) \leqslant \kappa \mathbf{P}(\xi_i > x_i)\mathbf{P}(\xi_j > x_j)$ for all $i \neq j$, then r.v.s $\xi_1, \xi_2, \dots$ are called *pairwise UEND*.

Similarly, a sequence of real-valued r.v.s $\xi_1, \xi_2, \dots$ is said to be *lower extended negatively dependent* (LEND) if there exists a positive constant κ such that, for each $n \geqslant 1$ and real $x_1, x_2, \dots, x_n$, it holds that

$$\mathbf{P}\left(\bigcap_{i=1}^{n}\{\xi_i \leqslant x_i\}\right) \leqslant \kappa \prod_{i=1}^{n}\mathbf{P}(\xi_i \leqslant x_i). \tag{3}$$

If $\mathbf{P}(\xi_i \leqslant x_i, \xi_j \leqslant x_j) \leqslant \kappa \mathbf{P}(\xi_i \leqslant x_i)\mathbf{P}(\xi_j \leqslant x_j)$ for all $i \neq j$ and some $\kappa > 0$, then r.v.s $\xi_1, \xi_2, \dots$ are called *pairwise* LEND.

A sequence $\xi_1, \xi_2, \dots$ is said to be *extended negatively dependent* (END) if it is both UEND and LEND. A sequence $\xi_1, \xi_2, \dots$ is said to be *pairwise END* if it is both pairwise UEND and pairwise LEND. The structure of END was introduced in [14]. Some useful properties of such structure were later analyzed in [4]. Note that these END structures cover certain positive dependence structures.

In the case when (2) or (3) are satisfied with $\kappa = 1$, we get a structure of upper negative dependence (UND) or lower negative dependence (LND), respectively. Analogously, if both (2) and (3) are satisfied, then we obtain the negative dependence (ND) structure. Such structures were introduced in [9] and [3]. For properties of UND and LND r.v.s, see [3] and [18].

3 Main results

The asymptotics of the probability $\mathbf{P}(Z_\tau > x) = \mathbf{P}(\theta_1 + \dots + \theta_\tau > x)$ with i.i.d. heavy-tailed r.v.s θ, θ_i, $i \geqslant 1$, was studied extensively in the literature, see [1, 8, 10, 15] and references therein. In particular, a well-known result (see [10, Thm. A3.20]) states that if $F_\theta \in \mathscr{S}$ and τ is light-tailed, then

$$\mathbf{P}(Z_\tau > x) \sim \mathbf{E}\tau \overline{F}_\theta(x). \tag{4}$$

If $F_\theta \in \mathscr{L} \cap \mathscr{D}$ and $\overline{F_\tau}(x) = o(\overline{F_\theta}(x))$, then relation (4) was obtained in [15]. If $F_\theta \in \mathscr{S}^*$, $F_\tau \in \mathscr{C}$ and $\overline{F_\theta}(x) = O(\overline{F_\tau}(x))$, then [8] proved that

$$P(Z_\tau > x) \sim \mathbf{E}\tau \overline{F_\theta}(x) + \overline{F_\tau}\left(\frac{x}{\mathbf{E}\theta}\right).$$

In case of some dependence structures within r.v.s $\theta_1, \theta_2, \dots$, similar results were obtained in [5, 14, 20, 24]. For the asymptotic results for tail probabilities of (weighted) sums of dependent subexponential r.v.s, see [22, 25].

We now introduce the following assumption.

Assumption A. Let $X, X_1, X_2, \dots$ be a sequence of UEND (with dominating constant $\kappa > 0$) real-valued r.v.s with common d.f. $F_X \in \mathscr{D}$ such that $J^-_{F_X} > 0$ and $F_X(-x) = o(\overline{F_X}(x))$; let $\theta, \theta_1, \theta_2, \dots$ be a sequence of nonnegative r.v.s (not necessarily independent and identically distributed) and let τ be a nondegenerate at zero nonnegative integer-valued r.v. with distribution function F_τ. $\{X, X_1, X_2, \dots\}$, $\{\theta, \theta_1, \theta_2, \dots\}$ and τ are mutually independent.

In addition, assume that there exists $\epsilon \in (0, J^-_{F_X})$ such that

$$\mathbf{E}(X^+)^{1+\epsilon} < \infty \tag{5}$$

and

$$\mathbf{E}\sum_{i=1}^{\tau} \theta_i^{J^-_{F_X}-\epsilon} < \infty, \qquad \mathbf{E}\sum_{i=1}^{\tau} \theta_i^{J^+_{F_X}+\epsilon} < \infty. \tag{6}$$

The following theorem was proved in [23].

Theorem 1. (*See* [23].) *Let Assumption* A *and conditions* (5), (6) *be satisfied. If*

$$\mathbf{P}(Z_\tau > x) = o\bigl(\overline{F_X}(x)\bigr), \tag{7}$$

then

$$L_{F_X}\mathbf{E}\sum_{i=1}^{\tau}\mathbf{P}(\theta_i X_i > x) \lesssim \mathbf{P}(M_\tau > x) \lesssim L^{-1}_{F_X}\mathbf{E}\sum_{i=1}^{\tau}\mathbf{P}(\theta_i X_i > x). \tag{8}$$

Remark. Under condition $\mathbf{E}\tau < \infty$, the first restriction in (6) can be dropped as

$$\begin{aligned}\mathbf{E}\sum_{i=1}^{\tau} \theta_i^{J^-_{F_X}-\epsilon} &= \mathbf{E}\Biggl(\sum_{i=1}^{\tau} \theta_i^{J^-_{F_X}-\epsilon}\mathbf{1}_{\{\theta_i \leqslant 1\}}\Biggr) + \mathbf{E}\Biggl(\sum_{i=1}^{\tau} \theta_i^{J^-_{F_X}-\epsilon}\mathbf{1}_{\{\theta_i > 1\}}\Biggr)\\ &\leqslant \mathbf{E}\tau + \mathbf{E}\sum_{i=1}^{\tau} \theta_i^{J^+_{F_X}+\epsilon}.\end{aligned}$$

Clearly, if the random series $Z_\infty := \theta_1 + \theta_2 + \cdots$ converges almost surely (it is typical in insurance mathematics, where X_i denotes the net loss over period i and θ_i represents the stochastic discount from time i to 0), then condition

$$\mathbf{P}(Z_\infty > x) = o\bigl(\overline{F_X}(x)\bigr) \tag{9}$$

is sufficient for relation (7) to hold. So that, the statement of Theorem 1 is valid if (7) is replaced by (9).

Corollary 1. *If Assumption* A*, conditions* (5), (6) *and* (9) *are satisfied, then relation* (8) *holds.*

Consider now the case $\mathbf{P}(Z_\infty = \infty) > 0$. For example, if $\theta, \theta_1, \theta_2, \dots$ are nonnegative independent r.v.s, then, according to the three series theorem, $\mathbf{P}(Z_\infty = \infty) = 1$ if and only if $\sum_{k=1}^{\infty}\mathbf{E}\min\{\theta_k, 1\} = \infty$. This fact can be extended for arbitrarily dependent nonnegative r.v.s as well, see [17]. If, additionally, r.v.s $\theta, \theta_1, \theta_2, \dots$ are identically distributed, then the last condition is equivalent to $\mathbf{E}\theta > 0$. Identically distributed weights are rather natural when studying the present value of investment portfolio of n risky assets with X_i, denoting the potential loss of ith asset over a period, and θ_i being the stochastic discount factor over the period. Clearly, in such a case, relation (9) does not hold and some other approaches must be used in order to obtain the asymptotics of $\mathbf{P}(Z_\tau > x)$.

Applying results in Section 4, which deal with case of identically distributed r.v.s. $\theta_1, \theta_2, \ldots$, we obtain the following theorems, which constitute the main results of this note.

Theorem 2. *Let r.v.s $\theta, \theta_1, \theta_2, \ldots$ be identically distributed and let Assumption* A *be satisfied. Assume that* (5) *and* $\mathbf{E}\theta^{J^+_{F_X}+\epsilon} < \infty$ *hold.*

(i) *If $F_\theta \in \mathscr{D}$ and either $\overline{F_\theta}(x) \sim c^* \overline{F_\tau}(x)$ for some $c^* > 0$ or $\overline{F_\tau}(x) = o(\overline{F_\theta}(x))$, then relation* (8) *holds*;

(ii) *If $F_\tau \in \mathscr{D}$, $\mathbf{E}\tau < \infty$ and $\overline{F_\theta}(x) = o(\overline{F_\tau}(x))$, $\overline{F_\tau}(x) = o(\overline{F_X}(x))$, then* (8) *holds.*

Proof. First note that condition $\mathbf{E}(X^+)^{1+\epsilon} < \infty$ implies $J^+_{F_X} \geqslant 1$ and, thus, $\mathbf{E}\theta < \infty$. Observe that, by Markov's inequality and Lemma 1,

$$\overline{F_\theta}(x) \leqslant x^{-(J^+_{F_X}+\epsilon)} \mathbf{E}\theta^{J^+_{F_X}+\epsilon} = o\big(\overline{F_X}(x)\big). \tag{10}$$

Hence, statement (i) of the theorem follows combining Proposition 1 below and Theorem 1 and noting that $\overline{F_\tau}(x/\mathbf{E}\theta) = o(\overline{F_X}(x))$ is equivalent to $\overline{F_\tau}(x) = o(\overline{F_X}(x))$ if $F_\tau \in \mathscr{D}$, $F_X \in \mathscr{D}$. Similar arguments apply to part (ii). $\square$

In the case of the strongly subexponential class $\mathscr{S}^*$, combining (10), Proposition 2 and Theorem 1, we obtain the following theorem.

Theorem 3. *Let $\theta, \theta_1, \theta_2, \ldots$ be i.i.d. r.v.s. and let Assumption* A *be satisfied. Assume that* (5) *and* $\mathbf{E}\theta^{J^+_{F_X}+\epsilon} < \infty$ *hold. If $F_\theta \in \mathscr{S}^*$ and there exists $c > \mathbf{E}\theta$ such that $\overline{F_\tau}(x) = o(\overline{F_\theta}(cx))$, then* (8) *holds.*

4 Asymptotics of $\mathrm{P}(Z_\tau > x)$

In this section, we study the asymptotics of $\mathbf{P}(Z_\tau > x)$ when $\theta_1, \theta_2, \ldots$ are identically distributed r.v.s. The next proposition is a modification of Theorem 1 in [21]. In this case, more general dependence structure of r.v.s $\theta_1, \theta_2, \ldots$ is considered.

Proposition 1. *Let $\theta, \theta_1, \theta_2, \ldots$ be nonnegative END r.v.s with common d.f. F_θ and finite positive mean $\mathbf{E}\theta$. Let τ be a nonnegative integer-valued r.v., independent of the sequence $\theta, \theta_1, \theta_2, \ldots$.*

(i) *If $F_\theta \in \mathscr{D}$ and $\overline{F_\theta}(x) \asymp \overline{F_\tau}(x)$, then $F_\tau \in \mathscr{D}$, $\mathbf{E}\tau < \infty$ and*

$$L_{F_\theta} \mathbf{E}\tau \overline{F_\theta}(x) + L_{F_\tau} \overline{F_\tau}\left(\frac{x}{\mathbf{E}\theta}\right) \lesssim \mathbf{P}(Z_\tau > x) \lesssim L^{-1}_{F_\theta} \mathbf{E}\tau \overline{F_\theta}(x) + L^{-1}_{F_\tau} \overline{F_\tau}\left(\frac{x}{\mathbf{E}\theta}\right); \tag{11}$$

(ii) *If $F_\theta \in \mathscr{D}$, $\overline{F_\tau}(x) = o(\overline{F_\theta}(x))$, then $\mathbf{E}\tau < \infty$ and*

$$L_{F_\theta} \mathbf{E}\tau \overline{F_\theta}(x) \lesssim \mathbf{P}(Z_\tau > x) \lesssim L^{-1}_{F_\theta} \mathbf{E}\tau \overline{F_\theta}(x); \tag{12}$$

(iii) *If* $F_\tau \in \mathscr{D}$, $\mathbf{E}\tau < \infty$ *and* $\overline{F_\theta}(x) = o(\overline{F_\tau}(x))$, *then*

$$L_{F_\tau}\overline{F_\tau}\left(\frac{x}{\mathbf{E}\theta}\right) \lesssim \mathbf{P}(Z_\tau > x) \lesssim L_{F_\tau}^{-1}\overline{F_\tau}\left(\frac{x}{\mathbf{E}\theta}\right). \tag{13}$$

For the upper asymptotic relations in (11)–(13)*, the assumption that* $\theta_1, \theta_2, \dots$ *are END can be replaced by weaker assumption that* $\theta_1, \theta_2, \dots$ *are UEND.*

Proof. We prove only upper bounds; the proof of lower bounds follow similarly as in [21]. In particular, it uses the strong law of large numbers for END random variables, see [4].

(i) As in the proof of Theorem 1 of [21], split

$$\begin{aligned} &P(Z_\tau > x) \\ &\quad = \left(\sum_{n=1}^{M} + \sum_{n=M+1}^{[(1-\epsilon)x(\mathbf{E}\theta)^{-1}]} + \sum_{n=[(1-\epsilon)x(\mathbf{E}\theta)^{-1}]+1}^{\infty}\right)\mathbf{P}(Z_n > x)\mathbf{P}(\tau = n) \\ &\quad =: K_1 + K_2 + K_3 \end{aligned} \tag{14}$$

for each triplet $\epsilon \in (0,1)$, $M \in \mathbb{N}$, $x > 0$ such that $[(1-\epsilon)x(\mathbf{E}\theta)^{-1}] \geqslant M + 1$. Clearly, by conditions of the proposition, $F_\tau \in \mathscr{D}$, because

$$\limsup_{x\to\infty}\frac{\overline{F_\tau}(xy)}{\overline{F_\tau}(x)} \leqslant \limsup_{x\to\infty}\frac{\overline{F_\tau}(xy)}{\overline{F_\theta}(xy)}\limsup_{x\to\infty}\frac{\overline{F_\theta}(xy)}{\overline{F_\theta}(x)}\limsup_{x\to\infty}\frac{\overline{F_\theta}(x)}{\overline{F_\tau}(x)} < \infty.$$

Moreover, conditions of the proposition imply the finiteness of $\mathbf{E}\tau$. Indeed, since $\limsup_{x\to\infty}\overline{F_\tau}(x)/\overline{F_\theta}(x) \leqslant c_1$ for some $c_1 > 0$, we obtain that $\mathbf{P}(\tau > x) \leqslant 2c_1 \times \mathbf{P}(\theta > x)$, $x \geqslant x^*$. Hence,

$$\mathbf{E}\tau = \int\limits_{[0,\infty)} \mathbf{P}(\tau > x)\,\mathrm{d}x \leqslant x^* + 2c_1 \int\limits_{[x^*,\infty)} \mathbf{P}(\theta > x)\,\mathrm{d}x \leqslant x^* + 2c_1\mathbf{E}\theta < \infty.$$

Using Lemma 2 below, for each fixed M, it holds

$$K_1 \lesssim \overline{F_\theta}(x)L_{F_\theta}^{-1}\sum_{n=1}^{M} n\mathbf{P}(\tau = n). \tag{15}$$

For the term K_2, write

$$K_2 \leqslant \sum_{n=M+1}^{[(1-\epsilon)x(\mathbf{E}\theta)^{-1}]} \mathbf{P}(Z_n - n\mathbf{E}\theta > \epsilon x)\mathbf{P}(\tau = n),$$

where, by Lemma 3, $\mathbf{P}(Z_n - n\mathbf{E}\theta > \epsilon x) \leqslant c_2 n\overline{F_\theta}(\epsilon x)$ for some $c_2 = c_2(\epsilon, \kappa, \mathbf{E}\theta)$. Hence, similarly to (3.3) in [21], it follows that

$$K_2 \lesssim c_3\overline{F_\theta}(x)\sum_{n=M+1}^{\infty} n\mathbf{P}(\tau = n) \tag{16}$$

with some $c_3 = c_3(\epsilon, \kappa, \mathbf{E}\theta)$. Finally,

$$K_3 \leqslant \overline{F_\tau}\big((1-\epsilon)x(\mathbf{E}\theta)^{-1}\big). \tag{17}$$

Relations (15)–(17) and (14) imply that, for all $\epsilon \in (0,1)$, $M \in \mathbb{N}$, and sufficiently large x,

$$\begin{aligned}
&\frac{\mathbf{P}(Z_\tau > x)}{L_{F_\theta}^{-1}\mathbf{E}\tau\overline{F_\theta}(x) + L_{F_\tau}^{-1}\overline{F_\tau}(x(\mathbf{E}\theta)^{-1})} \\
&\quad \leqslant \frac{K_2}{L_{F_\theta}^{-1}\mathbf{E}\tau\overline{F_\theta}(x)} + \max\biggl\{\frac{K_1}{L_{F_\theta}^{-1}\mathbf{E}\tau\overline{F_\theta}(x)}, \frac{K_3}{L_{F_\tau}^{-1}\overline{F_\tau}(x(\mathbf{E}\theta)^{-1})}\biggr\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\limsup_{x\to\infty}\frac{\mathbf{P}(Z_\tau > x)}{L_{F_\theta}^{-1}\mathbf{E}\tau\overline{F_\theta}(x) + L_{F_\tau}^{-1}\overline{F_\tau}(x(\mathbf{E}\theta)^{-1})} \\
&\quad \leqslant c_3 L_{F_\theta}\frac{\sum_{n=M+1}^{\infty} n\mathbf{P}(\tau = n)}{\mathbf{E}\tau} \\
&\qquad + \max\biggl\{\frac{\sum_{n=1}^{M} n\mathbf{P}(\tau = n)}{\mathbf{E}\tau}, L_{F_\tau}\limsup_{x\to\infty}\frac{\overline{F_\tau}((1-\epsilon)x)}{\overline{F_\tau}(x)}\biggr\}.
\end{aligned}$$

Letting $M \to \infty$ and $\epsilon \searrow 0$, we obtain the statement in case (i).

(ii) The proof of this part is identical to the proof of the same part in Theorem 1 of [21].

(iii) The proof is analogous to the proof of part (iii) in Theorem 1 of [21], noting that Lemma 1 therein is still valid if the UND structure is replaced, respectively, by the weaker, UEND, structure. □

Consider now the case when the summands $\theta_1, \theta_2, \ldots$ are i.i.d. strongly subexponential r.v.s.

Proposition 2. *Let $\theta, \theta_1, \theta_2, \ldots$ be a sequence of nonnegative independent r.v.s with common d.f. $F_\theta \in \mathscr{S}^*$ and finite positive mean $\mathbf{E}\theta$. Let τ be a nondegenerate nonnegative integer-valued r.v., independent of $\theta, \theta_1, \theta_2, \ldots$. If there exists $c > \mathbf{E}\theta$ such that $\overline{F_\tau}(x) = o(\overline{F_\theta}(cx))$, then $\mathbf{E}\tau < \infty$ and*

$$\mathbf{P}(Z_\tau > x) \sim \mathbf{E}\tau\overline{F_\theta}(x). \tag{18}$$

Proof. The proof follows from Theorem 1(ii) in [8]. □

Remark. Note that in more restrictive cases, the assumption of Proposition 2 can be simplified. For example, if the same main conditions of the proposition hold, $F_\theta \in \mathscr{L}\cap\mathscr{D}$ and $\overline{F_\tau}(x) = o(\overline{F_\theta}(x))$, then relation (18) holds (see [15, Thm. 2.3], [8, Thm. 8]). Thus, we strongly believe that, if $F_\theta \in \mathscr{L}\cap\mathscr{D}$, then the constant L_{F_θ} in Proposition 1(i), (ii) can be replaced by 1.

Remark. It is easy to see that, under the conditions of Proposition 2, the closure of the class $\mathscr{S}^*$ holds, i.e. $F_{Z_\tau} \in \mathscr{S}^*$ (see [13]).

5 Auxiliary lemmas

The first lemma is a well-known property of class $\mathscr{D}$ (see [19, Lemma 3.5]).

Lemma 1. *For a d.f. $F \in \mathscr{D}$ with its upper Matuszewska index J_F^+, it holds that*

$$x^{-p} = o\big(\overline{F}(x)\big) \quad \text{for any } p > J_F^+.$$

Next two lemmas are used in proving Proposition 1.

Lemma 2. *Let $\theta_1, \theta_2, \ldots$ be pairwise UEND r.v.s with common d.f. $F_\theta \in \mathscr{D}$. Then, for any fixed $n \geqslant 1$,*

$$\mathbf{P}(Z_n > x) \lesssim L_{F_\theta}^{-1} n \overline{F_\theta}(x). \tag{19}$$

Proof. It is obvious that inequality (19) holds for $n = 1$. If $n \geqslant 2$, then for any fixed $\epsilon \in (0,1)$, by inequality (2.3) in [21]) and the definition of pairwise UEND,

$$\begin{aligned}\mathbf{P}(Z_n > x) &\leqslant \sum_{1 \leqslant i < j \leqslant n} \mathbf{P}\left(\theta_i > \frac{\epsilon x}{n}, \theta_j > \frac{\epsilon x}{n}\right) + \sum_{j=1}^{n} \mathbf{P}\big(\theta_j > (1-\epsilon)x\big) \\ &\leqslant \kappa n^2 \left(\overline{F_\theta}\left(\frac{\epsilon x}{n}\right)\right)^2 + n\overline{F_\theta}\big((1-\epsilon)x\big).\end{aligned}$$

Here, since $F_\theta \in \mathscr{D}$, for any $n \geqslant 2$ and $\epsilon \in (0,1)$, it holds that $(\overline{F_\theta}(\epsilon x/n))^2 = o(\overline{F_\theta}(x))$. Hence,

$$\limsup_{x\to\infty} \frac{\mathbf{P}(Z_n > x)}{\overline{F_\theta}(x)} \leqslant n \lim_{\epsilon \searrow 0} \limsup_{x\to\infty} \frac{\overline{F_\theta}((1-\epsilon)x)}{\overline{F_\theta}(x)} = nL_{F_\theta}^{-1}. \qquad \square$$

The next lemma is a generalization of Corollary 3.1 in [18], where the structure UND has been used. The proof is almost identical and, thus, is omitted.

Lemma 3. *If $\theta_1, \theta_2, \ldots$ are UEND r.v.s with common d.f. $F_\theta \in \mathscr{D}$ and mean $\mathbf{E}\theta = 0$, then, for each $\gamma > 0$, there exists a constant $c_4 = c_4(\kappa, \gamma)$, irrespective to x and n, such that*

$$\mathbf{P}(Z_n > x) \leqslant c_4 n \overline{F_\theta}(x)$$

for all $x \geqslant \gamma n$ and $n \geqslant 1$.

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